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Research article

Blow-up of energy solutions for the semilinear generalized Tricomi equation with nonlinear memory term

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Abstract: In this paper, we investigate blow-up conditions for the semilinear generalized Tricomi equation with a general nonlinear memory term in $\mathbb{R}^n$ by using suitable functionals and employing iteration procedures. Particularly, a new combined effect from the relaxation function and the time-dependent coefficient is found.

Keywords: wave equation; semilinear hyperbolic equation; generalized Tricomi operator; blow-up; nonlinear memory term
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1. Introduction

In the present paper, we will investigate blow-up conditions for the following semilinear generalized Tricomi equation with a time-dependent speed of propagation:

$$\begin{cases} u_{tt} - t^{2\ell}\Delta u = g * |u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

with $p > 1$, where ℓ is a positive parameter and $g = g(t)$ denotes the relaxation function in the memory term (or the so-called memory kernel). Moreover, the right-hand side of the equation in the initial value problem (1) is given by

$$(g * |u|^p)(t, x) := \int_0^t g(t - \tau)|u(\tau, x)|^p d\tau,$$

in which the relaxation function satisfies some assumptions that will be specified later. Roughly speaking, our main purpose in this paper is to understand the interplay effect of the parameter ℓ and the relaxation function $g(t)$ to the blow-up condition (i.e. energy solutions for the Cauchy problem (1) blows up in finite time) for the power exponent p.

In order to introduce the background related to our model (1), let us recall some results for the limit case $\ell = 0$ in the Cauchy problem (1), i.e.

$$\begin{cases} u_{tt} - \Delta u = g * |u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{2}$$

Concerning the special case (the Riemann-Liouville fractional integral of $1 - \gamma$ order)

$$g(t) = \frac{1}{\Gamma(1-\gamma)} t^{-\gamma} \ \text{ with } \ \gamma \in (0, 1), \tag{3}$$

where Γ stands for the Euler integral of the second kind. The completed blow-up result has been firstly derived by Chen-Palmieri [6]. They proved blow-up of energy solutions to the semilinear wave equation with nonlinear memory carrying (3) if

$$1 < p \le p_0(n, \gamma) := \frac{n + 3 - 2\gamma + \sqrt{n^2 + (14 - 4\gamma)n + (3 - 2\gamma)^2 - 8}}{2(n-1)} \ \text{ for } \ n \ge 2,$$

and $1 < p < \infty$ for $n = 1$. Furthermore, $p_0(n, \gamma)$ is the greatest root of the equation

$$(n-1)p_0^2(n, \gamma) - (n + 3 - 2\gamma)p_0(n, \gamma) - 2 = 0. \tag{4}$$

Here, we should mention that the exponent $p_0(n, \gamma)$ is a natural generalization of the so-called Strauss exponent, since when $\gamma \uparrow 1$ the exponent $p_0(n, \gamma)$ tends to the Strauss exponent. Here, the Strauss exponent $p_{\text{Str}}(n)$ was proposed by Strauss [23] as critical exponent for the semilinear wave equation with power nonlinearity and it is the greatest root of

$$(n-1)p_{\text{Str}}^2(n) - (n+1)p_{\text{Str}}(n) - 2 = 0$$

In the same year, Chen-Palmieri [5] investigated the blow-up condition of local in time solution when $1 < p < p_{\text{Str}}(n)$ for the semilinear Moore-Gibson-Thompson (MGT) equation in the conservative case, which is somehow a particular case (the relaxation function $g(t)$ is an exponential decay function) of the Cauchy problem (2). More detailed explanations on the relation between the semilinear MGT equation and semilnear hyperbolic-like (or wave) model also shown in Chen-Ikehata [4]. Therefore, one may conjecture some influence of the relaxation function on the blow-up condition. Quite recently, Chen [3] answered this conjecture and developed a generalization of the result in [5] for a wider class of relaxation functions of the wave equation and it corresponding weakly coupled system with general nonlinear memory terms, where some new blow-up results combined with a new threshold in the subcritical case were derived. Although the global (in time) existence condition for the semilinear wave equation with nonlinear memory term is a completely open problem, there are some ideas and hints that would suggest the likelihood of the critical condition. All these results are based on the constant propagation speed. Clearly, when the model owns variable propagation speed, the shape of the light-cone depends on the speed of propagation. Concerning other studies for nonlinear memory in evolution equations, we refer readers to [2, 9–11].

Let us recall the results for the semilinear generalized Tricomi equation

$$\begin{cases} u_{tt} - t^{2\ell}\Delta u = |u|^p, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n. \end{cases} \tag{5}$$

In recent year, this problem caught a lot of attention. According to a series of work on global existence and blow-up from He-Witt-Yin [12–14] and Lin-Tu [18], one may conjecture the reasonable critical exponent for the Cauchy problem (5) is the greatest real root $p_1(n, \ell)$ of the quadratic equation

$$((\ell + 1)n - 1)p_1^2(n, \ell) - ((\ell + 1)n + 1 - 2\ell)p_1(n, \ell) - 2(\ell + 1) = 0. \tag{6}$$

Obviously, when $\ell = 0$ in the last equation, the greatest real root is the Strauss exponent. Therefore, it is a natural generalization of the Strauss exponent. In some sense, the power nonlinearity can be treated thanks to Hölder's inequality. However, when the nonlinear part has the memory effect, we need to find a suitable way to deal with relaxation function under suitable assumption. Other works for the semilinear Tricomi equation with derivative-type or mixed-type nonlinearity are refereed to Chen-Lucente-Palmieri [7], Lucente-Palmieri [19] and references therein.

From the above introduction, we found that these two generalized Strauss exponent are different, and they have been influenced by the property of the relaxation function or the time-dependent coefficient. Nevertheless, to the best of author's knowledge, the combined influence of the relaxation function and the time-dependent coefficient on the wave equation (especially, generalized Tricomi equation) is still unknown. In this paper, our purpose is to investigate the effects on the blow-up range for p due to the combined presence of the speed of propagation t^ℓ and of the relaxation function g. Moreover, we will consider a general assumption on the relaxation function whose form is not only including the Riemann-Liouville type but also some oscillations. For these reasons, we will face the treatment of a general memory term. Actually, we cannot use the classical Kato's lemma (see Kato [16]). To overcome this difficulty, we found that the iteration procedure (see, for example, Agemi-Kurokawa-Takamura [1], Lai-Takamura [17], Palmieri-Takamura [20–22], Chen-Reissig [8] and reference therein) will provide us a strong tool to deal with it.

After stating our main result, we will give some examples. For one thing, we will investigate the relation between our model and the memoryless case or the constant coefficient case. For another, we will show the wide applications of our result.

Throughout this paper, we denote $B_R(0)$ as the ball around the origin with radius R in the whole space $\mathbb{R}^n$; $f \lesssim g$ as $f \leq Cg$ with a positive constant C; $f \gtrsim g$ as $f \geq Cg$ with a positive constant C; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2. Main result

To begin with, following the idea of Chen-Palmieri [6], let us introduce a suitable definition of an energy solution for our main problem.

Definition 2.1. *Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. We say $u = u(t, x)$ is an energy solution to the Cauchy problem* (1) *if*

$$u \in C([0, T), H^1(\mathbb{R}^n)) \cap C^1([0, T), L^2(\mathbb{R}^n)) \ \text{ such that } \ g * |u|^p \in L^1_{\mathrm{loc}}((0, T) \times \mathbb{R}^n)$$

satisfies $u(0,\cdot) = u_0 \in H^1(\mathbb{R}^n)$ *and the next identity is fulfilled:*

$$\begin{aligned}&\int_{\mathbb{R}^n} u_t(t,x)\psi(t,x)dx - \int_0^t\int_{\mathbb{R}^n} u_t(\tau,x)\psi_t(\tau,x)dxd\tau + \int_0^t\int_{\mathbb{R}^n} \tau^{2\ell}\nabla u(\tau,x)\cdot\nabla\psi(\tau,x)dxd\tau\\ &= \int_{\mathbb{R}^n} u_1(x)\psi(0,x)dx + \int_0^t\int_{\mathbb{R}^n}(g*|u|^p)(\tau,x)\psi(\tau,x)dxd\tau\end{aligned} \tag{7}$$

for any test function $\psi \in C_0^\infty([0,T)\times\mathbb{R}^n)$ *and any* $t \in (0,T)$.

Applying integration by parts in (7), we get

$$\begin{aligned}&\int_{\mathbb{R}^n}(u_t(t,x)\psi(t,x) - u(t,x)\psi_t(t,x))dx + \int_0^t\int_{\mathbb{R}^n} u(\tau,x)(\psi_{tt}(\tau,x) - \tau^{2\ell}\Delta\psi(\tau,x))dxd\tau\\ &= \int_{\mathbb{R}^n}(u_1(x)\psi(0,x) - u_0(x)\psi_t(0,x))dx + \int_0^t\int_{\mathbb{R}^n}(g*|u|^p)(\tau,x)\psi(\tau,x)dxd\tau.\end{aligned}$$

Thanks to Definition 2.1, we can show the main result of the present paper. Before doing this, we introduce the exponent $p_2(n,\ell,\theta)$ given by the greatest root of the quadratic equation

$$((\ell+1)n-1)p_2^2(n,\ell,\theta) - ((\ell+1)n+1-2\ell+2(1-\theta))p_2(n,\ell,\theta) - 2(\ell+1) = 0. \tag{8}$$

We feel the combined effect of the speed of propagation and of the relaxation function in (8). Precisely, we denote

$$p_2(n,\ell,\theta) = \frac{((\ell+1)n+1-2\ell+2(1-\theta)) + \sqrt{((\ell+1)n+1-2\ell+2(1-\theta))^2 + 8(\ell+1)((\ell+1)n-1))}}{2((\ell+1)n-1)}.$$

The influence from the relaxation function (represented by θ) and the time-dependent coefficient (represented by ℓ) plays different crucial roles on the determination of the exponent $p_2(n,\ell,\theta)$. Indeed, this effect cannot be easily observed from the beginning of the model.

Theorem 2.1. *Let us consider the positive relaxation function* $g = g(t)$ *such that*

$$g(t-\tau) \geq h(t) > 0 \ \textit{ for any } \ \tau \in [0,t], \tag{9}$$

and the monotonously decreasing or constant function $h = h(t)$ *has the lower bound*

$$h(t) \geq Ct^{-\theta} \ \textit{ for any } \ t > 0, \tag{10}$$

where $\theta \geq 0$. *Let us suppose that* $u_0 \in H^1(\mathbb{R}^n)$ *and* $u_1 \in L^2(\mathbb{R}^n)$ *are nonnegative and compactly supported functions with supports contained in a ball* $B_R(0)$ *with some* $R > 0$ *such that* u_0, u_1 *are not identically zero. Let* $u = u(t,x)$ *be the local (in time) energy solution to the Cauchy problem* (1) *according to Definition 2.1. Providing that*

$$1 < p < p_2(n,\ell,\theta),$$

then a solution to (1) *according to Definition 2.1 that satisfies*

$$\textit{supp}\, u \subset \{(t,x) \in [0,T)\times\mathbb{R}^n : \ |x| \leq R + \phi(t)\}$$

blows up in finite time, i.e. $T < \infty$. *Here,* $\phi(t) := t^{\ell+1}/(\ell+1)$ *denotes the speed of propagation.*

Remark 2.1. *The condition* (9) *may be relaxed by some approaches developed in [3], where the trick slicing method need to be introduced. For our model, we need to investigate a new sequence in slicing procedure. In the forthcoming future, this relaxation of condition will be studied.*

Remark 2.2. *Let us take the first example when* $g(t)$ *is chosen so that* $g * |u|^p$ *is the Riemann-Liouville fractional integral of* $1-\gamma$ *order, i.e.* (3)*. Clearly,*

$$g(t-\tau) = \frac{1}{\Gamma(1-\gamma)}(t-\tau)^{-\gamma} \geq \frac{1}{\Gamma(1-\gamma)}t^{-\gamma} = h(t) \text{ for any } \tau \in [0,t],$$

where $h(t)$ *is a monotonously decreasing function with* $\gamma \in (0,1)$ *and it satisfies*

$$h(t) \geq Ct^{-\gamma} > 0 \text{ for any } t > 0.$$

So, the nontrivial energy solution to (1) *blows up in finite time provided* $1 < p < p_2(n,\ell,\gamma)$*. Let us consider the limit case* $\ell \downarrow 0$*. We observe from* (4) *and* (8) *that*

$$\lim_{\ell \to 0} p_2(n,\ell,\gamma) = p_0(n,\gamma)$$

for any $n \geq 1$ *and* $\gamma \in (0,1)$*. We recall*

$$\lim_{\gamma \uparrow 1} \frac{s_+^{-\gamma}}{\Gamma(1-\gamma)} = \delta_0(s) \text{ in the sense of distributions with } s_+^{-\gamma} := \begin{cases} s^{-\gamma} & \text{if } s > 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Thus, the blow-up result seems reasonable due to the fact that

$$\lim_{\gamma \uparrow 1} p_2(n,\ell,\gamma) = p_1(n,\ell)$$

for any $n \geq 1$ *and* $\ell > 0$*. One may see* (6) *and* (8)*. In this sense, we believe our result is reasonable.*

Remark 2.3. *Let us consider another example, in which we will show that the blow-up result not only holds for the polynomial decay relaxation function. We set* $g(t) = (2+\sin\omega(t))t^{-\gamma}$ *with* $\gamma \in (0,1)$ *and a continuous function* $\omega(t)$*. Then, we may find*

$$g(t-\tau) = (2+\sin\omega(t-\tau))\,(t-\tau)^{-\gamma} \geq Ct^{-\gamma} \text{ for any } \tau \in [0,t].$$

Thus, we still can prove blow-up of energy solutions if $1 < p < p_2(n,\ell,\gamma)$ *for* $\ell > 0$ *and* $\gamma \in (0,1)$*.*

Remark 2.4. *To determine the critical exponent for the Cauchy problem* (1)*, even in the special case of the Riemann-Liouville fractional integral type* (3)*, we still need to investigate the situation when* $p \geq p_2(n,\ell,\gamma)$*. Honestly, we conjecture that for the supercritical case* $p > p_2(n,\ell,\gamma)$*, under some suitable assumptions on small data, the solution exists globally in time. Concerning the limit case* $p = p_2(n,\ell,\gamma)$*, we expect every nontrivial solution blows up in finite time. Nevertheless, these questions are still open problems.*

Remark 2.5. *Concerning lifespan estimates, we consider initial data* $(u,u_t)(0,x) = \varepsilon(u_0,u_1)(x)$ *with the positive parameter* $\varepsilon > 0$*. Therefore, by following the same approach, the estimate* (17) *will be changed into*

$$F_0(t) \geq \exp\Big(p^j\Big(\log(M_0\varepsilon^p) + \Big(\alpha_0 + \frac{1}{p-1}\Big)\log h(t) + \Big(\beta_0 + \frac{3}{p-1}\Big)\log(t-t_0)$$

$$- (\gamma_0 + n(\ell+1))\log(R+t)\Big)\Big)(h(t))^{-\frac{1}{p-1}}(t-t_0)^{-\frac{3}{p-1}}(R+t)^{n(\ell+1)}.$$

By denoting the power of t in the exponential function by

$$\begin{aligned}\Omega(p,n,\ell,\theta) : &= -\alpha_0\theta - \frac{\theta}{p-1} + \beta_0 + \frac{3}{p-1} - \gamma_0 - n(\ell+1)\\ &= -\frac{1}{2(p-1)}\Big(((n-1)(\ell+1)+\ell)p^2 - ((n-1)(\ell+1)+\ell-2(\ell+\theta-2))p - 2\ell - 2\Big),\end{aligned}$$

we arrive at upper bound estimate for lifespan

$$T_\varepsilon \lesssim \varepsilon^{-\frac{2p(p-1)}{\Omega(p,n,\ell,\theta)}}$$

to the model (1) *in* $(0, T_\varepsilon)$. *The derivation of lifespan in this case is standard, and one may see, for example, the end of Section 2 of [6].*

3. Proof of the main theorem

3.1. Iteration frame and first lower bound estimate

Recalling some results from the paper He-Witt-Yin [12], we may define a test function $\Psi = \Psi(t,x)$ such that

$$\Psi(t,x) = \lambda(t)\Phi(x) > 0, \tag{11}$$

where the time-dependent test function $\lambda = \lambda(t)$ is given by

$$\lambda(t) := C_\ell t^{1/2} K_{1/(2\ell+2)}\left(\frac{1}{\ell+1}t^{\ell+1}\right)$$

with the positive constant C_ℓ such that $\lambda(0) = 1$, moreover, $K_\alpha(t)$ stands for the modified Bessel function of the second kind with the order α. The space-dependent test function was introduced by Yordanov-Zhang [24], namely,

$$\Phi(x) := \begin{cases} e^x + e^{-x} & \text{if } n = 1,\\ \displaystyle\int_{S^{n-1}} e^{x\cdot\omega} d\sigma_\omega & \text{if } n \geq 2,\end{cases}$$

where S^{n-1} denotes the $n-1$ dimensional sphere. Therefore, it holds that

$$\Delta\Phi(x) = \sum_{k=1}^{n} \partial_{x_k}^2 \Phi(x) = \Phi(x).$$

Due to the fact that $K_\alpha(\tau)$ with positive variable τ is the solution to the next differential equation of second-order:

$$\tau^2 K_\alpha''(\tau) + \tau K_\alpha'(\tau) - (\alpha^2+\tau^2)K_\alpha(\tau) = 0,$$

we may find from (11) that

$$\begin{aligned}\Psi_{tt} - t^{2\ell}\Delta\Psi &= \lambda''(t)\Phi(x) - t^{2\ell}\lambda(t)\Delta\Phi(x)\\ &= (\lambda''(t) - t^{2\ell}\lambda(t))\Phi(x) = 0,\end{aligned}$$

carrying its value $\Psi(0, x) = \Phi(x)$ and $\Psi(\infty, x) = 0$. In other words, the test function in (11) solves the adjoint equation of the homogeneous generalized Tricomi equation.

We now introduce two suitable functionals as follows:

$$\begin{aligned}F_0(t) &:= \int_{\mathbb{R}^n} u(t, x)dx,\\ F_1(t) &:= \int_{\mathbb{R}^n} u(t, x)\Psi(t, x)dx,\end{aligned}$$

where the test function $\Psi(t, x)$ was defined in (11). We should emphasize that the functional $F_0(t)$ plays a crucial role in the proof, and if $F_0(t)$ blows up then $u(t, x)$ will blow up in finite time. Our aim will be to show that $F_0(t)$ blows up in finite time. The auxiliary functional $F_1(t)$ will provide a first lower bound for $F_0(t)$.

Due to the support assumption in the statement of Theorem 2.1, we have

$$\operatorname{supp} u(t, \cdot) \subset B_{R+\phi(t)}(0) \ \text{ for any } \ t \in (0, T).$$

With the help of the previous statement, one may choose the test function in (7) by $\psi(\tau, x) \equiv 1$ for $|x| \le R + \phi(\tau)$, namely,

$$F_0'(t) = \int_{\mathbb{R}^n} u_1(x)dx + \int_0^t \int_{\mathbb{R}^n} (g * |u|^p)(\tau, x)dxd\tau.$$

Integrating the last one with respect to the time variable, it yields

$$F_0(t) = \int_{\mathbb{R}^n} (u_0(x) + u_1(x)t)dx + \int_0^t \int_0^\tau \int_{\mathbb{R}^n} \int_0^s g(s - \eta)|u(\eta, x)|^p d\eta dx ds d\tau. \tag{12}$$

The inequality (12) implies two facts. For one thing, the nonnegativity of $F_0(t)$ for any $t \ge 0$ can be asserted because of the assumption on initial data. For another, by using Hölder's inequality and the assumption on relaxation function (9), we get

$$\begin{aligned}F_0(t) &\ge \int_0^t \int_0^\tau \int_{\mathbb{R}^n} \int_0^s g(s - \eta)|u(\eta, x)|^p d\eta dx ds d\tau\\ &\ge \int_0^t \int_0^\tau h(s) \int_0^s \int_{\mathbb{R}^n} |u(\eta, x)|^p dx d\eta ds d\tau\\ &\ge \int_0^t \int_0^\tau h(s) \int_0^s |F_1(\eta)|^p \left(\int_{B_{R+\phi(\eta)}(0)} \Psi^{\frac{p}{p-1}}(\eta, x)dx\right)^{-(p-1)} d\eta ds d\tau.\end{aligned}$$

Employing the next estimate from He-Witt-Yin [12]:

$$\left(\int_{B_{R+\phi(\eta)}(0)} \Psi^{\frac{p}{p-1}}(\eta, x)dx\right)^{p-1} \lesssim \eta^{-\frac{\ell p}{2}}(R + \phi(\eta))^{(n-1)(p-1)-\frac{n-1}{2}p},$$

we see

$$F_0(t) \gtrsim h(t)(R+\phi(t))^{-(n-1)(p-1)} \int_0^t \int_0^\tau \int_0^s |F_1(\eta)|^p \eta^{\frac{\ell p}{2}} (R+\phi(\eta))^{\frac{n-1}{2}p} d\eta ds d\tau. \tag{13}$$

At the same time, the application of Hölder's inequality associated with nonnegativity of $F_0(t)$ shows

$$\begin{aligned} F_0(t) &\geq \int_0^t \int_0^\tau \int_0^s g(s-\eta) \int_{\mathbb{R}^n} |u(\eta,x)|^p dx d\eta ds d\tau \\ &\geq C_0 h(t)(R+t)^{-n(p-1)(\ell+1)} \int_0^t \int_0^\tau \int_0^s (F_0(\eta))^p d\eta ds d\tau, \end{aligned} \tag{14}$$

with a positive constant C_0, which provides the iteration frame for the functional F_0.

With the aim of deriving a first lower bound of $F_0(t)$, it is necessary for us to turn to the lower bound of $F_1(t)$. Thanks to nonnegativity of the nonlinear term, we may directly follow the same approach of Lemma 2.3 in He-Witt-Yin [12]. Then, there exists a constant $t_0 > 0$ such that

$$F_1(t) \gtrsim t^{-\ell} \ \text{ for } \ t \geq t_0. \tag{15}$$

Here, we have used the nontrivial assumption on u_0 and u_1 in deriving the last estimate. Thus, combining (13) with (15), we have

$$\begin{aligned} F_0(t) &\gtrsim h(t)(R+t)^{-\frac{\ell p}{2}-(n-1)(p-1)(\ell+1)} \int_{t_0}^t \int_{t_0}^\tau \int_{t_0}^s (\eta - t_0)^{\frac{n-1}{2}p(\ell+1)} d\eta ds d\tau \\ &\geq C_1 h(t)(t-t_0)^{\frac{n-1}{2}p(\ell+1)+3}(R+t)^{-\frac{\ell p}{2}-(n-1)(p-1)(\ell+1)} \end{aligned}$$

with a positive constant $C_1 > 0$, for any $t \geq t_0$. In other words, we have deduced the first lower bound estimate

$$F_0(t) \geq Q_0 (h(t))^{\alpha_0} (t-t_0)^{\beta_0} (R+t)^{-\gamma_0} \ \text{ for any } \ t \geq t_0,$$

where

$$Q_0 = C_1, \ \ \alpha_0 = 1, \ \ \beta_0 = \frac{n-1}{2}p(\ell+1)+3, \ \ \gamma_0 = \frac{\ell p}{2} + (n-1)(p-1)(\ell+1).$$

3.2. Iteration procedure

In the previous section, a first lower bound for the functional $F_0(t)$ has been established, and therefore, we will investigate a sequence of lower bounds for it by making use of the frame of iteration (14). To do that, we establish the sequence of lower bounds

$$F_0(t) \geq Q_j (h(t))^{\alpha_j} (t-t_0)^{\beta_j} (R+t)^{-\gamma_j} \ \text{ for any } \ t \geq t_0, \tag{16}$$

where $\{Q_j\}_{j\in\mathbb{N}_0}$, $\{\alpha_j\}_{j\in\mathbb{N}_0}$, $\{\beta_j\}_{j\in\mathbb{N}_0}$ and $\{\gamma_j\}_{j\in\mathbb{N}_0}$ are sequences of nonnegative real numbers which will be determined in the forthcoming part.

Let us assume that (16) holds for j. We need to prove that the lower bound estimate also holds for $j+1$. Let us plug (16) into the nonlinear integral inequality (14). Then, it holds

$$F_0(t) \geq C_0 Q_j^p h(t)(R+t)^{-n(p-1)(\ell+1)} \int_{t_0}^t \int_{t_0}^\tau \int_{t_0}^s (h(\eta))^{\alpha_j p} (\eta - t_0)^{\beta_j p} (R+\eta)^{-\gamma_j p} d\eta ds d\tau$$

$$\geq \frac{C_0 Q_j^p}{(\beta_j p+1)(\beta_j p+2)(\beta_j p+3)}(h(t))^{\alpha_j p+1}(t-t_0)^{\beta_j p+3}(R+t)^{-\gamma_j p-n(p-1)(\ell+1)}$$

for any $t \geq t_0$. So, the lower bound estimate (16) holds for $j+1$ providing that

$$\begin{aligned}
Q_{j+1} &= \frac{C_0 Q_j^p}{(\beta_j p+1)(\beta_j p+2)(\beta_j p+3)},\\
\alpha_{j+1} &= \alpha_j p+1,\\
\beta_{j+1} &= \beta_j p+3,\\
\gamma_{j+1} &= \gamma_j p+n(p-1)(\ell+1).
\end{aligned}$$

We now determine the sequence for $\alpha_j,\beta_j,\gamma_j$ firstly. We know that if

$$\delta_{j+1} = \delta_j p+m$$

with a constant m, then the sequence can be deduced by

$$\begin{aligned}
\delta_j &= \delta_{j-1}p+m = \delta_{j-2}p^2+m(1+p)\\
&= \cdots = \delta_0 p^j+m(1+p+\cdots+p^{j-1})\\
&= \left(\delta_0+\frac{m}{p-1}\right)p^j-\frac{m}{p-1}.
\end{aligned}$$

Hence, the elements in all sequences can be determined

$$\begin{aligned}
\alpha_j &= \left(\alpha_0+\frac{1}{p-1}\right)p^j-\frac{1}{p-1},\\
\beta_j &= \left(\beta_0+\frac{3}{p-1}\right)p^j-\frac{3}{p-1},\\
\gamma_j &= \left(\gamma_0+n(\ell+1)\right)p^j-n(\ell+1).
\end{aligned}$$

One derives

$$(\beta_j p+1)(\beta_j p+2)(\beta_j p+3) \leq (\beta_j p+3)^3 = \beta_{j+1}^3 \leq \left(\beta_0+\frac{3}{p-1}\right)^3 p^{3j},$$

which implies

$$Q_j \geq C_0\left(\beta_0+\frac{3}{p-1}\right)^{-3} p^{-3j}Q_{j-1}^p =: Mp^{-3j}Q_{j-1}^p.$$

In order to estimate Q_j from below, we apply the logarithmic function to both sides of the last inequality, obtaining

$$\begin{aligned}
\log Q_j &\geq \log M-3j\log p+p\log Q_{j-1}\\
&\geq (1+p)\log M-3(j+(j-1)p)\log p+p^2\log Q_{j-2}
\end{aligned}$$

$$\geq \cdots \geq (1 + p + \cdots + p^{j-1}) \log M - 3(j + (j-1)p + \cdots + p^{j-1}) \log p + p^j \log Q_0$$
$$= p^j \left(\frac{\log M}{p-1} - \frac{3p \log p}{(p-1)^2} + \log Q_0 \right) - \frac{\log M}{p-1} + \frac{3p \log p}{(p-1)^2} + \frac{3j \log p}{p-1}.$$

In the last chain, we used

$$\sum_{k=0}^{j-1} (j-k) p^k = \frac{1}{p-1} \left(\frac{p^{j+1} - p}{p-1} - j \right).$$

Let us choose $j \geq j_0$ such that

$$-\frac{\log M}{p-1} + \frac{3p \log p}{(p-1)^2} + \frac{3j \log p}{p-1} \geq 0.$$

Here, the positive integer j_0 can be fixed by

$$j_0 = \min \left\{ j_1 \in \mathbb{N} : \ j_1 \geq \frac{\log M}{3 \log p} - \frac{p}{p-1} \right\}.$$

Fixing

$$\log M_0 := \frac{\log M}{p-1} - \frac{3p \log p}{(p-1)^2} + \log Q_0,$$

in conclusion for $j \geq j_0$, the lower bound for $F_0(t)$ can be expressed by

$$\begin{aligned} F_0(t) &\geq \exp(p^j \log M_0) (h(t))^{(\alpha_0 + \frac{1}{p-1}) p^j} (t - t_0)^{(\beta_0 + \frac{3}{p-1}) p^j} (R + t)^{-(\gamma_0 + n(\ell+1)) p^j} \\ &\quad \times (h(t))^{-\frac{1}{p-1}} (t - t_0)^{-\frac{3}{p-1}} (R + t)^{n(\ell+1)} \\ &= \exp \Big(p^j \Big(\log M_0 + \Big(\alpha_0 + \frac{1}{p-1} \Big) \log h(t) + \Big(\beta_0 + \frac{3}{p-1} \Big) \log(t - t_0) \\ &\qquad - (\gamma_0 + n(\ell+1)) \log(R + t) \Big) \Big) (h(t))^{-\frac{1}{p-1}} (t - t_0)^{-\frac{3}{p-1}} (R + t)^{n(\ell+1)}. \end{aligned} \tag{17}$$

According to our assumption on the relaxation function, it is valid that

$$\log h(t) \geq -\theta \log t + \log C.$$

Concerning $j \geq j_0$ and $t \geq \max\{2t_0, R\}$, we arrive at

$$\begin{aligned} F_0(t) &\geq \exp \left(p^j \left(\log M_0 + M_1 + \left(-\alpha_0 \theta - \frac{\theta}{p-1} + \beta_0 + \frac{3}{p-1} - \gamma_0 - n(\ell+1) \right) \log t \right) \right) \\ &\quad \times (h(t))^{-\frac{1}{p-1}} (t - t_0)^{-\frac{3}{p-1}} (R + t)^{n(\ell+1)}, \end{aligned}$$

where

$$M_1 = \left(\alpha_0 + \frac{1}{p-1} \right) \log C - \left(\beta_0 + \frac{3}{p-1} + \gamma_0 + n(\ell+1) \right) \log 2.$$

The coefficient of $\log t$ is

$$\begin{aligned}
&-\alpha_0\theta - \frac{\theta}{p-1} + \beta_0 + \frac{3}{p-1} - \gamma_0 - n(\ell+1)\\
&= -\frac{p}{2}((n-1)(\ell+1)+\ell) - \ell + 2 - \theta + \frac{3-\theta}{p-1}\\
&= -\frac{1}{2(p-1)}\Big(((n-1)(\ell+1)+\ell)p^2 - ((n-1)(\ell+1)+\ell-2(\ell+\theta-2))p - 2\ell - 2\Big) > 0
\end{aligned}$$

if and only if

$$((n-1)(\ell+1)+\ell)p^2 - ((n-1)(\ell+1)+\ell-2(\ell+\theta-2))p - 2\ell - 2 < 0$$

that is equivalent to our assumption $1 < p < p_2(n, \ell, \theta)$. All in all, for $t \geq \max\{2t_0, R\}$ and letting $j \to \infty$, we can claim that the lower bound for the functional $F_0(t)$ blows up in finite time. The proof is complete.

4. Conclusions

In this paper, we found a new blow-up result for the semilinear generalized Tricomi equation with a general nonlinear memory term. Particularly, the combined effect from the "decay" property of the relaxation function and the propagation speed on the blow-up condition was investigated. Finally, it seems also interesting to derive interplay effect for the corresponding nonlinear system

$$\begin{cases}
u_{tt} - t^{2\ell_1}\Delta u = g_1 * |v|^p, & x \in \mathbb{R}^n,\ t > 0,\\
v_{tt} - t^{2\ell_2}\Delta v = g_2 * |u|^q, & x \in \mathbb{R}^n,\ t > 0,\\
(u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{cases}$$

where $\ell_1, \ell_2 > 0$ and g_1, g_2 satisfy suitable assumptions. The crucial point of interesting is the blow-up condition $\Omega(p, q) = \Omega(p, q; \ell_1, \ell_2, g_1, g_2) > 0$. The weakly coupled system of generalized Tricomi equations with power nonlinearities is studied by [15] recently. In the forthcoming future, we will study this problem.

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References

1. R. Agemi, Y. Kurokawa, H. Takamura, Critical curve for *p*-*q* systems of nonlinear wave equations in three space dimensions, *J. Differential Equations*, **167** (2000), 87–133.

2. T. Cazenave, F. Dickstein, F. B. Weissler, An equation whose Fujita critical exponent is not given by scaling, *Nonlinear Anal.*, **68** (2008), 862–874.
3. W. Chen, Interplay effects on blow-up of weakly coupled systems for semilinear wave equations with general nonlinear memory terms, *Nonlinear Anal.*, **202** (2021), 112160.
4. W. Chen, R. Ikehata, The Cauchy problem for the Moore-Gibson-Thompson equation in the dissipative case, *J. Differential Equations*, **292** (2021), 176–219.
5. W. Chen, A. Palmieri, Nonexistence of global solutions for the semilinear Moore-Gibson-Thompson equation in the conservative case, *Discrete Contin. Dyn. Syst.*, **40** (2020), 5513–5540.
6. W. Chen, A. Palmieri, Blow-up result for a semilinear wave equation with a nonlinear memory term, In: M. Cicognani, D. Del Santo, A. Parmeggiani, M. Reissig, Eds, Anomalies in Partial Differential Equations, Springer INdAM Series, vol 43. Springer, Cham (2021).
7. W. Chen, S. Lucente, A. Palmieri, A blow-up result for a semilinear generalized Tricomi equation with combined nonlinearity, *Nonlinear Anal. Real World Appl.*, **61** (2021), 103354.
8. W. Chen, M. Reissig, Blow-up of solutions to Nakao's problem via an iteration argument, *J. Differential Equations*, **275** (2021), 733–756.
9. M. D'Abbicco, A wave equation with structural damping and nonlinear memory, *Nonlinear Differ. Equ. Appl.*, **21** (2014), 751–773.
10. M. D'Abbicco, The influence of a nonlinear memory on the damped wave equation, *Nonlinear Anal.*, **95** (2014), 130–145.
11. M. D'Abbicco, S. Lucente, The beam equation with nonlinear memory, *Z. Angew. Math. Phys.*, **67** (2016), 18.
12. D. He, I. Witt, H. Yin, On the global solution problem for semilinear generalized Tricomi equations, *Calc. Var.*, **56** (2017), 21.
13. D. He, I. Witt, H. Yin, On semilinear Tricomi equations with critical exponents or in two space dimensions, *J. Differential Equations*, **263** (2017), 8102–8137.
14. D. He, I. Witt, H. Yin, On semilinear Tricomi equations in one space dimension, *Commum. Pure Appl. Anal.*, **19** (2020), 4817–4838.
15. M. I. keda, J. Lin, Z. Tu, Small data blow-up for the weakly coupled system of the generalized Tricomi equations with multiple propagation speeds, *J. Evol. Equ.*, (2021). doi:10.1007/s00028-021-00703-4.
16. T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations, *Commun. Pure Appl. Math.*, **33** (1980), 501–505.
17. N. A. Lai, H. Takamura, Blow-up for semilinear damped wave equations with subcritical exponent in the scattering case, *Nonlinear Anal.*, **168** (2018), 222–237.
18. J. Lin, Z. Tu, Lifespan of semilinear generalized Tricomi equation with Strauss type exponent, *Preprint* (2019). arXiv:1903.11351v2.
19. S. Lucente, A. Palmieri, A blow-up result for a generalized Tricomi equation with nonlinearity of derivative type, *Milan J. Math.*, **89** (2021), 45–57.

20. A. Palmieri, H. Takamura, Blow-up for a weakly coupled system of semilinear damped wave equations in the scattering case with power nonlinearities, *Nonlinear Anal.*, **187** (2019), 467–492.
21. A. Palmieri, H. Takamura, Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations of derivative type in the scattering case, *Mediterr. J. Math.*, **17** (2020), 20.
22. A. Palmieri, H. Takamura, Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations in the scattering case with mixed nonlinear terms, *NoDEA Nonlinear Differential Equations Appl.*, **27** (2020), 39.
23. W. A. Strauss, Nonlinear scattering theory at low energy, *J. Funct. Anal.*, **41** (1981), 110–133.
24. B. T. Yordanov, Q. S. Zhang, Finite time blow up for critical wave equations in high dimensions, *J. Funct. Anal.*, **231** (2006), 361–374.

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