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Research article

Generalized proportional fractional integral inequalities for convex functions

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Abstract: In this paper, we establish some inequalities for convex functions by applying the generalized proportional fractional integral. Some new results by using the linkage between the proportional fractional integral and the Riemann-Liouville fractional integral are obtained. Moreover, we give special cases of our reported results. Obtained results provide generalizations for some of the current results in the literature by applying some special values to the parameters.

Keywords: proportional fractional integral; fractional inequalities; convex function
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1. Introduction

The Chebyshev inequality, which has a notable spot in inequality theory, creates limit values and esteems for synchronous functions and assists in the reproduction of new variation inequalities of many various sorts. The foundation for this inequality lies in the following Chebyshev functional (see [1]):

$$T(f,g) = \frac{1}{b-a}\int_a^b f(\varkappa)g(\varkappa)d\varkappa - \left(\frac{1}{b-a}\int_a^b f(\varkappa)d\varkappa\right)\left(\frac{1}{b-a}\int_a^b g(\varkappa)d\varkappa\right),$$

where $T(f,g) \geq 0$ and f, g are integrable and synchronous functions on $[a,b]$, i.e.

$$(f(\varkappa_1) - f(\varkappa_2))(g(\varkappa_1) - g(\varkappa_2)) \geq 0, \text{ for } \varkappa_1, \varkappa_2 \in [a,b].$$

Many researchers have been done on the Chebyshev inequality and its generalizations, expansions, iterations, and adjustments for different classes of functions. They have established wide utilization in functional analysis, numerical analysis, and statistics; for these outcomes, we allude the reader to [1–3]. Another attractive and helpful inequality so-called the Pólya-Szegö inequality, which comprises the primary inspiration point in our investigation, which we can express as (see [4]):

$$\frac{\int_a^b f^2(\varkappa)d\varkappa \int_a^b g^2(\varkappa)d\varkappa}{\left(\int_a^b f(\varkappa)g(\varkappa)d\varkappa\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2,$$

where $m \leq f(\varkappa) \leq M$ and $n \leq g(\varkappa) \leq N$, for some $m, M, n, N \in \mathbb{R}$ and for each $\varkappa \in [a, b]$.

The authors in [5] employed the Pólya-Szegö inequality to prove the following inequality

$$|T(f,g)| \leq \frac{(M-m)(N-n)}{4(b-a)^2\sqrt{MmNn}} \int_a^b f(\varkappa)d\varkappa \int_a^b g(\varkappa)d\varkappa,$$

where $0 < m \leq f(\varkappa) \leq M < \infty$ and $0 < n \leq g(\varkappa) \leq N < \infty$, for $\varkappa \in [a, b]$.

Ngo et al. [6] presented the following integral inequalities

$$\int_0^1 \omega^{\mu-1}(\varkappa)d\varkappa \geq \int_0^1 \varkappa^{\mu}\omega(\varkappa)d\varkappa, \tag{1.1}$$

and

$$\int_0^1 \omega^{\mu-1}(\varkappa)d\varkappa \geq \int_0^1 \varkappa\omega^{\mu}(\varkappa)d\varkappa, \tag{1.2}$$

where $\mu > 0$ and ω is a positive continuous function on $[0, 1]$ with

$$\int_h^1 \omega(\varkappa)d\varkappa \geq \int_h^1 \varkappa d\varkappa, \ h \in [0, 1].$$

Liu et al. [7] introduced the following inequality

$$\int_a^b \omega^{\mu+\nu}(\varkappa)d\varkappa \geq \int_a^b (\varkappa - a)^{\mu}\omega^{\nu}(\varkappa)d\varkappa \tag{1.3}$$

where $\mu, \nu > 0$ and ω is a positive continuous function on $\mathfrak{J} :=[a, b]$, with

$$\int_a^b \omega^{\varsigma}(\varkappa)d\varkappa \geq \int_a^b (\varkappa - a)^{\varsigma}d\varkappa$$

and $\varsigma = \min(1, \nu)$, $\varkappa \in \mathfrak{J}$. Now, we state the following results, which were established by Liu et al. [8].

Theorem 1.1. *[8] Let $\varpi, \omega > 0$ be continuous functions on $\mathfrak{J}$ with $\varpi(\varkappa) \leq \omega(\varkappa)$ for all $\varkappa \in \mathfrak{J}$ and such that $\frac{\varpi}{\omega}$ is a decreasing function and ϖ is an increasing function. Suppose that Φ is a convex function with $\Phi(0) = 0$. Then*

$$\frac{\int_a^b \varpi(\varkappa)d\varkappa}{\int_a^b \omega(\varkappa)d\varkappa} \geq \frac{\int_a^b \Phi(\varpi(\varkappa))d\varkappa}{\int_a^b \Phi(\omega(\varkappa))d\varkappa}.$$

Theorem 1.2. *[8] Let $\varpi, z, \omega > 0$ be continuous functions on $\mathfrak{J}$ with $\varpi(\varkappa) \leq \omega(\varkappa)$ for all $\varkappa \in \mathfrak{J}$ and such that $\frac{\varpi}{\omega}$ is a decreasing function and ϖ, z are increasing functions. Assume that Φ is a convex function with $\Phi(0) = 0$. Then*

$$\frac{\int_a^b \varpi(\varkappa)d\varkappa}{\int_a^b \omega(\varkappa)d\varkappa} \geq \frac{\int_a^b \Phi(\varpi(\varkappa))z(\varkappa)d\varkappa}{\int_a^b \Phi(\omega(\varkappa))z(\varkappa)d\varkappa}.$$

On the other hand, the area of fractional calculus (FC) is concerned with integrals and derivatives of non-integer order. This field has a long-term history. The premise of it tends to be followed back to the message among Leibniz and L'Hôpital in 1695 [9]. Over the nears, many authors have dedicated themselves to the improvement of the theories of FC [10–16]. Moreover, the applications of FC are found in different fields [9, 10, 17]. In virtually, different types of fractional operators, e.g, Riemann-Liouville (R-L), Caputo [15, 16], and Hilfer [17] were presented. Recently, many authors have considered certain novel fractional operators and their potential applications in different fields of sciences and engineering [18, 19]. Abdeljawad and Baleanu [20] have studied the monotonicity results for difference fractional operators with discrete exponential kernels. They also have set up fractional operators with exponential kernel and their discrete versions [21]. Caputo and Fabrizio [23] distinguished by proposing a new fractional operator without a singular kernel. Atangana and Baleanu [22] introduced a novel fractional operator with the non-singular and non-local kernel. Some properties of these operators can be found in [24]. The generalized fractional operator generated by a class of local proportional derivatives are introduced by Jarad et al. [25].

In this regard, the fractional operator inequalities and their applications have likewise a basic job in applied mathematics, especially in the theory of differential equations. Countless a few of many interesting integral inequalities are set up by the analysts and researchers, e.g., inequalities involving R-L and generalized R-L integrals [26, 27], Grüss-type and weighted Grüss type inequalities involving the generalized R-L integrals and fractional integration [28,29], some inequalities involving the extended gamma function and confluent hypergeometric k-function [30], and generalizations of the generalized Gronwall type inequalities associated with k-fractional derivatives [31]. Some recent works on Chebyshev's inequalities involving various types of fractional operators can be found in [32–36].

For more survey of some recent and earlier expansions related to the Minkowski (Gronwall, Hermite-Hadamard, Grüss) inequalities, we point the readers to see also [37–47].

Motivated by the above works, in this paper, we establish some new inequalities for convex functions by applying the generalized proportional fractional (GPF) integral. These results are recent and provide the generalizations of some reported results [8,48,49] by applying some special values to the parameters.

2. Preliminaries

In this section, we provide some basic definitions and some properties of proportional fractional integrals.

Definition 2.1. *([15]). The R-L fractional integrals ${}_aI^{\alpha}$ and I_b^{α} are respectively given by*

$$({}_aI^{\alpha}\varpi)(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_a^{\varkappa}(\varkappa-\upsilon)^{\alpha-1}\varpi(\upsilon)d\upsilon,\ a<\varkappa, \tag{2.1}$$

and

$$(I_b^{\alpha}\varpi)(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{b}(\upsilon-\varkappa)^{\alpha-1}\varpi(\upsilon)d\upsilon,\ \varkappa<b, \tag{2.2}$$

where $\alpha\in\mathbb{C}$, $Re(\alpha)>0$ and

$$\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du.$$

Definition 2.2. *([50, 51]). Let $\mathfrak{J} \subset \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$ with $Re(\alpha) > 0$ and $Re(\beta) \geq 0$. Then the tempered fractional integrals ${}_aI^{\alpha,\beta}$ and $I_b^{\alpha,\beta}$ are respectively given by*

$$({}_aI^{\alpha,\beta}\varpi)(\varkappa) = e^{-\beta\varkappa}\, {}_a\mathfrak{J}^{\alpha}(e^{\beta\varkappa}\varpi(\varkappa)) = \frac{1}{\Gamma(\alpha)}\int_a^{\varkappa} \exp\left[-\beta(\varkappa-\upsilon)\right](\varkappa-\upsilon)^{\alpha-1}\varpi(\upsilon)d\upsilon,\ a<\varkappa, \tag{2.3}$$

and

$$(I_b^{\alpha,\beta}\varpi)(\varkappa) = e^{-\beta\varkappa}\, I_b^{\alpha}(e^{\beta\varkappa}\varpi(\varkappa)) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{b} \exp\left[-\beta(\upsilon-\varkappa)\right](\upsilon-\varkappa)^{\alpha-1}\varpi(\upsilon)d\upsilon,\ \varkappa<b. \tag{2.4}$$

Definition 2.3. *([25]). For $0 < \rho \leq 1$ and $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, the left and right GPF integrals of a function $\varpi \in L^1(\mathfrak{J})$ are respectively given by*

$${}^{GPF}I_{a^+}^{(\alpha,\rho)}\varpi(\varkappa) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\varkappa}(\varkappa-\upsilon)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\upsilon)\right]\varpi(\upsilon)d\upsilon,\ \varkappa\in\mathfrak{J}, \tag{2.5}$$

and

$${}^{GPF}I_{b^-}^{(\alpha,\rho)}\varpi(\varkappa) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_{\varkappa}^{b}(\upsilon-\varkappa)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\upsilon-\varkappa)\right]\varpi(\upsilon)d\upsilon,\ \varkappa\in\mathfrak{J}. \tag{2.6}$$

Remark 2.4.

1) *If we put $\rho = 1$ in Eq (2.5) and Eq (2.6), then Eq (2.1) and Eq (2.2) are obtained, respectively, i.e., the generalized proportional operators reduce to the R-L operators.*

2) *If we replace $\frac{\rho-1}{\rho}$ with $-\beta$ in Eq (2.5) and Eq (2.6), then we obtain the tempered fractional integral operators (2.3) and (2.4) respectively.*

Here are some important characteristics of GPF integrals.

Proposition 2.5. *([25]). For any $\rho \in (0, 1]$, we have*

$$\begin{aligned}\left({}^{GPF}I_{a^+}^{(\alpha,\rho)}e^{\frac{\rho-1}{\rho}s}(s-a)^{\delta-1}\right)(\varkappa) &= \frac{\Gamma(\delta)}{\rho^{\alpha}\Gamma(\delta+\alpha)}e^{\frac{\rho-1}{\rho}\varkappa}(\varkappa-a)^{\delta+\alpha-1},\\ \left({}^{GPF}I_{b^-}^{(\alpha,\rho)}e^{\frac{\rho-1}{\rho}(b-s)}(b-s)^{\delta-1}\right)(\varkappa) &= \frac{\Gamma(\delta)}{\rho^{\alpha}\Gamma(\delta+\alpha)}e^{\frac{\rho-1}{\rho}(b-\varkappa)}(b-\varkappa)^{\delta+\alpha-1},\end{aligned}$$

where $\alpha, \rho \in \mathbb{C}$, $Re(\alpha) \geq 0$ and $Re(\rho) \geq 0$.

Proposition 2.6. *([25]). For any continuous function ϖ, we have*

$${}^{GPF}I_{a^+}^{(\alpha,\rho)}\ {}^{GPF}I_{a^+}^{(\beta,\rho)}\varpi(\varkappa) = {}^{GPF}I_{a^+}^{(\alpha+\beta,\rho)}\varpi(\varkappa),$$

where $0 < \rho \leq 1$, $Re(\alpha) \geq 0$ and $Re(\beta) \geq 0$.

3. Main results

In this section, we provide some inequalities for convex functions by using the GPF integral.

Theorem 3.1. *Let $\varpi, \omega > 0$ be continuous functions on $\mathfrak{J}$ with $\varpi(\varkappa) \leq \omega(\varkappa)$ for all $\varkappa \in \mathfrak{J}$ and such that $\frac{\varpi}{\omega}$ is a decreasing function and ϖ is an increasing function on $\mathfrak{J}$. Then for any convex function Φ with $\Phi(0) = 0$, the inequality*

$$\frac{{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\varpi(\varkappa)]}{{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\omega(\varkappa)]} \geq \frac{{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\Phi(\varpi(\varkappa))]}{{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\Phi(\omega(\varkappa))]} \tag{3.1}$$

holds for the GPF integral (2.5).

Proof. By the hypotheses of theorem, $\Phi(\varkappa)$ is convex function with $\Phi(0) = 0$. Then $\frac{\Phi(\varkappa)}{\varkappa}$ is an increasing function. Since ϖ is an increasing function, thus $\frac{\Phi(\varpi)}{\varpi}$ is an increasing function, too.

Since, $\frac{\varpi}{\omega}$ is a decreasing function, therefore for each $\sigma \in \mathfrak{J}$, we have

$$\left(\frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)} - \frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\right)\left(\frac{\varpi(\varkappa)}{\omega(\varkappa)} - \frac{\varpi(\sigma)}{\omega(\sigma)}\right) \geq 0. \tag{3.2}$$

It follows that

$$\frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}\frac{\varpi(\varkappa)}{\omega(\varkappa)} + \frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\frac{\varpi(\sigma)}{\omega(\sigma)} - \frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\frac{\varpi(\varkappa)}{\omega(\varkappa)} - \frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}\frac{\varpi(\sigma)}{\omega(\sigma)} \geq 0. \tag{3.3}$$

Multiplying (3.3) by $\omega(\sigma)\omega(\varkappa)$, we obtain

$$\frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}\varpi(\varkappa)\omega(\sigma) + \frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\sigma)\omega(\varkappa) - \frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)\omega(\sigma) - \frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}\varpi(\sigma)\omega(\varkappa) \geq 0. \tag{3.4}$$

Multiplying (3.4) by $\frac{1}{\rho^\alpha\Gamma(\alpha)}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]$ and integrating (3.4) with respect to σ over $[a, \varkappa]$, $a < \varkappa \leq b$, we get

$$\begin{aligned}
&\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^\varkappa(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}\varpi(\varkappa)\omega(\sigma)d\sigma \\
&+\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^\varkappa(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\sigma)\omega(\varkappa)d\sigma \\
&-\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^\varkappa(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)\omega(\sigma)d\sigma \\
&-\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^\varkappa(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}\varpi(\sigma)\omega(\varkappa)d\sigma \geq 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\varpi(\varkappa)\,{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)\right) + \left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)\right)\,{}^{GPF}I_{a^+}^{(\alpha,\rho)}(\varpi(\varkappa)) \\
&-\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)\right)\,{}^{GPF}I_{a^+}^{(\alpha,\rho)}(\omega(\varkappa)) - \omega(\varkappa)\,{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)\right) \geq 0.
\end{aligned} \tag{3.5}$$

Again, multiplying (3.5) by $\frac{1}{\rho^{\alpha}\Gamma(\alpha)}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]$ and integrating (3.5) with respect to σ over $[a,\varkappa]$, $a<\varkappa\leq b$, we obtain

$$\begin{aligned}&{}^{GPF}I_{a^+}^{(\alpha,\rho)}\varpi(\varkappa){}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)\right)+{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)\right){}^{GPF}I_{a^+}^{(\alpha,\rho)}(\varpi(\varkappa))\\&\geq{}^{GPF}I_{a^+}^{(\alpha,\rho)}(\Phi(\varpi(\varkappa))){}^{GPF}I_{a^+}^{(\alpha,\rho)}\omega(\varkappa)+{}^{GPF}I_{a^+}^{(\alpha,\rho)}\omega(\varkappa){}^{GPF}I_{a^+}^{(\alpha,\rho)}(\Phi(\varpi(\varkappa)).\end{aligned}$$

Consequently, we have

$$\frac{{}^{GPF}I_{a^+}^{(\alpha,\rho)}\varpi(\varkappa)}{{}^{GPF}I_{a^+}^{(\alpha,\rho)}\omega(\varkappa)}\geq\frac{{}^{GPF}I_{a^+}^{(\alpha,\rho)}(\Phi(\varpi(\varkappa))}{{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)\right)}. \tag{3.6}$$

Since $\varpi(\varkappa)\leq\omega(\varkappa)$ for all $\varkappa\in\mathfrak{J}$ and the function defined by $\varkappa\rightarrow\frac{\Phi(\varkappa)}{\varkappa}$ is an increasing, thus for $\sigma\in[a,\varkappa]$, $a<\varkappa\leq b$, we have

$$\frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}\leq\frac{\Phi(\omega(\sigma))}{\omega(\sigma)}. \tag{3.7}$$

Multiplying both sides of (3.7) by $\frac{1}{\rho^{\alpha}\Gamma(\alpha)}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\omega(\sigma)$, then integrating with respect to σ over $[a,\varkappa]$, $a<\varkappa\leq b$, we get

$$\begin{aligned}&\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\varkappa}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}\omega(\sigma)d\sigma\\&\leq\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\varkappa}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\Phi(\omega(\sigma))\,d\sigma,\end{aligned} \tag{3.8}$$

In view of (2.5) we can write (3.8) as follows

$${}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)\right)\leq{}^{GPF}I_{a^+}^{(\alpha,\rho)}\Phi(\omega(\varkappa)). \tag{3.9}$$

Hence from (3.6) and (3.9), we obtain (3.1). □

Remark 3.2.

i) *When $\rho=1$ in Theorem 3.1 we obtain [49, Theorem 3].*

ii) *When $\alpha=\rho=1$ and $\varkappa=b$ in Theorem 3.1 we get Theorem 1.1.*

Theorem 3.3. *Let $\varpi,\omega>0$ be continuous functions on $\mathfrak{J}$ with $\varpi(\varkappa)\leq\omega(\varkappa)$ for all $\varkappa\in\mathfrak{J}$ and such that $\frac{\varpi}{\omega}$ is a decreasing function and ϖ is an increasing function on $\mathfrak{J}$. Then for any convex function Φ with $\Phi(0)=0$, the inequality*

$$\frac{{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\varpi(\varkappa)]{}^{GPF}I_{a^+}^{(\beta,\rho)}[\Phi(\omega(\varkappa))]+{}^{GPF}I_{a^+}^{(\beta,\rho)}[\varpi(\varkappa)]{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\Phi(\omega(\varkappa))]}{{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\omega(\varkappa)]{}^{GPF}I_{a^+}^{(\beta,\rho)}[\Phi(\varpi(\varkappa))]+{}^{GPF}I_{a^+}^{(\beta,\rho)}[\omega(\varkappa)]{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\Phi(\varpi(\varkappa))]}\geq1 \tag{3.10}$$

holds for the GPF integral (2.5).

Proof. By virtue of assumptions of the theorem, $\Phi(\varkappa)$ is convex function with $\Phi(0) = 0$. Thus, the function $\frac{\Phi(\varkappa)}{\varkappa}$ is an increasing. Moreover, from the increasingly of function ϖ, the function $\frac{\Phi(\varpi)}{\varpi}$ is an increasing. Since the function $\frac{\varpi}{\omega}$ is a decreasing, therefore, multiplying (3.5) by $\frac{1}{\rho^{\beta}\Gamma(\beta)}(\varkappa-\sigma)^{\beta-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]$ and integrating the resultant identity with respect to σ over $[a,\varkappa]$, $a < \varkappa \leq b$, we get

$$\begin{aligned}
&{}^{GPF}I_{a^+}^{(\beta,\rho)}\varpi(\varkappa)^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)\right)+^{GPF}I_{a^+}^{(\beta,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)\right){}^{GPF}I_{a^+}^{(\alpha,\rho)}(\varpi(\varkappa))\\
&\geq {}^{GPF}I_{a^+}^{(\beta,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)\right){}^{GPF}I_{a^+}^{(\alpha,\rho)}(\omega(\varkappa))\\
&+\ {}^{GPF}I_{a^+}^{(\beta,\rho)}\omega(\varkappa)\ {}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)\right). \qquad (3.11)
\end{aligned}$$

Hence, from (3.9) and (3.11), we obtain the relation (3.10). □

Remark 3.4. *Put $\alpha = \beta$ in Theorem 3.3 we get Theorem 3.1. Moreover, if $\rho = 1$ in Theorem 3.3 we get [49, Theorem 4].*

Theorem 3.5. *Let $\varpi, z, \omega > 0$ be continuous functions on $\mathfrak{J}$ with $\varpi(\varkappa) \leq \omega(\varkappa)$ for all $\varkappa \in \mathfrak{J}$ and such that $\frac{\varpi}{\omega}$ is a decreasing function and ϖ, z are increasing functions. Assume that Φ is a convex function with $\Phi(0) = 0$. Then the inequality*

$$\frac{{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\varpi(\varkappa)]}{{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\omega(\varkappa)]} \geq \frac{{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\Phi(\varpi(\varkappa))z(\varkappa)]}{{}^{GPF}I_{a^+}^{(\alpha,\rho)}[\Phi(\omega(\varkappa))z(\varkappa)]}$$

holds for the GPF integral (2.5).

Proof. Since $\Phi(\varkappa)$ is convex function with $\Phi(0) = 0$, the function $\frac{\Phi(\varkappa)}{\varkappa}$ is an increasing. Besides, from the increasing property of the function ϖ, the function $\frac{\Phi(\varpi(\cdot))}{\varpi(\cdot)}$ is an increasing. Since the function $\frac{\varpi(\cdot)}{\omega(\cdot)}$ is a decreasing, thus, for each $\sigma \in [a,\varkappa]$ and $a < \varkappa \leq b$, we obtain

$$\left(\frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}z(\sigma)-\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}z(\varkappa)\right)(\varpi(\varkappa)\omega(\sigma)-\varpi(\sigma)\omega(\varkappa)) \geq 0. \qquad (3.12)$$

It follows that

$$\begin{aligned}
&\frac{\Phi(\varpi(\sigma))z(\sigma)}{\varpi(\sigma)}\varpi(\varkappa)\omega(\sigma)+\frac{\Phi(\varpi(\varkappa))z(\varkappa)}{\varpi(\varkappa)}\varpi(\sigma)\omega(\varkappa)\\
&-\frac{\Phi(\varpi(\varkappa))z(\varkappa)}{\varpi(\varkappa)}\varpi(\varkappa)\omega(\sigma)-\frac{\Phi(\varpi(\sigma))z(\sigma)}{\varpi(\sigma)}\varpi(\sigma)\omega(\varkappa) \geq 0. \qquad (3.13)
\end{aligned}$$

Multiplying (3.13) by $\frac{1}{\rho^{\alpha}\Gamma(\alpha)}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]$ and integrating the resulting inequality with respect to σ over $[a,\varkappa]$, $a < \varkappa \leq b$, we obtain

$$\begin{aligned}
&\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\varkappa}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}\varpi(\varkappa)\omega(\sigma)z(\sigma)d\sigma\\
&+\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\varkappa}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\sigma)\omega(\varkappa)z(\varkappa)d\sigma
\end{aligned}$$

$$-\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\varkappa}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)\omega(\sigma)z(\varkappa)d\sigma$$
$$-\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\varkappa}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\frac{\Phi(\varpi(\sigma))}{\varpi(\sigma)}\varpi(\sigma)\omega(\varkappa)z(\sigma)d\sigma \geq 0.$$

Consequently,

$$\begin{aligned}&\varpi(\varkappa)^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)z(\varkappa)\right)+\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)z(\varkappa)\right){}^{GPF}I_{a^+}^{(\alpha,\rho)}(\varpi(\varkappa))\\&-\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)z(\varkappa)\right){}^{GPF}I_{a^+}^{(\alpha,\rho)}(\omega(\varkappa))-\omega(\varkappa)\,{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)z(\varkappa)\right)\\&\geq 0.\end{aligned}\tag{3.14}$$

By the same arguments as before on the inequality (3.14), we get

$$\begin{aligned}{}^{GPF}I_{a^+}^{(\alpha,\rho)}\quad&(\varpi(\varkappa))^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)z(\varkappa)\right)\\&+{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)z(\varkappa)\right){}^{GPF}I_{a^+}^{(\alpha,\rho)}(\varpi(\varkappa))\\\geq\quad&{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\Phi(\varpi(\varkappa))z(\varkappa)\right){}^{GPF}I_{a^+}^{(\alpha,\rho)}(\omega(\varkappa))\\&+{}^{GPF}I_{a^+}^{(\alpha,\rho)}(\omega(\varkappa))\,{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\Phi(\varpi(\varkappa))z(\varkappa)\right).\end{aligned}$$

This follows that

$$\frac{{}^{GPF}I_{a^+}^{(\alpha,\rho)}\varpi(\varkappa)}{{}^{GPF}I_{a^+}^{(\alpha,\rho)}\omega(\varkappa)}\geq\frac{{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\Phi(\varpi(\varkappa)z(\varkappa))\right)}{{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)z(\varkappa)\right)}.\tag{3.15}$$

Moreover, since $\varpi(\varkappa)\leq\omega(\varkappa)$ for all $\varkappa\in\mathfrak{J}$, then using the fact that the function $\varkappa\to\frac{\Phi(\varkappa)}{\varkappa}$ is an increasing, thus for $\sigma\in[a,\varkappa]$ we can write

$$\frac{\Phi\left(\varpi(\sigma)\right)}{\varpi(\sigma)}\leq\frac{\Phi\left(\omega(\sigma)\right)}{\omega(\sigma)}.\tag{3.16}$$

With the same technique as before, inequality (3.16) leads to

$$\begin{aligned}&\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\varkappa}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\frac{\Phi\left(\varpi(\sigma)\right)}{\varpi(\sigma)}\omega(\sigma)z(\sigma)d\sigma\\\leq\quad&\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\varkappa}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\Phi(\omega(\sigma))z(\sigma)d\sigma.\end{aligned}$$

In view of (2.5), the last inequality can be written as follows

$${}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi\left(\varpi(\varkappa)\right)}{\varpi(\varkappa)}\omega(\varkappa)z(\varkappa)\right)\leq^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\Phi\left(\omega(\varkappa)\right)z(\varkappa)\right).\tag{3.17}$$

From (3.17) and (3.15), we get the desired result. □

Remark 3.6.

1. *If* $\rho = 1$ *in Theorem 3.5, we obtain the result in [49, Theorem 5].*
2. *If* $\alpha = 1$, $\rho = 1$ *and* $\varkappa = b$ *in Theorem 3.5, we obtain Theorem 1.2.*

Theorem 3.7. *Let* $\varpi, z, \omega > 0$ *be continuous functions on* $\mathfrak{J}$ *with* $\varpi(\varkappa) \leq \omega(\varkappa)$ *for all* $\varkappa \in \mathfrak{J}$ *and such that* $\frac{\varpi}{\omega}$ *is a decreasing function, and* ϖ, z *are increasing functions on* $\mathfrak{J}$*. Then for any convex function* Φ *with* $\Phi(0) = 0$*, the inequality*

$$\frac{{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left[\varpi(\varkappa)\right]{}^{GPF}I_{a^+}^{(\beta,\rho)}\left[\Phi(\omega(\varkappa))z(\varkappa)\right]+{}^{GPF}I_{a^+}^{(\beta,\rho)}\left[\varpi(\varkappa)\right]{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left[\Phi(\omega(\varkappa))z(\varkappa)\right]}{{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left[\omega(\varkappa)\right]{}^{GPF}I_{a^+}^{(\beta,\rho)}\left[\Phi(\varpi(\varkappa))z(\varkappa)\right]+{}^{GPF}I_{a^+}^{(\beta,\rho)}\left[\omega(\varkappa)\right]{}^{GPF}I_{a^+}^{(\alpha,\rho)}\left[\Phi(\varpi(\varkappa))z(\varkappa)\right]}\geq 1 \tag{3.18}$$

holds for the GPF integral (2.5).

Proof. Multiplying both sides of (3.14) by $\frac{1}{\rho^\beta\Gamma(\beta)}(\varkappa-\sigma)^{\beta-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]$ then integrating the resulting inequality with respect to σ over $[a,\varkappa]$, $a<\varkappa\leq b$, we obtain

$$\begin{aligned}
&{}^{GPF}I_{a^+}^{(\beta,\rho)}\varpi(\varkappa){}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)z(\varkappa)\right)\\
&+\ {}^{GPF}I_{a^+}^{(\beta,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\omega(\varkappa)z(\varkappa)\right)\ {}^{GPF}I_{a^+}^{(\alpha,\rho)}(\varpi(\varkappa))\\
&\geq\ {}^{GPF}I_{a^+}^{(\beta,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)z(\varkappa)\right)\ {}^{GPF}I_{a^+}^{(\alpha,\rho)}(\omega(\varkappa))\\
&+\ {}^{GPF}I_{a^+}^{(\beta,\rho)}\omega(\varkappa)\ {}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi(\varpi(\varkappa))}{\varpi(\varkappa)}\varpi(\varkappa)z(\varkappa)\right).
\end{aligned} \tag{3.19}$$

Since $\varpi(\varkappa)\leq\omega(\varkappa)$ for all $\varkappa\in\mathfrak{J}$, then using the fact that the function $\varkappa\to\frac{\Phi(\varkappa)}{\varkappa}$ is an increasing, thus for $\sigma\in[a,\varkappa]$ and $\varkappa\in\mathfrak{J}$, we have

$$\frac{\Phi\left(\varpi(\sigma)\right)}{\varpi(\sigma)}\leq\frac{\Phi\left(\omega(\sigma)\right)}{\omega(\sigma)}. \tag{3.20}$$

Multiplying the last inequality by $\frac{1}{\rho^\alpha\Gamma(\alpha)}(\varkappa-\sigma)^{\alpha-1}\exp\left[\frac{\rho-1}{\rho}(\varkappa-\sigma)\right]\omega(\sigma)z(\sigma)$, then integrating the resulting inequality with respect to σ over $[a,\varkappa]$, $a<\varkappa\leq b$, we get

$${}^{GPF}I_{a^+}^{(\alpha,\rho)}\left(\frac{\Phi\left(\varpi(\varkappa)\right)}{\varpi(\varkappa)}\omega(\varkappa)z(\varkappa)\right)\leq{}^{GPF}\ I_{a^+}^{(\alpha,\rho)}\left(\Phi\left(\omega(\varkappa)\right)z(\varkappa)\right). \tag{3.21}$$

By following similar arguments as previously mentioned, we obtain

$${}^{GPF}I_{a^+}^{(\beta,\rho)}\left(\frac{\Phi\left(\varpi(\varkappa)\right)}{\varpi(\varkappa)}\omega(\varkappa)z(\varkappa)\right)\leq{}^{GPF}\ I_{a^+}^{(\beta,\rho)}\left(\Phi\left(\omega(\varkappa)\right)z(\varkappa)\right). \tag{3.22}$$

Hence, thanks to (3.19), (3.21) and (3.22), we get the desired inequality (3.18). □

Remark 3.8. *If in Theorem 3.7* $\alpha=\beta$*, then we obtain Theorem 3.5.*

Remark 3.9. *If in Theorem 3.7* $\rho=1$*, then we obtain [49, Theorem 5].*

4. Conclusions

In this work, we have established some inequalities for generalized proportional fractional integrals by means of convex functions. As well as we have established many new special results by using the relationship between the generalized proportional fractional integral and the R-L integral. The obtained results cover the given results by Dahmani [48] for $\rho = 1$, and Liu et al. [8, Theorems 9 and 10] for $\alpha = 1$ and $\rho = 1$.

Besides, if we replaced the generalized proportional fractional integral with the tempered fractional integral, then the acquired inequalities will reduce to the results of Rahman et al. [49].

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Conflict of interest

The authors declare that they have no competing interest.

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