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Research article

Estimate for Schwarzian derivative of certain close-to-convex functions

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Abstract: Let $f(z)$ be analytic in the unit disk with $f(0) = f'(0) - 1 = 0$. For the following close-to-convex subclasses: $\Re\{(1-z)f'(z)\} > 0$, $\Re\{(1-z^2)f'(z)\} > 0$, $\Re\{(1-z+z^2)f'(z)\} > 0$ and $\Re\{(1-z)^2 f'(z)\} > 0$, we investigate the bounds for the first two consecutive derivatives of higher order Schwarzian derivatives of $f(z)$.

Keywords: analytic functions; univalent functions; close-to-convex; higher order Schwarzian derivative

Mathematics Subject Classification: 30C45

1. Introduction

Denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$ and $\mathbb{T} = \partial\mathbb{D}$. Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$. Hence, for $f(z) \in \mathcal{A}$, they can be expanded as the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1.1}$$

The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Let $\mathcal{P}$ be the class of analytic functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,\ z \in \mathbb{D}, \tag{1.2}$$

having a positive real part.

An analytic function $f(z)$ is close-to-convex in $\mathbb{D}$ if there exists a convex function $g(z)$ such that $\Re\frac{f'(z)}{g'(z)} > 0$ holds for $z \in \mathbb{D}$. Each close-to-convex function is univalent (see [10]). Close-to-convex functions have been widely studied in recent years (see [8, 13, 15, 18, 19, 33, 35, 36]). Let $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ be the subclasses of close-to-convex functions of $\mathcal{S}$ satisfying

$$\Re\{(1-z)f'(z)\} > 0 \Leftrightarrow (1-z)f'(z) \in \mathcal{P}, \tag{1.3}$$

$$\Re\{(1-z^2)f'(z)\} > 0 \Leftrightarrow (1-z^2)f'(z) \in \mathcal{P}, \tag{1.4}$$

$$\Re\{(1-z+z^2)f'(z)\} > 0 \Leftrightarrow (1-z+z^2)f'(z) \in \mathcal{P}, \tag{1.5}$$

$$\Re\{(1-z)^2 f'(z)\} > 0 \Leftrightarrow (1-z)^2 f'(z) \in \mathcal{P}, \tag{1.6}$$

respectively.

The conditions (1.3), (1.4) and (1.6) were introduced by Ozaki [28] as univalent criteria. Recall that the classes $\mathcal{F}_2$ and $\mathcal{F}_4$ have elegant geometric properties. Such functions in $\mathcal{F}_2$ map univalently $\mathbb{D}$ onto a convex domain in the direction of imaginary axis (see [12]). The function in $\mathcal{F}_4$ maps univalently onto a convex domain in the direction of real axis (see [3]).

S. Ponnusamy [30] studied that the conditions on the parameters of the Gaussain Hypergeometric functions $F(a,b;c;z)$ are determined to show that the Alexander transform of $f(z) = zF(a,b;c;z)$ belongs to one of the above four families, in particular. A similar studies about Alexander transform are also considered in [29, 32]. Since the bounds of univalent functions or their subclasses are improtant, it is interesting to investigate these kinds of bounds for the subclasses. In this direction, recently, the subclasses of close-to-convex functions have been studied, in particular, the logarithmic coefficients, Fekete-Szegö problem and Hermitian-Toeplitz determinants for the subclasses of close-to-convex functions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ and $\mathcal{F}_4$ of $\mathcal{S}$ have been considered in [2,4,5,17,21–23].

The Schwarzian derivative of a locally univalent function $f(z)$ is defined by

$$S_f(z) = \left(\frac{f''}{f''}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

It is well known that $S_f(z)$ plays an important role in the study of univalent functions (see [1,25–27,31,37]). Using Schwarzian derivatives, Nehari [25] proved that if $|S_f(z)|(1-|z|^2)^2 \leq 2,\ z \in \mathbb{D}$, then $f(z)$ is univalent in $\mathbb{D}$. In addition, Nehari [25] proved that if $f(z) \in \mathcal{S}$, then $|S_f(z)|(1-|z|^2)^2 \leq 6,\ z \in \mathbb{D}$. Following the papers [14,34], let $n \geq 3$, define $\sigma_3(f)(z) = S_f(z)$ and

$$\sigma_{n+1}(f)(z) = \sigma_n'(f)(z) - (n-1)\sigma_n(f)(z)\frac{f''(z)}{f'(z)}. \tag{1.7}$$

Harmelin [14] proved that the higher order Schwarzian derivatives $\sigma_n(f)$ satisfies $\sigma_n(T \circ f) = \sigma_n(f)$, where T denotes Möbius transformation. Note that the class of convex functions is linearly invariant, there is no loss in restricting consideration to $\sigma_n(f)(0)$. Dorff and Szynal [11] researched the bounds of $\sigma_n(f)(0)$ for convex functions. And then, Cho *et al.* [6] investigated the bounds of $\sigma_n(f)(0)$ ($n = 3,4,5$) in general forms of these classes consisting of Janowski classes: $\mathcal{S}^*[A,B] = \{f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}\}$, $\mathcal{K}^*[A,B] = \{f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz}\}$, where $-1 \leq B < A \leq 1$, which generalized the results in [11]. In particular, let $A = 1$ and $B = -1$, $\mathcal{S}^*[A,B]$ is the class of starlike functions and $\mathcal{K}^*[A,B]$ is the class of convex functions. Recently, Kumar *et al.* [20] study the bounds on the first three consecutive higher order Schwarzian derivatives for the class: $\mathcal{S}_B^* = \{f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec e^{e^z-1}\}$. For more details about $\mathcal{S}_B^*$, one can refer to [20].

Let $\sigma_n(f)(0) = \mathbf{S}_n$. Combining (1.1) with (1.7), we see that

$$|\mathbf{S}_3| = 6|a_3 - a_2^2|, \tag{1.8}$$

$$|\mathbf{S}_4| = 24|a_4 - 3a_3a_2 + 2a_2^3|. \tag{1.9}$$

Remark 1. By [16], we see that $|\mathbf{S}_3| \le 5$ for $f \in \mathcal{F}_1$, but the constant 5 is not sharp. By Lemma 1 in [24], we know that the sharp inequality $|\mathbf{S}_3| \le 6$ for $f \in \mathcal{F}_2$. According to [17], we have the sharp inequality $|\mathbf{S}_3| \le 6$ for $f \in \mathcal{F}_4$. However, the sharp bounds of $|\mathbf{S}_3|$ for $f \in \mathcal{F}_1$ and $f \in \mathcal{F}_3$ are unknown.

In this paper, one of the aims is to consider the bounds of $|\mathbf{S}_3|$ for $f \in \mathcal{F}_1$, $f \in \mathcal{F}_3$ and the bounds of $|\mathbf{S}_4|$ for the four classes $f \in \mathcal{F}_i$, where $i = 1, 2, 3, 4$. We first consider a special case when a_2 is real for $\mathcal{F}_i$ $(i = 1, 2, 3, 4)$ in Theorem 1. Moreover, by Remark 1, on the upper bound of $|\mathbf{S}_3|$ we only consider the class $\mathcal{F}_1$ or $\mathcal{F}_3$. Now we state our results as follows.

Theorem 1. *Let* $f(z) \in \mathcal{A}$ *and* $a_2 \in \mathbb{R}$.

(1) *If* $f(z) \in \mathcal{F}_1$, *then*

$$|\mathbf{S}_3| = 6|a_3 - a_2^2| \le \frac{14}{3}, \tag{1.10}$$

$$|\mathbf{S}_4| = 24|a_4 - 3a_3a_2 + 2a_2^3| \le 24. \tag{1.11}$$

(2) *If* $f(z) \in \mathcal{F}_2$, *then*

$$|\mathbf{S}_4| = 24|a_4 - 3a_3a_2 + 2a_2^3| \le 24. \tag{1.12}$$

(3) *If* $f(z) \in \mathcal{F}_3$, *then*

$$|\mathbf{S}_4| = 24|a_4 - 3a_3a_2 + 2a_2^3| \le 36. \tag{1.13}$$

(4) *If* $f(z) \in \mathcal{F}_4$, *then*

$$|\mathbf{S}_4| = 24|a_4 - 3a_3a_2 + 2a_2^3| \le 48. \tag{1.14}$$

All estimates are sharp.

If we remove the condition $a_2 \in \mathbb{R}$ in Theorem 1, we have the following theorem.

Theorem 2. *Let* $f(z) \in \mathcal{A}$.

(1) *If* $f(z) \in \mathcal{F}_1$, *then*

$$|\mathbf{S}_3| = 6|a_3 - a_2^2| \le \frac{8 + \sqrt{2}}{2}, \quad |\mathbf{S}_4| = 24|a_4 - 3a_3a_2 + 2a_2^3| \le 12(1 + \sqrt{2}).$$

(2) *If* $f(z) \in \mathcal{F}_2$, *then*

$$|\mathbf{S}_4| = 24|a_4 - 3a_3a_2 + 2a_2^3| \le \frac{32\sqrt{6}}{3}.$$

(3) *If* $f(z) \in \mathcal{F}_3$, *then*

$$|\mathbf{S}_3| = 6|a_3 - a_2^2| \le 6, \; |\mathbf{S}_4| = 24|a_4 - 3a_3a_2 + 2a_2^3| \le 12\left(1 + \frac{8\sqrt{6}}{9}\right).$$

The constant 6 is sharp.

Remark 2. According to Theorem 1 and Theorem 2, we see that $|\mathbf{S}_3| \le 6$ for $\mathcal{F}_3$ and $|\mathbf{S}_3| \le \frac{14}{3}$ for $\mathcal{F}_1$ when a_2 is real. Also, we have $|\mathbf{S}_3| \le \frac{8+\sqrt{2}}{2}$ for $\mathcal{F}_1$, which improves the corresponding case in [16]. Moreover, when a_2 is real, we find the sharp upper bounds of $|\mathbf{S}_4|$ for $\mathcal{F}_i$, where $i = 1, 2, 3, 4$.

2. Preliminary

To prove our theorems, we need the following lemmas.

Lemma 1. *$p \in \mathcal{P}$ is of the form* (1.2)*, then*

$$c_1 = 2\zeta_1, \tag{2.1}$$

$$c_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2, \tag{2.2}$$

$$c_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3, \tag{2.3}$$

for some $\zeta_i \in \overline{\mathbb{D}}$, where $i = 1, 2, 3$.

(2.1) is due to Caratheodory [7]. (2.2) can be referred in [28]. In [5], Cho *et al.* showed the formula (2.3).

For $\zeta_1 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 as in (2.1), i.e.,

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \ z \in \mathbb{D}. \tag{2.4}$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (2.1) and (2.2), i.e.,

$$p(z) = \frac{1 + (\overline{\zeta_1}\zeta_2 + \zeta_1)z + \zeta_2 z^2}{1 + (\overline{\zeta_1}\zeta_2 - \zeta_1)z - \zeta_2 z^2}, \ z \in \mathbb{D}. \tag{2.5}$$

Lemma 2. ([9]) *Let $Y(a, b, c) = \max_{z\in\overline{\mathbb{D}}}(|a + bz + cz^2| + 1 - |z|^2)$. If $ac \geq 0$, then*

$$Y(a, b, c) = \begin{cases} |a| + |b| + |c|, & |b| \geq 2(1 - |c|), \\ 1 + |a| + \frac{b^2}{4(1-|c|)}, & |b| < 2(1 - |c|). \end{cases} \tag{2.6}$$

If $ac < 0$, then

$$Y(a, b, c) = \begin{cases} 1 - |a| + \frac{b^2}{4(1-|c|)}, & -4ac(c^{-2} - 1) \leq b^2; |b| < 2(1 - |c|), \\ 1 + |a| + \frac{b^2}{4(1+|c|)}, & b^2 < \min\{4(1 + |c|)^2, -4ac(c^{-2} - 1)\}, \\ R(a, b, c), & \text{otherwise}, \end{cases} \tag{2.7}$$

where

$$R(a, b, c) = \begin{cases} |a| + |b| - |c|, & |c|(|b| + 4|a|) \leq |ab|, \\ -|a| + |b| + |c|, & |ab| \leq |c|(|b| - 4|a|), \\ (|a| + |c|)\sqrt{1 - \frac{b^2}{4ac}}, & \text{otherwise}. \end{cases} \tag{2.8}$$

3. Proofs of main results

Let $f \in \mathcal{F}_1$, $f \in \mathcal{F}_2$, $f \in \mathcal{F}_3$ or $f \in \mathcal{F}_4$. Putting the series (1.1) and (1.2) into (1.3), (1.4), (1.5) or (1.6) by equating the coefficients, we respectively have

$$a_2 = \frac{1}{2}(1 + c_1), \ a_3 = \frac{1}{3}(1 + c_1 + c_2), \ a_4 = \frac{1}{4}(1 + c_1 + c_2 + c_3), \tag{3.1}$$

$$a_2 =\frac{1}{2}c_1,\ a_3 = \frac{1}{3}(1 + c_2),\ a_4 = \frac{1}{4}(c_1 + c_3), \tag{3.2}$$
$$a_2 =\frac{1}{2}(1 + c_1),\ a_3 = \frac{1}{3}(c_1 + c_2),\ a_4 = \frac{1}{4}(-1 + c_2 + c_3), \tag{3.3}$$
$$a_2 =\frac{1}{2}(2 + c_1),\ a_3 = \frac{1}{3}(3 + 2c_1 + c_2),\ a_4 = \frac{1}{4}(4 + 3c_1 + 2c_2 + c_3). \tag{3.4}$$

By the condition $a_2 \in \mathbb{R}$ in Theorem 1, we find that $\zeta_1 \in [-1, 1]$ from (3.1)–(3.4). Now using Lemma 1 and Lemma 2, we prove Theorem 1.

Proof of Theorem 1. (1) Let $f \in \mathcal{F}_1$ be of the form (1.1) with $a_2 \in \mathbb{R}$. By (1.8) and (3.1), we calculate

$$\begin{aligned}|\mathbf{S}_3| =&6|a_3 - a_2^2| = \frac{1}{2}|1 - 2c_1 + 4c_2 - 3c_1^2| = \frac{1}{2}|1 - 4\zeta_1 - 4\zeta_1^2 + 8(1 - \zeta_1^2)\zeta_2|\\ \le&\frac{1}{2}(|1 - 4\zeta_1 - 4\zeta_1^2| + 8(1 - \zeta_1^2)) =: \psi(\zeta_1).\end{aligned}$$

If $\zeta_1 \in (\frac{\sqrt{2}-1}{2}, 1]$, then

$$\psi(\zeta_1) = \frac{1}{2}(7 + 4\zeta_1 - 4\zeta_1^2) \le \varphi\left(\frac{1}{2}\right) = 4.$$

If $\zeta_1 \in [-1, \frac{\sqrt{2}-1}{2}]$, then

$$\psi(\zeta_1) = \frac{1}{2}(9 - 4\zeta_1 - 12\zeta_1^2) \le \varphi\left(-\frac{1}{6}\right) = \frac{14}{3}.$$

Obviously, we see that $|\mathbf{S}_3| \le \frac{14}{3}$.

From above analysis, we see that the equality in (1.10) holds when $\zeta_1 = -\frac{1}{6}$ and $\zeta_2 = 1$, combining (2.5), we conclude $p(z) = \frac{1-\frac{1}{3}z+z^2}{1-z^2}$, which implies that the extremal function of (1.10) is $f_0(z) = \int_0^z \frac{1-\frac{1}{3}t+t^2}{(1-t)^2(1+t)}dt$ by (1.3).

Substituting (3.1) into (1.9), and by Lemma 1, it follows that

$$\begin{aligned}|\mathbf{S}_4| =&6| - c_2 + c_3 + c_1^2 - 2c_1c_2 + c_1^3|\\ =&12|\zeta_1^2 + \zeta_1^3 - (1 - \zeta_1^2)(2\zeta_1 + 1)\zeta_2 - (1 - \zeta_1^2)\zeta_1\zeta_2^2 + (1 - \zeta_1^2)(1 - |\zeta_2|^2)\zeta_3|\\ \le&12(1 - \zeta_1^2)\Psi(A, B, C),\end{aligned} \tag{3.5}$$

where $\zeta_1 \in [-1, 1]$, $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$ and $\Psi(A, B, C) = |A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2$ with $A = \frac{\zeta_1^2+\zeta_1^3}{1-\zeta_1^2}$, $B = -2\zeta_1 - 1$, $C = -\zeta_1$. When $\zeta_1 \ne -1$, we have $A = \frac{\zeta_1^2}{1-\zeta_1}$.

So for $\zeta_1 = -1$ and $\zeta_1 = 1$, we respectively have $|\mathbf{S}_4| = 0$ and $|\mathbf{S}_4| = 24$. Notice that $AC < 0$ for $\zeta_1 \in (0, 1)$ and $AC \ge 0$ for $\zeta_1 \in (-1, 0]$. To prove that $|\mathbf{S}_4| \le 24$, we divide it into five cases.

Case 1. If $\zeta_1 \in (-\frac{3}{4}, 0]$, then $B^2 < 4(1 - |C|)^2$, which implies that $|B| < 2(1 - |C|)$. In view of (2.6) and (3.5), it follows

$$|\mathbf{S}_4| \le 12(1 - \zeta_1^2)\left(1 + |A| + \frac{B^2}{4(1 - |C|)}\right) = 3(5 + 3\zeta_1) \le 15.$$

Case 2. If $\zeta_1 \in (-1, -\frac{3}{4}]$, then $B^2 \geq 4(1-|C|)^2$, which means that $|B| \geq 2(1-|C|)$. By (2.6) and (3.5), we get the following

$$|\mathbf{S}_4| \leq 12(1-\zeta_1^2)(|A|+|B|+|C|) = 12(4\zeta_1^3 + 2\zeta_1^2 - 3\zeta_1 - 1) \leq \frac{33}{4}$$

by the fact that $4\zeta_1^3 + 2\zeta_1^2 - 3\zeta_1 - 1$ is increasing in $\zeta_1 \in (-1, -\frac{3}{4}]$.

Case 3. If $\zeta_1 \in (0, \frac{1}{4})$, we get $B^2 + 4AC(C^{-2} - 1) = 1 > 0$ and $|B| < 2(1-|C|)$. So by (2.7) and (3.5), we can obtain

$$|\mathbf{S}_4| \leq 12(1-\zeta_1^2)\left(1 - |A| + \frac{B^2}{4(1-|C|)}\right) = 15(1+\zeta_1) < \frac{75}{4}.$$

Case 4. If $\zeta_1 \in [\frac{1}{4}, \frac{\sqrt{2}}{4}]$, then $|AB| \leq |C|(|B| - 4|A|)$. Combining (2.8) and (3.5), it is easy to get

$$|\mathbf{S}_4| \leq 12(1-\zeta_1^2)(-|A|+|B|+|C|) = 12(-4\zeta_1^3 - 2\zeta_1^2 + 3\zeta_1 + 1).$$

Notice that $-4\zeta_1^3 - 2\zeta_1^2 + 3\zeta_1 + 1$ is increasing in $\zeta_1 \in [\frac{1}{4}, \frac{\sqrt{2}}{4}]$, so

$$|\mathbf{S}_4| \leq \frac{15\sqrt{2}+18}{2} < 24.$$

Case 5. If $\zeta_1 \in (\frac{\sqrt{2}}{4}, 1)$, direct calculations lead that A, B, C satisfy the third case of (2.8), so

$$|\mathbf{S}_4| \leq 12(1-\zeta_1^2)(|A|+|C|)\sqrt{1 - \frac{B^2}{4AC}} = 6(1+\zeta_1)\sqrt{\frac{3\zeta_1+1}{\zeta_1}} < 24.$$

In fact, we find that $(3\zeta_1+1)(1+\zeta_1)^2 - 16\zeta_1 < 0$ for $\zeta_1 \in (\frac{\sqrt{2}}{4}, 1)$. This means that $(1+\zeta_1)\sqrt{\frac{3\zeta_1+1}{\zeta_1}} < 4$ for $\zeta_1 \in (\frac{\sqrt{2}}{4}, 1)$.

Therefore, we establish the inequality (1.11). Next, we prove the sharpness. Let $f(z) = \int_0^z \frac{1+t}{(1-t)^2}dt$. It is clear that $f(z) = \int_0^z \frac{1+t}{(1-t)^2}dt \in \mathcal{F}_1$. In this case, direct calculations give $|\mathbf{S}_4| = 24$. The first part is complete.

(2) Let $f \in \mathcal{F}_2$ be of the form (1.1) with $a_2 \in \mathbb{R}$. Using (1.9), (2.1)–(2.3) and (3.2), we get

$$\begin{aligned}|\mathbf{S}_4| =& 6|-c_1 + c_3 - 2c_1c_2 + c_1^3| = 12|-\zeta_1 + \zeta_1^3 - 2(1-\zeta_1^2)\zeta_1\zeta_2 - (1-\zeta_1^2)\zeta_1\zeta_2^2 + (1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3| \\ \leq& 12(1-\zeta_1^2)\Psi(A,B,C),\end{aligned} \tag{3.6}$$

where $\zeta_1 \in [-1, 1]$, $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$ and $\Psi(A,B,C) = |A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2$, $A = \frac{-\zeta_1+\zeta_1^3}{1-\zeta_1^2}$, $B = -2\zeta_1$, $C = -\zeta_1$. If $\zeta_1 \neq \pm 1$, we find that $A = -\zeta_1$.

Taking $\zeta_1 = -1$ and $\zeta_1 = 1$ into account, it respectively follows $|\mathbf{S}_4| = 24$ and $|\mathbf{S}_4| = 0$. Note that $AC \geq 0$ for $\zeta_1 \in (-1, 1)$. Applying Lemma 2 and (3.6), we have:

Case 1. If $\zeta_1 \in (-\frac{1}{2}, 0)$, then $|B| < 2(1-|C|)$, it follows that

$$|\mathbf{S}_4| \leq 12(1-\zeta_1^2)\left(1 + |A| + \frac{B^2}{4(1-|C|)}\right) = 12(1-\zeta_1^2)\left(1 - \zeta_1 + \frac{\zeta_1^2}{1+\zeta_1}\right) = 12(-\zeta_1 + 1) < 18.$$

Case 2. If $\zeta_1 \in (-1, -\frac{1}{2}]$, then $|B| \geq 2(1 - |C|)$, and so we get

$$|\mathbf{S}_4| \leq 12(1 - \zeta_1^2)(|A| + |B| + |C|) = 12(4\zeta_1^3 - 4\zeta_1) \leq \frac{32\sqrt{3}}{3}.$$

Case 3. If $\zeta_1 \in [\frac{1}{2}, 1)$, then $|B| \geq 2(1 - |C|)$, we obtain

$$|\mathbf{S}_4| \leq 12(1 - \zeta_1^2)(|A| + |B| + |C|) = 12(1 - \zeta_1^2)4\zeta_1 \leq \frac{32\sqrt{3}}{3}.$$

Case 4. If $\zeta_1 \in [0, \frac{1}{2})$, then $|B| < 2(1 - |C|)$, we conclude

$$|\mathbf{S}_4| \leq 12(1 - \zeta_1^2)\left(1 + |A| + \frac{B^2}{4(1 - |C|)}\right) = 12(1 - \zeta_1^2)\left(1 + \zeta_1 + \frac{\zeta_1^2}{1 - \zeta_1}\right) = 12(1 + \zeta_1) < 18.$$

Hence, the inequality (1.12) is true. Equality in (1.12) holds for the function given by (1.4), where $p(z) = \frac{1-z}{1+z}$ is given by (2.4) with $\zeta_1 = -1$, namely for $f(z) = \int_0^z \frac{1}{(1+t)^2} dt = -\frac{1}{1+z}$. This completes the proof the second part.

(3) Let $f \in \mathcal{F}_3$ be of the form (1.1) with $a_2 \in \mathbb{R}$. Using the equalities (1.9), (2.1)–(2.3) and (3.3), we have

$$\begin{aligned}|\mathbf{S}_4| =& 6|c_1 - c_2 + c_3 + c_1^2 - 2c_1c_2 + c_1^3| \\ =& 12|\zeta_1 + \zeta_1^2 + \zeta_1^3 - (1 - \zeta_1^2)(2\zeta_1 + 1)\zeta_2 - (1 - \zeta_1^2)\zeta_1\zeta_2^2 + (1 - \zeta_1^2)(1 - |\zeta_2|^2)\zeta_3| \\ \leq& 12(1 - \zeta_1^2)\Psi(A, B, C),\end{aligned}$$

where $\zeta_1 \in [-1, 1]$, $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$ and $\Psi(A, B, C) = |A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2$ with $A = \frac{\zeta_1 + \zeta_1^2 + \zeta_1^3}{1 - \zeta_1^2}$, $B = -2\zeta_1 - 1$, $C = -\zeta_1$.

For $\zeta_1 = -1$ and $\zeta_1 = 1$, we respectively have $|\mathbf{S}_4| = 12$ and $|\mathbf{S}_4| = 36$. In addition, $\zeta_1 = 0$, we have

$$|\mathbf{S}_4| = 6|c_1 - c_2 + c_3 + c_1^2 - 2c_1c_2 + c_1^3| = 12| - \zeta_2 + (1 - |\zeta_2|^2)\zeta_3| \leq 12(|\zeta_2| + 1 - |\zeta_2|^2) \leq 15.$$

Note that $AC < 0$ and $B^2 + 4AC(C^{-2} - 1) = -3 < 0$ for $\zeta_1 \in (-1, 1)\backslash\{0\}$. Moreover,

$$B^2 - 4(1 + |C|)^2 = \begin{cases} 12\zeta_1 - 3, & \zeta_1 \in (-1, 0), \\ -4\zeta_1 - 3, & \zeta_1 \in (0, 1), \end{cases}$$

it follows that $B^2 - 4(1 + |C|)^2 < 0$ for $\zeta_1 \in (-1, 1)\backslash\{0\}$.

Assume first that $\zeta_1 \in (-1, 0)$, then by (2.8) in Lemma 2, we obtain

$$|\mathbf{S}_4| \leq 12(1 - \zeta_1^2)\left(1 - \frac{\zeta_1 + \zeta_1^2 + \zeta_1^3}{1 - \zeta_1^2} + \frac{4\zeta_1^2 + 1 + 4\zeta_1}{4(1 - \zeta_1)}\right) = 3(5 + \zeta_1) < 15.$$

Assume now that $\zeta_1 \in (0, 1)$, then using (2.8) in Lemma 2, we get

$$|\mathbf{S}_4| \leq 12(1 - \zeta_1^2)\left(1 + \frac{\zeta_1 + \zeta_1^2 + \zeta_1^3}{1 - \zeta_1^2} + \frac{4\zeta_1^2 + 1 + 4\zeta_1}{4(1 + \zeta_1)}\right) = 3(5 + 7\zeta_1) < 36.$$

Thus, we have (1.13). Equality in (1.13) holds for the function $f(z)$ given by (1.5), where $p(z) = \frac{1+z}{1-z}$ is given by (2.4) with $\zeta_1 = 1$, i.e., for the function $f(z) = \int_0^z \frac{1+t}{(1-t)(1-t+t^2)}dt$. This completes the proof.

(4) Let $f \in \mathcal{F}_4$ with $a_2 \in \mathbb{R}$. From (1.9), Lemma 1 and (3.4), we have

$$\begin{aligned}|\mathbf{S}_4| =&6|c_1 - 2c_2 + c_3 + 2c_1^2 - 2c_1c_2 + c_1^3|\\ =&12|\zeta_1 + 2\zeta_1^2 + \zeta_1^3 - 2(1-\zeta_1^2)(\zeta_1+1)\zeta_2 - (1-\zeta_1^2)\zeta_1\zeta_2^2 + (1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3|\\ \leq&12(1-\zeta_1^2)\Psi(A,B,C),\end{aligned}$$

where $\zeta_1 \in [-1,1]$, $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$ and $\Psi(A,B,C) = |A+B\zeta_2+C\zeta_2^2|+1-|\zeta_2|^2$ with $A = \frac{\zeta_1+2\zeta_1^2+\zeta_1^3}{1-\zeta_1^2}$, $B = -2\zeta_1-2$, $C = -\zeta_1$. In particular, if $\zeta_1 \neq -1$, then $A = \frac{\zeta_1(1+\zeta_1)}{1-\zeta_1}$.

Applying Lemma 2 and the processing methods in (1), (2) or (3), we can obtain that the inequality (1.14) is true, here we omit its details. The equality (1.14) holds when $\zeta_1 = 1$. By (2.4), we have $p(z) = \frac{1+z}{1-z}$, i.e., $f(z) = \int_0^z \frac{1+t}{(1-t)^3}dt = \frac{z}{(1-z)^2}$. This proof is completed. □

Proof of Theorem 2. (1) If $f(z) \in \mathcal{F}_1$, by (2.1), (2.2) and (3.1), we calculate

$$\begin{aligned}|\mathbf{S}_3| =&6|a_3 - a_2^2| = \frac{1}{2}|1-2c_1+4c_2-3c_1^2| = \frac{1}{2}|1-4\zeta_1-4\zeta_1^2+8(1-|\zeta_1|^2)\zeta_2|\\ \leq&\frac{1}{2}(|1-4\zeta_1-4\zeta_1^2| + 8(1-|\zeta_1|^2)).\end{aligned}$$

Let $\zeta_1 = re^{i\theta}$, where $r \in [0,1]$ and $\theta \in [0,2\pi)$, then

$$|1-4\zeta_1-4\zeta_1^2|^2 = \phi(\cos\theta) = \phi(x) = -16r^2x^2 + 8r(4r^2-1)x + 16r^4 + 24r^2 + 1, \ x \in [-1,1].$$

$\phi'(x) = 0$ holds when $x = x_0 = \frac{4r^2-1}{4r}$. It is obvious that $x_0 < 1$ for $r \in [0,1]$. So $\phi(x) \leq \phi(x_0) = 2(4r^2+1)^2$. It follows that $|\mathbf{S}_3| \leq \frac{8+\sqrt{2}}{2}$.

Substituting (3.1) into (1.9), combining Lemma 1, it follows that

$$\begin{aligned}|\mathbf{S}_4| =&6|-c_2 + c_3 + c_1^2 - 2c_1c_2 + c_1^3|\\ =&12|\zeta_1^2 + \zeta_1^3 - (1-|\zeta_1|^2)(2\zeta_1+1)\zeta_2 - (1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + (1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3|\\ \leq&12(1+x^3+(1-x^2)(2x+1)y-(1-x^2)(1-x)y^2) = F(x,y),\end{aligned}$$

where $\zeta_1, \zeta_2, \zeta_3 \in \overline{\mathbb{D}}$, $x = |\zeta_1|$ and $y = |\zeta_2|$.

Note first that $x = 1$, we have $F(1,y) = 24$. For $x \in [0,1)$, we get

$$\frac{\partial F}{\partial y} = 12(1-x^2)(2x+1-2(1-x)y) = 0 \ \Leftrightarrow \ y = \frac{2x+1}{2(1-x)} =: y_0.$$

If $x \in [\frac{1}{4},1)$, then $y_0 \geq 1$. it follows that $F(x,y) \leq F(x,1) = 12(1+3x-2x^3) \leq 12(1+\sqrt{2})$. If $x \in [0,\frac{1}{4})$, then $y_0 < 1$, it follows that $F(x,y) \leq F(x,y_0) = 15+15x+24x^2+24x^3 < \frac{165}{8}$. Hence, the inequality $|\mathbf{S}_4| \leq 12(1+\sqrt{2})$ holds.

(2) If $f(z) \in \mathcal{F}_2$, using (1.9), Lemma 1 and (3.2), we get

$$|\mathbf{S}_4| =6|-c_1 + c_3 - 2c_1c_2 + c_1^3|$$

$$
\begin{aligned}
&=12|-\zeta_1+\zeta_1^3-2(1-|\zeta_1|^2)\zeta_1\zeta_2-(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2+(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3| \\
&\le 12(1+x-x^2+x^3+2(1-x^2)xy-(1-x^2)(1-x)y^2),
\end{aligned}
$$

where $\zeta_1, \zeta_2, \zeta_3 \in \overline{\mathbb{D}}$, $x = |\zeta_1|$ and $y = |\zeta_2|$.

Similar to the processing methods in (1) of this theorem, it is easy to prove that $|\mathbf{S}_4| \le \frac{32\sqrt{6}}{3}$ is true.

(3) Let $f(z) \in \mathcal{F}_3$. Combining (2.1)–(3.3), and we have

$$
\begin{aligned}
|\mathbf{S}_3| &=6|a_3-a_2^2| = \frac{1}{2}|2c_1+4c_2-3c_1^2-3| = \frac{1}{2}|-3+4\zeta_1-4\zeta_1^2+8(1-|\zeta_1|^2)\zeta_2| \\
&\le \frac{1}{2}(|3-4\zeta_1+4\zeta_1^2|+8(1-|\zeta_1|^2)) \le \frac{1}{2}(11+4|\zeta_1|-4|\zeta_1|^2) \le 6.
\end{aligned}
$$

Equality holds when $\zeta_1 = -\frac{1}{2}$ and $\zeta_2 = -1$, by (1.5) and (2.5), it follows that the extremal function $f(z) = \int_0^z \frac{1-t^2}{(1-t+t^2)(1+t+t^2)}dt$.

Substituting (3.3) into (1.9), by Lemma 1, it follows that

$$
\begin{aligned}
|\mathbf{S}_4| &=6|c_1-c_2+c_3+c_1^2-2c_1c_2+c_1^3| \\
&=12|\zeta_1+\zeta_1^2+\zeta_1^3-(1-|\zeta_1|^2)(2\zeta_1+1)\zeta_2-(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2+(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3| \\
&\le 12(1+x+x^3+(1-x^2)(2x+1)y-(1-x^2)(1-x)y^2),
\end{aligned}
$$

where $\zeta_1, \zeta_2, \zeta_3 \in \overline{\mathbb{D}}$, $x = |\zeta_1|$ and $y = |\zeta_2|$.

Similar to the processing methods in (1) of this theorem, it is easy to prove $|\mathbf{S}_4| \le 12(1+\frac{8\sqrt{6}}{9})$. The proof is completed. □

4. Conclusions

Higher order Schwarzian derivatives for normalized univalent functions were first considered by Schippers [34], and those of convex functions were considered by Dorff and Szynal [11]. In the present investigation, higher order Schwarzian derivatives for the close-to-convex subclasses: $\Re\{(1-z)f'(z)\} > 0$, $\Re\{(1-z^2)f'(z)\} > 0$, $\Re\{(1-z+z^2)f'(z)\} > 0$ and $\Re\{(1-z)^2f'(z)\} > 0$ are considered, where $f(z)$ is analytic in the unit disk with $f(0) = f'(0)-1 = 0$. The bounds for the first two consecutive derivatives are investigated, which can enrich the research field of univalent analytic function.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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