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Research article

CKV-type B-matrices and error bounds for linear complementarity problems

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Abstract: In this paper, we introduce a new subclass of P-matrices called Cvetković-Kostić-Varga type B-matrices (CKV-type B-matrices), which contains DZ-type-B-matrices as a special case, and present an infinity norm bound for the inverse of CKV-type B-matrices. Based on this bound, we also give an error bound for linear complementarity problems of CKV-type B-matrices. It is proved that the new error bound is better than that provided by Li et al. [24] for DZ-type-B-matrices, and than that provided by M. García-Esnaola and J.M. Peña [10] for B-matrices in some cases. Numerical examples demonstrate the effectiveness of the obtained results.

Keywords: CKV-type B-matrices; linear complementarity problems; error bounds; infinity norm; P-matrices
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1. Introduction

Given an $n \times n$ real matrix A and a vector $q \in \mathbb{R}^n$, the linear complementarity problem is to find a vector $x \in \mathbb{R}^n$ satisfying

$$x \geq 0, Ax + q \geq 0, (Ax + q)^T x = 0 \tag{1.1}$$

or to show that no such vector x exists. We denote the problem (1.1) and its solution by LCP(A, q) and x^*, respectively. The LCP(A, q), as one of the fundamental problems in optimization and mathematical programming, has various applications in the quadratic programming, the optimal stopping, the Nash equilibrium point of a bimatrix game, the network equilibrium problem, the contact problem, and the free boundary problem for journal bearing, for details, see [1, 2, 26].

It is well known that the LCP(A, q) has a unique solution x^* for any $q \in \mathbb{R}^n$ if and only if A is a P-matrix [2]. Here, a real square matrix A is called a P-matrix if all its principal minors are positive. For this case, an important topic in the study of the LCP(A, q) concerns the bound of $\|x - x^*\|_\infty$, since it can be used as termination criteria for iterative algorithms and can be used to measure the sensitivity of the solution of LCP(A, q) in response to a small perturbation, e.g., [18, 19, 28, 34]. When the matrix A is a P-matrix, Chen and Xiang [3] gave the following error bound for the LCP(A, q):

$$\|x - x^*\|_\infty \leq \max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \|r(x)\|_\infty, \tag{1.2}$$

where $D = diag(d_i)$ with $0 \leq d_i \leq 1$ for each $i \in N := \{1, \dots, n\}$, $d = (d_1, d_2, \dots, d_n)^T \in [0, 1]^n$, and $r(x) = \min\{x, Ax + q\}$ in which the min operator denotes the componentwise minimum of two vectors. Furthermore, to avoid the high-cost computations of the inverse matrix in (1.2), some easily computable bounds for the LCP(A, q) were derived for the different subclass of P-matrices, such as, B-matrices [10, 20], doubly B-matrices [6], SB-matrices [7, 8], MB-matrices [4], B-Nekrasov matrices [11, 21], weakly chained diagonally dominant B-matrices [22,32,35], B^R_π-matrices [12, 13, 27], and so on [9, 14–17, 23, 36].

Recently, Li et al. [24] presented a new subclass of P-matrices called Dashnic-Zusmanovich type B-matrices (DZ-type-B-matrices), and provided an error bound for the LCP(A, q) when A is a DZ-type-B-matrix.

Definition 1.1. *[33] A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, with $n \geq 2$, is a DZ-type matrix if for each $i \in N$, there exists $j \in N$, $j \neq i$ such that*

$$\left(|a_{ii}| - r_i^j(A)\right)|a_{jj}| > |a_{ij}| r_j(A),$$

where $r_i^j(A) = r_i(A) - |a_{ij}|$ and $r_i(A) = \sum_{j\in N\setminus\{i\}} |a_{ij}|$.

Definition 1.2. *[24] Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ be written in the form*

$$A = B^+ + C, \tag{1.3}$$

where

$$B^+ = [b_{ij}] = \begin{bmatrix} a_{11} - r_1^+ & \cdots & a_{1n} - r_1^+ \\ \vdots & \ddots & \vdots \\ a_{n1} - r_n^+ & \cdots & a_{nn} - r_n^+ \end{bmatrix}, C = \begin{bmatrix} r_1^+ & \cdots & r_1^+ \\ \vdots & \ddots & \vdots \\ r_n^+ & \cdots & r_n^+ \end{bmatrix},$$

and $r_i^+ := \max\{0, a_{ij} | j \neq i\}$. Then, A is called a DZ-type-B-matrix if B^+ is a DZ-type matrix with all positive diagonal entries.

Theorem 1.1. *[24, Theorem 6] Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a DZ-type-B-matrix, and $B^+ = [b_{ij}]$ be the matrix of (1.3). Then,*

$$\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq (n - 1) \cdot \max_{i\in N} \min_{j\in\gamma_i(B^+)} \zeta_{ij}(B^+), \tag{1.4}$$

where

$$\gamma_i(B^+) := \left\{ j \in N\setminus\{i\} : \left(|b_{ii}| - r_i^j(B^+)\right)|b_{jj}| > |b_{ij}| r_j(B^+) \right\},$$

and

$$\zeta_{ij}(B^+) := \frac{\left(b_{ii} - r_i^j(B^+)\right) b_{jj} \max\left\{\frac{1}{b_{ii} - r_i^j(B^+)}, 1\right\} + b_{jj}|b_{ij}| \max\left\{\frac{1}{b_{jj}}, 1\right\}}{\left(b_{ii} - r_i^j(B^+)\right) b_{jj} - |b_{ij}| r_j(B^+)}.$$

Very recently, Cvetković et al. [5] proposed a new subclass of H-matrices called CKV-type matrices, which generalizes CKV matrices (also known as Σ-SDD matrices in the literature) and DZ-type matrices.

Definition 1.3. *[5] A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, with $n \geq 2$, is called a CKV-type matrix if $N_A = \emptyset$ or $S_i^\star(A)$ is not empty for all $i \in N_A$, where $N_A := \{i \in N : |a_{ii}| \leq r_i(A)\}$ and*

$$S_i^\star(A) := \Big\{S \in \Sigma(i) : |a_{ii}| > r_i^S(A), \text{ and for all } j \in \overline{S}$$
$$(|a_{ii}| - r_i^S(A))(|a_{jj}| - r_j^{\overline{S}}(A)) > r_i^{\overline{S}}(A) r_j^S(A)\Big\}$$

with $\Sigma(i) := \{S \subsetneq N : i \in S\}$ and $r_i^S(A) := \sum\limits_{j\in S\setminus\{i\}} |a_{ij}|$.

Motivated by the definition of DZ-type-B-matrices, two meaningful questions naturally arise: can we get a more general subclass of P-matrices using CKV-type matrices, and can we obtain a sharper error bound than the bound (1.4) for the linear complementarity problem of DZ-type-B-matrices? To answer these questions, in Section 2, we present a new class of matrices: CKV-type B-matrices, and prove that it is a subclass of P-matrices containing DZ-type-B-matrices and SB-matrices. Meanwhile, we give an upper bound for the infinity norm for the inverse of CKV-type B-matrices. In Section 3, we give an error bound for the LCP(A, q) when A is a CKV-type B-matrix, consequently, for the LCP(A, q) when A is a DZ-type-B-matrix, and some comparisons with other results are also discussed. Finally, in Section 4, numerical examples are given to illustrate the corresponding theoretical results.

2. CKV-type B-matrices

Using CKV-type matrices, we first give the definition of CKV-type B-matrices.

Definition 2.1. *A matrix $A \in \mathbb{R}^{n\times n}$ is called a CKV-type B-matrix if B^+ given by (1.3) is a CKV-type matrix with positive diagonal entries.*

To show that a CKV-type B-matrix is a P-matrix, we recall the following results.

Lemma 2.1. *[5, Theorem 6] Every CKV-type matrix is a nonsingular H-matrix.*

Lemma 2.2. *[29, Corollary 2.4] If A is a real nonsingular M-matrix and P is a nonnegative matrix with rank(P)=1, then A + P is a P-matrix.*

Proposition 2.1. *If A is a CKV-type B-matrix, then A is a P-matrix.*

Proof. Let A be written in the form $A = B^+ + C$ as shown in (1.3). It follows from (1.3) and Definition 2.1 that B^+ is a Z-matrix (all non-diagonal entries are non-positive [1]) with positive diagonal entries and C is a nonnegative matrix of rank 1. By Lemma 2.1, we know that B^+ is a nonsingular H-matrix, and thus the conclusion follows from Lemma 2.2. □

As shown in [5, 33], the relations of strictly diagonally dominant (SDD) matrices, doubly strictly diagonally dominant (DSDD) matrices, S-strictly diagonally dominant (S-SDD) matrices, DZ-type matrices, and CKV-type matrices are:

- {SDD} ⊆ {DSDD} ⊆ {DZ} ⊆ {S-SDD} and {SDD} ⊆ {DZ-type};
- {DSDD} ⊈ {DZ-type} and {DZ-type} ⊈ {DSDD};
- {DZ} ⊈ {DZ-type} and {DZ-type} ⊈ {DZ};
- {S-SDD}⊆ {CKV-type} and {DZ-type} ⊆ {CKV-type}.

According to [24] and the above relations, we give a figure to illustrate the relations among B-matrices, DZ-type-B-matrices, DB-matrices, SB-matrices, CKV-type B-matrices. Here, the notions of B-matrices, DB-matrices, and of SB-matrices are listed as follows. Let $A = B^+ + C \in \mathbb{R}^{n\times n}$, where B^+ is defined by (1.3). Then, A is called

- a B-matrix if B^+ is SDD with all positive diagonal entries [30];
- a DB-matrix if B^+ is DSDD with all positive diagonal entries [29];
- a SB-matrix if B^+ is S-SDD with all positive diagonal entries for a given non-empty proper subset S of N [25].

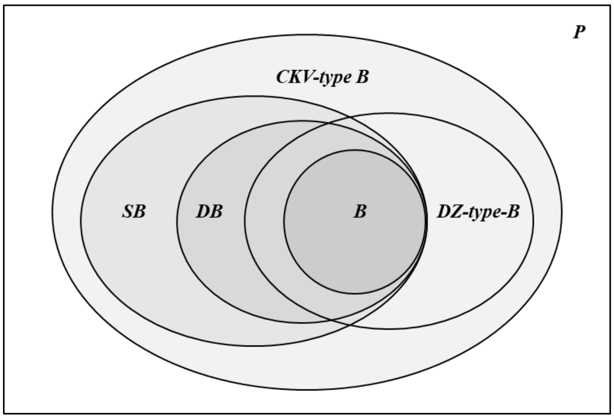

Figure 1. Relations of CKV-type B-matrices and some existing subclasses of P-matrices.

Next, we give a sufficient and necessary condition for a CKV-type B-matrix. Before that, a lemma is needed.

Lemma 2.3. *[5, Remark 9] A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, with $n \geq 2$, is called a CKV-type matrix if $S_i^{\star}(A)$ given by Definition 1.3 is not empty for all $i \in N$. Especially, if A is an SDD matrix, then for all $i \in N$, all proper subsets S containing i belong to $S_i^{\star}(A)$.*

Proposition 2.2. *Given any diagonal matrix $D = diag(d_1, d_2, \ldots, d_n)$ with $d_i \in [0, 1]$ for all $i \in N$, and let I be the identity matrix, then A is a CKV-type B-matrix if and only if $I - D + DA$ is a CKV-type B-matrix.*

Proof. Sufficiency is clearly established. We next show the necessity. Suppose $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a CKV-type B-matrix. Then, $A = B^+ + C$, where B^+ is the matrix of (1.3). Let $\overline{A} := I - D + DA = [\bar{a}_{ij}]$ and $\overline{A} := I - D + DA = \overline{B}^+ + \overline{C}$, where

$$\overline{B}^+ = \begin{bmatrix} \bar{a}_{11} - \bar{r}_1^+ & \cdots & \bar{a}_{1n} - \bar{r}_1^+ \\ \vdots & \ddots & \vdots \\ \bar{a}_{n1} - \bar{r}_n^+ & \cdots & \bar{a}_{nn} - \bar{r}_n^+ \end{bmatrix}, \overline{C} = \begin{bmatrix} \bar{r}_1^+ & \cdots & \bar{r}_1^+ \\ \vdots & \ddots & \vdots \\ \bar{r}_n^+ & \cdots & \bar{r}_n^+ \end{bmatrix},$$

and $\bar{r}_i^+ := \max\{0, \bar{a}_{ij} | j \neq i\}$.

Note that

$$\bar{a}_{ij} = \begin{cases} 1 - d_i + d_i a_{ii}, & i = j, \\ d_i a_{ij}, & i \neq j. \end{cases}$$

It follows that

$$\bar{r}_i^+ := \max\{0, \bar{a}_{ij} | j \neq i\} = \max\{0, d_i a_{ij} | j \neq i\} = d_i \max\{0, a_{ij} | j \neq i\} = d_i r_i^+, \tag{2.1}$$

and

$$\bar{a}_{ij} - \bar{r}_i^+ = \begin{cases} 1 - d_i + d_i(a_{ii} - r_i^+) & i = j, \\ d_i(a_{ij} - r_i^+), & i \neq j. \end{cases} \tag{2.2}$$

Since $A = B^+ + C$, it holds that

$$\overline{A} = I - D + DA = I - D + D(B^+ + C) = (I - D + DB^+) + DC.$$

Note that $B^+ = [b_{ij}]$ and $b_{ij} = a_{ij} - r_i^+$. Hence, from (2.1) and (2.2), we easily obtain that

$$\overline{B}^+ = I - D + DB^+ \text{ and } \overline{C} = DC.$$

Let $\overline{B}^+ = [\bar{b}_{ij}]$. Then,

$$\bar{b}_{ij} = \begin{cases} 1 - d_i + d_i b_{ii}, & i = j, \\ d_i b_{ij}, & i \neq j. \end{cases}$$

Since A is a CKV-type B-matrix, then $B^+ = [b_{ij}]$ is a CKV-type matrix with positive diagonal entries. Thus, by Lemma 2.3, it follows that for each $i \in N$, there exists $S \in S_i^\star(B^+)$, which implies that

$$\begin{cases} |b_{ii}| > r_i^S(B^+), \\ (|b_{ii}| - r_i^S(B^+))(|b_{jj}| - r_j^{\overline{S}}(B^+)) > r_i^{\overline{S}}(B^+) r_j^S(B^+) \text{ for all } j \in \overline{S}. \end{cases} \tag{2.3}$$

Hence, for each $i \in N$, it follows from (2.3) that

$$\bar{b}_{ii} - r_i^S(\overline{B}^+) = 1 - d_i + d_i\left(b_{ii} - r_i^S(B^+)\right) > 0,$$

and that for all $j \in \overline{S}$, if $d_i \neq 0$ and $d_j \neq 0$, then

$$\begin{aligned} \left(|\bar{b}_{ii}| - r_i^S(\overline{B}^+)\right)\left(|\bar{b}_{jj}| - r_j^{\overline{S}}(\overline{B}^+)\right) &= \left[1 - d_i + d_i\left(b_{ii} - r_i^S(B^+)\right)\right]\left[1 - d_j + d_j\left(b_{jj} - r_j^{\overline{S}}(B^+)\right)\right] \\ &\geq d_i\left(b_{ii} - r_i^S(B^+)\right) d_j\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) \\ &> d_i r_i^{\overline{S}}(B^+) d_j r_j^S(B^+) \\ &= r_i^{\overline{S}}(\overline{B}^+) r_j^S(\overline{B}^+), \end{aligned}$$

and if $d_i = 0$ or $d_j = 0$, then

$$\left(|\bar{b}_{ii}| - r_i^S(\overline{B}^+)\right)\left(|\bar{b}_{jj}| - r_j^{\overline{S}}(\overline{B}^+)\right) = \left[1 - d_i + d_i\left(b_{ii} - r_i^S(B^+)\right)\right]\left[1 - d_j + d_j\left(b_{jj} - r_j^{\overline{S}}(B^+)\right)\right]$$

$$
\begin{aligned}
&> 0 \\
&= r_i^{\overline{S}}(\overline{B}^+) r_j^S(\overline{B}^+).
\end{aligned}
$$

These mean that $S \in S_i^\star(\overline{B}^+)$ for each $i \in N$. Therefore, from Definition 1.3, $\overline{B}^+$ is a CKV-type matrix with positive diagonal entries, and consequently, $\overline{A} = I - D + DA$ is a CKV-type B-matrix from Definition 2.1. □

In the following, we give an infinity norm bound for the inverse of CKV-type B-matrices. First, two lemmas are listed.

Lemma 2.4. *[5] Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}, n \geq 2$, be a CKV-type matrix. Then*

$$
\|A^{-1}\|_\infty \leq \max_{i\in N} \min_{S\in S_i^\star(A)} \max_{j\in \overline{S}} \beta_{ij}^S(A),
$$

where $S_i^\star(A)$ is given by Definition 1.3, and

$$
\beta_{ij}^S(A) = \frac{|a_{jj}| - r_j^{\overline{S}}(A) + r_i^{\overline{S}}(A)}{\left(|a_{ii}| - r_i^S(A)\right)\left(|a_{jj}| - r_j^{\overline{S}}(A)\right) - r_i^{\overline{S}}(A) r_j^S(A)}.
$$

Lemma 2.5. *[11] Suppose $P = (p_1, \ldots, p_n)^T e$, where $e = (1, \ldots, 1)$ and $p_i \geq 0$ for all $i \in N$, then*

$$
\|(I + P)^{-1}\|_\infty \leq n - 1.
$$

Theorem 2.1. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}, n \geq 2$, be a CKV-type B-matrix, and $B^+ = [b_{ij}]$ be the matrix of (1.3). Then*

$$
\|A^{-1}\|_\infty \leq (n-1) \cdot \max_{i\in N} \min_{S\in S_i^\star(B^+)} \max_{j\in \overline{S}} \beta_{ij}^S(B^+),
$$

where $S_i^\star(B^+)$ is defined as in Definition 1.3, and

$$
\beta_{ij}^S(B^+) = \frac{|b_{jj}| - r_j^{\overline{S}}(B^+) + r_i^{\overline{S}}(B^+)}{\left(|b_{ii}| - r_i^S(B^+)\right)\left(|b_{jj}| - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)}.
$$

Proof. Since A is a CKV-type B-matrix, so B^+ is a CKV-type matrix with positive diagonal entries and also a Z-matrix. By Corollary 4 of [31], we know that B^+ is an M-matrix and thus $(B^+)^{-1}$ is nonnegative. Hence, from $A = B^+ + C$ in which B^+ and C are given by (1.3), we have

$$
A^{-1} = \left(B^+\left(I + (B^+)^{-1}C\right)\right)^{-1} = \left(I + (B^+)^{-1}C\right)^{-1}(B^+)^{-1},
$$

which implies that

$$
\|A^{-1}\|_\infty \leq \|(I + (B^+)^{-1}C)^{-1}\|_\infty \cdot \|(B^+)^{-1}\|_\infty. \tag{2.4}
$$

Note that $C = (r_1^+, \ldots, r_n^+)^T e$ is nonnegative. Therefore, $(B^+)^{-1}C$ can be written as $(p_1, \ldots, p_n)^T e$, where $p_i \geq 0$ for all $i \in N$. By Lemma 2.5, we get

$$
\|(I + (B^+)^{-1}C)^{-1}\|_\infty \leq n - 1. \tag{2.5}
$$

Since B^+ is a CKV-type matrix, it follows from Lemma 2.4 that

$$
\|(B^+)^{-1}\|_\infty \leq \max_{i\in N} \min_{S\in S_i^\star(B^+)} \max_{j\in \overline{S}} \beta_{ij}^S(B^+). \tag{2.6}
$$

Hence, from (2.4), (2.5), and (2.6), the conclusion follows. □

3. Error bounds for the linear complementarity problem

Based on Theorem 2.1, we give in this section an upper bound of $\max_{d\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty$ when A is a CKV-type B-matrix, and give some comparisons with other results. Before that, a useful lemma is needed.

Lemma 3.1. *[20, Lemma 3] Let $\gamma > 0$ and $\eta \geq 0$. Then for any $x \in [0,1]$,*

$$\frac{1}{1-x+\gamma x} \leq \frac{1}{\min\{\gamma, 1\}}$$

and

$$\frac{\eta x}{1-x+\gamma x} \leq \frac{\eta}{\gamma}.$$

Theorem 3.1. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}, n \geq 2$, be a CKV-type B-matrix, and $B^+ = [b_{ij}]$ be the matrix of (1.3). Then*

$$\max_{d\in[0,1]^n} \|(I-D+DA)^{-1}\|_\infty \leq (n-1)\cdot \max_{i\in N} \min_{S\in S_i^\star(B^+)} \max_{j\in\overline{S}} \alpha_{ij}^S(B^+), \tag{3.1}$$

where $S_i^\star(B^+)$ is defined as in Definition 1.3, and

$$\alpha_{ij}^S(B^+) = \frac{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right)\max\left\{\frac{1}{b_{ii}-r_i^S(B^+)}, 1\right\} + \left(b_{jj} - r_j^{\overline{S}}(B^+)\right) r_i^{\overline{S}}(B^+)\max\left\{\frac{1}{b_{jj}-r_j^{\overline{S}}(B^+)}, 1\right\}}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)}.$$

Proof. Since A is a CKV-type B-matrix, by Proposition 2.2, it follows that $I-D+DA$ is also a CKV-type B-matrix. Taking into account that $A = B^+ + C$ in which B^+ and C are defined as (1.3), then

$$I - D + DA = I - D + D(B^+ + C) = I - D + DB^+ + DC.$$

Denote $\overline{B}^+ := I - D + DB^+ = [\bar{b}_{ij}]$ and $\overline{C} = DC$. Then, from Theorem 2.1, we have

$$\|(I-D+DA)^{-1}\|_\infty \leq (n-1)\cdot \max_{i\in N} \min_{S\in S_i^\star(\overline{B}^+)} \max_{j\in\overline{S}} \beta_{ij}^S(\overline{B}^+). \tag{3.2}$$

By Lemma 3.1, it follows that for all $i \in N$, $j \in \overline{S}$,

$$\begin{aligned}
\beta_{ij}^S(\overline{B}^+) &= \frac{|\bar{b}_{jj}| - r_j^{\overline{S}}(\overline{B}^+) + r_i^{\overline{S}}(\overline{B}^+)}{\left(|\bar{b}_{ii}| - r_i^S(\overline{B}^+)\right)\left(|\bar{b}_{jj}| - r_j^{\overline{S}}(\overline{B}^+)\right) - r_i^{\overline{S}}(\overline{B}^+) r_j^S(\overline{B}^+)} \\
&= \frac{1 - d_j + d_j\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) + d_i r_i^{\overline{S}}(B^+)}{\left[1 - d_i + d_i\left(b_{ii} - r_i^S(B^+)\right)\right]\left[1 - d_j + d_j\left(b_{jj} - r_j^{\overline{S}}(B^+)\right)\right] - d_i r_i^{\overline{S}}(B^+) d_j r_j^S(B^+)} \\
&= \frac{\frac{1}{1-d_i+d_i(b_{ii}-r_i^S(B^+))} + \frac{d_i r_i^{\overline{S}}(B^+)}{[1-d_i+d_i(b_{ii}-r_i^S(B^+))][1-d_j+d_j(b_{jj}-r_j^{\overline{S}}(B^+))]}}{1 - \frac{d_i r_i^{\overline{S}}(B^+) d_j r_j^S(B^+)}{[1-d_i+d_i(b_{ii}-r_i^S(B^+))][1-d_j+d_j(b_{jj}-r_j^{\overline{S}}(B^+))]}}
\end{aligned}$$

$$\leq \frac{\max\left\{\frac{1}{b_{ii}-r_i^S(B^+)}, 1\right\} + \max\left\{\frac{1}{b_{jj}-r_j^{\overline{S}}(B^+)}, 1\right\} \frac{r_i^{\overline{S}}(B^+)}{b_{ii}-r_i^S(B^+)}}{1 - \frac{r_i^{\overline{S}}(B^+)}{b_{ii}-r_i^S(B^+)} \frac{r_j^S(B^+)}{b_{jj}-r_j^{\overline{S}}(B^+)}}$$

$$= \frac{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right)\max\left\{\frac{1}{b_{ii}-r_i^S(B^+)}, 1\right\} + \left(b_{jj} - r_j^{\overline{S}}(B^+)\right) r_i^{\overline{S}}(B^+)\max\left\{\frac{1}{b_{jj}-r_j^{\overline{S}}(B^+)}, 1\right\}}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)}$$

$$=: \alpha_{ij}^S(B^+).$$

Furthermore, by the proof of Proposition 2.2, $S_i^\star(B^+) \subseteq S_i^\star(\overline{B}^+)$ for each $i \in N$. Thus,

$$\max_{i\in N} \min_{S\in S_i^\star(\overline{B}^+)} \max_{j\in\overline{S}} \beta_{ij}^S(\overline{B}^+) \leq \max_{i\in N} \min_{S\in S_i^\star(B^+)} \max_{j\in\overline{S}} \alpha_{ij}^S(B^+). \tag{3.3}$$

Hence, the conclusion follows from (3.2) and (3.3). □

Remark 3.1. *Note that, if* $b_{ii} - r_i^S(B^+) \leq 1$ *and* $b_{jj} - r_j^{\overline{S}}(B^+) \leq 1$*, then*

$$\alpha_{ij}^S(B^+) = \frac{b_{jj} - r_j^{\overline{S}}(B^+) + r_i^{\overline{S}}(B^+)}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)};$$

If $b_{ii} - r_i^S(B^+) > 1$ *and* $b_{jj} - r_j^{\overline{S}}(B^+) \leq 1$*, then*

$$\alpha_{ij}^S(B^+) = \frac{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) + r_i^{\overline{S}}(B^+)}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)};$$

If $b_{ii} - r_i^S(B^+) \leq 1$ *and* $b_{jj} - r_j^{\overline{S}}(B^+) > 1$*, then*

$$\alpha_{ij}^S(B^+) = \frac{b_{jj} - r_j^{\overline{S}}(B^+) + \left(b_{jj} - r_j^{\overline{S}}(B^+)\right) r_i^{\overline{S}}(B^+)}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)};$$

If $b_{ii} - r_i^S(B^+) > 1$ *and* $b_{jj} - r_j^{\overline{S}}(B^+) > 1$*, then*

$$\alpha_{ij}^S(B^+) = \frac{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) + \left(b_{jj} - r_j^{\overline{S}}(B^+)\right) r_i^{\overline{S}}(B^+)}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)}.$$

Since a DZ-type-B-matrix is a CKV-type B-matrix, the bound (3.1) can also be used to estimate $\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty$ when A is a DZ-type-B-matrix. The following theorem provides that the bound (3.1) is better than the bound (1.4) in Theorem 1.1 (Theorem 6 of [24]).

Theorem 3.2. *Let* $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ *be a DZ-type-B-matrix, and* $B^+ = [b_{ij}]$ *be the matrix of (1.3). Then (3.1) holds. Furthermore,*

$$\max_{i\in N} \min_{S\in S_i^\star(B^+)} \max_{j\in\overline{S}} \alpha_{ij}^S(B^+) \leq \max_{i\in N} \min_{j\in\gamma_i(B^+)} \zeta_{ij}(B^+),$$

where $\gamma_i(B^+)$ *and* $\zeta_{ij}(B^+)$ *are given by Theorem 1.1,* $S_i^\star(B^+)$ *and* $\alpha_{ij}^S(B^+)$ *are defined in Theorem 3.1.*

Proof. For each $i \in N$, note that

$$\gamma_i(B^+) := \left\{ j \in N\backslash\{i\} : \left(|b_{ii}| - r_i^j(B^+)\right)|b_{jj}| > |b_{ij}|r_j(B^+) \right\},$$

and

$$S_i^\star(B^+) := \Big\{ S \in \Sigma(i) : |b_{ii}| > r_i^S(B^+), \text{ and for all } j \in \overline{S}, \\ \left(|b_{ii}| - r_i^S(B^+)\right)\left(|b_{jj}| - r_j^{\overline{S}}(B^+)\right) > r_i^{\overline{S}}(B^+) r_j^S(B^+) \Big\}$$

with $\Sigma(i) := \{S \subsetneq N : i \in S\}$. Take $S = N \setminus \{j\}$, $\overline{S} = \{j\}$, $j \neq i$, then

$$r_i^S(B^+) = r_i^j(B^+),\ r_i^{\overline{S}}(B^+) = |b_{ij}|,\ r_j^S(B^+) = r_j(B^+),\ r_j^{\overline{S}}(B^+) = 0,$$

which leads to

$$\begin{aligned}
\alpha_{ij}^S(B^+) &= \frac{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right)\max\left\{\frac{1}{b_{ii}-r_i^S(B^+)}, 1\right\} + \left(b_{jj} - r_j^{\overline{S}}(B^+)\right) r_i^{\overline{S}}(B^+)\max\left\{\frac{1}{b_{jj}-r_j^{\overline{S}}(B^+)}, 1\right\}}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)} \\
&= \frac{\left(b_{ii} - r_i^j(B^+)\right) b_{jj}\max\left\{\frac{1}{b_{ii}-r_i^j(B^+)}, 1\right\} + b_{jj}|b_{ij}|\max\left\{\frac{1}{b_{jj}}, 1\right\}}{\left(b_{ii} - r_i^j(B^+)\right) b_{jj} - |b_{ij}| r_j(B^+)} \\
&= \zeta_{ij}(B^+).
\end{aligned}$$

It is easy to see that $j \in \gamma_i(B^+)$ is equivalent to $S = N \setminus \{j\} \in S_i^\star(B^+)$. Therefore, for each $i \in N$,

$$\begin{aligned}
\min_{S \in S_i^\star(B^+)} \max_{j \in \overline{S}} \alpha_{ij}^S(B^+) &= \min\left\{ \min_{S = N\backslash\{j\} \in S_i^\star(B^+)} \max_{j\in\overline{S}} \alpha_{ij}^S(B^+), \min_{S \in S_i^\star(B^+)\backslash(N\backslash\{j\})} \max_{j\in\overline{S}} \alpha_{ij}^S(B^+) \right\} \\
&= \min\left\{ \min_{j \in \gamma_i(B^+)} \zeta_{ij}(B^+), \min_{S \in S_i^\star(B^+)\backslash(N\backslash\{j\})} \max_{j\in\overline{S}} \alpha_{ij}^S(B^+) \right\} \\
&\leq \min_{j \in \gamma_i(B^+)} \zeta_{ij}(B^+).
\end{aligned}$$

This completes the proof. □

Particularly, for B-matrices, as an important subclass of CKV-type B-matrices, we next show that the bound (3.1) is better than that given by García-Esnaola and Peña in [10] in some cases.

Theorem 3.3. *[10, Theorem 2.3] Let $A \in \mathbb{R}^{n\times n}$ be a B-matrix, and $B^+ = [b_{ij}]$ be the matrix of (1.3). Let $\beta_i := b_{ii} - r_i(B^+)$ and $\beta := \min_{i\in N}\{\beta_i\}$. Then*

$$\max_{d\in[0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq (n-1) \cdot \frac{1}{\min\{\beta, 1\}}. \tag{3.4}$$

Theorem 3.4. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a B-matrix, and $B^+ = [b_{ij}]$ be the matrix of (1.3). Let $\beta_i := b_{ii} - r_i(B^+)$, $\beta := \min_{i\in N}\{\beta_i\}$, and S be a nonempty proper subset of N. Then (3.1) holds. Furthermore, if $b_{ii} - r_i^S(B^+) \le 1$ for all $i \in N$ and $b_{jj} - r_j^{\overline{S}}(B^+) \le 1$ for all $j \in \overline{S}$, then,*

$$\max_{i\in N} \min_{S\in S_i^\star(B^+)} \max_{j\in\overline{S}} \alpha_{ij}^S(B^+) \le \frac{1}{\min\{\beta, 1\}}, \tag{3.5}$$

and if $\beta_i > 1$ for each $i \in N$, then

$$\max_{i\in N} \min_{S\in S_i^\star(B^+)} \max_{j\in\overline{S}} \alpha_{ij}^S(B^+) \ge \frac{1}{\min\{\beta, 1\}}, \tag{3.6}$$

where $\alpha_{ij}^S(B^+)$ is defined as in Theorem 3.1.

Proof. By the fact that a B-matrix is a CKV-type B-matrix, we know that (3.1) holds directly. We now prove that (3.5) and (3.6) hold. For each $i \in N$, $S \in S_i^\star(B^+)$, and $j \in \overline{S}$, if $b_{ii} - r_i^S(B^+) \le 1$ and $b_{jj} - r_j^{\overline{S}}(B^+) \le 1$, then from Remark 3.1 that

$$\alpha_{ij}^S(B^+) = \frac{b_{jj} - r_j^{\overline{S}}(B^+) + r_i^{\overline{S}}(B^+)}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)}.$$

If $b_{jj} - r_j(B^+) < b_{ii} - r_i(B^+)$, then

$$\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_j^S(B^+) < \left(b_{ii} - r_i^S(B^+)\right) - r_i^{\overline{S}}(B^+),$$

which implies that

$$\begin{aligned}\left(b_{jj} - r_j^{\overline{S}}(B^+)\right)^2 - \left(b_{jj} - r_j^{\overline{S}}(B^+)\right) r_j^S(B^+) + r_i^{\overline{S}}(B^+)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+) \\ < \left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+),\end{aligned}$$

i.e.

$$\left[\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_j^S(B^+)\right]\left[\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) + r_i^{\overline{S}}(B^+)\right] < \left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+).$$

It follows that

$$\begin{aligned}\frac{\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) + r_i^{\overline{S}}(B^+)}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)} &< \frac{1}{(b_{jj} - r_j^{\overline{S}}(B^+)) - r_j^S(B^+)} \\ &= \frac{1}{b_{jj} - r_j(B^+)} \\ &= \max\left\{\frac{1}{b_{ii} - r_i(B^+)}, \frac{1}{b_{jj} - r_j(B^+)}\right\} \\ &\le \frac{1}{\min\{\beta, 1\}}.\end{aligned} \tag{3.7}$$

If $b_{jj} - r_j(B^+) \geq b_{ii} - r_i(B^+)$, then

$$\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_j^S(B^+) \geq \left(b_{ii} - r_i^S(B^+)\right) - r_i^{\overline{S}}(B^+),$$

implying that

$$\begin{aligned}\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - \left(b_{jj} - r_j^{\overline{S}}(B^+)\right) r_i^{\overline{S}}(B^+) + \left(b_{ii} - r_i^S(B^+)\right) r_i^{\overline{S}}(B^+) - \left(r_i^{\overline{S}}(B^+)\right)^2 \\ \leq \left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+),\end{aligned}$$

i.e.

$$\left[\left(b_{ii} - r_i^S(B^+)\right) - r_i^{\overline{S}}(B^+)\right] \cdot \left[\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) + r_i^{\overline{S}}(B^+)\right] \leq \left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+).$$

It holds that

$$\begin{aligned}\frac{\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) + r_i^{\overline{S}}(B^+)}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)} &\leq \frac{1}{\left(b_{ii} - r_i^S(B^+)\right) - r_i^{\overline{S}}(B^+)} \\ &= \frac{1}{b_{ii} - r_i(B^+)} \\ &= \max\left\{\frac{1}{b_{ii} - r_i(B^+)}, \frac{1}{b_{jj} - r_j(B^+)}\right\} \\ &\leq \frac{1}{\min\{\beta, 1\}}. \end{aligned} \tag{3.8}$$

Hence, (3.5) follows from (3.7) and (3.8).

If $\beta_i := b_{ii} - r_i(B^+) > 1$ for each $i \in N$, then

$$b_{ii} - r_i^S(B^+) \geq b_{ii} - r_i(B^+) > 1 \text{ and } b_{jj} - r_j^{\overline{S}}(B^+) \geq b_{jj} - r_j(B^+) > 1.$$

By Remark 3.1, we can see that

$$\begin{aligned}\alpha_{ij}^S(B^+) &= \frac{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) + \left(b_{jj} - r_j^{\overline{S}}(B^+)\right) r_i^{\overline{S}}(B^+)}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)} \\ &\geq \frac{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) + r_j^S(B^+) r_i^{\overline{S}}(B^+)}{\left(b_{ii} - r_i^S(B^+)\right)\left(b_{jj} - r_j^{\overline{S}}(B^+)\right) - r_i^{\overline{S}}(B^+) r_j^S(B^+)} \\ &\geq 1.\end{aligned}$$

Therefore,

$$\max_{i \in N} \min_{S \in S_i^\star(B^+)} \max_{j \in \overline{S}} \alpha_{ij}^S(B^+) \geq 1 = \frac{1}{\min\{\beta, 1\}}.$$

The proof is complete. □

Remark from Theorem 3.4 that we can take the minimum of bounds (3.1) and (3.4) to estimate the error bound for the LCP(A, q) with A being a B-matrix, that is,

$$\max_{d \in [0,1]^n} \|(I - D + DA)^{-1}\|_\infty \leq (n-1) \cdot \min\left\{\frac{1}{\min\{\beta, 1\}}, \max_{i \in N} \min_{S \in S_i^\star(B^+)} \max_{j \in \overline{S}} \alpha_{ij}^S(B^+)\right\}.$$

4. Numerical examples

In this section, three examples are given to show the advantage of the bound (3.1) in Theorem 3.1.

Example 4.1. *Consider the following matrix*

$$A = \begin{bmatrix} 4 & 0 & -2 & -2 \\ 0 & 3 & -2 & -2 \\ -1 & -1 & 6 & -2 \\ -1 & -1 & -2 & 6 \end{bmatrix}.$$

Obviously, $B^+ = A$ and $C = 0$. It is easy to verify that B^+ is not a DZ-type matrix and an S-SDD matrix, consequently, not a SDD matrix and a DSDD matrix. Hence, A is not a DZ-type-B-matrix and an SB-matrix, and thus not a B-matrix and a DB-matrix. So we cannot use the error bounds in [6–8, 10, 20, 24] to estimate $\max_{d\in[0,1]^4} \|(I - D + DA)^{-1}\|_\infty$. However, by calculations, one has that B^+ is a CKV-type matrix with positive diagonal entries, and thus A is a CKV-type B-matrix. So by the bound (3.1) in Theorem 3.1, we get

$$\max_{d\in[0,1]^4} \|(I - D + DA)^{-1}\|_\infty \le 21.$$

Example 4.2. *Consider the following matrix*

$$A = \begin{bmatrix} 3 & 0 & -2 & -2 \\ 0 & 3 & -2 & -2 \\ -1 & -1 & 6 & 0 \\ -1 & -1 & 0 & 6 \end{bmatrix}.$$

Note that $r_i^+ := \max\{0, a_{ij} | j \ne i\} = 0$ for $i = 1, 2, 3, 4$. Hence, $B^+ = A$ and $C = 0$. By calculations, we have that A is a DZ-type-B-matrix, and thus it is a CKV-type B-matrix. By the bound (3.1) in Theorem 3.1, we have

$$\max_{d\in[0,1]^4} \|(I - D + DA)^{-1}\|_\infty \le 12.6,$$

while by the bound (1.4) in Theorem 1.1, it holds that

$$\max_{d\in[0,1]^4} \|(I - D + DA)^{-1}\|_\infty \le 27.$$

Obviously, the bound (3.1) is sharper than bound (1.4) in Theorem 1.1 (Theorem 6 of [24]).

Example 4.3. *Consider the B-matrix*

$$A = \begin{bmatrix} 3 & -1 & -1 & -\frac{1}{2} \\ -1 & 3 & -1 & -\frac{1}{2} \\ -1 & -1 & 3 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that $B^+ = A$, $C = 0$. Then, by the bound (3.4) in Theorem 3.3, we have

$$\max_{d\in[0,1]^4} \|(I - D + DA)^{-1}\|_\infty \le 6.$$

In addition, A is also a CKV-type B-matrix. By calculations, for $i = 1, 2, 3$, take $S = \{1, 2, 3\}$ and $\overline{S} = \{4\}$, it follows that $b_{ii} - r_i^S(B^+) \leq 1$ for all $i \in \{1, 2, 3, 4\}$ and $b_{44} - r_4^{\overline{S}}(B^+) \leq 1$; and for $i = 4$, take $S = \{4\}$ and $\overline{S} = \{1, 2, 3\}$, it follows that $b_{ii} - r_i^S(B^+) \leq 1$ for all $i \in \{1, 2, 3, 4\}$ and $b_{jj} - r_j^{\overline{S}}(B^+) \leq 1$ for all $j \in \overline{S}$, which satisfy the hypothesis of Theorem 3.4. Therefore, by Theorem 3.4, we get

$$\max_{d\in[0,1]^4} \|(I - D + DA)^{-1}\|_\infty \leq 4.5,$$

which is smaller than the bound (3.4) in Theorem 3.3 (Theorem 2.3 of [10]).

5. Conclusions

In this paper, on the basis of the class of CKV-type matrices, a new subclass of P-matrices: CKV-type B-matrices, containing B-matrices, DB-matrices, SB-matrices as well as DZ-type-B-matrices, is introduced, and an upper bound for the infinity norm for the inverse of CKV-type B-matrices is provided. Then, by this bound, an error bound for the corresponding LCP(A, q) is given. We also proved that the new error bound is sharper than those of [10] and [24] in some cases, and give numerical examples to show the advantage of our results.

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Conflict of interest

The authors declare no conflict of interest.

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