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Research article

Fast growth and fixed points of solutions of higher-order linear differential equations in the unit disc

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Abstract: In this paper, we investigate the fast growing solutions of higher-order linear differential equations where A_0, the coefficient of f, dominates other coefficients near a point on the boundary of the unit disc. We improve the previous results of solutions of the equations where the modulus of A_0 is dominant near a point on the boundary of the unit disc, and obtain extensive version of iterated order of solutions of the equations where the characteristic function of A_0 is dominant near the point. We also obtain a general result of the iterated exponent of convergence of the fixed points of the solutions of higher-order linear differential equations in the unit disc. This work is an extension and an improvement of recent results of Hamouda and Cao.

Keywords: linear differential equations; unit disc; iterated order; characteristic function; fixed points
Mathematics Subject Classification: 30D35, 34M10

1. Introduction

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [1–4]). As for the definition of the iterated order of meromorphic function, we know that for $r \in [0, 1)$, $\exp_1 r = e^r$ and $\exp_{n+1} r = \exp(\exp_n r)$, $n \in \mathbb{N}$, and for all r sufficiently large in $(0, 1)$, $\log_1 r = \log r$ and $\log_{n+1} r = \log(\log_n r)$, $n \in \mathbb{N}$. Moreover, we denote by $\exp_0 r = r$, $\log_0 r = r$, $\exp_{-1} r = \log_1 r$, $\log_{-1} r = \exp_1 r$. Then, let us recall the following definitions for $n \in \mathbb{N}$.

Definition 1 ([5, 6]). *Let f be a meromorphic function in D. Then the iterated n-order of f is defined by*

$$\sigma_n(f) = \overline{\lim_{r\to 1^-}} \frac{\log_n^+ T(r, f)}{-\log(1 - r)},$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{n+1}^+ x = \log^+(\log_n^+ x)$. *For* $n = 1$, $\sigma_1(f) = \sigma(f)$.

If f *is analytic in D, then the iterated n-order is defined by*

$$\sigma_{M,n}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{n+1}^+ M(r,f)}{-\log(1-r)}.$$

For $n = 1$, $\sigma_{M,1}(f) = \sigma_M(f)$.

Remark 1 ([5])**.** *It follows by M. Tsuji* [7] *that if* f *is an analytic function in* $\mathbb{D}$*, then*

$$\sigma_1(f) \le \sigma_{M,1}(f) \le \sigma_1(f) + 1,$$

which is the best possible in the sense that there are analytic functions g and h such that $\sigma_{M,1}(g) = \sigma_1(g)$ *and* $\sigma_{M,1}(h) = \sigma_1(h) + 1$, *see* [8]. *However, it follows by Proposition* 2.2.2 *in* [3] *that* $\sigma_{M,n}(f) = \sigma_n(f)$ *for* $n \ge 2$.

Definition 2 ([5])**.** *Let* f *be a meromorphic function in* $\mathbb{D}$*. Then the iterated n-convergence exponent of the sequence of zeros in* $\mathbb{D}$ *of* $f(z)$ *is defined by*

$$\lambda_n(f) = \overline{\lim_{r\to 1^-}} \frac{\log_n^+ N(r,\frac{1}{f})}{-\log(1-r)},$$

where $N(r,\frac{1}{f})$ *is the integrated counting function of zeros of* $f(z)$.

Similarly, the iterated n-convergence exponent of the sequence of distinct zeros in $\mathbb{D}$ *of* $f(z)$ *is defined by*

$$\bar{\lambda}_n(f) = \overline{\lim_{r\to 1^-}} \frac{\log_n^+ \overline{N}(r,\frac{1}{f})}{-\log(1-r)},$$

where $\overline{N}(r,\frac{1}{f})$ *is the integrated counting function of distinct zeros of* $f(z)$.

In [6], Heittokangas et al. investigated the fast growing solutions of the differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where the coefficient of f dominates other coefficients in the unit disc D. They proved the following result.

Theorem 1 ([6])**.** *Let* $n \in \mathbb{N}$ *and* $\alpha \ge 0$*. All solutions of* (1) *,where the coefficients* $A_0(z), ..., A_{k-1}(z)$ *are analytic in D, satisfy* $\sigma_{M,n+1}(f) \le \alpha$ *if and only if* $\sigma_{M,n}(A_j) \le \alpha$ *for all* $j = 0, ..., k-1$*. Moreover, if* $q \in \{0, ..., k-1\}$ *is the largest index for which* $\sigma_{M,n}(A_q) = \max_{0\le j\le k-1}\{\sigma_{M,n}(A_j)\}$*, then there are at least* $k-q$ *linearly independent solutions* f *of* (1.1) *such that* $\sigma_{M,n+1}(f) = \sigma_{M,n+1}(A_q)$.

As a result of Theorem 1, by comparing the iterated n-order of coefficients, they obtained that if $q = 0$, i.e., $\sigma_{M,n}(A_j) < \sigma_{M,n}(A_0)$, for all $j = 1, ..., k-1$, then all solutions $f \not\equiv 0$ of (1.1) satisfy $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$ (see [6, Theorem 1.2]).

Heittokangas et al. in [6] and others also investigated the fast growth of solutions by comparing the iterated n-type of coefficients if $\sigma_{M,n}(A_j) \le \sigma_{M,n}(A_0)$ for all $j = 1, ..., k-1$.

Cao and Yi obtained some results similar to Theorem 1 in [9], and in [5] Cao proved the following results in the cases of the modulus and characteristic function of the coefficient A_0 dominating, respectively, those of other coefficients.

Theorem 2 ([5]). *Let H be a set of complex numbers satisfying $\overline{dens}_D\{|z| : z \in H \subseteq D\} > 0$, and let $A_0,\ A_1, ..., A_{k-1}$ be analytic functions in D such that*

$$\max\{\sigma_{M,n}(A_i) : i = 1, 2, ..., k-1\} \leq \sigma_{M,n}(A_0) = \sigma < \infty,$$

and for some constants $0 \leq \beta < \alpha$ we have, for all $\varepsilon > 0$ sufficiently small,

$$|A_0(z)| \geq \exp_n\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

and

$$|A_i(z)| \leq \exp_n\left\{\beta\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\},\ i = 1, 2, ..., k-1,$$

as $|z| \to 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of (1.1) satisfies $\sigma_n(f) = \infty$ and $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$.

Theorem 3 ([5]). *Let H be a set of complex numbers satisfying $\overline{dens}_D\{|z| : z \in H \subseteq D\} > 0$, and let $A_0,\ A_1, ..., A_{k-1}$ be analytic functions in D such that*

$$\max\{\sigma_n(A_i) : i = 1, 2, ..., k-1\} \leq \sigma_n(A_0) = \sigma < \infty,$$

and for some constants $0 \leq \beta < \alpha$ we have, for all $\varepsilon > 0$ sufficiently small,

$$T(r, A_0) \geq \exp_{n-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

and

$$T(r, A_i) \leq \exp_{n-1}\left\{\beta\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\},\ i = 1, 2, ..., k-1,$$

as $|z| \to 1^-$ for $z \in H$. Then every solution $f \not\equiv 0$ of (1.1) satisfies $\sigma_n(f) = \infty$ and $\alpha_{M,n} \geq \sigma_{n+1}(f) \geq \sigma_n(A_0)$, where $\alpha_{M,n} = \max\{\sigma_{M,n}(A_j) : j = 0, 1, ..., k-1\}$.

A result similar to Theorem 3 is given in Corollary 1 and 2.

In the case of the modulus of A_0 dominating those of other coefficients, Hamouda obtained extensive version and improved Theorem 2 in [10] as follows.

Theorem 4 ([10]). *Let $A_0(z), ..., A_{k-1}(z)$ be meromorphic functions in the unit disc D. If there exist a point ω_0 on the boundary ∂D of the unit disc and a curve $\gamma \subset D$ tending to ω_0 such that*

$$\lim_{z\to\omega_0} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|(1-|z|)^{\mu}} = 0,\ \text{with } z \in \gamma,$$

for any $\mu > 0$, then every solution $f(z) \not\equiv 0$ of the differential Eq (1.1) is of infinite order.

Theorem 5 ([10]). *Let $A_0(z), ..., A_{k-1}(z)$ be meromorphic functions in the unit disc D. If there exist $\omega_0 \in \partial D$ and a curve $\gamma \subset D$ tending to ω_0 such that*

$$\lim_{z\to\omega_0} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_n\left(\frac{\lambda}{(1-|z|)^{\mu}}\right) = 0,\ \text{with } z \in \gamma,$$

where $n \geq 1$ is an integer, $(\exp_1(z) = \exp(z),\ \exp_{n+1}(z) = \exp\{\exp_p(z)\})$*, and $\lambda > 0,\ \mu > 0$ are real constants, then every solution $f(z) \not\equiv 0$ of the differential Eq* (1.1) *satisfies $\sigma_n(f) = \infty$, and furthermore $\sigma_{n+1}(f) \geq \mu$.*

There arises a natural question: can we improve Hamouda's results further, and what is the extensive version releted to coefficient characteristic functions. The first aim of this paper is to investigate this problem and obtain the following results of the fast growth of solutions of (1.1), where the modulus and characteristic function of A_0 dominates, respectively, those of other coefficients.

Theorem 6. *Let $A_0(z), ..., A_{k-1}(z)$ be meromorphic functions in the unit disc D. Suppose there exist a point ω_0 on the boundary ∂D of the unit disc and a curve $\gamma \subset D$ tending to ω_0 such that for any $\mu > 0$,*

$$\lim_{\substack{z \to \omega_0 \\ z \in \gamma}} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|(1 - |z|)^{\mu}} < 1. \tag{1.2}$$

If the Eq (1.1) *has a solution $f \not\equiv 0$, then f is of infinite order.*

Theorem 7. *Let $A_0(z), ..., A_{k-1}(z)$ be meromorphic functions in unit disc D. Suppose there exist $\omega_0 \in \partial D$ and a curve $\gamma \subset D$ tending to ω_0 such that*

$$\lim_{\substack{z \to \omega_0 \\ z \in \gamma}} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_n\left(\frac{\lambda}{(1 - |z|)^{\mu}}\right) < 1, \tag{1.3}$$

where $n \geq 1$ is an integer, and $\lambda > 0,\ \mu > 0$ are real constants. If the Eq (1.1) *has a solution $f \not\equiv 0$, then f satisfies $\sigma_n(f) = \infty$ and $\sigma_{n+1}(f) \geq \mu$.*

Remark 2. *Obviously, Theorems 4 and 5 are direct results of Theorems 6 and 7.*

Theorem 8. *Let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in the unit disc D. If there exist $\omega_0 \in \partial D$ and a curve $\gamma \subset D$ tending to ω_0 such that for any $\mu > 0$,*

$$\lim_{\substack{z \to \omega_0 \\ z \in \gamma}} \frac{\prod_{j=1}^{k-1} e^{T(r,A_j)}}{e^{T(r,A_0)}(1 - |z|)^{\mu}} < 1, \tag{1.4}$$

then every solution $f \not\equiv 0$ of (1.1) *is of infinite order.*

Theorem 9. *Let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in the unit disc D such that*

$$\max\{\sigma_{M,n}(A_i) : i = 1, 2, ..., k-1\} \leq \sigma_{M,n}(A_0) = \mu\ (0 < \mu < \infty).$$

If there exist $\omega_0 \in \partial D$ and a curve $\gamma \subset D$ tending to ω_0 such that

$$\lim_{\substack{z \to \omega_0 \\ z \in \gamma}} \frac{\prod_{j=1}^{k-1} e^{T(r,A_j)}}{e^{T(r,A_0)}} \exp_n\left(\frac{\lambda}{(1 - |z|)^{\mu}}\right) < 1, \tag{1.5}$$

where $n \geq 1$ is an integer, and $\lambda > 0$ is a real constant, then every solution $f \not\equiv 0$ of (1.1) *satisfies $\sigma_n(f) = \infty$ and $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$.*

Corollary 1. *Let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in the unit disc D such that*

$$\max\{\sigma_{M,n}(A_i) : i = 1, 2, ..., k-1\} \leq \sigma_{M,n}(A_0) = \sigma \ (0 < \sigma < \infty).$$

If there exist $\omega_0 \in \partial D$ and a curve $\gamma \subset D$ tending to ω_0 such that for some constants $0 \leq \beta < \alpha$ and any given ε $(0 < \varepsilon < \sigma)$, we have

$$T(r, A_0) \geq \exp_{n-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

and

$$T(r, A_i) \leq \exp_{n-1}\left\{\beta\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}, \ i = 1, 2, ..., k-1,$$

as $z \to \omega_0$ for $z \in \gamma$, then every solution $f \not\equiv 0$ of (1.1) *satisfies $\sigma_n(f) = \infty$ and $\sigma_{n+1}(f) = \sigma_{M,n}(A_0)$.*

In fact, from the assumption of Corollary 1, for any given ε $(0 < \varepsilon < \sigma)$, taking $0 < \lambda < \alpha - \beta$, we can easily obtain

$$\lim_{\substack{z\to\omega_0 \\ z\in\gamma}} \frac{\prod_{j=1}^{k-1} e^{T(r,A_j)}}{e^{T(r,A_0)}} \exp_n\left(\frac{\lambda}{(1-|z|)^{\mu}}\right) = 0 < 1.$$

Since ε is arbitrary, we can substitute μ by $\sigma - \varepsilon$ in the assumption (1.5) and the proof of Theorem 9, and then easily obtain the result.

Corollary 2. *Let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in the unit disc D such that*

$$\max\{\sigma_n(A_i) : i = 1, 2, ..., k-1\} \leq \sigma_n(A_0) = \sigma \ (0 < \sigma < \infty).$$

If there exist $\omega_0 \in \partial D$ and a curve $\gamma \subset D$ tending to ω_0 such that for some constants $0 \leq \beta < \alpha$ and any given ε $(0 < \varepsilon < \sigma)$, we have

$$T(r, A_0) \geq \exp_{n-1}\left\{\alpha\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}$$

and

$$T(r, A_i) \leq \exp_{n-1}\left\{\beta\left(\frac{1}{1-|z|}\right)^{\sigma-\varepsilon}\right\}, \ i = 1, 2, ..., k-1,$$

as $z \to \omega_0$ for $z \in \gamma$, then every solution $f \not\equiv 0$ of (1.1) *satisfies $\sigma_n(f) = \infty$ and $\alpha_{M,n} \geq \sigma_{n+1}(f) \geq \sigma_n(A_0)$, where $\alpha_{M,n} = \max\{\sigma_{M,n}(A_j) : j = 0, 1, ..., k-1\}$.*

Remark 3. *Corollary 1 improves Theorems 3 with an accurate value of $\sigma_{n+1}(f)$ instead of a range of it.*

Example 1. *Consider the following differential equation*

$$f'' + H_1(z)\exp_2\left\{\left(\frac{1}{1-z}\right)^2\right\} f' + 2H_0(z)\exp\left\{2\exp\left\{\left(\frac{1}{1-z}\right)^2\right\}\right\} f = 0,$$

where $H_0(z)$ and $H_1(z)$ are meromorphic functions in the unit disc D and analytic at the point $\omega_0 = 1$. We choose the curve γ to be the ray $\arg z = 0$ in D.

We note that if $\max\{\sigma_2(H_0),\ \sigma_2(H_1)\} < 2$, then the coefficients have the same 2-order and 2-type. It is easy to see by Theorem 6, every solution $f \not\equiv 0$ of this equation satisfies $\sigma_2(f) = \infty$ and $\sigma_3(f) \geq 2$.

We also have that if $1 \leq |H_1(z)| \leq |H_0(z)|$, then every solution $f \not\equiv 0$ of this equation satisfies $\sigma(f) = \sigma_2(f) = \infty$ and $\sigma_3(f) \geq 2$. For example, if $H_0(z) = H_1(z) = \frac{1}{z}$, then the assumption (1.3) holds since

$$\lim_{\substack{z\to\omega_0\\ z\in\gamma}} \frac{|A_1(z)| + 1}{|A_0(z)|} \exp_2\left\{\frac{1}{(1-|z|)^2}\right\} = \frac{1}{2} < 1.$$

Therefore, from Theorem 7, every solution $f \not\equiv 0$ of this equation satisfies $\sigma_2(f) = \infty$ and $\sigma_3(f) \geq 2$.

If $H_0(z)$ and $H_1(z)$ above are analytic functions in D, then, from Theorems 8 and 9, the same results hold, and we can easily obtain $\sigma_3(f) = 2$ further.

In addition, Cao also investigated the fixed points of homogeneous linear differential equations in D in [5].

Theorem 10 ([5])**.** *Under the hypothesis of one of Theorems 2 and 3, if $A_1(z) + zA_0(z) \not\equiv 0$, then every solution $f \not\equiv 0$ of (1.1) satisfies $\overline{\lambda}_{n+1}(f - z) = \sigma_{n+1}(f)$.*

The second aim of this paper is to investigate the fixed points of solutions of higher-order equation further. We obtain the following result.

Theorem 11. *Let $A_0(z), ..., A_{k-1}(z)$ be finite iterated n-order analytic (or meromorphic) functions in the unit disc D. If all non-trivial solutions f of (1.1) satisfy $\sigma_n(f) = \infty$ and $\sigma_{n+1}(f) < \infty$, then $\overline{\lambda}_n(f - z) = \sigma_n(f) = \infty$ and $\overline{\lambda}_{n+1}(f - z) = \sigma_{n+1}(f)$.*

Remark 4. *By removing the condition $A_1(z) + zA_0(z) \not\equiv 0$, and as a general result, Theorems 11 improves Theorems 10.*

Corollary 3. *Assume that the assumptions of one of Theorems 2, 3, 9, Corollaries 1 and 2 hold. Then every solution $f \not\equiv 0$ of (1.1) satisfies $\overline{\lambda}_{n+1}(f - z) = \sigma_{n+1}(f)$.*

2. Preliminary Lemmas

Lemma 1 ([8])**.** *Let k and j be integers satisfying $k > j \geq 0$, and let $\varepsilon > 0$ and $d \in (0, 1)$. If f is a meromorphic function in D such that $f^{(j)}$ does not vanish identically, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{k-j}, \quad |z| \notin E,$$

where $E \subset [0, 1)$ with finite logarithmic measure $\int_E \frac{dr}{1-r} < \infty$ and $s(|z|) = 1 - d(1 - |z|)$. Moreover, if $\sigma_1(f) < \infty$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{(k-j)(\sigma_1(f)+2+\varepsilon)}, \quad |z| \notin E,$$

while if $\sigma_n(f) < \infty$ for $n \geq 2$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \exp_{n-1}\left\{\left(\frac{1}{1-|z|}\right)^{\sigma_n(f)+\varepsilon}\right\}, \quad |z| \notin E.$$

Lemma 2. *Let $f : D \to \mathbb{R}$ be an analytic or meromorphic function in the unit disc D. If there exist a point $\omega_0 \in \partial D$ and a curve $\gamma \subset D$ tending to ω_0 such that*

$$\lim_{\substack{z\to\omega_0\\ z\in\gamma}} f(z) < a,\ a \in \mathbb{R},$$

then there exists a set $E \subset [0, 1)$ with infinite logarithmic measure $\int_E \frac{dr}{1-r} = \infty$ such that for all $|z| \in E$, we have $f(z) < a$.

Proof of Lemma 2. Set

$$\lim_{\substack{z\to\omega_0\\ z\in\gamma}} f(z) = b < a,\ a,\ b \in \mathbb{R}.$$

Then for any $\varepsilon = a - b > 0$, there exists $\delta > 0$ such that for all $z \in \gamma$ and $0 < |z - \omega_0| < \delta$, we have $f(z) < b + \varepsilon = a$. Let $g : z \to |z|$, $z \in \gamma$ and $E = \{|z| : z \in \gamma \cap D^o(\omega_0, \delta)\}$. It is easy to see that g is continuous and $E \subset [0, 1)$ is of infinite logarithmic measure. For all $|z| \in E$, we have $z \in \gamma$ and $0 < |z - \omega_0| < \delta$. Hence, for all $|z| \in E$, we have $f(z) < a$. □

Lemma 3 ([4]). *Let f be a meromorphic function in the unit disc, and let $k \geq 1$ be an integer. Then*

$$m(r, \frac{f^{(k)}}{f}) = S(r, f),$$

where $S(r, f) = O(\log^+ T(r, f) + \log(\frac{1}{1-r}))$, possibly outside a set $E \subset [0, 1)$ with $\int_E \frac{dr}{1-r} < +\infty$. If f is of finite order of growth, then

$$m(r, \frac{f^{(k)}}{f}) = O\left(\log\left(\frac{1}{1-r}\right)\right).$$

Lemma 4 ([11]). *Let f be a meromorphic function in the unit disc D for which $i(f) = p > 1$ and $\sigma_p(f) = \beta < +\infty$, and let $k \geq 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m(r, \frac{f^{(k)}}{f}) = O\left(\exp_{p-2}\left(\frac{1}{1-r}\right)^{\beta+\varepsilon}\right),$$

holds for all r outside a set $E \subset [0, 1)$ with $\int_E \frac{dr}{1-r} < \infty$.

Lemma 5 ([12]). *Let f be a solution of* (1.1) *where the coefficients $A_j(z)$ $(j = 0, ..., k-1)$ are analytic functions in the disc $D_R = \{z \in \mathbb{C} : |z| < R\}$, $0 < R \leq \infty$, let $n_c \in \{1, ..., k\}$ be the number of nonzero coefficients $A_j(z)$, $j = 0, ..., k-1$, and let $\theta \in [0, 2\pi)$ and $\varepsilon > 0$. If $z_\theta = \nu e^{i\theta} \in D_R$ is such that $A_j(z_\theta) \neq 0$ for some $j = 0, ..., k-1$, then for all $\nu < r < R$,*

$$|f(re^{i\theta})| \leq C \exp\left(n_c \int_\nu^r \max_{j=0,...,k-1} |A_j(te^{i\theta})|^{1/(k-j)} dt\right),$$

where $C > 0$ is a constant satisfying

$$C \leq (1+\varepsilon) \max_{j=0,...,k-1}\left(\frac{|f^{(j)}(z_\theta)|}{(n_c)^j \max\limits_{j=0,...,k-1} |A_n(z_\theta)|^{j/(k-n)}}\right).$$

The next lemma follows by Lemma 5.

Lemma 6 ([5,10]). *Let $n \in \mathbb{N}$. If the coefficient $A_0(z),\ A_1(z), \cdots, A_{k-1}(z)$ are analytic in the unit disc D, then all solutions of* (1.1) *satisfy* $\sigma_{M,n+1}(f) \le \max\{\sigma_{M,n}(A_j) : j = 0, \cdots, k-1\}$.

Lemma 7 ([9]). *If f and g are meromorphic functions in the unit disc D, $n \in \mathbb{N}$, then we have*
(i) $\sigma_n(f) = \sigma_n(1/f),\ \sigma_n(a \cdot f) = \sigma_n(f)\ (a \in \mathbb{C} - \{0\})$;
(ii) $\sigma_n(f) = \sigma_n(f')$;
(iii) $\max\{\sigma_n(f+g),\ \sigma_n(f \cdot g)\} \le \max\{\sigma_n(f),\ \sigma_n(g)\}$;
(iv) if $\sigma_n(f) < \sigma_n(g)$, *then* $\sigma_n(f+g) = \sigma_n(g),\ \sigma_n(f \cdot g) = \sigma_n(g)$.

Lemma 8 ([11]). *Let $A_0,\ A_1, ..., A_{k-1}$ and $F\ (\not\equiv 0)$ be finite iterated p-order analytic functions in the unit disc D. If f is a solution with $\sigma_p(f) = \infty$ and $\sigma_{p+1}(f) = \rho < \infty$ of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then $\overline{\lambda}_p(f) = \lambda_p(f) = \sigma_p(f) = \infty$ *and* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \rho$.

Lemma 9 ([13]). *Let $A_0, A_1, ..., A_{k-1}$ and $F \not\equiv 0$ be meromorphic functions in the unit disc D and let f be a meromorphic solution of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z),$$

such that

$$\max\{\sigma_p(F),\ \sigma_p(A_j)\ (j = 0, 1, ..., k-1)\} < \sigma_p(f).$$

Then $\overline{\lambda}_p(f) = \lambda_p(f) = \sigma_p(f)$.

3. Proofs of Theorems

Proof of Theorem 6. Suppose that $f \not\equiv 0$ is a solution of (1.1) with finite order $\sigma(f) = \sigma < \infty$. From Lemma 1, for a given $\varepsilon > 0$ there exists a set $E_1 \subset [0, 1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$, such that for all $z \in D$ satisfying $|z| \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le \frac{1}{(1-|z|)^{j(\sigma+2+\varepsilon)}}\ (j = 1, ..., k) \tag{3.1}$$

From (1.1) we can write

$$|A_0(z)| \le \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)| \left|\frac{f'(z)}{f(z)}\right| \tag{3.2}$$

By (3.1) and (3.2), for all $z \in D$ satisfying $|z| \notin E_1$, we have

$$|A_0(z)| \le (\sum_{j=1}^{k-1} |A_j(z)| + 1) \frac{1}{(1-|z|)^{k(\sigma+2+\varepsilon)}}. \tag{3.3}$$

By the assumption (1.2) and Lemma 2, for any $\mu > 0$, there exists a set $E_2 \subset [0,1)$ with infinite logarithmic measure $\int_{E_2} \frac{dr}{1-r} = \infty$ such that for all $|z| \in E_2$, we have

$$\frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|(1-|z|)^{\mu}} < 1.$$

It yields that for any $\mu > 0$,

$$|A_0(z)| > \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{(1-|z|)^{\mu}} \tag{3.4}$$

as $|z| \in E_2 \setminus E_1$, where $E_2 \setminus E_1$ is of infinite logarithmic measure. Obviously, (3.4) contradicts (3.3) in $\{z \in D : |z| \in E_2 \setminus E_1\}$. □

Proof of Theorem 7. Suppose that $f \not\equiv 0$ is a solution of (1.1) with $\sigma_n(f) = \sigma_n < \infty$. From Lemma 1, for a given $\varepsilon > 0$ there exists a set $E_3 \subset [0,1)$ with $\int_{E_3} \frac{dr}{1-r} < \infty$, such that for all $z \in D$ satisfying $|z| \notin E_3$, if $n = 1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \frac{1}{(1-|z|)^{j(\sigma_1+2+\varepsilon)}} \ (j = 1, ..., k); \tag{3.5}$$

and if $n \geq 2$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \exp_{n-1}\left\{\left(\frac{1}{1-|z|}\right)^{\sigma_n+\varepsilon}\right\} \ (j = 1, ..., k). \tag{3.6}$$

Using (3.5) and (3.6) in (3.2), we have

$$|A_0(z)| \leq (\sum_{j=1}^{k-1} |A_j(z)| + 1)\frac{1}{(1-|z|)^{k(\sigma_1+2+\varepsilon)}} \ (n = 1), \tag{3.7}$$

and

$$|A_0(z)| \leq (\sum_{j=1}^{k-1} |A_j(z)| + 1) \exp_{n-1}\left\{\left(\frac{1}{1-|z|}\right)^{\sigma_n+\varepsilon}\right\} \ (n \geq 2), \tag{3.8}$$

for all $z \in D$ satisfying $|z| \notin E_3$. By the assumption (1.3) and Lemma 2, there exists a set $E_4 \subset [0,1)$ with infinite logarithmic measure $\int_{E_4} \frac{dr}{1-r} = \infty$ such that for all $|z| \in E_4$, we have

$$\frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_n\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\} < 1.$$

It yields that

$$|A_0(z)| > (\sum_{j=1}^{k-1} |A_j(z)| + 1) \exp_n\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\}, \ |z| \in E_4. \tag{3.9}$$

Obviously, (3.9) contradicts both (3.7) and (3.8) in $\{z \in D : |z| \in E_4 \setminus E_3\}$. So, $\sigma_n(f) = \infty$. Now by Lemma 1 and since $\sigma_n(f) = \infty$ $(n \geq 1)$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{k-j}, \ |z| \notin E_5, \tag{3.10}$$

where $E_5 \subset [0,1)$ with finite logarithmic measure and $s(|z|) = 1 - d(1-|z|)$ ($d \in (0,1)$). By (3.2) and (3.10), for all $z \in D$ satisfying $|z| \notin E_5$, we have

$$|A_0(z)| \leq (\sum_{j=1}^{k-1} |A_j(z)| + 1)\left(\frac{1}{1-|z|}\right)^{k(2+\varepsilon)} (T(s(|z|), f))^k. \tag{3.11}$$

By (3.9) and (3.11), for all $z \in D$ satisfying $|z| \in E_4 \setminus E_5$, we have

$$\exp_n\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\}(1-|z|)^{k(2+\varepsilon)} < (T(s(|z|), f))^k. \tag{3.12}$$

Set $s(|z|) = R$. We have $1 - |z| = \frac{1}{d}(1-R)$ and $\int_{E_5} \frac{dr}{1-r} < \infty$. So, (3.12) becomes

$$\exp_n\left\{\frac{\lambda d^{\mu}}{(1-R)^{\mu}}\right\}(\frac{1-R}{d})^{k(2+\varepsilon)} < (T(R, f))^k,\ R \in d(E_4 \setminus E_5) + 1 - d. \tag{3.13}$$

Obviously, $d(E_4 \setminus E_5) + 1 - d$ is of infinite logarithmic measure. Then by (3.13), we get

$$\sigma_{n+1}(f) = \overline{\lim_{R \to 1^-}} \frac{\log^+_{n+1} T(R, f)}{-\log(1-R)} \geq \mu.$$

□

Proof of Theorem 8. Suppose that $f \not\equiv 0$ is a solution of (1.1) with finite order $\sigma(f) = \sigma < \infty$. From Lemma 3, we have

$$m(r, \frac{f^{(j)}}{f}) = O\left(\log\left(\frac{1}{1-r}\right)\right)\ (j = 1, ..., k). \tag{3.14}$$

From (1.1), we can write

$$-A_0 = \frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f}.$$

It follows that

$$m(r, A_0) \leq \sum_{j=1}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m(r, \frac{f^{(j)}}{f}) + O(1). \tag{3.15}$$

By the assumption (1.4) and Lemma 2, for any $\mu > 0$, there exists a set $E_6 \subset [0,1)$ with infinite logarithmic measure such that for all $|z| \in E_6$, we have

$$\frac{\prod_{j=1}^{k-1} e^{T(r,A_j)}}{e^{T(r,A_0)}(1-|z|)^{\mu}} < 1.$$

It yields that for any $\mu > 0$,

$$\sum_{j=1}^{k-1} T(r, A_j) - T(r, A_0) + \mu \log\left(\frac{1}{1-|z|}\right) < 0,\ |z| \in E_6. \tag{3.16}$$

Using (3.14) and (3.15), we get

$$T(r, A_0) = m(r, A_0) \leq \sum_{j=1}^{k-1} T(r, A_j) + O\left(\log\left(\frac{1}{1-|z|}\right)\right). \tag{3.17}$$

It is easy to see (3.17) contradicts (3.16) in $\{z \in D : |z| \in E_6\}$. Therefore, $\sigma(f) = \infty$. □

Proof of Theorem 9. Suppose that $f \not\equiv 0$ is a solution of (1.1) with $\sigma_n(f) < \infty$. If $n = 1$, we have 3.17. If $n \geq 2$, from Lemma 4, for any $\varepsilon > 0$, we have

$$m(r, \frac{f^{(j)}}{f}) = O\left(\exp_{n-2}\left(\frac{1}{1-r}\right)^{\sigma_n(f)+\varepsilon}\right) \ (j = 1, ..., k), \tag{3.18}$$

holds for all r outside a set $E_7 \subset [0, 1)$ with $\int_{E_7} \frac{dr}{1-r} < \infty$. By (3.15) and (3.18), we have

$$T(r, A_0) \leq \sum_{j=1}^{k-1} T(r, A_j) + O\left(\exp_{n-2}\left(\frac{1}{1-r}\right)^{\sigma_n(f)+\varepsilon}\right), \ r \notin E_7, \ n \geq 2. \tag{3.19}$$

By the assumption (1.5) and Lemma 2, there exists a set $E_8 \subset [0, 1)$ with infinite logarithmic measure such that for all $|z| \in E_8$, we obtain

$$\frac{\prod_{j=1}^{k-1} e^{T(r,A_j)}}{e^{T(r,A_0)}} \exp_n\left(\frac{\lambda}{(1-|z|)^{\mu}}\right) < 1.$$

It yields that

$$T(r, A_0) - \sum_{j=1}^{k-1} T(r, A_j) > \exp_{n-1}\left(\frac{\lambda}{(1-|z|)^{\mu}}\right), \ |z| = r \in E_8. \tag{3.20}$$

Obviously, (3.20) contradicts (3.17) and (3.19) in $\{z \in D : |z| \in E_8 \setminus E_7\}$. So, $\sigma_n(f) = \infty$. Now by Lemma 3 and since $\sigma_n(f) = \infty$ $(n \geq 1)$, we have

$$m(r, \frac{f^{(k)}}{f}) = O(\log^+ T(r, f) + \log(\frac{1}{1-r})), \tag{3.21}$$

possibly outside a set $E_9 \subset [0, 1)$ with $\int_{E_9} \frac{dr}{1-r} < \infty$. Using (3.21) in (3.15), we obtain

$$T(r, A_0) \leq \sum_{j=1}^{k-1} T(r, A_j) + O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right), \ r \notin E_9. \tag{3.22}$$

By (3.20) and (3.22), we have

$$\exp_{n-1}\left(\frac{\lambda}{(1-|z|)^{\mu}}\right) < O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right), \ |z| = r \in E_8 \setminus E_9. \tag{3.23}$$

Obviously, $E_8 \setminus E_9$ is of infinite logarithmic measure. Then by (3.23), we get

$$\sigma_{n+1}(f) = \overline{\lim_{r\to 1^-}} \frac{\log^+_{n+1} T(r, f)}{-\log(1-r)} \geq \mu.$$

By Lemma 6, we obtain $\sigma_{n+1}(f) \le \sigma_{M,n}(A_0) = \mu$. Therefore, $\sigma_{n+1}(f) = \sigma_{M,n}(A_0) = \mu$. We complete the proof. □

Proof of Theorem 11. Suppose that $f \not\equiv 0$ is a solution of (1.1). By the assumption, we have

$$\sigma_n(f) = \infty,\ \sigma_{n+1}(f) < \infty. \tag{3.24}$$

Set $g(z) = f(z) - z,\ z \in D$. Then by (3.24), we get

$$\sigma_n(g) = \sigma_n(f) = \infty,\ \sigma_{n+1}(g) = \sigma_{n+1}(f),\ \overline{\lambda}_{n+1}(g) = \overline{\lambda}_{n+1}(f - z). \tag{3.25}$$

Substituting $f = g + z$ into (1.1), we get

$$g^{(k)} + A_{k-1}g^{(k-1)} + \cdots + A_1 g' + A_0 g = -A_1 - zA_0. \tag{3.26}$$

Next we prove that $-A_1 - zA_0 \not\equiv 0$. Suppose that $-A_1 - zA_0 \equiv 0$. Hence Eq (1.1) has a solution f_1 satisfying $f_1 = -z$ and $\sigma_n(f_1) < \infty$. This contradicts (3.24). Hence, $-A_1 - zA_0 \not\equiv 0$. By Lemma 7, we have

$$\max\{\sigma_n(-A_1 - zA_0),\ \sigma_n(A_j)\ (j = 0, 1, ..., k-1)\} < \infty.$$

Hence, by (3.25), (3.26) and Lemma 8 or Lemma 9, we deduce that $\overline{\lambda}_n(g) = \sigma_n(g) = \infty,\ \overline{\lambda}_{n+1}(g) = \sigma_{n+1}(g)$. Therefore, we obtain

$$\overline{\lambda}_n(f - z) = \overline{\lambda}_n(g) = \sigma_n(g) = \sigma_n(f) = \infty,$$

$$\overline{\lambda}_{n+1}(f - z) = \overline{\lambda}_{n+1}(g) = \sigma_{n+1}(g) = \sigma_{n+1}(f).$$

□

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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