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Research article

Existence and stability of solutions of ψ-Hilfer fractional functional differential inclusions with non-instantaneous impulses

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Abstract: In this paper, we prove two existence results of solutions for an ψ-Hilfer fractional non-instantaneous impulsive differential inclusion in the presence of delay in an infinite dimensional Banah spaces. Then, by using the multivalued weakly Picard operator theory, we study the stability of solutions for the considered problem in the sense of ψ-generalized Ulam-Hyers. To achieve our aim, we present a relation between any solution of the considered problem and the corresponding fractional integral equation. The given problem here is new because it contains a delay and non-instantaneous impulses effect. Examples are given to clarify the possibility of applicability our assumptions.

Keywords: non-instantaneous impulsive differential inclusion; $\psi-$Hilfer fractional derivative; Ulam-Hyers-Rassias stability; muktivalued weakly Picard operators
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1. Introduction

A non-instantaneous impulsive differential equation is due to Hernándaz et. al. [1], and is used to describe impulsive action, which stays active on a finite time interval. Hilfer [2] introduced a fractional derivative, which is a generalization for Riemann-Liouville fractional derivative and Caputo fractional derivative. Many works have been appeared studying various models involving fractional differential with instantaneous and non-instantaneous impulses and providing solutions to those models. For example, Saravanakumar et al. [3] analyzed the existence of mild solution of non instantaneous impulsive for Hilfer fractional stochastic differential equations driven by fractional Brownian motion,. Shu et al. [4] presented a right formula of mild solutions to a fractional semilinear evolution equation generated by a sectorial operator, and its order belongs to the intervals $(0, 1)$ and $(1, 2)$, Wang et al. [5] studied the global attracting solutions to non-instantaneous impulsive differential inclusions containing

Hilfer fractional, and Ngo et al. [6] presented a formula of solution for a non-instantaneous impulsive differential equation containing $\psi-$Hilfer derivative with lower limit of the fractional derivative at zero. For more works on non-instantaneous impulsive differential equations and inclusions, we refer to [7–13].

Moreover, Ulam problem [14] has been attracted by many researchers. We highlight some recent works on the existence and Hyers-Ulam stability of solutions for fractional differential equations. Guo et al. [15] investigated the existence and Hyers-Ulam stability of mild solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order between one and two, Guo et al. [16] proved the existence and Hyers-Ulam stability of the almost periodic solution to fractional differential equations with impulse involving fractional Brownian, Wang et al. [11] presented the generalized Ulam-Hyers stability for a non-instantaneous impulsive differential inclusions containing the Caputo derivative and Vanterler et al. [17] studied, in finite dimensional Banach spaces, the stability of a Volterra integro-differential equation containing $\psi-$Hilfer derivative in the sense of Ulam-Hyers. More recently, Vanterler et al. [18] investigated, in finite dimensional Banach spaces, the $\delta-$Ulam-Hyers-Rassias stability for a non-instantaneous impulsive fractional differential equation containing $\psi-$Hilfer derivative, Benchohra et al. [19] established, in finite dimensional spaces, the existence and stability of solutions for an implicit fractional differential equations with Riemann-Liouville fractional derivative, and Kumar et al. [20] studied the existence and stability of solution for a fractional differential equation with non-instantaneous integral impulses. Very recently, Ben Mahlouf et al. [21] given sufficient conditions to guarantee the existence and stability of solutions for generalized nonlinear fractional differential equations of order $\alpha \in (1,2)$, Elsayed et al. [22] established the existence and stability of boundary value problem for differential equation with Hilfer-Katugampola fractional derivative. For more papers on Ulam-Hyers stability of solutions, we refer to [23–31].

It is worth noting that, when the considered problem contains non-instantaneous impulses, there are two approaches in the literature, one by keeping the lower limit of the fractional derivative at zero [6, 11, 17, 18], and the other by switching it at the impulsive points [5, 10] Motivated by the above cited work, we prove two existence results of solutions, in infinite dimensional Banach spaces, for a non-instantaneous impulsive fractional differential inclusion involving $\psi-$Hilfer derivative with delay and we switch the lower limit of the fractional derivative at the impulsive points, and then we study the $\psi-$generalized Ulam-Hyers stability.

Let E be a real Banach space, $J = [0,b]$, $b > 0$, $J^* = (0,b]$, $r > 0$, $0 < \vartheta < 1$, $0 \leq \nu \leq 1$, $\mu = \vartheta + \nu - \vartheta\nu, \Psi : [-r,0] \to E$ a continuous function except a finite number of discontinuity points $s \neq 0$ such that $\Psi(0) = 0$, all values $x(s^+)$, and $x(s^-)$ are finite, $\psi \in C^1([0,b],\mathbb{R})$ be increasing, $\psi'(\varrho) > 0, ; \in J$, and $D^{\vartheta,\nu,\psi}_{s_i+}$ be the $\psi-$Hilfer derivative with lower limit at s_i of orderϑ and type ν. Moreover, $0 = s_0 < \varrho_1 < s_1 < \varrho_2 < \cdots < \varrho_n < s_n < \varrho_{n+1} = b$, and $I^{1-\mu,\psi}_{s_i^+} x(s_i^+) = \lim_{\varrho \to s_i^+} I^{1-\mu,\psi}_{s_i^+} x(\varrho)$, $F : J \times E \to 2^E - \{\phi\}$ is a multifunction, and $g_i : [\varrho_i , s_i] \times E \longrightarrow E\, i = 1, 2, \ldots, n$. Finally, for any $\varrho \in J$, $\tau(\varrho) : \mathcal{H} \to \Theta$, $x(\theta) = x(\varrho + \theta), \theta \in [-r, 0]\, x \in \mathcal{H}$, where Θ and $\mathcal{H}$ will be introduced in the next section. In this paper, we establish existence results of solutions of the following $\psi-$Hilfer fractional non-instantaneous impulsive differential inclusions with delay:

$$\begin{cases} D_{s_i^+}^{\vartheta,\nu,\psi} x(\varrho) \in F(\varrho, \tau(\varrho)x)), \ a.e. \ \varrho \in \cup_{i=0}^{i=n}(s_i, \ \varrho_{i+1}] \\ x(\varrho) = \Psi(\varrho), \varrho \in [-r, 0] \\ I_{0^+}^{1-\mu,\psi} x(0^+) = \Psi(0), \\ x(\varrho_i^+) = g_i(\varrho_i, x(\varrho_i^-)), i = 1, ...n, \\ x(\varrho) = g_i(\varrho, x(\varrho_i^-)), \varrho \in (\varrho_i \ s_i], i = 1, ...n, \\ I_{s_i^+}^{1-\mu,\psi} x(s_i^+) = g_i(s_i, x(\varrho_i^-)), i = 1, ...n. \end{cases} \tag{1}$$

Then, we investigate the ψ-generalized Ulam-Hyers stability of Problem (1). To achieve our aim, we present a relation between the solutions of this problem and the corresponding fractional integral equation (Lemma5).

To make a comparison between the present paper objectives and other relevant recent papers, we refer to the following:

1-Abbas et al. [31] proved the existence of solutions and studied Ulam-Hyers-Rassias stability of problem (1) in the absence of both delay and impulses effect, $E = \mathbb{R}$, $\psi(\varrho) = \ln \varrho, \varrho \in [1, b]$, F is a single-valued function.

2-Benchohra [19] investigated the existence and Ulam stability for nonlinear implicit differential equations with Riemann-Liouville derivative, which is including in Hilfer derivative

3-Ngo et al. [6] presented a formula of solution for a non-instantaneous impulsive differential equation containing $\psi-$Hilfer derivative with lower limit of the fractional derivative at zero and in the absence of delay.

4-Vanterler el al. [17] established the existence and stability of solutions for Problem (1) in the absence of delay. and when $E = \mathbb{R}$, the lower limit of the fractional derivative at zero F is a single-valued function

5-Wang et al. [10] considered Problem (1) in the absence of delay, when $\psi(\varrho) = \varrho$ and without studying the stability of solutions.

6-Wang et al. [11] consider a non-instantaneous impulsive semilinear differential inclusions containing Caputo derivative and in the absence of delay.

To clarify the novelty and contribution of this study, we refer to, in this paper, we present a relation between a solution of Problem (1) and the corresponding fractional integral equation (Lemma5), provide two methods to demonstrate the existence of solutions for Problem (1), then, investigate the $\psi-$generalized Ulam-Hyers stability of solutions. Because our considered problem contains $\psi-$Hilfer fractional derivative, non-instantaneous impulses with the lower limit of the fractional derivative switches at the impulsive points, presence of delay, and the right hand side is a multi-valued function, therefore, this study generalize recent results, as it is shown above, such as [6, 10, 11, 17, 19, 31]. In addition, there isn't work in the literature, on $\psi-$Hilfer fractional non-instantaneous impulsive differential inclusions, in infinite dimensional spaces, in the presence of delay, and the lower limit of the fractional derivative switches at the impulsive points. Moreover, the technique presented in this paper can be used to study the existence and Ulam-Hyers stability of solutions or mild solutions for the problems considered in [3, 4, 15, 16, 20–22] to the case when, there are impulses and delay on the system, the right hand side is a multi-valued function and involving $\psi-$Hilfer fractional derivative.

In section 2, we prove some properties for $\psi-$fractional integral and $\psi-$fractional derivative, then we present, in Lemma 5, a relation between any solution of problem (1) and the corresponding fractional integral equation. In section 3, we prove an existence result of Problem (1). In section 4, we give another existence result of (1), then we investigate the $\psi-$generalized Ulam-Hyers stability of solutions. In the last section, examples are given to clarify the possibility of applicability of our assumptions.

2. Preliminaries and notations

Let $P_{ck}(E)$ be the family of non-empty convex and compact subsets of E . Since the given problem containing Hilfer derivative we need to the the spaces:

$$C_{1-\mu,\psi}(J,E) := \{x \in C(J^*,E) : (\psi(.)-\psi(a))^{1-\mu}x(.) \in C([a,b],E)\},$$

and

$$C^n_{1-\mu,\psi}(J,E) := \{x \in C^{n-1}(J,E) : [\frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho}]^n\ x \in C_{1-\mu,\psi}(J,E), n \in \mathbb{N}.$$

Obviously $C_{1-\mu,\psi}(J,E)$ and $C^n_{1-\mu,\psi}(J,E)$ are Banach spaces with norms

$$\|x\|_{C_{1-\mu,\psi}(J,E)} := \sup_{\varrho\in J}\|\ (\psi(\varrho)\ -\psi(0))^{1-\mu}x(\varrho)\|,$$

and

$$\|x\|_{C^n_{1-\mu,\psi}(J,E)} := \sum_{k=1}^{k=n-1}\|x^{(k)}\|_{C(J,E)} + \|[\frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho}]^n\ x\|_{C_{1-\mu,\psi}(J,E)}.$$

Because Problem (1) involving impulses effect we recall the Banach space:

$$\begin{aligned} PC_{1-\mu,\psi}(J,E) &: = \{x : J^* \to E,\ (\psi(.)-\psi(s_k))^{1-\mu}x(.) \in C(J_k,E) \\ ,\ k &= 0,1,..,n,\ \lim_{\varrho\to s_k^+}(\psi(\varrho)-\psi(s_k))^{1-\mu}x(\varrho) \text{ exists, } x \in C(\varrho_i,E), \\ \text{and } \lim_{\varrho\to\varrho_i^+}x(\varrho) \text{ exist, } i &= 1,2,..,n\}, \end{aligned}$$

endowed with the norm

$$\|x\|_{PC_{1-\mu,\psi}(J,E)} := \max\{\sup_{\substack{\varrho\in\overline{J_k}\\ k=0,1,,,n,}}(\psi(\varrho)-\psi(s_k))^{1-\mu}\|x(\varrho)\|_E,\ \sup_{\substack{\varrho\in\overline{\varrho_i}\\ i=1,...,n}}\|x(\varrho)\|_E\},$$

where $J_k = (s_k, \varrho_{k+1}]$, $\overline{J_k} = [s_k, \varrho_{k+1}]$ $(k = 0,1,..,n), \varrho_i = (\varrho_i, s_i]$ and $\overline{\varrho_i} = [\varrho_i, s_i]$ $(i = 1,2,..,n)$,
Next, the function $\chi_{PC_{1-\mu}(J,E)} : P_b(PC_{1-\mu,\psi}(J,E)) \to [0,\infty)$,given by by

$$\chi_{PC_{1-\mu,\psi}(J,E)}(D) := \max\{\max_{k=0,1,..,n}\chi_{C(\overline{J_k},E)}(D_{|\overline{J_k}}), \max_{i=1,..,n}\chi_{C(\overline{\varrho_i},E)}(D_{|\overline{\varrho_i}})\}$$

is a measure of noncompactness on $PC_{1-\mu}(J,E)$, where

$$\begin{aligned} D_{|\overline{J_k}} &: = \{h^* \in C(\overline{J_k},E) : h^*(\varrho) = (\psi(\varrho)-\psi(s_k))^{1-\mu}h(\varrho), \varrho \in J_k, \\ h^*(s_k) &= \lim_{\varrho\to s_k^+}h^*(\varrho), h \in D\}, \end{aligned}$$

and

$$D_{|\overline{\varrho_i}} := \{h^* \in C(\overline{\varrho_i},E) : h^*(\varrho) = h(\varrho), \varrho \in \varrho_i, h^*(\varrho_i) = h(\varrho_i^+), h \in D\}.$$

In the sequel, $I_{a+}^{q,\psi}$ denotes to the $\psi-$Riemann-Liouville fractional integral operator of order q with the lower limit at a , $D_{a+}^{\vartheta,\psi}f$ to the $\psi-$Riemann-Liouville fractional derivative operator of order ϑ with the lower limit at a and ${}^{c}D_{a+}^{\vartheta,\psi}f$ to the $\psi-$Caputo fractional derivative of order ϑ with the lower limit at a for $f \in AC^{1,\psi}([a,b],E)$, where

$$AC^{1,\psi}([a,b],E) := \{x : J \to E, [\frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho}]\, x \in AC(J,E)\}.$$

If $f \in C^{1,\psi}([a,b],E) := \{x \in C(J,E) : [\frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho}]\, x \in C(J,E)\}$, then

$${}^{c}D_{a+}^{\vartheta,\psi}f(\varrho) := I_{a+}^{1-\vartheta,\psi}[\frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho}]\, f(\varrho), \varrho \in [a,b].$$

If $\psi(\varrho) = \varrho$, we obtain the Caputo fractional derivative, and if $\psi(\varrho) = \ln\varrho$, we obtain the Caputo-Hadamard fractional derivative. The following remark and more information about $\psi-$fractional integral and derivative can be found in [32–35]

Remark 2.1. *If $q = 1$, then $I_{a+}^{1,\psi}f(\varrho) = \int_a^\varrho \psi\prime(s)f(s)\,ds$, and hence*

$$\frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho}\, I_{a+}^{1,\psi}f(\varrho) = \frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho}\int_a^\varrho \psi\prime(s)f(s)\,ds = f(\varrho), a.e. for\ \varrho \in [a,b].$$

In the following lemma we give an important for $I_{a+}^{\vartheta,\psi}$, which we need later.

Lemma 2.1. *Let $0 < \eta \leq \vartheta$. Then $I_{a+}^{\vartheta,\psi}$ is bounded from $C_{\eta,\psi}([a,b],E)$ into $C([a,b],E)$ and*

$$I_{a+}^{\vartheta,\psi}f(a) = \lim_{\varrho\to a+} I_{a+}^{\vartheta,}f(\varrho) = 0,\ \forall f \in C_{\eta,\psi}([a,b],E).$$

Proof. The assumption $f \in C_{\eta,\psi}([a,b],E)$ leads to $(\psi(\varrho)-\psi(a))^\eta f(\varrho)$ is continuous on $[a,b]$, and hence there is $M > 0$ such that $\|(\psi(\varrho)-\psi(a))^\eta f(\varrho)\| \leq M, \forall\varrho \in [a,b]$. As a consequence, $\|f(\varrho)\| \leq (\psi(\varrho)-\psi(a))^{-\eta}M, \forall\varrho \in (a,b]$. Then

$$\begin{aligned}\|I_{a+}^{\vartheta,\psi}f(\varrho)\| &= \|\int_a^\varrho \frac{(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)}{\Gamma(\vartheta)}f(s)\,ds\| \\ &\leq \frac{M}{\Gamma(\vartheta)}\,|I_{a+}^{\vartheta,\psi}(\psi(\varrho)-\psi(a))^{-\eta}|.\end{aligned}$$

In virtu of Lemma2 in [35], $I_{a+}^{\vartheta,\psi}(\psi(\varrho)-\psi(a))^{-\eta} = \frac{\Gamma(1-\eta)}{\Gamma(1-\eta+\vartheta)}(\psi(\varrho)-\psi(a))^{\vartheta-\eta}$, and hence

$$\|I_{a+}^{\vartheta,\psi}f(\varrho)\| \leq \frac{M}{\Gamma(\vartheta)}\frac{\Gamma(1-\eta)}{\Gamma(1-\eta+\vartheta)}(\psi(\varrho)-\psi(a))^{\vartheta-\eta}$$

Since $0 < \eta \leq \vartheta$, we get $\lim_{\varrho\to a+} I_{a+}^{\vartheta,\psi}f(\varrho) = 0$, which means that $I_{a+}^{\vartheta,\psi}f(\varrho)$ is continuous. Moreover, $I_{a+}^{\vartheta,\psi}$ is bounded from $C_{\eta,\psi}([a,b],E)$ into $C([a,b],E)$. □

Let us recall the definition of $\psi-$Hilfer fractional derivative.

Definition 1. *[2] Let $f \in L^1([a,b],E)$ be such that $I_{a+}^{(1-\nu)(1-\vartheta),\psi} f \in AC^{1,\psi}([a,b],E)$. The $\psi-$Hilfer fractional derivative of order* $0 < \vartheta < 1$*and type* $0 \leq \nu \leq 1$ *and with lower limit at a for a function* $f : [a,b] \to E$ *is defined by*

$$D_{a+}^{\vartheta,\nu,\psi} f(\varrho) = I_{a+}^{\nu(1-\vartheta),\psi} [\frac{1}{\psi\prime(\varrho)} \frac{d}{d\varrho}] \, (I_{a+}^{(1-\nu)(1-\vartheta),\psi} f)(\varrho), \varrho \in [a,b],$$

Denote

$$C_{1-\mu,\psi}^{\mu}(J,E) := \{x \in C_{1-\mu,\psi}(J,E), D_{a+}^{\mu,\psi} x \in C_{1-\mu,\psi}(J,E)\},$$

$$C_{1-\mu,,\psi}^{\vartheta,\nu}(J,E) := \{x \in C_{1-\mu}(J,E), D_{a+}^{\vartheta,\nu,\psi} x \in C_{1-\mu,\psi}(J,E)\},$$

$$PC_{1-\mu,\psi}^{\mu}(J,E) := \{x \in PC_{1-\mu}(J,E), D_{s_k^+}^{\mu,\psi} x_{\,|J_k} \in C_{1-\mu,\psi}(J_k,E), k = 0,1,..,n\},$$

and

$$PC_{1-\mu,\psi}^{\vartheta,\nu}(J,E) := \{x \in PC_{1-\mu,\psi}(J,E), D_{s_k^+}^{\vartheta,\nu,\psi} x_{\,|J_k} \in C_{1-\mu,\psi}(J_k,E),\ k = 0,1,..,n\}.$$

Notice that, the operator $D_{a+}^{\vartheta,\nu,\psi}$ can be written as:
Let

$$\begin{aligned} D_{a+}^{\vartheta,\nu,\psi} f(\varrho) &= I_{a+}^{\nu(1-\vartheta),\psi} [\frac{1}{\psi\prime(\varrho)} \frac{d}{d\varrho}] (I_{a+}^{1-\mu} f(\varrho) \\ &= I_{a+}^{\nu(1-\vartheta),\psi} D_{a+}^{\mu,\psi} f(\varrho),\ \mu = \vartheta + \nu - \vartheta\nu. \end{aligned}$$

So, if $f \in C_{1-\mu,\psi}^{\mu}([a,b],E)$, then by Lemma 1, $D_{a+}^{\vartheta,\nu,\psi} f(\varrho)$ exists $\varrho \in [a,b]$.

Remark 2.2. *Since* $D_{a+}^{\vartheta,\nu,\psi} x = (I_{a+}^{\nu(1-\vartheta),\psi} D_{a+}^{\mu,\psi} x)(\varrho)$*, it follows from lemma1. that* $C_{1-\mu,\psi}^{\mu}([a,b],E) \subseteq C_{1-\mu,\psi}^{\vartheta,\nu}([a,b],E)$*. Similarly,* $PC_{1-\mu,\psi}^{\mu}([a,b],E) \subseteq PC_{1-\mu,\psi}^{\vartheta,\nu}([a,b],E)$.

Now, since our considered problem contains a delay we need to present the following spaces:
1- The normed space

$$\begin{aligned} \Theta &= \{z : [-r,0] \to E \text{ such that } z \text{ has a finite number of discontinuity} \\ \text{points } s &\neq 0,\ \text{all values } z(s^+), \text{and } z(s^-) \text{ are finite}\}, \end{aligned}$$

endowed with the norm:

$$\|z\|_\Theta = \int_{-r}^{0} \|z(s)\| ds.$$

2-The metric space (the space of solutions)

$$H : = \{x : [-r,b] \to E\ ,\ x_{|_{[-r,0]}} = \Psi, x_{|J^*} \in PC_{1-\mu,\psi}(J,E)\},$$

where the metric function is given by:

$$d_H(x,y) := sup_{\varrho \in J} \|x(\varrho) - y(\varrho)\|.$$

3- The Banach space

$$\mathcal{H} := \{x : [-r, b] \to E \text{ such that } x(\varrho) = 0, \forall \varrho \in [-r, 0],\ x_{|J^*} \in PC_{1-\mu,\psi}(J, E)\},$$

endowed with the norm:

$$\|x\|_{\mathcal{H}} := \|x_{|J^*}\|_{PC_{1-\mu,\psi}(J,E)} + \|x_{|[-r,0]}\|_{\Theta} = \|x_{|J^*}\|_{PC_{1-\mu,\psi}(J,E)}.$$

Remark 2.3. *(i) If $x \in \mathcal{H}$, then $x(0^-) = 0$ and $x(0^+) = \lim_{\varrho\to 0^+}(\psi(\varrho) - \psi(0))^{1-\mu} x(\varrho)$.*

(ii) If $x \in H$, then $x(0^-) = \Psi(0)$ and $x(0^{+-}) = \lim_{\varrho\to 0^+}(\psi(\varrho) - \psi(0))^{1-\mu} x(\varrho)$. So, if $\Psi(0) = 0$, then x will be continuous at zero.

It is easily seen that the function:

$$\chi_{\mathcal{H}}(B) := \chi_{PC_{1-\mu}(J,E)}\{x|_J : x \in B\},$$

define a measure of noncompactness on $\mathcal{H}$, where B is a bounded subset of $\mathcal{H}$.

We need to the following fixed point for multi-valued functions.

Lemma 2.2. *([36], Theorem 3.1) Let W be a closed convex subset of a Banach space X and $\varrho : W \to P_c(W)$. Suppose that ϱ is closed, $\varrho(D)$ is relatively compact, whenever D is compact, and that, for some $x_0 \in W$, one has*

$$\begin{aligned} B \quad &\subseteq \quad W,\ B = conv(\{x_0\} \cup \varrho(B)),\ \overline{B} = \overline{C} \text{ with } C \subseteq B \text{ countable} \\ &\Longrightarrow \quad B \text{ is relatively compact.} \end{aligned}$$

Then, there is a fixed point forϱ.

3. Existence results of solution for (1).

In this section, we demonstrate the existence of solutions of Problem (1). For any $x \in H$ let

$$S^1_{F(.,\tau(.)x)} = \{z \in L^1(J, E) : z(\varrho) \in F(\varrho, \tau(.)x), a.e. \text{ for } \varrho \in J\}.$$

and

$$\begin{aligned} I^{\nu(1-\vartheta),\psi}(PC_{1-\mu,\psi}(J, E)) &= \{f : J \to E, \text{ there is } h \in PC_{1-\mu,\psi}(J, E) \text{ such that} \\ f(\varrho) &= I^{\nu(1-\vartheta),\psi}_{s_i+} h(\varrho), \varrho \in J_k,\ k = 0, 1, 2, .., n\}. \end{aligned}$$

and for any $x \in \mathcal{H}$ let

$$\overline{x}(\varrho) := \begin{cases} \Psi(\varrho), \varrho \in [-r, 0], \\ x(\varrho), \varrho \in (0, b]. \end{cases}$$

In order to derive the relation between any solution for Problem (1) and the corresponding fractional integral equation, we need to the following essential Lemmas.

Lemma 3.1. *Let $0 < \vartheta < 1, \eta \in [0, 1)$.If $f \in C_{\eta,\psi}([a, b], E)$and $I^{1-\vartheta,\psi}_{a+} f \in C^1_{\eta,\psi}([a, b], E)$, then*

$$I^{\vartheta,\psi}_{a+} D^{\vartheta,\psi}_{a+} f(\varrho) = f(\varrho) - \frac{(\psi(\varrho) - \psi(a))^{\vartheta-1} I^{1-\vartheta,\psi}_{a+} f(a)}{\Gamma(\vartheta)}, \varrho \in [a, b].$$

Proof. Since $f \in C_{\eta,\psi}([a,b],E)$, then $I_{a+}^{1-\vartheta,\psi} f(\varrho)$ is defined for $\varrho \in [a,b]$. Moreover, the assumption $I_{a+}^{1-\vartheta,\psi} f \in C^1_{\eta,\psi}([a,b],E)$ implies to $D_{a+}^{\vartheta,\psi} f(\varrho) \in C_{\eta,\psi}([a,b],E)$, and hence $I_{a+}^{\vartheta,\psi} D_{a+}^{\vartheta,\psi} f(\varrho)$ and $I_{a+}^{\vartheta+1,\psi} D_{a+}^{\vartheta,\psi} f(\varrho)$ are well defined for $\varrho \in [a,b]$. Observe that

$$\begin{aligned}
& I_{a+}^{\vartheta,\psi} D_{a+}^{\vartheta,\psi} f(\varrho) \\
= & \frac{1}{\psi\prime(\varrho)} \frac{d}{d\varrho} (\int_a^\varrho \psi\prime(s) I_{a+}^{\vartheta,\psi} D_{a+}^{\vartheta,\psi} f(s) ds) \\
= & \frac{1}{\psi\prime(\varrho)} \frac{d}{d\varrho} I_{a+}^{1,\psi} I_{a+}^{\vartheta,\psi} D_{a+}^{\vartheta,\psi} f(\varrho) \\
= & \frac{1}{\psi\prime(\varrho)} \frac{d}{d\varrho} I_{a+}^{\vartheta+1,\psi} D_{a+}^{\vartheta,\psi} f(\varrho), \varrho \in [a,b].
\end{aligned}$$

Now,

$$\begin{aligned}
I_{a+}^{\vartheta+1,\psi} D_{a+}^{\vartheta,\psi} f(\varrho) &= \frac{1}{\Gamma(\vartheta+1)} \int_a^\varrho (\psi(\varrho)-\psi(s))^\vartheta \psi\prime(s) (\frac{1}{\psi\prime(s)} \frac{d}{ds})(I_{a+}^{1-\vartheta,\psi} f(s)) ds \\
&= \frac{1}{\Gamma(\vartheta+1)} \int_a^\varrho (\psi(\varrho)-\psi(s))^\vartheta \frac{d}{ds} (I_{a+}^{1-\vartheta,\psi} f(s)) ds.
\end{aligned}$$

By integration by parts, we get

$$\begin{aligned}
I_{a+}^{\vartheta+1,\psi} D_{a+}^{\vartheta,\psi} f(\varrho) &= -\frac{(\psi(\varrho)-\psi(a))^\vartheta I_{a+}^{1-\vartheta,\psi} f(a)}{\Gamma(\vartheta+1)} \\
&\quad + \frac{1}{\Gamma(\vartheta)} \int_a^\varrho (\psi(\varrho)-\psi(s))^{\vartheta-1} \psi\prime(s) I_{a+}^{1-\vartheta,\psi} f(s) ds \\
&= -\frac{(\psi(\varrho)-\psi(a))^\vartheta I_{a+}^{1-\vartheta,\psi} f(a)}{\Gamma(\vartheta+1)} + I_{a+}^{\vartheta,\psi} I_{a+}^{1-\vartheta,\psi} f(\varrho) \\
&= I_{a+}^{1,\psi} f(\varrho) - \frac{(\psi(\varrho)-\psi(a))^\vartheta I_{a+}^{1-\vartheta,\psi} f(a)}{\Gamma(\vartheta+1)}.
\end{aligned}$$

It follows from Remark 1, that

$$\begin{aligned}
& I_{a+}^{\vartheta,\psi} D_{a+}^{\vartheta,\psi} f(\varrho) \\
= & \frac{1}{\psi\prime(\varrho)} \frac{d}{d\varrho} I_{a+}^{1,\psi} f(\varrho) - \frac{1}{\psi\prime(\varrho)} \frac{d}{d\varrho} \frac{(\psi(\varrho)-\psi(a))^\vartheta (I_{a+}^{1-\vartheta,\psi} f(a))}{\Gamma(\vartheta+1)}] \\
= & f(\varrho) - \frac{(\psi(\varrho)-\psi(a))^{\vartheta-1} I_{a+}^{1-\vartheta,\psi} f(a)}{\Gamma(\vartheta)}.
\end{aligned}$$

□

Lemma 3.2. *Let* $\alpha \in (0,1)$, $\eta > 0$ *and* $f \in C_{\eta,\psi}([a,b],E)$*. Then* $D_{a+}^{\alpha,\psi} I_{a+}^{\alpha,\psi} f(\varrho) = f(\varrho), a.e. for\ \varrho \in [a,b]$.

Proof. In view of Remark 1, we get for $a.e. \varrho \in [a,b]$

$$\begin{aligned}
D_{a+}^{\alpha,\psi} I_{a+}^{\alpha,\psi} f(\varrho) &= [\frac{1}{\psi\prime(\varrho)} \frac{d}{d\varrho}]\, I_{a+}^{1-\alpha,\psi} I_{a+}^{\alpha,\psi} f(\varrho) \\
&= [\frac{1}{\psi\prime(\varrho)} \frac{d}{d\varrho}]\, I_{a+}^{1,\psi} f(\varrho) = f(\varrho).
\end{aligned}$$

□

Now, we give in the following lemma the relation between any solution for Problem (1) and the corresponding fractional integral equation.

Lemma 3.3. *Let* $0 < \vartheta < 1$*,* $0 \le \nu \le 1$*,* $\mu = \vartheta + \nu - \vartheta\nu$ *and* $x \in PC^{\nu(1-\vartheta),\psi}_{1-\mu,\psi}([0,b],E)$*,* $g_i : (\varrho_i, s_i] \to E, i = 1, ...n$*, is continuous. The following hold.*

(1) The function $y : (0,b] \to E$ given by

$$y(\varrho) = \begin{cases} \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) + I^{\vartheta,\psi}_{0+}x(\varrho), \varrho \in (0,\varrho_1] \\ g_i(\varrho, x(\varrho_i^-)), \varrho \in (\varrho_i, s_i], i = 1, ...n, \\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)} g_i(s_i, x(\varrho_i^-)) \\ +I^{\vartheta,\psi}_{s_i+}x(\varrho), \varrho \in (s_i\ \varrho_{i+1}], i = 1, ...n, \end{cases} \tag{2}$$

belongs to $PC^{\mu}_{1-\mu,\psi}([0,b],E)$, $D^{\vartheta,\nu,\psi}_{s_i+}y(\varrho)$ exists for any $\varrho \in (s_i\ \varrho_{i+1}]$,$i = 0, 1, ...n$, and verifies the $\psi-$Hilfer fractional problem:

$$\begin{cases} D^{\vartheta,\nu,\psi}_{s_i^+}y(\varrho) = x(\varrho),\ \varrho \in \cup_{i=0}^{i=n}(s_i,\ \varrho_{i+1}] \\ I^{1-\mu}_{0^+}y(0^+) = \Psi(0), \\ y(\varrho_i^+) = g_i(\varrho_i, x(\varrho_i^-)), i = 1, ...n, \\ y(\varrho) = g_i(\varrho, x(\varrho_i^-)), \varrho \in (\varrho_i\ s_i], i = 1, ...n, \\ I^{1-\mu}_{s_i^+}y(s_i^+) = g_i(s_i, x(\varrho_i^-)), i = 1, ...n, \end{cases} \tag{3}$$

(2) If $y \in PC^{\mu}_{1-\mu,\psi}([0,b],E)$ is a solution of (3), then y satisfies (2).

Proof. Let $\varrho \in (0,\varrho_1]$. In view of (2)

$$y(\varrho) = \frac{(\psi(\varrho) - \psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) + I^{\vartheta,\psi}_{0+}x(\varrho). \tag{4}$$

According to Lemma 3 in [35], $D^{\mu,\psi}_{0^+}(\psi(\varrho) - \psi(a))^{\mu-1} = 0$. Then, by applying $D^{\mu,\psi}_{a+}$ to both side of (4), it yields

$$\begin{aligned} D_{0^+}y(\varrho) &= (\frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho})I^{(1-\mu),\psi}_{0^+}I^{\vartheta,\psi}_{0^+}x(\varrho) \\ &= (\frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho})I^{1-(\mu-\vartheta),\psi}_{0^+}x(\varrho) \\ &= D^{\mu-\vartheta,\psi}_{0^+}x(\varrho) = D^{\nu(1-\vartheta),\psi}_{0^+}x(\varrho). \end{aligned} \tag{5}$$

Observe that the assumption $x \in PC^{\nu(1-\vartheta),\psi}_{1-\mu,\psi}([0,b],E)$ implies to $D^{\nu(1-\vartheta),\psi}_{0+}x_{\ |J_0} \in C_{1-\mu,\psi}([0,\varrho_1],E)$. It follows from (5) that $D^{\mu,\psi}_{a+}y \in C_{1-\mu,\psi}([0,\varrho_1],E)$ and hence $y|_{J_0} \in C^{\mu}_{1-\mu,\psi}([0,\varrho_1],E)$. Consequently $y|_{J_0} \in C^{\vartheta,\nu}_{1-\mu,\psi}([0,\varrho_1],E)$ (see, Remark 2), and this assures that $D^{\vartheta,\nu,\psi}_{0^+}y(\varrho)$ is well defined for $\varrho \in J_0$. Now, since $D^{\nu(1-\vartheta),\psi}_{0+}x_{\ |J_0} \in C_{1-\mu,\psi}([0,\varrho_1],E)$, then $(\frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho})I^{1-\nu(1-\vartheta),\psi}_{0^+}x_{\ |J_0} \in C_{1-\mu,\psi}([0,\varrho_1],E)$. So, $I^{1-\nu(1-\vartheta),\psi}_{0^+}x_{\ |J_0} \in C^1_{1-\mu,\psi}([0,\varrho_1],E)$. As a result from Lemma (3) one obtains

$$D^{\vartheta,\nu,\psi}_{0^+}y(\varrho) \ = \ I^{\nu(1-\vartheta)}_{0^+}D^{\mu,\psi}_{0^+}y(\varrho) = I^{\nu(1-\vartheta)}_{0^+}D^{\nu(1-\vartheta),\psi}_{0^+}x(\varrho)$$

$$= \quad x(\varrho) - \frac{(\psi(\varrho)-\psi(0))^{\nu(1-\vartheta)-1}}{\Gamma(\nu(1-\vartheta))}(I_{0^+}^{1-\nu(1-\vartheta),\psi}x)(0).$$

Since $I_{0^+}^{1-\nu(1-\vartheta),\psi}x \in C_{1-\mu,\psi}([0,b],E)$ and $1-\mu < 1-\nu(1-\vartheta)$, then, by Lemma(1) $(I_{0^+}^{1-\nu(1-\vartheta),\psi}x)(0) = 0$. So, $D_{0^+}^{\vartheta,\nu,\psi}y(\varrho) = x(\varrho)$, $\varrho \in [0,\varrho_1]$. It remains to demonstrate that y satisfies $I_{0^+}^{1-\mu}y(0^+) = y_a$. To do this, apply $I_{a+}^{1-\mu,\psi}$ to both side of (4)

$$\begin{aligned} I_{0^+}^{1-\mu,\psi}y(\varrho) &= \frac{\Psi(0)}{\Gamma(\mu)}I_{0^+}^{1-\mu,\psi}(\psi(\varrho)-\psi(0))^{\mu-1}(0) + I_{0^+}^{1-\mu,\psi}I_{a+}^{\vartheta,\psi}x(\varrho) \\ &= \Psi(0) + I_{0^+}^{1-\mu+\vartheta,\psi}I_{a+}^{\vartheta,\psi}x(\varrho). \end{aligned}$$

Because $x \in C_{1-\mu,\psi}([0,\varrho_1],E)$ and $1-\mu < 1-\mu+\vartheta$, then, by Lemma(1), $I_{0^+}^{1-\mu+\vartheta,\psi}I_{0^+}^{\vartheta,\psi}x(0) = 0$. So, $I_{0^+}^{1-\mu,\psi}y(0) = \Psi(0)$.

Similarly, we can show that for $\varrho \in (s_i\ \varrho_{i+1}]$, $i = 1, ...n$,we have $D_{s_i^+}^{\vartheta,\nu,\psi}y(\varrho) = x(\varrho)$ and $I_{s_i^+}^{1-\mu}y(s_i^+) = g_i(s_i, x(\varrho_i^-))$, and hence y is a solution for(3).

(2) Let $y \in PC_{1-\mu,\psi}^{\mu}(J,E)$ be a solution of (3). Let $i = 0$. Then $y \in C_{1-\mu,\psi}(J_0,E)$ and $(\frac{1}{\psi\prime(\varrho)}\frac{d}{d\varrho})I_{0^+}^{1-\mu,\psi}y_{\ |J_0} \in C_{1-\mu,\psi}(J_0,E)$. Thus, $I_{s_i}^{1-\mu,\psi}y_{\ |J_0} \in C_{1-\mu,\psi}^{1}(J_0,E)$. By applying Lemma (3) it yields

$$\begin{aligned} I_{0^+}^{\mu,\psi}D_{0^+}^{\mu,\psi}y(\varrho) &= y(\varrho) - \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}(I_{0^+}^{1-\mu,\psi}x)(0) \\ &= y(\varrho) - \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0), \varrho \in (0,\varrho_1). \end{aligned} \tag{6}$$

Next, applying $I_{0^+}^{\vartheta,\psi}$ to both side of the equation $D_{0^+}^{\vartheta,\nu,\psi}y(\varrho) = x(\varrho)$, we get from (6)

$$\begin{aligned} I_{0^+}^{\vartheta,\psi}x(\varrho) &= I_{0^+}^{\vartheta,\psi}D_{0^+}^{\vartheta,\nu,\psi}y(\varrho) \\ &= I_{0^+}^{\vartheta,\psi}I_{0^+}^{\mu-\vartheta,\psi}D_{0^+}^{\mu,\psi}y(\varrho) \\ &= I_{0^+}^{\mu,\psi}D_{0^+}^{\mu,\psi}y(\varrho) \\ &= y(\varrho) - \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0), \varrho \in (0,\varrho_1]. \end{aligned}$$

So,

$$y(\varrho) = \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) + I_{0^+}^{\vartheta,\psi}x(\varrho), \varrho \in (0,\varrho_1].$$

Similarly, we can show that for $i = 1, ...n$,

$$y(\varrho) = \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}g_i(s_i, x(\varrho_i^-)) + I_{0^+}^{\vartheta,\psi}x(\varrho), \varrho \in (s_i\ \varrho_{i+1}].$$

□

Now, based on Lemma 5, we can give the concept of solutions for problem (1).

Definition 2. *A function $\overline{x} \in H$ is called a mild solution of* (1) *if there is $f \in S^1_{F(,\tau(\varrho)\overline{x})}$ such that $\overline{x}$ satisfies the integral equation*

$$\overline{x}(\varrho)=\begin{cases}\Psi(\varrho),\varrho\in[-r,0]\\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0)+I_{0^+}^{\vartheta,\psi}f(\varrho),\varrho\in(0,\varrho_1]\\ g_i(\varrho,x(\varrho_i^-)),\varrho\in(\varrho_i,s_i],i=1,......n,\\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}g_i(s_i,x(\varrho_i^-))\\ +I_{s_i^+}^{\vartheta,\psi}f(\varrho),\varrho\in(s_i\,\varrho_{i+1}],i=1,......n.\end{cases}$$

Remark 3.1. *The function $\overline{x}$ is not necessarily continuous at the points $s_i, i=0,1,......n$. But if $\Psi(0)=0$, then it will be continuous at s_0, and if $g_i(s_i,x(\varrho_i^-))=0$, it will be continuous at $s_i, i=1,......n$.*

In the following, we present our first existence result of solutions for Problem (1).

Theorem 3.1. *Let $F: J\times\Theta\to P_{ck}(E)$ be a multifunction, $\Psi\in\Theta$, and $g_i:[\varrho_i,s_i]\times E\to E$ $(i=1,2,....n)$. We assume the following conditions*

(F_1) For every $z\in H$, $S^1_{F(.,\tau(.)z)}$ is not empty subset of $I^{\nu(1-\vartheta),\psi}(PC_{1-\mu,\psi}(J,E))$ and for almost every $\varrho\in J, z\longrightarrow F(\varrho,\tau(\varrho)z)$ is upper semicontinuous .

(F_2) There is a $\varphi\in L^{\frac{1}{q}}(I,\mathbb{R}^+),(0<q<\vartheta)$ such that for any $z\in\mathcal{H}$

$$\|F(\varrho,\tau(\varrho)\overline{z}\|\le\varphi(\varrho)(1+\|z\|_{\mathcal{H}}),a.e,\varrho\in J$$

(F_3) There is a $\varsigma\in L^{\frac{1}{q}}(I,\mathbb{R}^+),(0<q<\vartheta)$ with the property that for any bounded subset $D\subseteq\mathcal{H}$, any $k=0,1,2,..,n$, and $a.e.$, for $\varrho\in J_k$

$$\begin{aligned}\chi_E(F(\varrho,\{\tau(\varrho)\overline{x} &: x\in D))\\ &\le (\psi(\varrho)-\psi(s_k))^{1-\mu}\varsigma(\varrho)\sup_{\theta\in[-r,0]}\chi_E\{(\tau(\varrho))\overline{x}(\theta):x\in D\},\end{aligned}$$

and

$$\frac{2\kappa_1}{\Gamma(\vartheta)}\,\eta\|\varsigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)}<1, \tag{7}$$

where $\eta=\delta^q\frac{(\psi(b)-\psi(0))^{\vartheta-q}}{(\frac{\vartheta-q}{1-q})^{1-q}}$, $\delta=\max_{s\in J}\psi\prime(s)$, $\kappa_1=(\psi(b)-\psi(0))^{1-\mu}$ and χ is the Hausdorff measure of noncompactness on E .

(H_1) for every $i=1,2,\dots,n$, $g_i:[\varrho_i,s_i]\times E\to E$ is uniformly continuous on bounded sets and for any $\varrho\in J$, $g_i(\varrho,.)$ maps any bounded subset of E to a relatively compact subset and there exists a positive constant h_i such that for any $x\in E$

$$\|g_i(\varrho,x))\|\le h_i(\psi(\varrho_i)-\psi(s_{i-1}))^{1-\mu}\|x\|,\ \varrho\in(\varrho_i,s_i],x\in E.$$

Then Problem (1) has a mild solution provided that

$$\frac{\kappa_1}{\Gamma(\vartheta)}\|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\eta+h^*+\frac{h^*}{\Gamma(\mu)}<1, \tag{8}$$

where, $h^*=\sum_{i=0}^{i=n}h_i$.

Proof. We define a multioperator $\Phi : \mathcal{H} \to P(\mathcal{H})$ as follows: let $x \in \mathcal{H}$, then due to (F_1) there is $f \in S^1_{F(.,\tau(.)\overline{x})}$, and hence we can define $y \in \Phi(x)$ if and only if

$$y(\varrho) = \begin{cases} 0, \varrho \in [-r, 0] \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) + I^{\vartheta,\psi}_{0^+} f(\varrho), \varrho \in (0, \varrho_1] \\ g_i(\varrho, x(\varrho_i^-)), \varrho \in (\varrho_i, s_i], i = 1, ...n, \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)} g_i(s_i, x(\varrho_i^-)) \\ +I^{\vartheta,\psi}_{s_i^+} f(\varrho), \varrho \in (s_i\ \varrho_{i+1}], i = 1, ...n. \end{cases} \tag{9}$$

Let us clarify that a point x is a fixed point for Φ if and only if $\overline{x}$ is a solution for (1). Let x be a fixed point to Φ. Then

$$x(\varrho) = \begin{cases} 0, \varrho \in [-r, 0] \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) + I^{\vartheta,\psi}_{0^+} f(\varrho), \varrho \in (0, \varrho_1] \\ g_i(\varrho, x(\varrho_i^-)), \varrho \in (\varrho_i, s_i], i = 1, ...n, \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)} g_i(s_i, x(\varrho_i^-)) \\ +I^{\vartheta,\psi}_{s_i^+} f(\varrho), \varrho \in (s_i\ \varrho_{i+1}], i = 1, ...n. \end{cases} \tag{10}$$

where $f \in S^1_{F(.,\tau(.)\overline{x})}$. Therefore,

$$\overline{x}(\varrho) = \begin{cases} \Psi(\varrho), \varrho \in [-r, 0] \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) + I^{\vartheta,\psi}_{0^+} f(\varrho)ds, \varrho \in (0, \varrho_1] \\ g_i(\varrho, x(\varrho_i^-)), \varrho \in (\varrho_i, s_i], i = 1, ...n, \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)} g_i(s_i, x(\varrho_i^-)) \\ +I^{\vartheta,\psi}_{s_i^+} f(\varrho), \varrho \in (s_i\ \varrho_{i+1}], i = 1, ...n. \end{cases}$$

This means $\overline{x}$ satisfies (4), and hence it is a solution for (1). Similarly, it is easy to see that if $\overline{x}$ satisfies (4), then x is a fixed point for Φ. So we prove, by application Lemma 5, that Φ has a fixed point. Obviously the values of Φ are convex.

Step1. We demonstrate that there is a $n \in \mathbb{N}$ with $\Phi(B_n) \subseteq B_n$,where $B_n = \{x \in \mathcal{H} : \|x\|_{\mathcal{H}} \leq n\}$. Suppose that for any natural number n, there are $x_n, y_n \in \mathcal{H}$ with $y_n \in \Phi(x_n)$, $\|x_n\|_{\mathcal{H}} \leq n$ and $\|y_n\|_{\mathcal{H}} > n$. Then, there is $f_n \in S^1_{F(.,\tau(.)\overline{x}_n)}$, $n \geq 1$ such that

$$y_n(\varrho) = \begin{cases} 0, \varrho \in [-r, 0] \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) + I^{\vartheta,\psi}_{0^+} f_n(\varrho) \in (0, \varrho_1] \\ g_i(\varrho, x_n(\varrho_i^-)), \varrho \in (\varrho_i, s_i], i = 1, ...n, \\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)} g_i(s_i, x_n(\varrho_i^-)) \\ +I^{\vartheta,\psi}_{s_i^+} f_n(\varrho), \varrho \in (s_i\ \varrho_{i+1}], i = 1, ...n. \end{cases}$$

In view of (F_2), we get for almost $\varrho \in J$

$$\begin{aligned} \|f_n(\varrho)\| &\leq \varphi(\varrho)(1 + \|x_n\|_{\mathcal{H}}) \\ &\leq \varphi(\varrho)(1 + n). \end{aligned} \tag{11}$$

Then, if for almost $\varrho \in (0, \varrho_1]$, we get from Holder's inequality and (12)

$$\begin{aligned} & (\psi(\varrho) - \psi(0))^{1-\mu} \|y_n\ (\varrho)\| \\ \leq\ & \frac{\Psi(0)}{\Gamma(\mu)} + \frac{(1+n)(\psi(b) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)} \int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\vartheta-1} \psi\prime(s)\varphi(\varrho) ds \end{aligned}$$

Now, from Holder's inequality

$$\begin{aligned} & \int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\vartheta-1} \psi\prime(s)\varphi(\varrho) ds \\ \leq\ & \|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} [\int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\frac{\vartheta-1}{1-q}} (\psi\prime(s))^{\frac{1}{1-q}} ds]^{1-q} \\ \leq\ & \|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} [\int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\frac{\vartheta-1}{1-q}} \psi\prime(s)(\psi\prime(s))^{\frac{q}{1-q}} ds]^{1-q} \\ \leq\ & \delta^q \|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} [\int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\frac{\vartheta-1}{1-q}} \psi\prime(s) ds]^{1-q} \\ \leq\ & \delta^q \frac{(\psi(b) - \psi(0))^{\vartheta-q}}{(\frac{\vartheta-q}{1-q})^{1-q}} = \eta. \end{aligned} \tag{12}$$

Then,

$$\begin{aligned} & (\psi(\varrho) - \psi(0))^{1-\mu} \|y_n\ (\varrho)\| \\ \leq\ & \frac{\Psi(0)}{\Gamma(\mu)} + \frac{(1+n)(\psi(b) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)} \int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\vartheta-1} \psi\prime(s)\varphi(\varrho) ds \\ \leq\ & \frac{\Psi(0)}{\Gamma(\mu)} + \frac{\kappa_1}{\Gamma(\vartheta)} \|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} (1+n)\eta. \end{aligned} \tag{13}$$

If $\varrho \in (\varrho_i\ s_i], = 1, 2, \ldots, n$, then by (H_1)

$$\begin{aligned} \|y_n(\varrho)\| &= \sup_{\varrho\in(\varrho_i s_i]} \|g_i(\varrho, x_n(\varrho_i^-))\| \\ &\leq h^*(\psi(\varrho_i)\ - \psi(s_{i-1}))^{1-\mu} \|x_n(\varrho_i^-)\| \\ &\leq h^* \|x_n\|_{\mathcal{H}} \leq hn. \end{aligned} \tag{14}$$

Similar as in (13), we get for almost $\varrho \in (s_i\ \varrho_{i+1}], i = 1, 2, \ldots, n$.

$$\begin{aligned} & (\psi(\varrho) - \psi(s_i))^{1-\mu} \|y_n(\varrho)\| \\ \leq\ & \frac{\|g_i(s_i, x_n(\varrho_i^-))\|}{\Gamma(\mu)} \\ & + \frac{\kappa_1}{\Gamma(\vartheta)} \|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} (1+n)\eta \\ \leq\ & \frac{h^* n}{\Gamma(\mu)} + \frac{\kappa_1}{\Gamma(\vartheta)} \|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} (1+n)\eta. \end{aligned}$$

It follows from this inequality, (13) and (14), that

$$n < \|y_n\|_{\mathcal{H}} \leq \frac{\Psi(0)}{\Gamma(\mu)} + h^* n + \frac{h^* n}{\Gamma(\mu)}$$

$$+\frac{\kappa_1}{\Gamma(\vartheta)}\|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}(1+n)\eta.$$

By dividing both side by n and taking the limit as $n \to \infty$, one obtains

$$1 < \frac{\kappa_1}{\Gamma(\vartheta)}\|\varphi\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\eta + h^* + \frac{h^*}{\Gamma(\mu)},$$

which contradicts with (8). Thus, there is a natural number n_0 such that $\Phi(B_{n_0}) \subseteq \Phi(B_{n_0})$.

Step2. Φ is closed on B_{n_0}.

Let $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$be two sequences in B_{n_0} with $x_n \to x$ in $B_{n_0}, y_n \to y$ in $\mathcal{H}$ and $y_n \in \Phi(x_n); n \geq 1$. Then, there is $f_n \in S^1_{F(.,\tau(.)\overline{x}_n)}$ such that

$$y_n(\varrho) = \begin{cases} 0, \varrho \in [-r, 0] \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) \\ +\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)f_n(s)ds, \varrho \in (0, \varrho_1] \\ g_i(\varrho, x_n(\varrho_i^-)), \varrho \in (\varrho_i, s_i], i = 1, ...n, \\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}g_i(s_i, x_n(\varrho_i^-)) \\ +\frac{1}{\Gamma(\vartheta)}\int_{s_i}^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)f_n(s), \varrho \in (s_i\ \varrho_{i+1}], i = 1, ...n. \end{cases} \tag{15}$$

According to (11), it follows that $\|f_n(\varrho)\| \leq \varphi(\varrho)(1+n), a.e.\varrho \in J$, and hence $(f_n)_{n\geq 1}$ is bounded in $L^{\frac{1}{q}}(J, E)$. Since $L^{\frac{1}{q}}(J, E)$ is reflexive, then, by using Mazur's Lemma, there is sequence, $(z_j)_{j\geq 1}$, of convex combinations of $(f_n)_{n\geq 1}$ converging strongly to f in $L^1(J, E)$ as $j \to \infty$. Notice that, by (F_2) again, for every $\varrho \in J$, $s \in (0, \varrho]$ and every $n \geq 1$

$$\|(\varrho - s)^{\vartheta-1}f_n(s)\| \leq |\varrho - s)|^{\vartheta-1}\ \varphi(\varrho)(1+n) \in L^1((0, \varrho], \mathbb{R}^+).$$

Let

$$\widetilde{y}_n(\varrho) = \begin{cases} 0, \varrho \in [-r, 0] \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) \\ +\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)z_n(s)ds, \varrho \in (0, \varrho_1] \\ g_i(\varrho, x_n(\varrho_i^-)), \varrho \in (\varrho_i, s_i], i = 1, ...n, \\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}g_i(s_i, x_n(\varrho_i^-)) \\ +\frac{1}{\Gamma(\vartheta)}\int_{s_i}^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)z_n(s), \varrho \in (s_i\ \varrho_{i+1}], i = 1, ...n. \end{cases} \tag{16}$$

Clearly $\widetilde{y}_n(\varrho) \to y(\varrho)$,$\varrho \in J$ and $z_n(\varrho) \to f(\varrho)$, for almost $\varrho \in J$. Also, $\tau(\varrho)\overline{x}_n \to \tau(\varrho)\overline{x}; \varrho \in J$, and hence, by the upper semicontinuity of $F(\varrho, .); a.e.\ \varrho \in J$, it follows that $f(\varrho) \in F(\varrho, \tau(\varrho)\overline{x}), a.e.$[[37]. Theorem1, Sec. 4, Ch.1]. Therefore, by, the uniform continuity of $g_i(s_i, .)$ on bounded subsets and by passing to the limit as $n \to \infty$ in (15) we obtain, from the Lebesgue dominated convergence theorem $y \in \Phi(x)$.

Step3. We show that $M_{|\overline{J_k}}$ $(k = 0, 1, ..., n)$ and $M_{|\overline{\varrho_i}}$ $(i = 1, 2, ..., n)$ are equicontinuous, where

$$\begin{aligned} M_{|\overline{J_k}} &= \{z : \overline{J_k} \to E,\ z(\varrho) = (\psi(\varrho) - \psi(s_k))^{1-\mu}y(\varrho), \varrho \in J_k, \\ z(s_k) &= \lim_{\varrho \to s_k}(\psi(\varrho) - \psi(s_k))^{1-\mu}y(\varrho),\ y \in \Phi(x), x \in B_{n_0}\}, \end{aligned}$$

and

$$\begin{aligned} M_{|\overline{\varrho_i}} &= \{y^* \in C(\overline{\varrho_i}, E) : y^*(\varrho) = y(\varrho), \varrho \in (\varrho_i, s_i], \\ y^*(\varrho_i) &= y(\varrho_i^+), y \in \Phi(x), x \in B_{n_0}\}. \end{aligned}$$

Case 1. let $z \in M_{|\overline{J_0}}$.Then there are $x \in B_{n_0}$and $y \in \Phi(x)$ such that for $\varrho \in (0, \varrho_1]$,

$$y(\varrho) = \frac{(\psi(\varrho) - \psi(0))^{1-\mu}\Psi(0)}{\Gamma(\mu)} + \frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho) - \psi(s))^{\vartheta-1}\psi\prime(s)f(s)ds,$$

$z(\varrho) = (\psi(\varrho) - \psi(0))^{1-\mu}y(\varrho)$ and $z(0) = \lim_{\varrho\to 0+}(\psi(\varrho) - \psi(0))^{1-\mu}y(\varrho)$, where $f \in S^1_{F(.,\tau(.)\overline{x})}$. It follows

$$\begin{aligned} \lim_{\delta\to 0+} z(0+\delta) &= \lim_{\delta\to 0+} z(\delta) \\ &= \lim_{\delta\to 0+}(\psi(\delta) - \psi(0))^{1-\mu}y(\delta) = z(0). \end{aligned}$$

Let ϱ_1, ϱ_2 be two points in $(0, \varrho_1]$ be such that $\varrho_1 < \varrho_2$. Then,

$$\begin{aligned} &\|z(\varrho_2) - z(\varrho_1)\| \\ \leq\ &\|\frac{(\psi(\varrho_2) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)}\int_0^{\varrho_2}(\psi(\varrho_2) - \psi(s))^{\vartheta-1}\psi\prime(s)f(s)ds \\ &-\frac{(\psi(\varrho_1) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)}\int_0^{\varrho_1}(\psi(\varrho_1) - \psi(s))^{\vartheta-1}\psi\prime(s)f(s)ds\| \\ \leq\ &\frac{(\psi(\varrho_2) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)}\|\int_{\varrho_1}^{\varrho_2}(\psi(\varrho_2) - \psi(s))^{\vartheta-1}\psi\prime(s)f(s)ds\| \\ &+\|\int_0^{\varrho_1}(\psi(\varrho_1) - \psi(s))^{\vartheta-1}\psi\prime(s)\|f(s)\|ds \times \\ &\left|\frac{(\psi(\varrho_2) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)} - \frac{(\psi(\varrho_1) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)}\right| \\ &+\frac{(\psi(\varrho_1) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)} \times \\ &\|\int_0^{\varrho_1}[(\psi(\varrho_2) - \psi(s))^{\vartheta-1}\psi\prime(s) - (\psi(\varrho_1) - \psi(s))^{\vartheta-1}\psi\prime(s)]\|f(s)\|ds\|. \end{aligned}$$

By the absolute integral of the Lebesgue integral and Holder's inequality, it yields from (12)

$$\begin{aligned} &\lim_{\varrho_2\to\varrho_1}\frac{(\psi(\varrho_2) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)}\int_{\varrho_1}^{\varrho_2}(\psi(\varrho_2) - \psi(s))^{\vartheta-1}\psi\prime(s)\|f(s)\|ds \\ \leq\ &\frac{\kappa_1}{\Gamma(\vartheta)}(1+n_0)\lim_{\varrho_2\to\varrho_1}\int_{\varrho_1}^{\varrho_2}(\psi(\varrho_2) - \psi(s))^{\vartheta-1}\psi\prime(s)\varphi(s)ds \\ \leq\ &\frac{\kappa_1}{\Gamma(\vartheta)}(1+n_0) = 0, \end{aligned}$$

independent of x.

Since ψ is continuous, we get by (12)

$$\begin{aligned}
& \| \int_0^{\varrho_1} \frac{(\psi(\varrho_1)-\psi(s))^{\vartheta-1}\psi\prime(s)}{\Gamma(\vartheta)} f(s)ds\| \times \\
& \left|(\psi(\varrho_2)-\psi(0))^{1-\mu}-(\psi(\varrho_1)-\psi(0))^{1-\mu}\right| \\
\leq\ & (1+n_0)\int_0^{\varrho_1} \frac{(\psi(\varrho_1)-\psi(s))^{\vartheta-1}\psi\prime(s)}{\Gamma(\vartheta)} \varphi(s)ds \times \\
& \lim_{\varrho_2\to\varrho_1} \left|(\psi(\varrho_2)-\psi(0))^{1-\mu}-(\psi(\varrho_1)-\psi(0))^{1-\mu}\right| \\
=\ & 0
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \lim_{\varrho_2\to\varrho_1} \frac{(\psi(\varrho_1)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)} \times \\
& \| \int_0^{\varrho_1} [(\psi(\varrho_2)-\psi(s))^{\vartheta-1}\psi\prime(s)-(\psi(\varrho_1)-\psi(s))^{\vartheta-1}\psi\prime(s)] f(s)ds\| \\
\leq\ & \frac{\kappa_1}{\Gamma(\vartheta)}(1+n_0)\times \\
& \lim_{\varrho_2\to\varrho_1} \int_0^{\varrho_1} [(\psi(\varrho_2)-\psi(s))^{\vartheta-1}\psi\prime(s)-(\psi(\varrho_1)-\psi(s))^{\vartheta-1}\psi\prime(s)]\varphi(s)ds \\
\leq\ & \frac{\kappa_1}{\Gamma(\vartheta)}(1+n_0)\|\varphi\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)} \times \\
& \lim_{\varrho_2\to\varrho_1} [\int_0^{\varrho_1} [(\psi(\varrho_2)-\psi(s))^{\vartheta-1}\psi\prime(s)-(\psi(\varrho_1)-\psi(s))^{\vartheta-1}\psi\prime(s)]^{\frac{1}{1-q}} ds]^{1-q}
\end{aligned}$$

Put $\overline{\omega}=\frac{\vartheta-1}{1-q}\in(-1,0)$, then for $s<\varrho_1<\varrho_2$, we have $(\psi(\varrho_1)-\psi(s))^{\overline{\omega}}\geq(\psi(\varrho_2)-\psi(s))^{\overline{\omega}}$. By applying Lemma 3 in [43] and taking into account $1-q\in(0,1)$, we get

$$\begin{aligned}
& |\left[(\psi(\varrho_1)-\psi(s))^{\overline{\omega}}\right]^{1-q}-\left[(\psi(\varrho_2)-\psi(s))^{\overline{\omega}}\right]^{1-q}| \\
\leq\ & \left[(\psi(\varrho_1)-\psi(s))^{\overline{\omega}}-(\psi(\varrho_2)-\psi(s))^{\overline{\omega}}\right]^{1-q}.
\end{aligned}$$

Then,

$$\begin{aligned}
& |(\psi(\varrho_1)-\psi(s))^{\vartheta-1}-(\psi(\varrho_2)-\psi(s))^{\vartheta-1}| \\
\leq\ & \left[(\psi(\varrho_1)-\psi(s))^{\overline{\omega}}-(\psi(\varrho_2)-\psi(s))^{\overline{\omega}}\right]^{1-q}.
\end{aligned}$$

This leads to

$$\begin{aligned}
& |(\psi(\varrho_1)-\psi(s))^{\vartheta-1}-(\psi(\varrho_2)-\psi(s))^{\vartheta-1}|^{\frac{1}{1-q}} \\
\leq\ & (\psi(\varrho_1)-\psi(s))^{\overline{\omega}}-(\psi(\varrho_2)-\psi(s))^{\overline{\omega}}.
\end{aligned}$$

Therefore,

$$\lim_{\varrho_2\to\varrho_1} [\int_0^{\varrho_1} [(\psi(\varrho_2)-\psi(s))^{\vartheta-1}\psi\prime(s)-(\psi(\varrho_1)-\psi(s))^{\vartheta-1}\psi\prime(s)]^{\frac{1}{1-q}} ds]^{1-q}$$

$$
\begin{aligned}
&\le \lim_{\varrho_2\to\varrho_1}[\int_0^{\varrho_1}[(\psi(\varrho_2)-\psi(s))^{\vartheta-1}-(\psi(\varrho_1)-\psi(s))^{\vartheta-1}]^{\frac{1}{1-q}}(\psi\prime(s))^{\frac{1}{1-q}}ds]^{1-q}\\
&\le \lim_{\varrho_2\to\varrho_1}[\int_0^{\varrho_1}[(\psi(\varrho_1)-\psi(s))^{\overline{\omega}}-(\psi(\varrho_2)-\psi(s))^{\overline{\omega}}](\psi\prime(s))^{\frac{1}{1-q}}ds]^{1-q}\\
&\le \delta^q\lim_{\varrho_2\to\varrho_1}[\int_0^{\varrho_1}[(\psi(\varrho_1)-\psi(s))^{\overline{\omega}}-(\psi(\varrho_2)-\psi(s))^{\overline{\omega}}]\psi\prime(s)ds]^{1-q}\\
&= \delta^q\lim_{\varrho_2\to\varrho_1}[\frac{(\psi(\varrho_1)-\psi(s))^{\overline{\omega}+1}}{\overline{\omega}+1}-\frac{(\psi(\varrho_2)-\psi(s))^{\overline{\omega}+1}}{\overline{\omega}+1}]^{1-q}=0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\lim_{\varrho_2\to\varrho_1}\frac{(\psi(\varrho_1)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)}\times\\
&\|\int_0^{\varrho_1}[(\psi(\varrho_2)-\psi(s))^{\vartheta-1}\psi\prime(s)-(\psi(\varrho_1)-\psi(s))^{\vartheta-1}\psi\prime(s)]f(s)ds\|\\
=\ &0
\end{aligned}
$$

independent of x.

Case 2. Let $i\in\{1,2,..,n\}$ be fixed and set $K_i=\{x(\varrho_i^-):x\in B_{n_0}\}$.Obviously$K_i$ is bounded subset of E.

Let $y\in M_{|\varrho_i}$. Then, there is $x\in B_{n_0}$ such that

$$y(\varrho)=g_i(\varrho,x(\varrho_i^-)),\varrho\in(\varrho_i,s_i]$$

It follows, from the uniform continuity of g_i on the bounded set $[\varrho_i,s_i]\times K_i$, that for $\varrho_1,\varrho_2\in(\varrho_i,s_i]$

$$\lim_{\varrho_2\to\varrho_1}\|y(\varrho_2)-y(\varrho_1)\|=\lim_{\varrho_2\to\varrho_1}\|g_i(\varrho_2,x(\varrho_i^-))-g_i(\varrho_1,x(\varrho_i^-))\|=0,$$

independent of x. When $\varrho=\varrho_i$, let $\delta>0$ be such that $\varrho_i+\delta\in(\varrho_i,s_i]$ and $\lambda>0$ with $\varrho_i<\lambda<\varrho_i+\delta\le s_i$. Then, we have

$$\lim_{\delta\to0+}\|y^*(\varrho_i+\delta)-y^*(\varrho_i)\|=\lim_{\delta\to0+}\lim_{\lambda\to\varrho_i^+}\|y(\varrho_i+\delta)-y(\lambda)\|=0.$$

Case 3. let $k=1,\dots,n$ be fixed ,$z\in M_{|\overline{J_k}}$. Then, there are $x\in B_{n_0}$and $y\in\Phi(x)$ such that for $\varrho\in\varrho\in(s_k,\varrho_{k+1}]$,

$$
\begin{aligned}
y(\varrho)\ &=\ \frac{(\psi(\varrho)-\psi(s_k))^{\mu-1}}{\Gamma(\mu)}g_k(s_k,x_n(\varrho_k^-))\\
&\quad+\frac{1}{\Gamma(\vartheta)}\int_{s_k}^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)z_n(s),
\end{aligned}
$$

$z(\varrho)=(\psi(\varrho)-\psi(s_k))^{1-\mu}y(\varrho)$ and $z(s_k)=\lim_{\varrho\to s_k+}(\psi(\varrho)-\psi(s_k))^{1-\mu}y(\varrho)$, where $f\ \in S^1_{F(.,\tau(.)\overline{x})}$. It follows

$$
\begin{aligned}
\lim_{\delta\to0+}z(s_k+\delta)\ &=\ \lim_{\delta\to0+}(\psi(s_k+\delta)-\psi(s_k))^{1-\mu}y(s_k+\delta)\\
&=\ \lim_{\varrho\to s_k+}(\psi(\varrho)-\psi(s_k))^{1-\mu}y(\varrho)=z(s_k).
\end{aligned}
$$

Next, let $\varrho_1, \varrho_2 \in (s_k, \varrho_{k+1}](\varrho_1 < \varrho_2)$.By arguing as in case 1, one can show that

$$\lim_{\varrho_2 \to \varrho_1} \|z(\varrho + \delta) - z(\varrho)\| = 0,$$

independent of x.

Step 4. Let $K \subseteq B_{n_0}$, $K = conv(\{0\} \cup \Phi(K))$, $\overline{Z} = \overline{C}$ with $C \subseteq K$ countable. We have to show that K is relatively compact in $\mathcal{H}$ Let $D = \{y_n : n \geq 1\} \subseteq \Phi(K)$ with $C \subseteq conv(\{x_0\} \cup D)$, $x_n \in K$ with $y_n \in \Phi(x_n)$.This means that, there is $f_n \in S^1_{F(.,\tau(.)\overline{x}_n)}$ such that the relation (15) holds. Observe that, from (F_3) it holds for $a.e.s \in J_0$

$$\begin{aligned} \chi_E\{f_n(s) \quad &: \quad n \geq 1\} \leq (\psi(\varrho) - \psi(0))^{1-\mu} \varsigma(\varrho) \chi\{F(s, \tau(s)\overline{x}_n) : n \geq 1\} \\ &\leq \zeta(s)(\psi(\varrho) - \psi(0))^{1-\mu} \sup_{\theta \in [-r,0]} \chi\{\overline{x}_n(s + \theta) : n \geq 1\} \\ &\leq \zeta(s)(\psi(\varrho) - \psi(0))^{1-\mu} \sup_{\delta \in [-r,s]} \chi\{\overline{x}_n(\delta) : n \geq 1\}. \end{aligned}$$

Because $\sup_{\delta \in [-r,0]} \chi\{\overline{x}_n(\delta) : n \geq 1\} = \sup_{\delta \in [-r,0]} \chi\{\Psi(\delta)\} = 0$. Thus,

$$\begin{aligned} \chi_E\{f_n(s) \quad &: \quad n \geq 1\} \\ &\leq \zeta(s)(\psi(\varrho) - \psi(0))^{1-\mu} \sup_{\delta \in [0,s]} \chi\{x_n(\delta) : n \geq 1\} \\ &= \zeta(s) \sup_{\delta \in [0,s]} \chi\{(\psi(\varrho) - \psi(0))^{1-\mu} x_n(\delta) : n \geq 1\} \\ &\leq \zeta(s) \chi_{PC_{1-\mu,\psi}(J,E)}\{x_n : n \geq 1\} \\ &\leq \zeta(s) \chi_{\mathcal{H}}(B_{n-1}). \end{aligned} \tag{17}$$

According to the definition $\chi_{\mathcal{H}}(Z)$, one obtains

$$\begin{aligned} \chi_{\mathcal{H}}(Z) &= \chi_{\mathcal{H}}(\overline{Z}) = \chi_{\mathcal{H}}(\overline{C}) \\ &= \chi_{\mathcal{H}}(C)) \leq \chi_{\mathcal{H}}(conv(\{x_0\} \cup D)) \\ &= \chi_{\mathcal{H}}(D) \\ &= \max\{\max_{k=0,1,..,n} \chi_{C(\overline{J_k},E)}(D_{|\overline{J_k}}), \max_{i=1,..,n} \chi_{C(\overline{\varrho_i},E)}(D_{|\overline{\varrho_i}})\}. \end{aligned} \tag{18}$$

In view of Step 3, $D_{|\overline{J_i}}$ and $D_{|\overline{\varrho_i}}$ are equicontinuous, consequently

$$\begin{aligned} &\chi_{\mathcal{H}}(K) \\ \leq \quad &\max\{\max_{i=0,1,..,n} \max_{\varrho \in \overline{J_k}} \chi\{y_n^*(\varrho) : n \geq 1\}, \max_{i=1,..,n} \max_{\varrho \in \overline{\varrho_i}} \chi\{y_n^*(\varrho) : n \geq 1\}\}, \end{aligned} \tag{19}$$

Let $\varrho \in (0, \varrho_1]$. Then,

$$y_n^*(\varrho) = \frac{\Psi(0)}{\Gamma(\mu)} + \frac{(\psi(\varrho) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)} \int_0^{\varrho} (\psi(\varrho) - \psi(0))^{\vartheta-1} \psi\prime(s) f_n(s) ds,$$

which yields with (17)

$$\chi\{y_n^*(\varrho) \quad : \quad n \geq 1\}$$

$$\begin{aligned}
&\leq \frac{(\psi(\varrho)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(0))^{\vartheta-1}\psi\prime(s)\chi\{f_n(s):n\geq 1\}ds\\
&\leq \frac{(\psi(b)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)}\chi_{\mathcal{H}}(B_{n-1})\int_0^{\varrho}(\psi(\varrho)-\psi(0))^{\vartheta-1}\psi\prime(s)\zeta(s)ds\\
&\leq \frac{\kappa_1}{\Gamma(\vartheta)}\,\eta\|\varsigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)}\chi_{\mathcal{H}}(B_{n-1}). \qquad (20)
\end{aligned}$$

Notice that

$$\chi\{y_n^*(0):n\geq 1\}=\chi\{\lim_{\varrho\to 0^+}y_n^*(\varrho):n\geq 1\}=\chi\{\frac{\Psi(0)}{\Gamma(\mu)}\}=0. \qquad (21)$$

Moreover, from the fact that $\|x_n\|_{\mathcal{H}}\leq n_0$,the set $\{x_n(\varrho_i^-):n\geq 1\}$ is bounded for every $i=1,2,...,n$, and hence from the assumption (H_1), we get

$$\chi\{g_i(\varrho,x_n(\varrho_i^-)),n\geq 1\}=0,\varrho\in(\varrho_i,s_i],i=1,...,n. \qquad (22)$$

and

$$\chi\{g_i(\varrho_i,x_n(\varrho_i^-)),n\geq 1\}=\chi\{g_i(s_i,x_n(\varrho_i^-)):n\geq 1\}=0. \qquad (23)$$

Similarly,

$$\begin{aligned}
\chi\{y_n^*(s_i) \quad &: \quad n\geq 1\}=\chi\{\lim_{\varrho\to s_i^+}(\psi(\varrho)-\psi(s_i))^{1-\mu}y_n(\varrho),:n\geq 1\}\\
&= \chi\{\frac{1}{\Gamma(\mu)}g_i(s_i,x_n(\varrho_i^-)):n\geq 1\}=0,i=1,2,..,n. \qquad (24)
\end{aligned}$$

By arguing as in (20), one can show that

$$\begin{aligned}
&\max_{\varrho\in\overline{J_k}}\chi\{y_n^*(\varrho) \quad : \quad n\geq 1\}\\
&\qquad \leq \frac{\kappa_1}{\Gamma(\vartheta)}\,\eta\|\varsigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)}\chi_{\mathcal{H}}(B_{n-1}),k=1,2,..n. \qquad (25)
\end{aligned}$$

From the relations (7), (19)–(25), it follows that

$$\begin{aligned}
&\chi_{\mathcal{H}}(K)\\
&\leq \chi_{\mathcal{H}}(K)\frac{2\kappa_1}{\Gamma(\vartheta)}\,\eta\|\varsigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)}\\
&< \chi_{\mathcal{H}}(K).
\end{aligned}$$

So, $\chi_{PC_{1-\mu}(J,E)}(K)=0$, and hence K is relatively compact.

Step5. Φ maps compact sets into relatively compact sets.

Let B be a compact subset of B_{n_0},$\{y_n,:n\geq 1\}\subseteq\Phi(B)$ Then, there is $x_n\in B,n\geq 1$, such that $y_n\in\Phi(x_n)$. So, there is $f_n\in S^1_{F(.,\overline{x_n}(.))}$ such that (15) holds. We have to show that the set $Z=\{y_n:n\geq 1\}$ is relatively compact in $\mathcal{H}$. Note that, since B is compact in $\mathcal{H}$, then from (F_3) we get for $a.e.$ $s\in J_0$,

$$\begin{aligned}
\chi_E\{f_n(s) \quad &: \quad n\geq 1\}\leq(\psi(\varrho)-\psi(0))^{1-\mu}\varsigma(\varrho)\chi\{F(s,\tau(s)\overline{x}_n):n\geq 1\}\\
&\leq \zeta(s)(\psi(\varrho)-\psi(0))^{1-\mu}\sup_{\theta\in[-r,0]}\chi\{\overline{x}_n(s+\theta):n\geq 1\}
\end{aligned}$$

$$\begin{aligned}
&\leq \zeta(s)(\psi(\varrho)-\psi(0))^{1-\mu}\sup_{\delta\in[-r,s]}\chi\{\overline{x}_n(\delta):n\geq 1\}\\
&\leq \zeta(s)(\psi(\varrho)-\psi(0))^{1-\mu}\sup_{\delta\in[0,s]}\chi\{x_n(\delta):n\geq 1\}\\
&= \zeta(s)\sup_{\delta\in[0,s]}\chi\{(\psi(\varrho)-\psi(0))^{1-\mu}x_n(\delta):n\geq 1\}\\
&\leq \zeta(s)\chi_{PC_{1-\mu,\psi}(J,E)}\{x_n:n\geq 1\}\\
&\leq \zeta(s)\chi_{\mathcal{H}}(B)=0.
\end{aligned}$$

By the same reasons, one can show that for *a.e.* $s\in J_k$, $k=1,2,..,n$

$$\chi_E\{f_n(s):n\geq 1\}=0.$$

By arguing as in the previous step one can show that Z is relatively compact, and hence $\Phi(B)$ is relatively compact.

Now, by applying Lemma 8, there is $x\in\mathcal{H}$ and$f\in S^1_{F(.,\tau(.)\overline{x})}$ such that

$$x(\varrho)=\begin{cases}0,\varrho\in[-r,0]\\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0)\\ +\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)f(s)ds,\varrho\in(0,\varrho_1]\\ g_i(\varrho,x(\varrho_i^-)),\varrho\in(\varrho_i,s_i],i=1,......n,\\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}g_i(s_i,x(\varrho_i^-))\\ +\frac{1}{\Gamma(\vartheta)}\int_{s_i}^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)f(s),\varrho\in(s_i\,\varrho_{i+1}],i=1,......n.\end{cases}$$

Next, in view of (F_1), there is $h\in PC_{1-\mu,\psi}(J,E)$ such that $f(\varrho)=I_{s_i+}^{\nu(1-\vartheta),\psi}h(\varrho),\varrho\in J_k,k=0,..,n$, and hence, from Lemma 5,

$$D_{s_i+}^{\nu(1-\vartheta),\psi}f(\varrho)=D_{s_i+}^{\nu(1-\vartheta),\psi}I_{s_i+}^{\nu(1-\vartheta)\psi}h(\varrho)=h(\varrho),\varrho\in J_k,k=0,..,n.$$

This yields that $f\in PC_{1-\mu,\psi}^{\nu(1-\vartheta)}(J,E)$. Then, the function

$$\overline{x}(\varrho)=\begin{cases}\Psi(\varrho),\varrho\in[-r,0]\\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0)+\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)f(s)ds,\varrho\in(0,\varrho_1]\\ g_i(\varrho,x(\varrho_i^-)),\varrho\in(\varrho_i,s_i],i=1,......n,\\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}g_i(s_i,x(\varrho_i^-))\\ +\frac{1}{\Gamma(\vartheta)}\int_{s_i}^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s),\varrho\in(s_i\,\varrho_{i+1}],i=1,......n.\end{cases}\tag{26}$$

belongs to H and in view of Lemma(6) it is a solution for (1). This completes the proof. □

Remark 3.2. *Theorem 1 remains true if condition (F_2) is replaced by the following assumption:*

$(F_2)^*$There is a $\varphi\in L^{\frac{1}{q}}(I,\mathbb{R}^+)$ such that for any $z\in\Theta$

$$\|F(\varrho,z)\|\leq\begin{cases}\varphi(\varrho)\ (1+(\psi(\varrho)-\psi(s_i))^{1-\mu}\,\|z(0)\|_E),a.e.\varrho\in\cup_{i=0}^{i=n}(s_i\,\varrho_{i+1}],\\ \varphi(\varrho)\ (1+\|z(0)\|_E),a.e.\varrho\in\cup_{i=1}^{i=n}(\varrho_i,s_i].\end{cases}$$

In fact, condition (F_2) is used only to prove relation (11). We show this relation by using $(F_2)^*$. Let $f_n \in S^1_{F(.,\tau(.)\overline{x}_n)}$, $n \geq 1$ Then, by $(F_2)^*$, for almost $\varrho \in \cup_{i=0}^{i=n}(s_i\ \varrho_{i+1}]$ we get

$$\begin{aligned}
\|f_n(\varrho)\| &\leq \varphi(\varrho)(1 + (\psi(\varrho) - \psi(s_i))^{1-\mu}\|\tau(\varrho)\overline{x}_n(0)\|)\\
&\leq \varphi(\varrho)(1 + (\psi(\varrho) - \psi(s_i))^{1-\mu}\|\overline{x}_n(\varrho)\|)\\
&\leq \varphi(\varrho)(1 + (\psi(\varrho) - \psi(s_i))^{1-\mu}\|x_n(\varrho)\|\\
&\leq \varphi(\varrho)(1 + \|x_n\|_{\mathcal{H}}).
\end{aligned}$$

and for almost $\varrho \in \cup_{i=1}^{i=n}(\varrho_i\ s_i]$

$$\begin{aligned}
\|f_n(\varrho)\| &\leq \varphi(\varrho)(1 + \|\tau(\varrho)\overline{x}_n(0)\|_E)\\
&\leq \varphi(\varrho)(1 + \overline{x}_n(\varrho))\\
&\leq \varphi(\varrho)(1 + \|x_n(\varrho)\|)\\
&\leq \varphi(\varrho)(1 + \|x_n\|_{\mathcal{H}}).
\end{aligned}$$

4. $\psi-$generalized Ulam-Hyers stability of Problem (1).

In this section, we give another version for the existence of solutions and investigate the generalized $\psi-$generalized Ulam-Hyers stability of problem (1). For basic information about multivalued weakly Picard operators we refer to [38].

Definition 3. *[39, 40] A An increasing function $\varsigma : [0,\infty) \to [0,\infty)$ is called comparison function if* $\lim_{n\to\infty} \varsigma^n(s) = 0,\ \forall s \in [0,\infty)$,*where $\varsigma^n(s) = \varsigma^{n-1}(\varsigma(s))$.If in addition $\sum_{n=1}^{\infty} \varsigma^n(\varrho) < \infty, \forall s \in (0,\infty)$, then $\varsigma : [0,\infty) \to [0,\infty)$ is called strictly comparison.*

Remark 4.1. *[41]*

1- If $\varsigma : [0,\infty) \to [0,\infty)$ is comparisons, then $\varsigma(s) < s, \forall s > 0, \varsigma(0) = 0$ and ς is continuous at 0.
2-the functions $\varsigma_1(s) = c\varrho; c \in [0,1)$ and $\varsigma_2(s) = \frac{s}{s+1} : s \in [0,\infty)$ are strictly comparison
Let $M := \{y \in H : y_{|J} \in PC^{\mu}_{1-\mu,\psi}(J,E),\ D^{\vartheta,\nu,\psi}_{s_i+}y(\varrho)$ exists for any $\varrho \in (s_i\ \varrho_{i+1}]$,$i = 0,1,......n,\}$

Definition 4. *[12] Problem (1) is called $\psi-$generalized Ulam- Hyers stable if there is a continuous function $\theta : [0,\infty) \to [0,\infty)$ and $\theta(0) = 0$ such that for each $\epsilon > 0$ and each solution $y \in M$ of the inequality*

$$\begin{cases}
d(D^{\vartheta,\nu,\psi}_{s_i^+}y(\varrho), F(\varrho,\tau(\varrho)y)) \leq \epsilon,\ a.e.\ \varrho \in (s_i,\ \varrho_{i+1}], i = 0,1,....,n,\\
\|y(\varrho) - g_i(\varrho, y(\varrho_i^-))\| \leq \epsilon, \varrho \in (\varrho_i\ s_i], i = 1,......n,,\\
\|y(\varrho_i^+) - g_i(\varrho_i, y(\varrho_i^-))\| \leq \epsilon, i = 1,......n,
\end{cases} \tag{27}$$

there is a solution $x \in H$ for (1) with

$$\sup_{\varrho\in J}\|x(\varrho) - y(\varrho)\| \leq \theta(\epsilon). \tag{28}$$

In the following theorem, we establish the existence and generalized $\psi-$Ulam-Hyers stability of solutions Problem (1).

Theorem 4.1. *Let $F : J \times \Theta \to P_{ck}(E)$, $\Psi \in \Theta$ and $g_i : [\varrho_i, s_i] \times E \to E$ $(i = 1, 2, ...n)$be such that:*

(F_4) For every $z \in H$, $S^1_{F(.,\tau(.)z)}$ is a non-empty subset of $I^{\nu(1-\vartheta),\psi}(PC_{1-\mu,\psi}(J, E))$.

(F_5) There is a function $\sigma \in L^{\frac{1}{q}}(I, \mathbb{R}^+), 0 < q < \vartheta$ and a strict comparison function $\varsigma : [0, \infty) \to [0, \infty)$ such that

(i) For every $x_1, x_2 \in \mathcal{H}$

$$h(F(\varrho, \tau(\varrho)\overline{x}_1), F(\varrho, \tau(\varrho)\overline{x}_2)) \leq \sigma(\varrho)\varsigma(\|x_1 - x_2\|)_{\mathcal{H}}, \ \forall \varrho \in J.$$

(ii)

$$Sup\{ \|y\| \ : \ y \in F(\varrho, \tau(\varrho)\overline{x}_0)\} \leq \sigma(\varrho), \text{ for } a.e. \ \varrho \in J,$$

where $\overline{x}_0(s) = \Psi(s); s \in [-r, 0]$ and $\overline{x}_0(s) = 0; s \in (0, b]$.

(H_2)For any$i = 1, 2, ...n$, there $\xi_i > 0$ such that for any $\varrho \in [\varrho_i, s_i]$

$$\|g_i(\varrho, x) - g_i(\varrho, y)\| \leq \xi_i \ \varsigma(\|x - y\|_E), \forall x, y \in E.$$

Then, there is solution for Problem(1) provided that

$$\frac{\xi}{\Gamma(\mu)} + \eta\delta^q \frac{(\psi(b) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)} \|\sigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)} < 1, \tag{29}$$

where $\xi = \sum_{i=1}^{i=n} \xi_i$.Moreover, if, in addition there, is a $c > 1$such that $\varsigma(c\varrho) \leq c\ \varsigma(\varrho); \varrho \in [0, \infty)$ and $\varrho = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varsigma^n(\varrho)$, then problem (1) is a $\psi-$generalized Ulam-Hyers stable.

Proof. Condition (F_4) allows to define a multifunction $\Phi : \mathcal{H} \to P(\mathcal{H})$ as in (9) Note that by (F_5), for every $x \in \mathcal{H}$, and for a.e.$\varrho \in J$

$$\begin{aligned}
\|F(\varrho, \tau(\varrho)\overline{x})\| &= h(F(\varrho, \tau(\varrho)\overline{x}), \{0\}) \\
&\leq h(F(\varrho, \tau(\varrho)\overline{x}), F(\varrho, \tau(\varrho)\overline{x}_0)) + h(F(\varrho, \tau(\varrho)\overline{x}_0), \{0\}) \\
&\leq \sigma(\varrho)\varsigma(\|x\|_{\mathcal{H}}) + \sigma(\varrho) \\
&\leq \sigma(\varrho)(1 + \varsigma(\|x\|_{\mathcal{H}})), \text{for } a.e.\varrho \in J.
\end{aligned}$$

Then, as the reasons of Theorem 1, we can show that the values of Φ are closed. We show that Φ is $\varsigma-$contraction. Let $x_2, x_1 \in \mathcal{H}$ and $y_1 \in \Phi(x_1)$. Then, there exists $f \in S^1_{F(.,\tau(.)\overline{x}_1)}$such that

$$y_1(\varrho) = \begin{cases} 0, \varrho \in [-r, 0] \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) + \frac{1}{\Gamma(\vartheta)} \int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\vartheta-1}\psi\prime(s) f(s)ds, \varrho \in (0, \varrho_1] \\ g_i(\varrho, x(\varrho_i^-)), \varrho \in (\varrho_i, s_i], i = 1, ...n, \\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)} g_i(s_i, x(\varrho_i^-)) \\ +\frac{1}{\Gamma(\vartheta)} \int_{s_i}^{\varrho} (\psi(\varrho) - \psi(s))^{\vartheta-1}\psi\prime(s) f(s), \varrho \in (s_i\ \varrho_{i+1}], i = 1, ...n. \end{cases}$$

Define $\Pi : J \to 2^E$ as

$$\Pi(\varrho) = \{u \in E : \|f\ (\varrho) - u\ \| \ \leq \sigma(\varrho)\varsigma(\|x_1 - x_2\|_{\mathcal{H}})\}.$$

From $(F_5)(i)$, we have

$$\begin{aligned} & h(F(\varrho,\tau(\varrho)\overline{x}_1),F(\varrho,\tau(\varrho)\overline{x}_2)) \\ \leq\ & \sigma(\varrho)\varsigma(\|x_1-x_2\|_{\mathcal{H}}),\varrho\in J, \end{aligned}$$

Since the values of F are compact, there exists $u_\varrho \in F(\varrho,\tau(\varrho)x_2)$ such that

$$\|u_\varrho - f(\varrho)\| \ \leq \sigma(\varrho)\varsigma(\|x_1-x_2\|_{\mathcal{H}}),$$

as a consequence $\Pi(\varrho);\varrho\in J$ is not empty. Furthermore, because f,σ,x_1,x_2 are measurable and E is separable, it follows from [[42], Theorem III.41], the multifunction $s\rightarrow\Pi(s)\cap F(s,\tau(s)\overline{x}_2)$ is measurable and since its values are non-empty and compact, there is $h\in S^1_{F(\varrho,\tau(\varrho)\overline{x}_2)}$ such that

$$\|h(\varrho)-f(\varrho)\| \ \leq\ \sigma(\varrho)\varsigma(\|x_1-x_2\|_{\mathcal{H}}), a.e.\varrho\in J.$$

Set

$$y_2(\varrho)=\begin{cases} 0,\varrho\in[-r,0] \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0)+\frac{1}{\Gamma(\vartheta)}\int_0^\varrho(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)h(s)ds,\varrho\in(0,\varrho_1] \\ g_i(\varrho,x_2(\varrho_i^-)),\varrho\in(\varrho_i,s_i],i=1,......n, \\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}g_i(s_i,x_2(\varrho_i^-)) \\ +\frac{1}{\Gamma(\vartheta)}\int_{s_i}^\varrho(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)h(s),\varrho\in(s_i\ \varrho_{i+1}],i=1,......n. \end{cases}$$

Notice that, $y_2\in\ \Phi(x_2)$ and if $\varrho\in J_{0,}$, we get from Holder's inequality

$$\begin{aligned} & (\psi(\varrho)-\psi(0))^{1-\mu}\|y_2(\varrho)-y_1(\varrho)\| \\ \leq\ & \frac{(\psi(\varrho)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)}\int_0^\varrho(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)\sigma(s)\ \|h(s)-f(s)\|ds \\ \leq\ & \frac{(\psi(\varrho)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)}\varsigma(\|x_1-x_2\|_{\mathcal{H}})\int_0^\varrho(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)\sigma(s)ds \\ \leq\ & \frac{(\psi(\varrho)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)}\ \varsigma(\|x_1-x_2\|_{\mathcal{H}})\|\sigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)}\delta^q\frac{(\psi(b)-\psi(0))^{\vartheta-q}}{(\frac{\vartheta-q}{1-q})^{1-q}} \\ \leq\ & \eta\delta^q\frac{(\psi(b)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)}\|\sigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)}\varsigma(\|x_1-x_2\|_{\mathcal{H}}). \end{aligned} \tag{30}$$

Similarly, if $\varrho\in(s_i\ \varrho_{i+1}],i=1,......,n$, we get

$$\begin{aligned} & (\psi(\varrho)-\psi(s_i))\|y_2(\varrho)-y_1(\varrho)\| \\ \leq\ & \frac{1}{\Gamma(\mu)}\|g_i(s_i,x_1(\varrho_i^-))-g_i(s_i,x_2(\varrho_i^-))\| \\ & +\eta\delta^q\frac{(\psi(b)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)}\|\sigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)}\varsigma(\|x_1-x_2\|_{\mathcal{H}}). \\ \leq\ & \frac{\xi}{\Gamma(\mu)}\varsigma(\|x_1-x_2\|_{\mathcal{H}})+\eta\delta^q\frac{(\psi(b)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)}\|\sigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)}\varsigma(\|x_1-x_2\|_{\mathcal{H}}). \\ \leq\ & \varsigma(\|x_1-x_2\|_{\mathcal{H}})[\frac{\xi}{\Gamma(\mu)}+\ \eta\delta^q\frac{(\psi(b)-\psi(0))^{1-\mu}}{\Gamma(\vartheta)}\|\sigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)}], \end{aligned} \tag{31}$$

where $\xi = \sum_{i=1}^{i=n} \xi_i$. Furthermore, If $\varrho \in (\varrho_i, s_i], i = 1, ...n$, then by (H_2) one obtains

$$\begin{aligned} \|y_2(\varrho) - y_1(\varrho)\| &\leq \|g_i(\varrho, x(\varrho_i^-)) - g_i(\varrho, x(\varrho_i^-))\| \\ &\leq \xi\varsigma(\|x_1(\varrho_i^-) - x_2(\varrho_i^-)\|) \\ &\leq \xi\ \varsigma(\|x_1 - x_2\|_{\mathcal{H}}). \end{aligned} \tag{32}$$

It follows from (30)–(32),

$$\begin{aligned} & h_{\mathcal{H}}(\Phi(x_2), \Phi(x_1)) \\ < \ & \varsigma(\|x_1 - x_2\|_{\mathcal{H}})[\frac{\xi}{\Gamma(\mu)} + \ \eta\delta^q \frac{(\psi(\varrho) - \psi(0))^{1-\mu}}{\Gamma(\vartheta)} \|\sigma\|_{L^{\frac{1}{q}}(I,\mathbb{R}^+)}], \end{aligned}$$

where, $h_{\mathcal{H}}$ is the Hausdorff distance in $\mathcal{H}$. This inequality and (29) imply that Φ is $\varsigma-$contraction and thus by Theorem 3.1(i) in [41], Φ has a fixed point and as in Theorem 1, this fixed point is a solution for Problem (1).

Now, in order to demonstrate that Problem(1) is a $\psi-$generalized Ulam-Hyers stable. Let $\epsilon > 0$ and $y \in M$ be a solution of inequality (27). Because the values of F are compact, there $f \in S^1_{F(.,\tau(\varrho)y)}$ such that

$$\|D^{\vartheta,\nu,\psi}_{s_i^+} y(\varrho) - f(\varrho)\| = d(D^{\vartheta,\nu,\psi}_{s_i^+} y(\varrho), F(\varrho, \tau(\varrho)y)), a.e.\ \varrho \in (s_i,\ \varrho_{i+1}], i = 0, 1, ..., n.$$

Then, for almost everywhere $\varrho \in \cup_{i=0}^{i=n}(s_i, \varrho_{i+1}]$

$$D^{\vartheta,\nu,\psi}_{s_i^+} y(\varrho) = w(\varrho) + f(\varrho),$$

where $w \in PC^{\nu(1-\vartheta)}_{1-\mu,\psi}(J, E)$ and $\|w(\varrho)| \leq \epsilon, \forall\varrho \in J$. Furthermore, according to (F_4), there is $h \in PC_{1-\mu,\psi}(J, E)$ such that $f(\varrho) = I^{\nu(1-\vartheta),\psi}_{s_i+} h(\varrho), \varrho \in J_k,\ k = 0, 1, 2, .., n$, and hence, from Lemma 4,

$$D^{\nu(1-\vartheta),\psi}_{s_i+} f(\varrho) = D^{\nu(1-\vartheta),\psi}_{s_i+} I^{\nu(1-\vartheta)\psi}_{s_i+} h(\varrho) = h(\varrho), \varrho \in J_k,\ k = 0, 1, 2, .., n.$$

This yields that $f \in PC^{\nu(1-\vartheta)}_{1-\mu,\psi}(J, E)$.Therefore, $y(\varrho) = \Psi(\varrho), \varrho \in [-r, 0]$ and

$$\begin{cases} D^{\vartheta,\nu,\psi}_{s_i^+} y(\varrho) = f(\varrho) + \vartheta(\varrho),\ a.e.\ \varrho \in (s_i,\ \varrho_{i+1}], i = 0, 1, ..., n, \\ y(\varrho) = g_i(\varrho, y(\varrho_i^-)) + \epsilon, \varrho \in (\varrho_i\ s_i], i = 1, ...n, , \\ y(\varrho_i^+) = g_i(\varrho_i, y(\varrho_i^-)) + \epsilon, i = 1, ...n, \end{cases} \tag{33}$$

In view of the second assertion of Lemma 5, relation (33) one obtains

$$y(\varrho) = \begin{cases} \Psi(\varrho), \varrho \in [-r, 0] \\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0) + \frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho) - \psi(s))^{\vartheta-1} f(s)ds \\ +\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho) - \psi(s))^{\vartheta-1}\vartheta(\varrho)ds, \varrho \in (0, \varrho_1], \\ g_i(\varrho, y(\varrho_i^-)) + \epsilon, \varrho \in (\varrho_i\ s_i], i = 1, 2, \dots, n, \\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)} y(s_i) + \frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho) - \psi(s))^{\vartheta-1}\psi\prime(s)(f(s))ds \\ +\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho) - \psi(s))^{\vartheta-1}\psi\prime(s)\vartheta(\varrho)ds, \varrho \in (s_i\ \varrho_{i+1}], i = 1, 2, \dots, n. \end{cases} \tag{34}$$

Let

$$z(\varrho)=\begin{cases}0,\varrho\in[-r,0]\\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0)+\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)f(s)ds,\varrho\in(0,\varrho_1],\\ g_i(\varrho,y(\varrho_i^-)),\varrho\in(\varrho_i\ s_i],i=1,2,\dots,n,\\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}g_i(s_i,y(\varrho_i^-))+\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)(f(s))ds|\\ +\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)\vartheta(\varrho)ds,\varrho\in(s_i\ \varrho_{i+1}],i=1,2,\dots,n.\end{cases}$$

Obviously, $z\in\Phi(y^*)$,where

$$y^*(\varrho)=\begin{cases}0,\varrho\in[-r,0]\\ \frac{(\psi(\varrho)-\psi(0))^{\mu-1}}{\Gamma(\mu)}\Psi(0)+\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)f(s)ds\\ +\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\vartheta(\varrho)ds,\varrho\in(0,\varrho_1],\\ g_i(\varrho,y(\varrho_i^-))+\epsilon,\varrho\in(\varrho_i\ s_i],i=1,2,\dots,n,\\ \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}y(s_i)+\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)(f(s))ds|\\ +\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)\vartheta(\varrho)ds,\varrho\in(s_i\ \varrho_{i+1}],i=1,2,\dots,n.\end{cases}$$

Moreover, from (33) and (34) we get for $\varrho\in(0,\varrho_1]$

$$\begin{aligned}\|z(\varrho)-y^*(\varrho)\| &\le \frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)\|\vartheta(\varrho)\|ds\\ &\le \frac{\epsilon(\psi(b)-\psi(0))^{\vartheta}}{\Gamma(\vartheta+1)}.\end{aligned}$$

If$\varrho\in(\varrho_i\ s_i]$, then$\|\overline{z}(\varrho)-y(\varrho)\|\le\epsilon$.If $\varrho\in(s_i\ \varrho_{i+1}]$, then

$$\begin{aligned}\|z(\varrho)-y^*(\varrho)\| &\le \|\frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}g_i(s_i,y(\varrho_i^-))\frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}y(s_i)\|\\ &\quad+\frac{1}{\Gamma(\vartheta)}\int_0^{\varrho}(\psi(\varrho)-\psi(s))^{\vartheta-1}\psi\prime(s)\|\vartheta(\varrho)\|ds\\ &\le \frac{(\psi(\varrho)-\psi(s_i))^{\mu-1}}{\Gamma(\mu)}\|g_i(s_i,y(\varrho_i^-))-y(s_i)\|\\ &\quad+\frac{\epsilon(\psi(b)-\psi(0))^{\vartheta}}{\Gamma(\vartheta+1)}\\ &\le \epsilon\frac{(\psi(b)-\psi(0))^{\mu-1}}{\Gamma(\mu)}+\frac{\epsilon(\psi(b)-\psi(0))^{\vartheta}}{\Gamma(\vartheta+1)}\\ &= \epsilon[\frac{(\psi(b)-\psi(0))^{\mu-1}}{\Gamma(\mu)}+\frac{(\psi(b)-\psi(0))^{\vartheta}}{\Gamma(\vartheta+1)}+1]\end{aligned}$$

As a consequence,

$$\sup_{\varrho\in J}\|y^*(\varrho)-z(\varrho)\|\le\epsilon[\frac{(\psi(b)-\psi(0))^{\mu-1}}{\Gamma(\mu)}+\frac{(\psi(b)-\psi(0))^{\vartheta}}{\Gamma(\vartheta+1)}+1]. \tag{35}$$

On the other hand, from the facts that Φ is ς- contraction, ς is strictly comparison, $\varsigma(c\varrho)\le c\,\varsigma(\varrho)(c>1)$ for every$\varrho\in[0,\infty)$ and $\varrho=0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty}\varsigma^n(\varrho)$, it yields from Theorem3.1(ii) [41], that Φ is $\rho-$multivalued weakly Picard operator, where $\rho(\varrho) = \varrho + \sum_{n=1}^{\infty} \varsigma^n(\varrho)$. Then, the function $\Phi^{\infty} : Graph(\Phi) \to Fix(\Phi)$ is well defined and

$$\|y^* - \Phi^{\infty}(y^*, z)\|_{\mathcal{H}} \leq \rho(\|y^* - z\|_{\mathcal{H}}). \tag{36}$$

Put $x = \Phi^{\infty}(y^*, z)$. So, $x \in \Phi(x)$ and from (35) and (36) we get

$$\begin{aligned} d_H(y, \overline{x}) &\leq \|y^* - \Phi^{\infty}(y^*, z)\|_{\mathcal{H}} \leq \rho(\|y^* - z\|_{\mathcal{H}}) \\ &= \rho(\epsilon[\frac{(\psi(b) - \psi(0))^{\mu-1}}{\Gamma(\mu)} + \frac{(\psi(b) - \psi(0))^{\vartheta}}{\Gamma(\vartheta + 1)} + 1]) = \theta(\epsilon), \end{aligned}$$

where $\theta(\varrho) = \rho(\varrho[\frac{(\psi(b)-\psi(0))^{\mu-1}}{\Gamma(\mu)} + \frac{(\psi(b)-\psi(0))^{\vartheta}}{\Gamma(\vartheta+1)} + 1])$. Consequently, Problem (1) is $\psi-$generalized Ulam-Hyers stable. □

5. Examples

In this section we give examples to clarify the possibility of applicability our assumptions.

Example 1 *Let E be a Hilbert space, K a non-empty convex compact subset of $E, r = 1, s_0 = 0, \varrho_1 = 1, s_1 = 2, \varrho_2 = 3, \vartheta = \frac{1}{2}, \nu = \frac{1}{4}$ and $\mu = \vartheta + \nu - \vartheta\nu = \frac{5}{8}$. Let $F : J \times \Theta \to P_{ck}(E)$ be defined as follows:*

$$F(\varrho, z) = \begin{cases} (\psi(\varrho) - \psi(0))^{\frac{3}{8}} \|z(0)\| K, \ \varrho \in [0, 1], \\ \|z(0)\| K, \varrho \in (1, \ 2], \\ (\psi(\varrho) - \psi(2))^{\frac{3}{8}} \|z(0)\| K, \varrho \in (2, 3]. \end{cases} \tag{37}$$

Clearly, for almost every $\varrho \in J, z \longrightarrow F(\varrho, \tau(\varrho)z)$ is upper semicontinuous. Set $\lambda = \sup\{\|x\| : x \in K\}$. Then, for any $(\varrho, z) \in J \times \Theta$ and any $y \in F(\varrho, z)$ we have

$$\|y(\varrho)\| \leq \begin{cases} (1 + (\psi(\varrho) - \psi(0))^{\frac{3}{8}} \|z(0)\|) \lambda, \ \varrho \in [0, 1], \\ \|z(0)\| \lambda, \varrho \in (1, \ 2], \\ (1 + (\psi(\varrho) - \psi(2))^{\frac{3}{8}} \|z(0)\|) \lambda, \varrho \in (2, 3]. \end{cases}$$

Then condition $(F_2)$of Theorem1 is verified with $\varphi(\varrho) = \lambda; \varrho \in J$. Moreover, if $f \in S^1_{F(.,z)}; z \in \Theta$, then

$$f(\varrho) = \begin{cases} (\psi(\varrho) - \psi(0))^{\frac{3}{8}} \|z(0)\| y_0, \ \varrho \in [0, 1], \\ y_1, \varrho \in (1, \ 2], \\ (\psi(\varrho) - \psi(2))^{\frac{3}{8}} y_2, \varrho \in (2, 3]. \end{cases} \tag{38}$$

We define

$$h_{x_1,x_2,x_3}(\varrho) = \begin{cases} \frac{\Gamma(\frac{11}{8})}{\Gamma(\frac{10}{8})} (\psi(\varrho) - \psi(0))^{\frac{2}{8}} x_1, \ \varrho \in [0, 1], \\ x_2, \varrho \in (1, \ 2], \\ \frac{\Gamma(\frac{11}{8})}{\Gamma(\frac{10}{8})} (\psi(\varrho) - \psi(2))^{\frac{2}{8}} x_3, \varrho \in (2, 3]. \end{cases} \tag{39}$$

Obviously $h_{x_1,x_2,x_3} \in PC_{1-\mu,\psi}(J, E)$ and from Lemma 2, we get $I^{\nu(1-\vartheta),\psi}_{s_i^+} h_{x_1,x_2,x_3}(\varrho) = f(\varrho); i = 0, 1.$

Now, let D be a bounded subset of $\mathcal{H}$, z_1,$z_2 \in D$, $\varrho \in (0,1]$, $x \in F(\varrho.\tau(\varrho)\overline{z_1})$ and $y \in F(\varrho.\tau(\varrho)\overline{z_2})$. Then $x = (\psi(\varrho) - \psi(0))^{\frac{3}{8}}\|z_1(\varrho)\|x^*$ and $y = (\psi(\varrho) - \psi(0))^{\frac{3}{8}}\|z_2(\varrho)\|y^*$, where $x^*, y^* \in K$. It follows that

$$\|x - y\| \leq (\psi(\varrho) - \psi(0))^{\frac{3}{8}}\|x^*\|z_1(\varrho)\| - \|z_2(\varrho)\|y^*\|,$$

As a result,

$$\begin{aligned}
& \inf_{y\in F(\varrho.\tau(\varrho)\overline{z_2})} \|x - y\| \\
\leq\ & (\psi(\varrho) - \psi(0))^{\frac{3}{8}}\|x^*\|z_1(\varrho)\| - \|z_2(\varrho)\|x^*\| \\
=\ & \|x^*\|(\psi(\varrho) - \psi(0))^{\frac{3}{8}}\|\|z_1(\varrho)\| - \|z_2(\varrho)\| \\
\leq\ & \lambda((\psi(\varrho) - \psi(0))^{\frac{3}{8}}\|z_1(\varrho) - z_2(\varrho)\|,
\end{aligned}$$

which means that

$$h(F(\varrho.\tau(\varrho)\overline{z_1}), F(\varrho.\tau(\varrho)\overline{z_2}) \leq \lambda((\psi(\varrho) - \psi(0))^{\frac{3}{8}}\|z_1(\varrho) - z_2(\varrho)\|. \tag{40}$$

Then, for $\varrho \in (0, \varrho_1]$

$$\begin{aligned}
\chi_E(F(\varrho, \{\tau(\varrho)\overline{x} \ &:\ x \in D)) \\
&\leq \lambda(\psi(\varrho) - \psi(0))^{1-\mu}\varsigma(\varrho)\chi_E\{x(\varrho) : x \in D\} \\
&\leq \lambda(\psi(\varrho) - \psi(0))^{1-\mu}\varsigma(\varrho) \sup_{\theta\in[-r,0]} \chi_E\{(\tau(\varrho))\overline{x}(\theta) : x \in D\},
\end{aligned}$$

Similarly, one can show that if for $\varrho \in (s_2, \varrho_3]$

$$\chi_E(F(\varrho, D)) \leq \lambda(\psi(\varrho) - \psi(s_2))^{1-\mu} \sup_{\theta\in[-r,0]} \chi_E\{(\tau(\varrho))\overline{x}(\theta) : x \in D\}$$

and consequently, by choosing λ small enough such that the relation (7) becomes realized with $\varsigma(\varrho) = \lambda; \varrho \in J$. Next, let $g_1 : [1,2] \times E \to E$ such that

$$g(\varrho, x) = \rho(\psi(1) - \psi(0))^{\frac{3}{8}}\varrho(x), \tag{41}$$

where, ρ is a positive number andϱ is a linear, bounded and compact operator on E. So, condition (H_1) is satisfied. As a consequence, from Theorem (1), the problem (1) has a solution where F and g are given by (47) and (50) and $\Psi \in \Theta$ provided that

$$\frac{3\lambda\kappa_1}{\Gamma(\frac{1}{2})}\eta + \rho + \frac{\rho}{\Gamma(\frac{5}{8})} < 1, \tag{42}$$

where, $\eta = \delta^q \frac{(\psi(b)-\psi(0))^{\vartheta-q}}{(\frac{\vartheta-q}{1-q})^{1-q}}$, $\delta = \max_{s\in J} \psi\prime(s)$, $q = \frac{1}{4}$ and $\kappa_1 = (\psi(b) - \psi(0))^{1-\mu}$. By choosing ρ small enough relation (42) becomes realized.

Example 2 *Let E, K, r, s_0, ϱ_1, s_1, ϱ_2, ϑ, ν, μ, $F : J \times \Theta \to P_{ck}(E)$ and $g_1 : [1,2] \times E \to E$ be as in Example 1. Then assumption (F_4) is satisfied. Moreover, in view of (40) for any $z_1, z_1 \in \mathcal{H}$*

$$\begin{aligned}
h(F(\varrho.\tau(\varrho)\overline{z_1}), F(\varrho.\tau(\varrho)\overline{z_2}) &\leq \lambda((\psi(\varrho) - \psi(0))^{\frac{3}{8}}\|z_1(\varrho) - z_2(\varrho)\| \\
&\leq \lambda\|z_1 - z_2\|_{\mathcal{H}} \\
&= \sigma(\varrho)\varsigma(\|z_1 - z_2\|_{\mathcal{H}}),
\end{aligned}$$

where, $\sigma(\varrho) = 2\lambda$ and $\varsigma(\varrho) = \frac{\varrho}{2}; \varrho \in J$. Observe that ς is strictly comparison and $\varsigma(c\varrho) = c\ \varsigma(\varrho)$, for every $c > 0$, every $\varrho \in [0, \infty)$ and $\varrho = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varsigma^n(\varrho)$, and hence assumption $(F_5)$is satisfied. Now, for any $\varrho \in [1, 2]$ and any $x, y \in E$

$$\begin{aligned} \|g_1(\varrho, x) - g_1(\varrho, y)\| &\leq \rho(\psi(1) - \psi(0))^{\frac{3}{8}} \|\varrho(x) - \varrho(y)\| \\ &\leq \rho\|\varrho\|(\psi(1) - \psi(0))^{\frac{3}{8}} \|x - y\|_E \\ &= \xi\varsigma(\|x - y\|_E), \end{aligned}$$

where $\xi = 2\rho\|\varrho\|(\psi(1) - \psi(0))^{\frac{3}{8}}$. It follows that (H_2) is satisfied. By applying Theorem 2, Problem (1) has a solution and it is $\psi-$generalized Ulam-Hyers stable, where F and g are given by (37) and (40) and $\Psi \in \Theta$ provided that

$$\frac{2\rho\|\varrho\|(\psi(1) - \psi(0))^{\frac{3}{8}}}{\Gamma(\mu)} + \frac{2\rho\|\varrho\|(\psi(1) - \psi(0))^{\frac{3}{8}}}{\Gamma(\mu)} + \eta\|\sigma\|_{L^{\frac{1}{q}}(I, \mathbb{R}^+)} < 1.$$

By choosing ρ small enough this inequality becomes realized.

6. Conclusions

A relation between a solution of the considered problem and the corresponding fractional integral equation is given, then two existence results of solutions for an ψ-Hilfer fractional non-instantaneous impulsive differential inclusion in the presence of delay in an infinite dimensional Banah spaces are obtained. Moreover, by using the multivalued weakly Picard operator theory, the stability of solutions for the considered problem in the sense of generalized Ulam-Hyers is studied. This work generalizes many recent results in the literature, for example [6, 10, 11, 17, 19, 31]. Moreover, our technique can be used to study the existence and Ulam-Hyers stability of solutions or mild solutions for the problems considered in [3, 4, 15, 16, 20–22] to the case when, there are impulses and delay on the system, the right hand side is a multi-valued function and involving $\psi-$Hilfer fractional derivative. There are many directions for future work, for example: Generalize the obtained results in [3, 4, 15, 16] when, the considered problems in these works involving ψ-Hilfer fractional derivative.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

Authors's contributions

All authors contributed equality and read and approved the final version of the manuscript.

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