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Research article

Parametric inference of Akash distribution for Type-II censoring with analyzing of relief times of patients

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Abstract: In this paper, the problem of estimating the parameter of Akash distribution applied when the lifetime of the product follow Type-II censoring. The maximum likelihood estimators (MLE) are studied for estimating the unknown parameter and reliability characteristics. Approximate confidence interval for the parameter is derived under the s-normal approach to the asymptotic distribution of MLE. The Bayesian inference procedures have been developed under the usual error loss function through Lindleys technique and Metropolis-Hastings algorithm. The highest posterior density interval is developed by using Metropolis-Hastings algorithm. Finally, the performances of the different methods have been compared through a Monte Carlo simulation study. The application to set of real data is also analyzed using proposed methods.

Keywords: Akash distribution; Bayes estimators; MLEs; Metropolis-Hastings algorithm; Type-II censoring
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1. Introduction

Firstly, Akash distribution, a continuous one parameter distribution which has been introduced by Shanker [1]. Akash distribution has many real life application in studies pertaining to medical, engineering and survival analysis. Akash distribution has been proved useful in making statistical inferences for such situations. They studied the shapes of the density moments, distribution of order statistics, Renyi entropy measure, skewness and kurtosis. In particular, the author have obtained reliability characteristics and stress-strength reliability of the proposed model. They have computed the MLE, method of moment estimate for the unknown parameter and studied their behavior numerically. Two real lifetime data sets are analysed from medical science.

Akash distribution is described respectively by the probability density function (PDF) and cumulative distribution function (CDF), take the following forms:

$$f_X(x) = \frac{\beta^3}{(\beta^2+2)}(1+x^2)\,e^{-\beta x}\ ,\ x>0,\ \beta>0, \tag{1.1}$$

$$F_X(x) = 1-\left(1+\frac{\beta x(\beta x+2)}{(\beta^2+2)}\right)e^{-\beta x}\ ,\ x>0. \tag{1.2}$$

This distribution can be easily expressed as a combination of exponential (β) and gamma ($3,\beta$) with their mixing proportions $\frac{\beta^2}{\beta^2+2}$ and $\frac{2}{\beta^2+2}$ respectively.

Consequently, the hazard rate, reliability and mean residual life functions take the following forms

$$h(t) = \frac{\beta^2(t+1)}{(\beta t+\beta+1)},\ \ t>0. \tag{1.3}$$

$$R(t) = \left(1+\frac{\beta t(\beta t+2)}{(\beta^2+2)}\right)e^{-\beta t},\ \ t>0, \tag{1.4}$$

$$m(t) = \frac{(\beta t+\beta+2)}{\beta(\beta t+\beta+1)},\ \ t>0. \tag{1.5}$$

In last few years, many authors have studied inference methods for Akash distribution. Using one parameter family of distribution such as Akash, exponential and Lindley distributions have been innovated and considered the concept of modeling of lifetime data by Shanker et al. [2]. They have also obtained some inferential problems. Many real life data sets are used to reflected its exibility over the exponential distribution. Shanker and Shukla [3] studied two-parameter Akash distribution and calculated its different statistical properties, estimation problem and application to this distribution. Shanker et al. [4] has introduced a generalized Akash distribution and studied their statistical aspects in this paper. The method of moments and the maximum likelihood method for estimating its parameters have been obtained. Also for real data, the author have fitted this distribution and compareed with other several distribution. Abushal [5] studied the classical and Bayesian property of the unknown parameters and reliability characteristic of Shanker distribution. However, two real data sets are studied and the applicability of Shanker distribution have been presented. Owing to this we can treat Akash distribution as an alternative lifetime model in reliability analysis.

In the literature, Several censoring schemes have been discussed. Even though, Type-I and Type-II censoring schemes are most popular censoring. Consider a life test where n independent units taken from a Akash distribution are placed under observation and failure time of each unit is recorded. Suppose that the test is terminated when r^{th}, $(1 \le r \le n,\ r$ is prefixed) unit fails. These observed failure times, say $(X_1, X_2, \ldots, X_r);\ X_1 \le X_2 \le \cdots \le X_r$, is a Type-II censored sample of size r. In this censoring scheme $n-r$ units remain unobserved and survive beyond the time of termination. In Type-II censoring the time of termination is a random variable. The likelihood function based on $X_1, X_2, \ldots, X_r$ is given by (see, Cohen [6] for detail)

$$L(|\ \underline{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_{(i)})\left[1-F(x_{(r)})\right]^{n-r} \tag{1.6}$$

The number of failure times r is fixed in Type-II censoring whereas the random observation is the endpoint $X_{(r)}$. One may refer to the book by Lawless [7] for extensive literature and applications of this censoring.

The current paper is concerned with estimating the unknown parameter β, the reliability function $R(t)$, the hazard function $h(t)$ and the mean residual life function $m(t)$ of an Akash distribution grounded on Type-II censored sample. Section 2, displays the estimates of the unknown parameter using maximum likelihood estimate. Approximate confidence intervals (CIs) are also derived. Section 3, considered the Bayesian estimates and Highest posterior density (HPD) interval are discussed. Furthermore, a Simulation study is presented using Monte Carlo simulations in section 4 and section 6, presents a real life data to explain all the methods discussed. At last, the conclusion shown in section 7.

2. Maximum likelihood estimators

Assume that n independent observed values taken of Akash distribution as presented in (1.1) are put on a test. Using the Type-II censoring, we obtained the ordered r failures. If the ordered r failures are $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ then the likelihood function of β, under Type-II censored data drawn of an Akash distribution, is obtained as follows:

$$L(\beta \mid \underline{x}) \propto \frac{\beta^{3r}}{(\beta^2+2)^n}\, e^{-\beta s}\, (\beta^2 + 2 + 2\beta\, x_{(r)} + \beta^2\, x_{(r)}^2)^{n-r}, \tag{2.1}$$

where $s = \sum_{i=1}^{r} x_i + x_{(r)}\,(n-r)$, $\underline{x} = (x_{(1)}, x_{(2)}, \ldots, x_{(r)})$; $x_{(1)} \le x_{(2)} \le \cdots \le x_{(r)}$.

We consider the logarithm of likelihood function (2.1) as

$$\log L(\beta) \propto 3\,r \log\beta - n\log(\beta^2+2) - \beta\, s + (n-r)\log(\beta^2\, x_{(r)}^2 + 2\beta\, x_{(r)} + \beta^2 + 2). \tag{2.2}$$

MLEs of β is a solution of Eq (2.2) accomplished by addressing the first partial derivatives of the total log-likelihood to be zero. So, we consider the equation as follows,

$$\frac{d\log L}{d\beta} = \frac{3\,r}{\beta} - \frac{2\beta\, n}{(\beta^2+2)} - s + \frac{(n-r)(2\beta\, x_{(r)}^2 + 2\, x_{(r)} + 2\beta)}{(\beta^2\, x_{(r)}^2 + 2\beta\, x_{(r)} + \beta^2 + 2)} = 0. \tag{2.3}$$

The closed form solutions to the nonlinear Eq (2.3) is difficult to reach and a numerical method must be applied to solve these simultaneous equation for obtaining the MLE $\hat{\beta}$ of β. Iterative techniques like Newton-Raphson and Broydan used to obtain the desired estimate. In particular, the nonlinear equations are solved using *nleqslv* package of the *R* statistical software. We refer to Pradhan and Kundu [8] for a detailed discussion for the initial guess and convergence of the iterative process. We have taken true parameter values as our initial guess.

Note that (2.3) becomes in the form:

$$\beta = h(\beta), \tag{2.4}$$

where,

$$h(\beta) = 3\,r\left[\frac{2\,n\beta}{(\beta^2+2)} + s - \frac{(n-r)(2\beta\,x_{(r)}^2 + 2\,x_{(r)} + 2\beta)}{(\beta^2\,x_{(r)}^2 + 2\beta\,x_{(r)} + \beta^2 + 2)}\right]^{-1}.$$

We plan a simple iterative techniques to resolve (2.4) for β. Start with an initial guess say $\beta^{(0)}$ and obtain $\beta^{(1)} = h(\beta^{(0)})$ and continuance this, we get $\beta^{(k)} = h(\beta^{(k-1)})$. When we get $|\beta^{(k)} - \beta^{(k-1)}| < \upsilon$, we stop this proceeding.

Finally, We have obtained the MLE of reliability characteristic as follows:

$$\hat{R}(t) = \left(1 + \frac{\hat{\beta}\,t\,(\hat{\beta}\,t+2)}{(\hat{\beta}^2+2)}\right)e^{-\hat{\beta}\,t}, \qquad \hat{h}(t) = \frac{\hat{\beta}^2(t+1)}{(\hat{\beta}\,t + \hat{\beta} + 1)}, \quad \text{and} \quad \hat{m}(t) = \frac{(\hat{\beta}\,t + \hat{\beta} + 2)}{\hat{\beta}(\hat{\beta}\,t + \hat{\beta} + 1)} \quad t > 0.$$

The interval estimation for the parameter that gets in the observed Fisher's information $I(\hat{\beta})$ for the likelihood is obtain by $I(\hat{\beta}) = -\left.\frac{d^2 \log L}{d\beta^2}\right|_{\beta=\hat{\beta}}$.

Frequently, following the asymptotic variance of MLE for Akash distribution parameter is computed using observed Fisher's information i.e., $Var(\hat{\beta}) = [I_X(\beta)]^{-1}$.

The sampling distribution of $\frac{(\hat{\beta}-\beta)}{\sqrt{Var(\hat{\beta})}}$ can be approximating by a standard normal distribution. That is, the approximate of a two sided $100(1-\eta)\%$ confidence intervals for β given by $\hat{\beta} \pm z_{\frac{\eta}{2}}\sqrt{Var(\hat{\beta})}$, where $0 < \eta < 1$ and $z_{\frac{\eta}{2}}$ is the $\frac{\eta}{2}$th percentile of standard normal distribution. The coverage probability using simulation is presented by

$$P\left[\left|\frac{(\hat{\beta}-\beta)}{\sqrt{Var(\hat{\beta})}}\right| \le z_{\frac{\eta}{2}}\right] = (1-\eta).$$

3. The Bayesian estimation

In this section, we have derived Bayes estimates for parameter β and reliability characteristics of an Akash distribution. We consider the Bayesian inference under squared error loss function which is also called the quadratic loss function. This is symmetrical loss function and it has been considered equal important. This loss function may be mathematically expressed as: $L(d(\mu), \hat{d}(\mu)) = (\hat{d}(\mu) - d(\mu))^2$ with $\hat{d}(\mu)$ being an estimate for $d(\mu)$. Here $d(\mu)$ denotes some parametric function of μ. thus, the Bayesian estimate; denoted by $\hat{d}(\mu)$ can be determined by the posterior mean of $d(\mu)$.

Suppose $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ be a Type-II censored order statistics of a random sample of size n, of Akash distribution. Any conjugate and noninformative prior distribution can used if unknown parameter does not exit. Kundu and Pradhan [9] discussed in detail that Gamma distribution can accommodate variety of shapes depending upon parameter values. Thus the family of gamma distributions is highly flexible in nature and can be considered as suitable prior β. It is assumed that β has Gamma (a, b) prior i.e.,

$$\pi(\beta) \propto \beta^{a-1}\,e^{-b\beta} \;\; \beta > 0,\;\; a > 0,\, b > 0. \tag{3.1}$$

Based on the prior, the posterior distribution of β is derived as

$$\pi(\beta\,|\,\underline{\mathrm{x}}) \propto \frac{\beta^{3\,r+a-1}}{(\beta^2+2)^n}\, e^{-\beta\,(b+s)}\;(\beta^2\,x_{(r)}^2+2\beta\,x_{(r)}+\beta^2+2)^{(n-r)}\,, \tag{3.2}$$

Hence, $\underline{\mathrm{x}} = (x_{(1)}, x_{(2)}, \ldots, x_{(r)})$.

Therefore, the Bayes estimate of β under the square error loss function is computed as,

$$\tilde{\beta} = E[\beta\,|\,\underline{\mathrm{x}}\,] = \frac{1}{k}\int_0^\infty \frac{\beta^{3\,r+a}}{(\beta^2+2)^n}\, e^{-\beta\,(b+s)}\;(\beta^2\,x_{(r)}^2+2\beta\,x_{(r)}+\beta^2+2)^{(n-r)}\,d\beta,$$

$$k = \int_0^\infty \frac{\beta^{3\,r+a-1}}{(\beta^2+2)^n}\, e^{-\beta\,(b+s)}\;(\beta^2\,x_{(r)}^2+2\beta\,x_{(r)}+\beta^2+2)^{(n-r)}\,d\beta.$$

Next, the Bayes estimate of reliability characteristic can be addressed as

$$\tilde{R}(t) = \frac{1}{k}\int_0^\infty \frac{\beta^{3\,r+a-1}\,(\beta^2\,t^2+2\beta\,t+\beta^2+2)}{(\beta^2+2)^{n+1}}\, e^{-\beta\,(b+s+t)}\;(\beta^2\,x_{(r)}^2+2\beta\,x_{(r)}+\beta^2+2)^{(n-r)}\,d\beta,$$

$$\tilde{h}(t) = \frac{1}{k}\int_0^\infty \frac{\beta^{3\,r+a+1}\,(t+1)}{(\beta\,t+\beta+1)\,(\beta^2+2)^n}\, e^{-\beta\,(b+s)}\;(\beta^2\,x_{(r)}^2+2\beta\,x_{(r)}+\beta^2+2)^{(n-r)}\,d\beta,$$

$$\tilde{m}_s(t) = \frac{1}{k}\int_0^\infty \frac{\beta^{3\,r+a-2}\,(\beta\,t+\beta+2)}{(\beta\,t+\beta+1)\,(\beta^2+2)^n}\, e^{-\beta\,(b+s)}\;(\beta^2\,x_{(r)}^2+2\beta\,x_{(r)}+\beta^2+2)^{(n-r)}\,d\beta,$$

Generally speaking, the ratio of the above integrals of all Bayes estimates of the parameter and reliability characteristic is not to be achieved in a closed forms. Therefore, the next section focuses on employing two famouse approximation techniques.

3.1. Lindley's approximation

It may be noted that the above Bayes estimates takes a ratio of two integrals can't be reduced to a closed form. Lindley [10] introduced an alternative method to approximate these integrals into a finite values. Consequently, suppose $I(\underline{\mathrm{x}})$, indicates the posterior exception of β and reliability characteristic gathering with the posterior distribution, that is expressed in the following form:

$$I(\underline{\mathrm{x}}) = \frac{\displaystyle\int_\beta g(\beta)\, e^{l(\beta)+\rho(\beta)}\,d\beta}{\displaystyle\int_\beta e^{l(\beta)+\rho(\beta)}\,d\beta}\,, \tag{3.3}$$

here $g(\beta)$ is any function of β. and $\rho(\beta) = \log\pi(\beta)$ and $l(\beta)$ is the log-likelihood. Lindley's approximation schemas, $I(\underline{\mathrm{x}})$ becomes

$$I(\underline{\mathrm{x}}) = g(\hat{\beta}) + \frac{1}{2}\Big[\Big(\hat{g}_{\beta\beta} + 2\,\hat{g}_\beta\,\hat{\rho}_\beta\Big)\,\hat{\sigma}_{\beta\beta} + \hat{g}_\beta\,\hat{\sigma}^2_{\beta\beta}\,\hat{l}_{\beta\beta\beta}\Big],$$

here $g_{\beta\beta}$ represents the second derivative of $g(\beta)$ with respect to β and $\hat{g}_{\beta\beta}$ denotes the expression obtained for $\beta = \hat{\beta}$. All other terms coming concerning $I(\underline{x})$ can be addressed as

$$\hat{l}_{\beta\beta} = \frac{\partial^2 l}{\partial \beta^2}\bigg|_{\beta=\hat{\beta}} = -\frac{3\,r}{\hat{\beta}^2} + \frac{2\,n\,(\beta^2-2)}{(\beta^2+2)^2} - \frac{(n-r)(2\beta^2\,x_{(r)}^4 + 4\beta^2\,x_{(r)}^2 + 4\beta\,x_{(r)}^3 + 4\beta\,x_{(r)} + 2\beta^2 - 4)}{(\beta^2\,x_{(r)}^2 + 2\beta\,x_{(r)} + \beta^2 + 2)^2},$$

$$\hat{l}_{\beta\beta\beta} = \frac{\partial^3 l}{\partial \beta^3}\bigg|_{\beta=\hat{\beta}} = \frac{6\,r}{\hat{\beta}^3} - \frac{4\,n\beta\,(\beta^2-6)}{(\beta^2+2)^3} - \frac{(n-r)(4\beta\,x_{(r)}^4 + 8\beta\,x_{(r)}^2 + 4c^3 + 4x_{(r)} + 4\beta)}{(\beta^2\,x_{(r)}^2 + 2\beta\,x_{(r)} + \beta^2 + 2)^2}$$
$$-\frac{2(n-r)(2\beta^2\,x_{(r)}^4 + 4\beta^2\,x_{(r)}^2 + 4\beta\,x_{(r)}^3 + 4\beta\,x_{(r)} + 2\beta^2 - 4)(2\beta\,x_{(r)}^2 + 2\,x_{(r)} + 2\,\beta)}{(\beta^2\,x_{(r)}^2 + 2\beta\,x_{(r)} + \beta^2 + 2)^3},$$

$$\hat{\sigma}_{\beta\beta} = -\frac{1}{\hat{l}_{\beta\beta}}, \qquad \hat{\rho}_{\beta} = \frac{(a-1)}{\hat{\beta}} - b.$$

The corresponding Bayes estimate of β, we have

$$g(\beta) = \beta,\;\; g_{\beta} = 1,\;\; g_{\beta\beta} = 0,\;\; \tilde{\beta}_l = \hat{\beta} + \left[\hat{\rho}_{\beta}\,\hat{\sigma}_{\beta\beta} + 0.5\,\hat{\sigma}_{\beta\beta}^2\,\hat{l}_{\beta\beta\beta}\right].$$

Consequently, we use the square error loss function, to compute the Bayes estimates for reliability characteristic in a similar manner.

We adopt the Metropolis-Hastings (MH) algorithm and derive Bayes estimates in the next part. Focuses on Hastings [11] and Metropolis et al. [12].

3.2. *Metropolis-Hastings algorithm*

Therefore, in the following algorithm, we employ the well-known Metropolis-Hastings MH algorithm gathering normal proposal distribution to generate samples from this distributions.
Step 1: Let initial value of the parameter to be β_0 and set l=1.
Step 2: Generate β_k using the proposal distribution $N(\beta_{l-1}, \sigma^2)$.
Step 3: Obtain $\lambda = \frac{\pi(\beta_k|x)}{\pi(\beta_{l-1}|x)}$.
Step 4: We obtain a sample α with the help of uniform distribution with parameter 0 and 1.
Step 5: If $\alpha \le \lambda$ then set

$$\beta_l \leftarrow \beta_k; \quad \text{otherwise} \qquad \beta_l \leftarrow \beta_{l-1}.$$

Step 6: Repeat steps (2–5) B times and we get adequate number of replicates.

We are able to obtain sample from the posterior distribution of β by this way. Let us assume that B represent the total number of generated sample and B_0 represent the initial burn-in sample.

The Bayes estimate of β and R(t) against SELF are computed by:

$$\tilde{\beta}_m = \frac{1}{B-B_0}\sum_{j=B_0+1}^{B}\beta_j,$$

$$\hat{R}_m(t) = \frac{1}{B-B_0}\sum_{j=B_0+1}^{B}\frac{(\beta_j^2 + \beta_j\,t + 1)}{(\beta_j^2+1)}\,e^{-\beta_i\,t}.$$

We get the Bayes estimates of $h(t)$ and $m(t)$ under square error loss function in a similar manner.

By the idea of Chen and Shao [13], we compute HPD interval of β using the sample generated from the above algorithm. Order $\beta_1, \beta_2, ..., \beta_B$, then consider the $100(1-\eta)\%$ symmetric credible interval is computed as $(\beta_1, \beta_{\lfloor(1-\eta)B+1\rfloor})....(\beta_{\lfloor B\eta\rfloor}, \beta_B)$. Here, $\lfloor\delta\rfloor$ represent the largest integer less than or equal to δ. The shortest one in among all such credible intervals is the HPD interval.

4. Simulation study

The aim of the simulation is to discuss the properties of the derived estimators. Consequently, we use Monte Carlo simulation study to compare the different estimators discussed in section 2 and section 3. The true value of parameter can be taken such as $\beta = 1.5$ and samples are generated from Akash distribution under Type-II censoring. We carry out the simulation procedures for different combinations of (n,r) and Monte-Carlo simulation study of 10000 samples. For $n = 30$ the r is $20, 24, 28, 30$ and for $n = 50$ the r is $40, 44, 48, 50$. We calculated behavior of the MLE and Bayes estimates for different combinations of n and r. The hyper-parameter assigned for the non prior density is $a = 0$ and $b = 0$ and prior density is $a = 3$ and $b = 2$. we set $Q = 10000$ replications to compute the Bayes estimates and HPD interval utilizing MH algorithm.

The performances of the different estimators are compared regarding their average estimates and their mean squared errors(MSEs) of $\hat{\beta}$, $\tilde{\beta}_l$, $\tilde{\beta}_m$, based on 10000 simulated samples of Akash distribution are written in Tables 1 and 2. Thus, based on the simulated sampes the coverage probabilities and average lengths of CI and HPD interval Considering 95% of the true coverage probability obtained for comparison purposes. In Table 9, the coverage probabilities and the corresponding average lengths are computed. By the Tables 1–9, the following conclusions obtained based on calculated values of the average estimates, estimated MSE, coverage probability and average length.

From the simulation of different estimates established in Tables 1–8, it can be seen that the MLE and Bayes estimates of unknown parameters and the reliability characteristics of Akash distribution are very good in terms of minimum MSEs of serval values for n and r. The Bayes estimates using informative prior (IP) and non informative prior (NIP) are dscussed. The Bayes estimates using the MH algorithm for gamma informative prior (IP) are preferable as they include prior information than MLE. In most cases, it is noticed that the Bayes estimates using the MH algorithm have performance better than Lindley estimates on the basis of minimum MSEs. For fixed n, the MSE of all estimates become a little bit small by increasing r. That is, when sample size n is increase then expected MSE values of all estimate decrease.

Table 1. Results of simulation study of average estimates and MSEs values for several values of r and $n = 30$.

			$\tilde{\beta}_l$				$\tilde{\beta}_m$			
	$\hat{\beta}$		NIP		IP		NIP		IP	
r	EV	MSE	EV	MSE	EV	MSE	EV	MSE	EV	MSE
20	1.53205	0.04037	1.53591	0.041629	1.53051	0.036926	1.52295	0.027582	1.51237	0.026748
24	1.52638	0.033763	1.53051	0.034599	1.52813	0.031367	1.51943	0.026091	1.50237	0.024648
28	1.52313	0.029556	1.52577	0.030213	1.52195	0.027746	1.51066	0.024752	1.49882	0.023551
30	1.52231	0.028092	1.52479	0.028684	1.52014	0.026459	1.50942	0.024922	1.50261	0.023342

Table 2. Results of simulation study of average estimates and MSEs values for several values of r and $n = 50$.

	$\hat{\beta}$		$\tilde{\beta}_l$				$\tilde{\beta}_m$			
			NIP		IP		NIP		IP	
r	EV	MSE	EV	MSE	EV	MSE	EV	MSE	EV	MSE
40	1.52537	0.019118	1.52774	0.019419	1.51793	0.018704	1.50177	0.018136	1.50301	0.016349
44	1.51499	0.018777	1.5162	0.018027	1.51351	0.017455	1.50395	0.017684	1.50487	0.015001
48	1.51343	0.01668	1.51551	0.016897	1.51189	0.016136	1.50477	0.015685	1.50346	0.013733
50	1.51321	0.016337	1.51524	0.016541	1.50465	0.015819	1.50337	0.013476	1.50541	0.010477

Table 3. Results of simulation study of average estimates and MSEs values for several values of r and T when $n = 30$.

	T = 0.5					T = 1.25				
		$\tilde{R}_l(t)$		$\tilde{R}_m(t)$			$\tilde{R}_l(t)$		$\tilde{R}_m(t)$	
r	$\hat{R}(t)$	NIP	IP	NIP	IP	$\hat{R}(t)$	NIP	IP	NIP	IP
20	0.69285	0.692352	0.693274	0.701443	0.701602	0.40833	0.412488	0.413217	0.421595	0.421993
	0.003816	0.003773	0.003373	0.002693	0.002617	0.006567	0.006294	0.005706	0.005245	0.005124
24	0.694112	0.693672	0.694332	0.702515	0.701554	0.408177	0.4118	0.412379	0.420086	0.420801
	0.003231	0.003196	0.002912	0.002558	0.002419	0.005866	0.005643	0.005187	0.004923	0.004759
28	0.695335	0.694929	0.695429	0.702109	0.702647	0.41129	0.414505	0.414877	0.421179	0.420015
	0.002854	0.002824	0.002603	0.002424	0.002316	0.005272	0.005104	0.004741	0.004703	0.004657
30	0.695235	0.694848	0.695315	0.700923	0.701451	0.411028	0.41412	0.414472	0.421004	0.419626
	0.002721	0.002693	0.002492	0.002438	0.002291	0.005059	0.004901	0.004564	0.004669	0.004399

Table 4. Results of simulation study of average estimates and MSEs values for several values of r and T when $n = 50$.

	T = 0.5					T = 1.25				
		$\tilde{R}_l(t)$		$\tilde{R}_m(t)$			$\tilde{R}_l(t)$		$\tilde{R}_m(t)$	
r	$\hat{R}(t)$	NIP	IP	NIP	IP	$\hat{R}(t)$	NIP	IP	NIP	IP
40	0.697192	0.696904	0.697134	0.701623	0.701221	0.411659	0.413789	0.413984	0.418495	0.418666
	0.002076	0.001864	0.001772	0.001875	0.001747	0.003564	0.003387	0.003225	0.003375	0.003021
44	0.697592	0.69732	0.697516	0.700929	0.700616	0.413274	0.415253	0.41539	0.418866	0.419011
	0.001949	0.001738	0.001656	0.001740	0.001624	0.003311	0.003148	0.003007	0.003158	0.002812
48	0.697742	0.697484	0.697658	0.70064	0.701049	0.411356	0.413245	0.413408	0.418818	0.417977
	0.001744	0.001634	0.001562	0.001643	0.001539	0.003263	0.002999	0.002872	0.003082	0.002526
50	0.6978	0.69755	0.697716	0.701074	0.700429	0.412884	0.414718	0.414843	0.416558	0.418149
	0.001712	0.001602	0.001533	0.001604	0.001513	0.003195	0.002899	0.002781	0.002932	0.002234

Table 5. Results of simulation study of average estimates and MSEs values of proposed estimates of $h(t)$ for several values of r and T when $n = 30$.

	T = 0.5					T = 20				
		$\tilde{h}_l(t)$		$\tilde{h}_m(t)$			$\tilde{h}_l(t)$		$\tilde{h}_m(t)$	
n	$\hat{h}(t)$	NIP	IP	NIP	IP	$\hat{h}(t)$	NIP	IP	NIP	IP
20	1.05635	1.06123	1.05905	1.04273	1.04216	1.48506	1.48997	1.4866	1.45381	1.45269
	0.029725	0.026146	0.023932	0.022619	0.021923	0.049293	0.040621	0.036032	0.027151	0.026473
24	1.05653	1.06102	1.0593	1.03947	1.04205	1.48356	1.48773	1.48516	1.45657	1.45454
	0.02641	0.023837	0.021983	0.02137	0.020209	0.038342	0.035286	0.031945	0.025472	0.024598
28	1.05516	1.05923	1.0579	1.0405	1.03879	1.47473	1.47838	1.47661	1.45352	1.45612
	0.025639	0.021992	0.02044	0.020313	0.019288	0.032497	0.030143	0.027722	0.024193	0.024244
30	1.05612	1.06005	1.05879	1.04391	1.04226	1.47491	1.47839	1.47674	1.45388	1.45642
	0.023024	0.02137	0.019904	0.020466	0.019147	0.030399	0.028988	0.026753	0.024142	0.022825

Table 6. Results of simulation study of average estimates and MSEs values for several values of r and T when $n = 50$.

	T = 0.5					T = 20				
		$\tilde{h}_l(t)$		$\tilde{h}_m(t)$			$\tilde{h}_l(t)$		$\tilde{h}_m(t)$	
r	$\hat{h}(t)$	NIP	IP	NIP	IP	$\hat{h}(t)$	NIP	IP	NIP	IP
40	1.05215	1.05508	1.0543	1.04122	1.04232	1.46985	1.47223	1.47141	1.45743	1.45707
	0.02057	0.015448	0.014647	0.015508	0.014397	0.022849	0.019149	0.018655	0.019043	0.018106
44	1.05117	1.0539	1.05324	1.04316	1.04398	1.46566	1.46787	1.46723	1.45622	1.45549
	0.019271	0.014506	0.013806	0.014459	0.01341	0.020418	0.017656	0.016801	0.017521	0.016437
48	1.05083	1.05341	1.05282	1.04385	1.04267	1.46965	1.47175	1.47109	1.45589	1.45772
	0.016786	0.013692	0.013271	0.013595	0.013061	0.019576	0.016802	0.016038	0.016733	0.015469
50	1.05077	1.05329	1.05272	1.04258	1.04442	1.46595	1.46798	1.46743	1.46097	1.45712
	0.015886	0.013482	0.012885	0.013355	0.01236	0.018889	0.016085	0.015676	0.016012	0.015181

Table 7. Results of simulation study of average estimates and MSEs values for several values of r and T when $n = 30$.

	T = 0.5					T = 20				
		$\tilde{m}_l(t)$		$\tilde{m}_m(t)$			$\tilde{m}_l(t)$		$\tilde{m}_m(t)$	
n	$\hat{m}(t)$	NIP	IP	NIP	IP	$\hat{m}(t)$	NIP	IP	NIP	IP
20	0.868747	0.882565	0.883024	0.88823	0.888171	0.683928	0.69282	0.693223	0.699105	0.699429
	0.020739	0.016366	0.014921	0.013875	0.013493	0.008495	0.007734	0.007043	0.006654	0.006501
24	0.869294	0.88136	0.881657	0.889799	0.886822	0.683318	0.691069	0.691391	0.697211	0.697778
	0.018654	0.014137	0.013052	0.013253	0.012248	0.007671	0.006841	0.006308	0.006224	0.005982
28	0.870462	0.881351	0.881534	0.887861	0.888675	0.686337	0.693357	0.693504	0.698128	0.696764
	0.01524	0.012652	0.011778	0.012118	0.011067	0.007079	0.006257	0.005823	0.005898	0.005786
30	0.869691	0.880141	0.880329	0.885346	0.885828	0.685875	0.692613	0.692761	0.697852	0.696091
	0.014646	0.012006	0.011209	0.012108	0.010475	0.006283	0.005939	0.005542	0.005816	0.005412

Table 8. Results of simulation study of average estimates and MSEs values for several values of r and T when $n = 50$.

	T = 0.1					T = 10				
		$\tilde{m}_l(t)$		$\tilde{m}_m(t)$			$\tilde{m}_l(t)$		$\tilde{m}_m(t)$	
n	$\hat{m}(t)$	NIP	IP	NIP	IP	$\hat{m}(t)$	NIP	IP	NIP	IP
40	0.870406	0.877636	0.877738	0.883827	0.882675	0.685461	0.690113	0.690211	0.694124	0.694373
	0.009138	0.008312	0.007926	0.008282	0.007631	0.004926	0.003992	0.003806	0.004008	0.003491
44	0.870736	0.87751	0.877588	0.881893	0.881141	0.686999	0.691367	0.691416	0.694441	0.694465
	0.008611	0.007767	0.007429	0.007746	0.007252	0.004636	0.003705	0.003544	0.003689	0.002922
48	0.87063	0.877041	0.87711	0.880719	0.881794	0.684863	0.688983	0.68907	0.694191	0.693233
	0.008182	0.00732	0.007019	0.007268	0.006831	0.004461	0.003506	0.003361	0.004053	0.002659
50	0.87062	0.876888	0.876954	0.881611	0.880187	0.686416	0.690454	0.690505	0.691706	0.693321
	0.008045	0.007176	0.006887	0.007038	0.006207	0.004244	0.003399	0.003262	0.003254	0.002338

Table 9. Estimated CP (in %) and AL of interval estimates of β for several values of n and r.

				HPD			
		Asy CI		NIP		IP	
n	r	CP	AL	CP	AL	CP	AL
30	20	0.75196	0.744171	0.7638	0.313666	0.9076	0.311396
	24	0.75132	0.691009	0.7818	0.309595	0.9194	0.306128
	28	0.78176	0.653071	0.8092	0.303242	0.9572	0.303592
	30	0.80374	0.639791	0.8328	0.304515	0.9842	0.303166
50	40	0.75178	0.529169	0.7802	0.292102	0.9276	0.28978
	44	0.75168	0.511364	0.8134	0.28818	0.9434	0.286294
	48	0.8516	0.497134	0.8604	0.285026	0.9798	0.284143
	50	0.85118	0.491318	0.8912	0.285257	0.9854	0.283814

In Table 9, we have calculated several 95% interval estimates of β for several values of n and r. The asymptotic intervals, informative and noninformative HPD intervals of the unknown parameter have been derived. By Table 9 we can say that informative HPD intervals have a better results compared to the other intervals in terms of the average length (AL) of these intervals. In addition, the asymptotic intervals compete good when compared with noninformative HPD intervals. Also observed that the ALs of all intervals tend to be decreased as n increases. For all intervals, the Coverage probabilities (CP) is reasonably vary about the nominal level.

Therefore, the Bayes estimate of Akash distribution of the parameter and reliability characteristics with MH algorithm are recommended.

5. Relief times of patients data

This section focuses on a data set from Gross and Clark [14]. The observed data of the lifetimes data relating to relief times (in minutes) of patients receiving an analgesic are 1.4, 1.3, 1.9, 2.2, 1.1, 1.7, 2.7, 1.7, 1.8, 1.5, 1.2, 1.6, 1.4, 4.1, 3.0, 1.8, 1.7, 2.3, 1.6, 2.0.

This data is related to 20 patents and also recently considered by Shanker et al. [2]. In this paper the author fitted this real data to Akash distribution and obtain that this data is best fit for Akash distribution. They also computed mathematical and inference problems for this distribution. Based on several combinations of (n, r), we considered different Type-II censoring schemes and obtained all estimates. We Obtained MLEs and Bayes estimates with SELF using Lindley's approximation and MH algorithm of β and reliability characteristic. Because there is no prior infromation so we considered the non-infromative prior (NIP), i.e., a=b=0. In Table 10, the MLEs and Bayes estimates of β and reliability characteristic are computed. In Table 11, based on true coverage probability of 95%, we derived the approximate CI and non informative HPD intervals. We found that all the estimates the parameter β and reliability characteristic are close to each other in Table 10. By the Table 11, we can see that among all the intervals, HPD interval performs better in respect of length.

Table 10. Point estimates of $\beta, R(t), h(t)$ and $m(t)$ for different choices of r and t from real data.

	r = 10			r = 15			r = 20		
	MLE	LI	MH	MLE	LI	MH	MLE	LI	MH
β	0.935317	0.934252	0.936705	1.08922	1.09025	1.10583	1.15692	1.1586	1.1527
$R(0.5)$	0.877947	0.876016	0.876284	0.832351	0.83072	0.826337	0.811294	0.809901	0.811675
$h(0.1)$	0.474309	0.476356	0.476982	0.593698	0.596849	0.608663	0.647851	0.651193	0.646741
$m(0.1)$	1.59613	1.64777	1.62246	1.33576	1.36631	1.33592	1.2447	1.26754	1.2791
$R(2)$	0.541957	0.547095	0.531825	0.436647	0.442626	0.439404	0.394519	0.400149	0.385593
$h(20)$	0.890005	0.888994	0.907152	1.04359	1.04466	1.04773	1.11119	1.11289	1.13335
$m(20)$	1.12095	1.15243	1.1117	0.956548	0.975131	0.967881	0.898533	0.91242	0.891254

Table 11. Interval estimation for β based on generated Type-II censored samples from real data.

r	Asy CI	HPD
10	(0.646794, 1.22384)	(0.476982, 1.62246)
15	(0.795101, 1.38333)	(0.608663, 1.33592)
20	(0.871638, 1.44221)	(0.646741, 1.2796)

6. Conclusions

This paper discussed the classical and Bayesian inferential methods for the Type-II censored data from Akash distribution. Both MLE and Bayes estimates are provided. Also we have derived the CIs and HPD interval of the parameter and reliability characteristic. we calculated numerical MLE and Bayes estimators of parameter β and reliability characteristic and compared these estimates in terms of their MSE by using Monte Carlo simulation. Although the consequences have been derived under the Type-II censoring scheme, similar methods can be applied to the rest of censoring schemes. We believe that more work can be done in these contexts. Therefore, additional research is needed.

Remarks: We can use Tierney-Kadane's [15] approximation rather than Lindley's method to obtain Bayes estimates with minor mean squared errors, see Abdel-Hamid and AL-Hussaini [16].

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Conflict of interest

The author declares no conflicts of interest.

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