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Research article

Concise high precision approximation for the complete elliptic integral of the first kind

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Abstract: In this paper, we obtain a concise high-precision approximation for $\mathcal{K}(r)$:

$$\frac{2}{\pi}\mathcal{K}(r) > \frac{22\left(r'\right)^2 + 84r' + 22}{7\left(r'\right)^3 + 57\left(r'\right)^2 + 57r' + 7},$$

which holds for all $r \in (0, 1)$, where $\mathcal{K}(r)$ is complete elliptic integral of the first kind and $r' = \sqrt{1 - r^2}$.

Keywords: simple bounds; rational approximation; complete elliptic integral of the first kind
Mathematics Subject Classification: 33E05, 26E60

1. Introduction

For $r \in (0, 1)$, Legendre's complete elliptic integrals of the first kind (see [1, 2]), denoted $\mathcal{K}(r)$, is defined by

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2(t)}},$$

which is the particular case of the Gaussian hypergeometric function and can be a special power series ([3–9]):

$$\begin{aligned} \mathcal{K}(r) &= \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} r^{2n} \\ &= \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 r^{2n} =: \frac{\pi}{2} \sum_{n=0}^{\infty} W_n^2 r^{2n}. \end{aligned} \tag{1.1}$$

Many researchers have obtained the upper and lower bounds of this special function ([10–18]). Let's assume that $r' = \sqrt{1 - r^2}$. In this paper, we use the method similar to Padé approximation to obtain the following rational functions of the argument r'

$$\begin{aligned}
\mathcal{F}_{1,0}(r') &= \frac{3-r'}{2}, \\
\mathcal{F}_{0,1}(r') &= \frac{2}{r'+1}, \\
\mathcal{F}_{1,1}(r') &= \frac{r'+7}{5r'+3}, \\
\mathcal{F}_{2,1}(r') &= \frac{-3\,(r')^2+22r'+61}{8\,(7r'+3)}, \\
\mathcal{F}_{1,2}(r') &= \frac{8\,(r'+1)}{3\,(r')^2+10r'+3}, \\
\mathcal{F}_{2,2}(r') &= \frac{5\,(r')^2+126r'+61}{61\,(r')^2+110r'+21}, \\
\mathcal{F}_{3,2}(r') &= \frac{5579r'+495\,(r')^2-35\,(r')^3+1769}{2\left(2050r'+1567\,(r')^2+287\right)}, \\
\mathcal{F}_{2,3}(r') &= \frac{22\,(r')^2+84r'+22}{7\,(r')^3+57\,(r')^2+57r'+7}
\end{aligned} \tag{1.2}$$

to approximate this special function $\mathcal{K}(r)$, and get the following series of results:

$$\begin{aligned}
\frac{2}{\pi}\mathcal{K}(r)-\mathcal{F}_{1,0}(r') &= O(r^4),\ \frac{2}{\pi}\mathcal{K}(r)-\mathcal{F}_{0,1}(r')=O(r^4), \\
\frac{2}{\pi}\mathcal{K}(r)-\mathcal{F}_{1,1}(r') &= O(r^6), \\
\frac{2}{\pi}\mathcal{K}(r)-\mathcal{F}_{2,1}(r') &= O(r^8),\ \frac{2}{\pi}\mathcal{K}(r)-\mathcal{F}_{1,2}(r')=O(r^8), \\
\frac{2}{\pi}\mathcal{K}(r)-\mathcal{F}_{2,2}(r') &= O(r^{10}), \\
\frac{2}{\pi}\mathcal{K}(r)-\mathcal{F}_{3,2}(r') &= O(r^{12}),\ \frac{2}{\pi}\mathcal{K}(r)-\mathcal{F}_{2,3}(r')=O(r^{12}).
\end{aligned}$$

From the last formula above, it is not difficult to find that the bound $\mathcal{F}_{2,3}(r')$ is quite sharp. The smaller r is, the smaller the error between $2\mathcal{K}(r)/\pi$ and this bound $\mathcal{F}_{2,3}(r')$ is. The following is the error analysis table of $\delta(r)=2\mathcal{K}(r)/\pi-\mathcal{F}_{2,3}(r')$:

r	0.1	0.2	0.3	0.4	0.5	0.99
$\delta(r)$	4.680×10^{-18}	2.118×10^{-14}	3.262×10^{-12}	1.327×10^{-10}	2.747×10^{-9}	1.969×10^{-2}

which shows that $\mathcal{F}_{2,3}(r')$ has a high precision approximation to $2\mathcal{K}(r)/\pi$.

Since

$$\mathcal{F}_{2,3}(r')-\mathcal{F}_{3,2}(r') = \frac{245}{2}\frac{(1-r')^6}{(7r'+1)\,(r'+7)\,(r'+1)\,(2050r'+1567(r')^2+287)}>0,$$

$$
\begin{aligned}
\mathcal{F}_{3,2}(r') - \mathcal{F}_{2,2}(r') &= \frac{2135}{2}\frac{(1-r')^5}{(110r' + 61(r')^2 + 21)(2050r' + 1567(r')^2 + 287)} > 0,\\
\mathcal{F}_{2,2}(r') - \mathcal{F}_{1,2}(r') &= 15\frac{(1-r')^4}{(3r'+1)(r'+3)(110r' + 61(r')^2 + 21)} > 0,\\
\mathcal{F}_{1,2}(r') - \mathcal{F}_{2,1}(r') &= \frac{9}{8}\frac{(1-r')^4}{(7r'+3)(r'+3)(3r'+1)} > 0,\\
\mathcal{F}_{2,1}(r') - \mathcal{F}_{1,1}(r') &= \frac{15}{8}\frac{(1-r')^3}{(5r'+3)(7r'+3)} > 0,\\
\mathcal{F}_{1,1}(r') - \mathcal{F}_{0,1}(r') &= \frac{(1-r')^2}{(r'+1)(5r'+3)} > 0,\\
\mathcal{F}_{0,1}(r') - \mathcal{F}_{1,0}(r') &= \frac{1}{2}\frac{(1-r')^2}{r'+1} > 0,
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathcal{F}_{2,3}(r') &> \mathcal{F}_{3,2}(r') > \mathcal{F}_{2,2}(r') > \mathcal{F}_{1,2}(r') > \mathcal{F}_{2,1}(r')\\
&> \mathcal{F}_{1,1}(r') > \mathcal{F}_{0,1}(r') > \mathcal{F}_{1,0}(r'). \qquad (1.3)
\end{aligned}
$$

We first show the following result:

Theorem 1.1. The inequality

$$
\frac{2}{\pi}\mathcal{K}(r) > \frac{22(r')^2 + 84r' + 22}{7(r')^3 + 57(r')^2 + 57r' + 7} = \mathcal{F}_{2,3}(r') \qquad (1.4)
$$

holds for all $r \in (0,1)$, where $r' = \sqrt{1-r^2}$.

Then by (1.3) and the above theorem, we have

Theorem 1.2. Let $\mathcal{F}_{i,j}(r')$ be defined as (1.2). Then the following inequality chain

$$
\begin{aligned}
\frac{2}{\pi}\mathcal{K}(r) &> \mathcal{F}_{2,3}(r') > \mathcal{F}_{3,2}(r')\\
&> \mathcal{F}_{2,2}(r') > \mathcal{F}_{1,2}(r') > \mathcal{F}_{2,1}(r') \qquad (1.5)\\
&> \mathcal{F}_{1,1}(r') > \mathcal{F}_{0,1}(r') > \mathcal{F}_{1,0}(r')
\end{aligned}
$$

holds for all $r \in (0,1)$, where $r' = \sqrt{1-r^2}$.

2. Lemmas

In order to prove our main results we need following lemmas.

Lemma 2.1.

$$
H(n) =: W_n - \frac{3}{8}\frac{(2n-1)(2n-3)(2n-5)\left(111n^2 + 401n - 1600\right)}{n^2(49n^4 + 2010n^3 - 13\,763n^2 + 28\,212n - 17\,804)} > 0 \qquad (2.1)
$$

holds for all $n \geq 35$.

Proof. It is not difficult to verify that the inequality (2.1) holds when $n = 35$. If the inequality (2.1) holds for $n = m > 35$, i.e

$$W_m > \frac{3}{8}\frac{(2m-1)(2m-3)(2m-5)\left(111m^2+401m-1600\right)}{m^2\left(49m^4+2010m^3-13\,763m^2+28\,212m-17\,804\right)}.$$

So

$$\begin{aligned} W_{m+1} &= \frac{(2m+1)!!}{(2m+2)!!} = \frac{2m+1}{2m+2}W_m \\ &> \left(\frac{2m+1}{2m+2}\right)\frac{3}{8}\frac{(2m-1)(2m-3)(2m-5)\left(111m^2+401m-1600\right)}{m^2\left(49m^4+2010m^3-13\,763m^2+28\,212m-17\,804\right)}. \end{aligned}$$

In order to complete the proof of (2.1) it suffices to show that

$$\begin{aligned} &\left(\frac{2m+1}{2m+2}\right)\frac{3}{8}\frac{(2m-1)(2m-3)(2m-5)\left(111m^2+401m-1600\right)}{m^2\left(49m^4+2010m^3-13\,763m^2+28\,212m-17\,804\right)} \\ > &\ \frac{3}{8}\frac{(2(m+1)-1)(2(m+1)-3)(2(m+1)-5)\left(111(m+1)^2+401(m+1)-1600\right)}{(m+1)^2\left[49(m+1)^4+2010(m+1)^3-13\,763(m+1)^2+28\,212(m+1)-17\,804\right]} \\ = &\ \frac{3}{8}\frac{(2m-1)(2m+1)(2m-3)\left(623m+111m^2-1088\right)}{(m+1)^2\left(6912m-7439m^2+2206m^3+49m^4-1296\right)}, \end{aligned}$$

which is

$$\begin{aligned} \frac{A}{B} &=: \frac{(2m-5)\left(111m^2+401m-1600\right)}{(2m+2)m^2\left(49m^4+2010m^3-13\,763m^2+28\,212m-17\,804\right)} \\ &> \frac{\left(623m+111m^2-1088\right)}{(m+1)^2\left(6912m-7439m^2+2206m^3+49m^4-1296\right)} =: \frac{C}{D}. \end{aligned}$$

In fact,

$$\begin{aligned} AD-BC &= 5439m^7-202\,236m^6+2504\,574m^5-14\,753\,604m^4+45\,551\,571m^3 \\ &\quad -72\,508\,896m^2+51\,673\,680m-10\,368\,000 \\ &= 9308\,275\,200+137\,097\,875\,040(m-20)+47\,752\,255\,764(m-20)^2 \\ &\quad +6984\,199\,251(m-20)^3+545\,207\,796(m-20)^4+23\,923\,854(m-20)^5 \\ &\quad +559\,224(m-20)^6+5439(m-20)^7 \\ &> 0. \end{aligned}$$

□

Lemma 2.1.([17, 18]) Let $\{a_k\}_{k=0}^{\infty}$ be a nonnegative real sequence with $a_m > 0$ and $\sum_{k=m+1}^{\infty} a_k > 0$, and

$$S(t) = -\sum_{k=0}^{m} a_k t^k + \sum_{k=m+1}^{\infty} a_k t^k$$

be a convergent power series on the interval $(0, r)$ $(r > 0)$. Then the following statements are true:

(1) If $S(r^-) \leq 0$, then $S(t) < 0$ for all $t \in (0, r)$;

(2) If $S(r^-) > 0$, then there exists $t_0 \in (0, r)$ such that $S(t) < 0$ for $t \in (0, t_0)$ and $S(t) > 0$ for $t \in (t_0, r)$.

3. Proof of Theorem 1.1

Let

$$\begin{aligned} f(r) &= \left[49(r')^6 - 2451(r')^4 + 2451(r')^2 - 49\right]\frac{2}{\pi}\mathcal{K}(r) \\ &\quad -\left[154(r')^5 - 666(r')^4 - 3380(r')^3 + 3380(r')^2 + 666(r') - 154\right]. \end{aligned}$$

Then

$$\begin{aligned} f(r) &= \left[49(r')^6 - 2451(r')^4 + 2451(r')^2 - 49\right]\frac{2}{\pi}\mathcal{K}(r) \\ &\quad -\left[666\sqrt{1-r^2} - 3380\left(1-r^2\right)^{\frac{3}{2}} + 154\left(1-r^2\right)^{\frac{5}{2}} - 2048r^2 - 666r^4 + 2560\right] \\ &= \left(-49r^6 - 2304r^4 + 2304r^2\right)\sum_{n=0}^{\infty} W_n^2 r^{2n} - 666\sum_{n=7}^{\infty}\frac{\left(-\frac{1}{2}\right)_n}{n!}r^{2n} + 3380\sum_{n=7}^{\infty}\frac{\left(-\frac{3}{2}\right)_n}{n!}r^{2n} \\ &\quad -154\sum_{n=7}^{\infty}\frac{\left(-\frac{5}{2}\right)_n}{n!}r^{2n} + 2048r^2 + 666r^4 - 2560 \\ &= -49\sum_{n=4}^{\infty} W_n^2 r^{2n+6} - 2304\sum_{n=5}^{\infty} W_n^2 r^{2n+4} + 2304\sum_{n=6}^{\infty} W_n^2 r^{2n+2} \\ &\quad -666\sum_{n=7}^{\infty}\frac{\left(-\frac{1}{2}\right)_n}{n!}r^{2n} + 3380\sum_{n=7}^{\infty}\frac{\left(-\frac{3}{2}\right)_n}{n!}r^{2n} - 154\sum_{n=7}^{\infty}\frac{\left(-\frac{5}{2}\right)_n}{n!}r^{2n} \end{aligned}$$

$$\begin{aligned} &= -49\sum_{n=7}^{\infty} W_{n-3}^2 r^{2n} - 2304\sum_{n=7}^{\infty} W_{n-2}^2 r^{2n} + 2304\sum_{n=7}^{\infty} W_{n-1}^2 r^{2n} \\ &\quad -666\sum_{n=7}^{\infty}\frac{\left(-\frac{1}{2}\right)_n}{n!}r^{2n} + 3380\sum_{n=7}^{\infty}\frac{\left(-\frac{3}{2}\right)_n}{n!}r^{2n} - 154\sum_{n=7}^{\infty}\frac{\left(-\frac{5}{2}\right)_n}{n!}r^{2n} \\ &= -49\sum_{n=7}^{\infty} W_{n-3}^2 r^{2n} - 2304\sum_{n=7}^{\infty} W_{n-2}^2 r^{2n} + 2304\sum_{n=7}^{\infty} W_{n-1}^2 r^{2n} \\ &\quad +3380\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\sum_{n=7}^{\infty}\frac{\left(\frac{1}{2}\right)_{n-2}}{n!}r^{2n} - 666\left(-\frac{1}{2}\right)\sum_{n=7}^{\infty}\frac{\left(\frac{1}{2}\right)_{n-1}}{n!}r^{2n} \\ &\quad -154\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\sum_{n=7}^{\infty}\frac{\left(\frac{1}{2}\right)_{n-3}}{n!}r^{2n} \end{aligned}$$

$$
\begin{aligned}
&= -49\sum_{n=7}^{\infty}W_{n-3}^2 r^{2n} - 2304\sum_{n=7}^{\infty}W_{n-2}^2 r^{2n} + 2304\sum_{n=7}^{\infty}W_{n-1}^2 r^{2n} + 2535\sum_{n=7}^{\infty}\frac{\left(\frac{1}{2}\right)_{n-2}}{n!}r^{2n} \\
&\quad +333\sum_{n=7}^{\infty}\frac{\left(\frac{1}{2}\right)_{n-1}}{n!}r^{2n} + \frac{1155}{4}\sum_{n=7}^{\infty}\frac{\left(\frac{1}{2}\right)_{n-3}}{n!}r^{2n} \\
&=: \sum_{n=7}^{\infty}c_n r^{2n},
\end{aligned}
$$

where

$$
c_n = -49W_{n-3}^2 - 2304W_{n-2}^2 + 2304W_{n-1}^2 + 2535\frac{\left(\frac{1}{2}\right)_{n-2}}{n!} + 333\frac{\left(\frac{1}{2}\right)_{n-1}}{n!} + \frac{1155}{4}\frac{\left(\frac{1}{2}\right)_{n-3}}{n!}.
$$

By

$$
W_{n-1} = \frac{2n}{2n-1}W_n \text{ and } \left(\frac{1}{2}\right)_n = \left(\frac{1}{2}\right)_{n-1}\left(\frac{1}{2}+(n-1)\right)
$$

we have

$$
\begin{aligned}
c_n &= -49\left(\frac{2n-4}{2n-5}\frac{2n-2}{2n-3}\frac{2n}{2n-1}W_n\right)^2 - 2304\left(\frac{2n-2}{2n-3}\frac{2n}{2n-1}W_n\right)^2 + 2304\left(\frac{2n}{2n-1}W_n\right)^2 \\
&\quad + \frac{333}{\left(\frac{1}{2}+(n-1)\right)}\frac{\left(\frac{1}{2}\right)_n}{n!} + \frac{2535}{\left(\frac{1}{2}+(n-1)\right)\left(\frac{1}{2}+(n-2)\right)} \\
&\quad + \frac{1155}{4\left(\frac{1}{2}+(n-1)\right)\left(\frac{1}{2}+(n-2)\right)\left(\frac{1}{2}+(n-3)\right)}\frac{\left(\frac{1}{2}\right)_n}{n!} \\
&= -64n^2\frac{28\,212n - 13\,763n^2 + 2010n^3 + 49n^4 - 17\,804}{(2n-3)^2(2n-5)^2(2n-1)^2}W_n^2 + 24\frac{401n + 111n^2 - 1600}{(2n-3)(2n-5)(2n-1)}W_n \\
&= -64n^2\frac{28\,212n - 13\,763n^2 + 2010n^3 + 49n^4 - 17\,804}{(2n-3)^2(2n-5)^2(2n-1)^2}W_n H(n).
\end{aligned}
$$

It's not difficult to verify that $c_n > 0$ for $n = 7, 8, \cdots, 34$, and according to Lemma 2.1, $c_n < 0$ for all $n \geq 35$.

So those coefficients in power series of $-f(r)$ satisfy the conditions of Lemma 2.2, and $-f(1^-) = \infty$. From Lemma 2.2, it follows that there is a unique $r^* \in (0,1)$ such that $-f(r) < 0$ for $r \in (0, r^*)$ and $-f(r) > 0$ for $r \in (r^*, 1)$, that is, $f(r) > 0$ for $r \in (0, r^*)$ and $f(r) < 0$ for $r \in (r^*, 1)$, and r^* is the unique zero of $f(r)$ on $(0,1)$. At the same time, since

$$
\begin{aligned}
f(r) &= \left[7(r')^3 - 57(r')^2 + 57(r') - 7\right]\times \\
&\quad \left\{\left[7(r')^3 + 57(r')^2 + 57(r') + 7\right]\frac{2}{\pi}\mathcal{K}(r) - \left[22(r')^2 + 84(r') + 22\right]\right\},
\end{aligned}
$$

and the function

$$
g(r) =: 7(r')^3 - 57(r')^2 + 57(r') - 7
$$

$$
\begin{aligned}
&= \frac{49\,(r')^6 - 2451\,(r')^4 + 2451\,(r')^2 - 49}{7\,(r')^3 + 57\,(r')^2 + 57\,(r') + 7} \\
&= \frac{-49r^6 - 2304r^4 + 2304r^2}{7\,(r')^3 + 57\,(r')^2 + 57\,(r') + 7} \\
&= \frac{-49r^2\left(r - 4\sqrt{3}/7\right)\left(r + 4\sqrt{3}/7\right)\left(r^2 + 48\right)}{7\,(r')^3 + 57\,(r')^2 + 57\,(r') + 7}
\end{aligned}
$$

has a unique zero $4\sqrt{3}/7$ on $(0, 1)$, which leads to $r^* = 4\sqrt{3}/7$. Clearly, $g(r) > 0$ for $r \in (0, r^*)$ and $g(r) < 0$ for $r \in (r^*, 1)$. That is to say, the two functions $f(r)$ and $g(r)$ have the same sign on both sides of the point r^*, so we have

$$
\begin{aligned}
&\frac{2}{\pi}\mathcal{K}(r) - \frac{22\,(r')^2 + 84\,(r') + 22}{7\,(r')^3 + 57\,(r')^2 + 57\,(r') + 7} \\
= &\frac{f(r)}{g(r)\left[7\,(r')^3 + 57\,(r')^2 + 57\,(r') + 7\right]} > 0
\end{aligned}
$$

holds for all $r \in (0, 1)$ with $r \neq r^*$. But the continuity at $r = r^*$ of the functions $\mathcal{K}(r)$ and $\mathcal{F}_{2,3}(r)$ ensures that the inequality (1.4) also holds for $r = r^*$.

The proof of Theorem 1.1 is complete.

4. Remark

Recently, Z. H. Yang, J. F. Tian and Y. R. Zhu [18] presented a new sharp lower bound for the complete elliptic integral of the first kind as follows:

$$
\frac{2}{\pi}\mathcal{K}(r) > \frac{5\,(r')^2 + 126r' + 61}{61\,(r')^2 + 110r' + 21} = \mathcal{F}_{2,2}(r'), \tag{4.1}
$$

where $r \in (0, 1)$ and $r' = \sqrt{1 - r^2}$. It is not difficult to find that the above conclusion is a direct corollary of Theorem 1.2.

5. Conclusions

This paper obtains a new lower bound for $2\mathcal{K}(r)/\pi$:

$$
\frac{2}{\pi}\mathcal{K}(r) > \frac{22\,(r')^2 + 84r' + 22}{7\,(r')^3 + 57\,(r')^2 + 57r' + 7}.
$$

This lower bound approximates $2\mathcal{K}(r)/\pi$ with high accuracy.

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Conflict of interest

The author declares that he has no conflict of interest.

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