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Research article

Some new Hermite-Hadamard type inequalities for generalized harmonically convex functions involving local fractional integrals

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Abstract: In this paper, we establish a new integral identity involving local fractional integral on Yang's fractal sets. Using this integral identity, some new generalized Hermite-Hadamard type inequalities whose function is monotonically increasing and generalized harmonically convex are obtained. Finally, we construct some generalized special means to explain the applications of these inequalities.

Keywords: Hermite-Hadamard type inequality; generalized harmonically convex function; Yang's fractal sets; local fractional integral
Mathematics Subject Classification: 26D15, 26A51, 26D10

1. Introduction

Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $\alpha, \beta \in I$ with $\alpha < \beta$, then the following inequality holds,

$$\phi(\frac{\alpha+\beta}{2}) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\phi(x)dx \leq \frac{\phi(\alpha)+\phi(\beta)}{2}, \tag{1.1}$$

which is well known as Hermite-Hadamard's inequality [1] for convex functions. Both inequalities hold in the reversed direction if ϕ is concave.

Convex function is an important function in mathematical analysis and has been applied in many aspects [2, 3]. With the extension of the definition of convex function, Hermite-Hadamard's inequality has been deeply studied. Some improvement and generalizations for Hermite-Hadamard's inequality (1.1) can been found in the references [4–12].

In [11], İşcan gave the definition of harmonically convexity as follows:

Definition 1. Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $\phi : I \to \mathbb{R}$ is said to be harmonically convex, if

$$\phi\left(\frac{xy}{tx+(1-t)y}\right) \leq t\phi(y) + (1-t)\phi(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.2) is reversed, then ϕ is said to be harmonically concave.

In recent years, many researchers presented many kinds of fractional calculus by different methods and explored their applications. For example, Riemann-Liouville fractional integrals and its applications in inequalities [13–16]. Recently, Yang stated the theory of local fractional calculus on Yang's fractal sets systematically in [17–19]. Local fractional calculus can explain the behavior of continuous but nowhere differentiable function. In view of the special advantages of local fractional calculus, more and more researchers extended their studies to Yang's fractal space, see [20–29].

In [22], Sun introduced the definition of the generalized harmonically convex function on Yang's fractal sets as follows:

Definition 2. Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $\phi : I \to \mathbb{R}^{\epsilon}(0 < \epsilon \leq 1)$ is said to be generalized harmonically convex, if

$$\phi\left(\frac{xy}{tx+(1-t)y}\right) \leq t^{\epsilon}\phi(y) + (1-t)^{\epsilon}\phi(x) \tag{1.3}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.3) is reversed, then ϕ is said to be generalized harmonically concave. The sign ϵ represents the fractal dimension.

Example 1. Let $\phi : (0, \infty) \to \mathbb{R}^{\epsilon}$ and $\psi : (-\infty, 0) \to \mathbb{R}^{\epsilon}$, then $\phi(x) = x^{\epsilon}$ is a generalized harmonically convex function and $\psi(x) = x^{\epsilon}$ is a generalized harmonically concave function.

The following result related to Hermite-Hadamard's inequalities holds.

Theorem 1. [22] Let $\phi : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}^{\epsilon}$ be a generalized harmonically convex function on fractal space and $\alpha, \beta \in I$ with $\alpha < \beta$. If $\phi(x) \in I_x^{(\epsilon)}[\alpha, \beta]$, then

$$\frac{1}{\Gamma(1+\epsilon)}\phi(\frac{2\alpha\beta}{\alpha+\beta}) \leq \frac{\alpha^{\epsilon}\beta^{\epsilon}}{(\beta-\alpha)^{\epsilon}} {}_{\alpha}I_{\beta}^{(\epsilon)}\frac{\phi(x)}{x^{2\epsilon}} \leq \frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\left[\phi(\alpha) + \phi(\beta)\right]. \tag{1.4}$$

Based on the theory of local fractional calculus and the definition of the generalized harmonically convex function on Yang's fractal sets, the main aim of this paper is using a new integral identity and monotonicity of functions to establish some new Hermite-Hadamard type inequalities involving local fractional calculus.

2. Preliminaries

Let $\mathbb{R}^{\epsilon}(0 < \epsilon \leq 1)$ be ϵ-type set of the real line numbers on Yang's fractal sets, and give the following operation rules, see[17,18]. The sign ϵ represents the fractal dimension, not the exponential sign.

If $\alpha^{\epsilon}, \beta^{\epsilon}, \gamma^{\epsilon} \in \mathbb{R}^{\epsilon}$, then addition and multiplication operations satisfy

(a) $\alpha^{\epsilon} + \beta^{\epsilon} \in \mathbb{R}^{\epsilon}$, $\alpha^{\epsilon}\beta^{\epsilon} \in \mathbb{R}^{\epsilon}$,

(b) $\alpha^{\epsilon} + \beta^{\epsilon} = \beta^{\epsilon} + \alpha^{\epsilon} = (\alpha + \beta)^{\epsilon} = (\beta + \alpha)^{\epsilon}$,

(c) $\alpha^{\epsilon} + (\beta^{\epsilon} + \gamma^{\epsilon}) = (\alpha + \beta)^{\epsilon} + \gamma^{\epsilon}$,

(d) $\alpha^{\epsilon}\beta^{\epsilon} = \beta^{\epsilon}\alpha^{\epsilon} = (\alpha\beta)^{\epsilon} = (\beta\alpha)^{\epsilon}$,
(e) $\alpha^{\epsilon}(\beta^{\epsilon}\gamma^{\epsilon}) = (\alpha^{\epsilon}\beta^{\epsilon})\gamma^{\epsilon}$,
(f) $\alpha^{\epsilon}(\beta^{\epsilon} + \gamma^{\epsilon}) = \alpha^{\epsilon}\beta^{\epsilon} + \alpha^{\epsilon}\gamma^{\epsilon}$,
(g) $\alpha^{\epsilon} + 0^{\epsilon} = \alpha^{\epsilon}$, $\alpha^{\epsilon} + (-\alpha)^{\epsilon} = 0^{\epsilon}$ and $\alpha^{\epsilon}1^{\epsilon} = 1^{\epsilon}\alpha^{\epsilon} = \alpha^{\epsilon}$,
(h) $(\alpha - \beta)^{\epsilon} = \alpha^{\epsilon} - \beta^{\epsilon}$.

Definition 3. [18, 19] If there exists the relation

$$|\phi(x) - \phi(x_0)| < \varepsilon^{\epsilon}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in \mathbb{R}$ .Then the function $\phi(x)$ is called local fractional continuous at x_0. If $\phi(x)$ is local fractional continuous on (α,β) , we denote by $\phi(x) \in C_{\epsilon}(\alpha,\beta)$.

Definition 4. [17, 19] Supposing that $\phi(x) \in C_{\epsilon}(\alpha,\beta)$, the local fractional derivative of $\phi(x)$ of order ϵ at $x = x_0$ is defined by

$$\phi^{(\epsilon)}(x_0) = \left.\frac{d^{\epsilon}\phi(x)}{dx^{\epsilon}}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Gamma(\epsilon+1)(\phi(x) - \phi(x_0))}{(x - x_0)^{\epsilon}}.$$

For any $x \in (\alpha,\beta)$, there exists $\phi^{(\epsilon)}(x) = D_x^{(\epsilon)}$, denoted by $\phi^{(\epsilon)}(x) \in D_x^{(\epsilon)}(\alpha,\beta)$. $D_{\epsilon}(\alpha,\beta)$ is called ϵ-local fractional derivative set. If there exits $\phi^{((n+1)\epsilon)}(x) = \overbrace{D_x^{\epsilon} \cdots D_x^{\epsilon}}^{(n+1)\text{times}} \phi(x)$ for any $x \in I \subseteq \mathbb{R}$, then we denote $\phi \in D_{(n+1)\epsilon}(I)$, where $n = 0, 1, 2, \cdots$

Definition 5. [17, 19] Let $\phi(x) \in C_{\epsilon}[\alpha,\beta]$. The local fractional integral of function $\phi(x)$ of order ϵ is defined by

$${}_{\alpha}I_{\beta}^{(\epsilon)}\phi(x) = \frac{1}{\Gamma(\epsilon+1)}\int_{\alpha}^{\beta}\phi(t)(dt)^{\epsilon} = \frac{1}{\Gamma(\epsilon+1)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\epsilon},$$

where $\alpha = t_0 < t_1 < \cdots < t_{N-1} < t_N = \beta, [t_j, t_{j+1}]$ is a partition of the interval $[\alpha,\beta]$, $\Delta t_j = t_{j+1} - t_j, \Delta t = \max\{\Delta t_0, \Delta t_1 \cdots \Delta t_{N-1}\}$.

Note that ${}_{\alpha}I_{\alpha}^{(\epsilon)}\phi(x) = 0$, and ${}_{\alpha}I_{\beta}^{(\epsilon)}\phi(x) = -{}_{\beta}I_{\alpha}^{(\epsilon)}\phi(x)$ if $\alpha < \beta$. We denote $\phi(x) \in I_x^{(\epsilon)}[\alpha,\beta]$ if there exits ${}_{\alpha}I_x^{(\epsilon)}\phi(x)$ for any $x \in (\alpha,\beta)$.

Lemma 1. [17]

(1) Suppose that $\phi(x) = \varphi^{(\epsilon)}(x) \in C_{\epsilon}[\alpha,\beta]$, then

$${}_{\alpha}I_{\beta}^{(\epsilon)}\phi(x) = \varphi(\beta) - \varphi(\alpha).$$

(2) (Local fractional integration by parts)
Suppose that $\phi(x), \varphi(x) \in D_{\epsilon}(\alpha,\beta)$, and $\phi^{(\epsilon)}(x), \varphi^{(\epsilon)}(x) \in C_{\epsilon}[\alpha,\beta]$, then

$${}_{\alpha}I_{\beta}^{(\epsilon)}\phi(x)\varphi^{(\epsilon)}(x) = [\phi(x)\varphi(x)]\Big|_{\alpha}^{\beta} - {}_{\alpha}I_{\beta}^{(\epsilon)}\phi^{(\epsilon)}(x)\varphi(x).$$

Lemma 2. [17] Suppose that $\phi(x) \in C_{\epsilon}[\alpha,\beta]$ and $\alpha < \gamma < \beta$, then

$${}_{\alpha}I_{\beta}^{(\epsilon)}\phi(x) = {}_{\alpha}I_{\gamma}^{(\epsilon)}\phi(x) + {}_{\gamma}I_{\beta}^{(\epsilon)}\phi(x).$$

Lemma 3. [17]

$$\frac{d^\epsilon x^{k\epsilon}}{dx^\epsilon}=\frac{\Gamma(1+k\epsilon)}{\Gamma(1+(k-1)\epsilon)}x^{(k-1)\epsilon};$$

$$\frac{1}{\Gamma(\epsilon+1)}\int_\alpha^\beta x^{k\epsilon}(dx)^\epsilon=\frac{\Gamma(1+k\epsilon)}{\Gamma(1+(k+1)\epsilon)}(\beta^{(k+1)\epsilon}-\alpha^{(k+1)\epsilon}),k>0.$$

Lemma 4. [18, 30] (Generalized Hölder's inequality) Let $\phi,\varphi\in C_\epsilon[\alpha,\beta],p,q>1$, with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\frac{1}{\Gamma(\epsilon+1)}\int_\alpha^\beta|\phi(x)\varphi(x)|(dx)^\epsilon\leq\left(\frac{1}{\Gamma(\epsilon+1)}\int_\alpha^\beta|\phi(x)|^p(dx)^\epsilon\right)^{1/p}\left(\frac{1}{\Gamma(\epsilon+1)}\int_\alpha^\beta|\varphi(x)|^q(dx)^\epsilon\right)^{1/q}.$$

Lemma 5. [17]

$$_\alpha I_\beta^{(\epsilon)}1^\epsilon=\frac{(\beta-\alpha)^\epsilon}{\Gamma(1+\epsilon)}.$$

Let us introduce the special functions on Yang's fractal sets as follows:

(1) The generalized Beta function is given by

$$B_\epsilon(x,y)=\frac{1}{\Gamma(1+\epsilon)}\int_0^1 t^{(x-1)\epsilon}(1-t)^{(y-1)\epsilon}(dt)^\epsilon,x>0,y>0,$$

(2) The generalized hypergeometric function is given by

$$_2F_1^\epsilon(\alpha,\beta;\gamma;z)=\frac{1}{B_\epsilon(\beta,\gamma-\beta)}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 t^{(\beta-1)\epsilon}(1-t)^{(\gamma-\beta-1)\epsilon}(1-zt)^{-\alpha\epsilon}(dt)^\epsilon,\gamma>\beta>0,|z|<1.$$

3. Main results

For convenience, we use the symbol A_t to denote $t\alpha+(1-t)\beta$ in the following sections.

Lemma 6. Let $I\subset(0,\infty)$ be an interval, $\phi:I^\circ\to\mathbb{R}^\epsilon$ (I° is the interior of I) such that $\phi\in D_\epsilon(I^\circ)$ and $\phi^{(\epsilon)}\in C_\epsilon(\alpha,\beta)$ for $\alpha,\beta\in I^\circ$ with $\alpha<\beta$. Then the following equality holds

$$\phi\Big(\frac{2\alpha\beta}{\alpha+\beta}\Big)-\Gamma(1+\epsilon)\frac{\alpha^\epsilon\beta^\epsilon}{(\beta-\alpha)^\epsilon}{}_\alpha I_\beta^{(\epsilon)}\frac{\phi(x)}{x^{2\epsilon}}=I_1+I_2+I_3,\tag{3.1}$$

where

$$I_1=\frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^{1/2}\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\frac{(dt)^\epsilon}{(A_t)^{2\epsilon}},$$

$$I_2=-\frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_{1/2}^1\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\frac{(dt)^\epsilon}{(A_t)^{2\epsilon}},$$

$$I_3=-\frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1(1-2t)^\epsilon\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\frac{(dt)^\epsilon}{(A_t)^{2\epsilon}}.$$

Proof. Calculating I_1, I_2, from Lemma 1(1), we get

$$\begin{aligned} I_1 =& \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^{1/2}\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\frac{(dt)^\epsilon}{(A_t)^{2\epsilon}} \\ =& \frac{1^\epsilon}{2^\epsilon}\phi\left(\frac{\alpha\beta}{A_t}\right)\Bigg|_0^{1/2} = \frac{1^\epsilon}{2^\epsilon}\Big[\phi\left(\frac{2\alpha\beta}{\alpha+\beta}\right)-\phi(\alpha)\Big] \end{aligned}$$

and

$$\begin{aligned} I_2 =& -\frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_{1/2}^{1}\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\frac{(dt)^\epsilon}{(A_t)^{2\epsilon}} \\ =& -\frac{1^\epsilon}{2^\epsilon}\phi\left(\frac{\alpha\beta}{A_t}\right)\Bigg|_{1/2}^{1} = \frac{1^\epsilon}{2^\epsilon}\Big[\phi\left(\frac{2\alpha\beta}{\alpha+\beta}\right)-\phi(\beta)\Big]. \end{aligned}$$

Calculating I_3, by the local fractional integration by parts, we have

$$\begin{aligned} I_3 =& \frac{\alpha^\epsilon\beta^\epsilon(\alpha-\beta)^\epsilon}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1\frac{(1-2t)^\epsilon}{(A_t)^{2\epsilon}}\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)(dt)^\epsilon \\ =& \frac{(2t-1)^\epsilon}{2^\epsilon}\phi\left(\frac{\alpha\beta}{A_t}\right)\Bigg|_0^1 - \frac{1}{\Gamma(1+\epsilon)}\int_0^1\Gamma(1+\epsilon)\phi\left(\frac{\alpha\beta}{A_t}\right)(dt)^\epsilon \\ =& \frac{\phi(\alpha)+\phi(\beta)}{2^\epsilon} - \frac{\Gamma(1+\epsilon)}{\Gamma(1+\epsilon)}\int_0^1\phi\left(\frac{\alpha\beta}{t\alpha+(1-t)\beta}\right)(dt)^\epsilon. \end{aligned}$$

Using changing variable with $x = \frac{\alpha\beta}{A_t}$, we have

$$\begin{aligned} I_3 =& \frac{\phi(\alpha)+\phi(\beta)}{2^\epsilon} - \Gamma(1+\epsilon)\left(\frac{\alpha\beta}{\beta-\alpha}\right)^\epsilon\frac{1}{\Gamma(1+\epsilon)}\int_\alpha^\beta\frac{\phi(x)}{x^{2\epsilon}}(dx)^\epsilon \\ =& \frac{\phi(\alpha)+\phi(\beta)}{2^\epsilon} - \Gamma(1+\epsilon)\frac{\alpha^\epsilon\beta^\epsilon}{(\beta-\alpha)^\epsilon}{}_\alpha I_\beta^{(\epsilon)}\frac{\phi(x)}{x^{2\epsilon}}. \end{aligned}$$

Adding $I_1 - I_3$, the desired result is obtained. This completes the proof. □

Theorem 2. Let $I \subset (0,\infty)$ be an interval, $\phi : I^\circ \to \mathbb{R}^\epsilon$ (I° is the interior of I) is an increasing function on I° such that $\phi \in D_\epsilon(I^\circ)$ and $\phi^{(\epsilon)} \in C_\epsilon[\alpha,\beta]$ for $\alpha,\beta \in I^\circ$ with $\alpha < \beta$. If $|\phi^{(\epsilon)}|^q$ is generalized harmonically convex on $[\alpha,\beta]$ for some fixed $q > 1$, then for all $x \in [\alpha,\beta]$, the following local fractional integrals inequality holds,

$$\begin{aligned} &\left|\phi\Big(\frac{2\alpha\beta}{\alpha+\beta}\Big) - \Gamma(1+\epsilon)\frac{\alpha^\epsilon\beta^\epsilon}{(\beta-\alpha)^\epsilon}{}_\alpha I_\beta^{(\epsilon)}\frac{\phi(x)}{x^{2\epsilon}}\right| \\ \leq& \frac{\phi(\beta)-\phi(\alpha)}{2^\epsilon} + \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\Big(K_1^\epsilon(\alpha,\beta)\Big)^{1-\frac{1}{q}}\Big(K_2^\epsilon(\alpha,\beta)\Big|\phi^{(\epsilon)}(\alpha)\Big|^q + K_3^\epsilon(\alpha,\beta)\Big|\phi^{(\epsilon)}(\beta)\Big|^q\Big)^{\frac{1}{q}}, \end{aligned} \tag{3.2}$$

where

$$K_1^{\epsilon}(\alpha,\beta) = \beta^{-2\epsilon}\Big[{}_2F_1^{\epsilon}\Big(2,1;3;\frac{1}{2}(1-\frac{\alpha}{\beta})\Big)\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)} + {}_2F_1^{\epsilon}\Big(2,2;3;1-\frac{\alpha}{\beta}\Big)\frac{2^{\epsilon}\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)} - \frac{{}_2F_1^{\epsilon}\Big(2,1;2;1-\frac{\alpha}{\beta}\Big)}{\Gamma(1+\epsilon)}\Big],$$

$$K_2^{\alpha}(\alpha,\beta) = \beta^{-2\epsilon}\Big[\frac{1}{2^{\epsilon}}{}_2F_1^{\epsilon}\Big(2,2;4;\frac{1}{2}(1-\frac{\alpha}{\beta})\Big)\Big(\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)} - \frac{\Gamma(1+2\epsilon)}{\Gamma(1+3\epsilon)}\Big) + 2^{\epsilon}{}_2F_1^{\epsilon}\Big(2,3;4;1-\frac{\alpha}{\beta}\Big)\frac{\Gamma(1+2\epsilon)}{\Gamma(1+3\epsilon)} - {}_2F_1^{\epsilon}\Big(2,2;3;1-\frac{\alpha}{\beta}\Big)\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\Big],$$

$$K_3^{\epsilon}(\alpha,\beta) = \beta^{-2\epsilon}\Big[{}_2F_1^{\epsilon}\Big(2,1;3;\frac{1}{2}(1-\frac{\alpha}{\beta})\Big)\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)} + 2^{\epsilon}{}_2F_1^{\epsilon}\Big(2,2;4;1-\frac{\alpha}{\beta}\Big)\Big(\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)} - \frac{\Gamma(1+2\epsilon)}{\Gamma(1+3\epsilon)}\Big) - {}_2F_1^{\epsilon}\Big(2,1;3;1-\frac{\alpha}{\beta}\Big)\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\Big].$$

Proof. Since ϕ is an increasing function on I°, and $0 < \alpha < \frac{2\alpha\beta}{\alpha+\beta} < \beta$, we can obtain

$$\phi(\alpha) < \phi(\frac{2\alpha\beta}{\alpha+\beta}) < \phi(\beta).$$

From the proof of Lemma 6, we have

$$|I_1| + |I_2| = \frac{\phi(\beta)-\phi(\alpha)}{2^{\epsilon}}. \tag{3.3}$$

Taking modulus in equality (3.1), we obtain

$$\begin{aligned}\left|\phi\Big(\frac{2\alpha\beta}{\alpha+\beta}\Big) - \Gamma(1+\epsilon)\frac{\alpha^{\epsilon}\beta^{\epsilon}}{(\beta-\alpha)^{\epsilon}}{}_{\alpha}I_{\beta}^{(\epsilon)}\frac{\phi(x)}{x^{2\epsilon}}\right| &\le |I_1| + |I_2| + |I_3| \\ &= \frac{\phi(\beta)-\phi(\alpha)}{2^{\epsilon}} + |I_3|. \end{aligned}\tag{3.4}$$

From Lemma 6, using the property of the modulus and the generalized Hölder's inequality, we have

$$\begin{aligned}|I_3| &= \left|\frac{\alpha^{\epsilon}\beta^{\epsilon}(\beta-\alpha)^{\epsilon}}{2^{\epsilon}}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 (1-2t)^{\epsilon}\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\frac{(dt)^{\epsilon}}{(A_t)^{2\epsilon}}\right| \\ &\le \frac{\alpha^{\epsilon}\beta^{\epsilon}(\beta-\alpha)^{\epsilon}}{2^{\epsilon}}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left|\frac{(1-2t)^{\epsilon}}{A_t^{2\epsilon}}\right|\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|(dt)^{\epsilon} \\ &= \frac{\alpha^{\epsilon}\beta^{\epsilon}(\beta-\alpha)^{\epsilon}}{2^{\epsilon}}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left|\frac{(1-2t)^{\epsilon}}{A_t^{2\epsilon}}\right|^{(1-\frac{1}{q})+\frac{1}{q}}\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|(dt)^{\epsilon} \\ &\le \frac{\alpha^{\epsilon}\beta^{\epsilon}(\beta-\alpha)^{\epsilon}}{2^{\epsilon}}\left[\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left|\frac{(1-2t)^{\epsilon}}{A_t^{2\epsilon}}\right|(dt)^{\epsilon}\right]^{1-\frac{1}{q}}\left[\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left|\frac{(1-2t)^{\epsilon}}{A_t^{2\epsilon}}\right|\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|^{q}(dt)^{\epsilon}\right]^{\frac{1}{q}}. \end{aligned}\tag{3.5}$$

Since $|\phi^{(\epsilon)}|^q$ is generalized harmonically convex on $[\alpha,\beta]$, thus

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left|\frac{(1-2t)^{\epsilon}}{A_t^{2\epsilon}}\right|\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|^q (dt)^{\epsilon}\\
\leq&\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left|\frac{(1-2t)^{\epsilon}}{A_t^{2\epsilon}}\right|\Big(t^{\epsilon}\left|\phi^{(\epsilon)}(\beta)\right|^q+(1-t)^{\epsilon}\left|\phi^{(\epsilon)}(\alpha)\right|^q\Big)(dt)^{\epsilon}\\
=&\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{|1-2t|^{\epsilon}t^{\epsilon}}{A_t^{2\epsilon}}(dt)^{\epsilon}\right)\left|\phi^{(\epsilon)}(\alpha)\right|^q\\
&+\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{|1-2t|^{\epsilon}(1-t)^{\epsilon}}{A_t^{2\epsilon}}(dt)^{\epsilon}\right)\left|\phi^{(\epsilon)}(\beta)\right|^q.
\end{aligned}
\tag{3.6}
$$

By calculating, we get

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{|1-2t|^{\epsilon}}{A_t^{2\epsilon}}(dt)^{\epsilon}\\
=&\frac{1}{\Gamma(1+\epsilon)}\int_0^{\frac{1}{2}} \frac{(1-2t)^{\epsilon}}{A_t^{2\epsilon}}(dt)^{\epsilon}+\frac{1}{\Gamma(1+\epsilon)}\int_{\frac{1}{2}}^1 \frac{(2t-1)^{\epsilon}}{A_t^{2\epsilon}}(dt)^{\epsilon}\\
=&\frac{2^{\epsilon}}{\Gamma(1+\epsilon)}\int_0^{\frac{1}{2}} \frac{(1-2t)^{\epsilon}}{A_t^{2\epsilon}}(dt)^{\epsilon}\\
&+\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{(2t)^{\epsilon}}{A_t^{2\epsilon}}(dt)^{\epsilon}-\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{1^{\epsilon}}{A_t^{2\epsilon}}(dt)^{\epsilon}\\
=&\beta^{-2\epsilon}\Bigg[\frac{1}{\Gamma(1+\epsilon)}\int_0^1 (1-u)^{\epsilon}\Big(1-\frac{u}{2}(1-\frac{\alpha}{\beta})\Big)^{-2\epsilon}(du)^{\epsilon}\\
&+\frac{2^{\epsilon}}{\Gamma(1+\epsilon)}\int_0^1 t^{\epsilon}\Big(1-(1-\frac{\alpha}{\beta})t\Big)^{-2\epsilon}(dt)^{\epsilon}\\
&-\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \Big(1-(1-\frac{\alpha}{\beta})t\Big)^{-2\epsilon}(dt)^{\epsilon}\Bigg]\\
=&\beta^{-2\epsilon}\Big[{}_2F_1^{\epsilon}\Big(2,1;3;\frac{1}{2}(1-\frac{\alpha}{\beta})\Big)B_{\epsilon}(1,2)+2^{\epsilon}{}_2F_1^{\epsilon}\Big(2,2;3;1-\frac{\alpha}{\beta}\Big)B_{\epsilon}(2,1)\\
&-{}_2F_1^{\epsilon}\Big(2,1;2;1-\frac{\alpha}{\beta}\Big)B_{\epsilon}(1,1)\Big]\\
=&\beta^{-2\epsilon}\Bigg[{}_2F_1^{\epsilon}\Big(2,1;3;\frac{1}{2}(1-\frac{\alpha}{\beta})\Big)\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}+{}_2F_1^{\epsilon}\Big(2,2;3;1-\frac{\alpha}{\beta}\Big)\frac{2^{\epsilon}\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\\
&-\frac{{}_2F_1^{\epsilon}\Big(2,1;2;1-\frac{\alpha}{\beta}\Big)}{\Gamma(1+\epsilon)}\Bigg]\\
=&K_1^{\epsilon}(\alpha,\beta).
\end{aligned}
\tag{3.7}
$$

Similarly, we get

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{|1-2t|^\epsilon t^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon\\
=&\frac{1}{\Gamma(1+\epsilon)}\int_0^{\frac{1}{2}} \frac{(1-2t)^\epsilon t^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon+\frac{1}{\Gamma(1+\epsilon)}\int_{\frac{1}{2}}^1 \frac{(2t-1)^\epsilon t^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon\\
=&\frac{2^\epsilon}{\Gamma(1+\epsilon)}\int_0^{\frac{1}{2}} \frac{t^\epsilon(1-2t)^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon+\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{2^\epsilon t^{2\epsilon}}{A_t^{2\epsilon}}(dt)^\epsilon-\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{t^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon\\
=&\beta^{-2\epsilon}\Big[\frac{1}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 u^\epsilon(1-u)^\epsilon\Big(1-\frac{u}{2}(1-\frac{\alpha}{\beta})\Big)^{-2\epsilon}(du)^\epsilon+\frac{2^\epsilon}{\Gamma(1+\epsilon)}\int_0^1 t^{2\epsilon}\Big(1-(1-\frac{\alpha}{\beta})t\Big)^{-2\epsilon}(dt)^\epsilon\\
&-\frac{1}{\Gamma(1+\epsilon)}\int_0^1 t^\epsilon\Big(1-(1-\frac{\alpha}{\beta})t\Big)^{-2\epsilon}(dt)^\epsilon\Big]\\
=&\beta^{-2\epsilon}\Big[\frac{1}{2^\epsilon}{}_2F_1^\epsilon\Big(2,2;4;\frac{1}{2}(1-\frac{\alpha}{\beta})\Big)\Big(\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}-\frac{\Gamma(1+2\epsilon)}{\Gamma(1+3\epsilon)}\Big)\\
&+2^\epsilon{}_2F_1^\epsilon\Big(2,3;4;1-\frac{\alpha}{\beta}\Big)\frac{\Gamma(1+2\epsilon)}{\Gamma(1+3\epsilon)}-{}_2F_1^\epsilon\Big(2,2;3;1-\frac{\alpha}{\beta}\Big)\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\Big]=K_2^\epsilon(\alpha,\beta),
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{|1-2t|^\epsilon(1-t)^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon\\
=&\frac{2^\epsilon}{\Gamma(1+\epsilon)}\int_0^{\frac{1}{2}} \frac{(1-2t)^\epsilon(1-t)^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon+\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{(2t-1)^\epsilon(1-t)^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon\\
\leq&\frac{2^\epsilon}{\Gamma(1+\epsilon)}\int_0^{\frac{1}{2}} \frac{(1-2t)^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon+\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{2^\epsilon t^\epsilon(1-t)^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon\\
&-\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{(1-t)^\epsilon}{A_t^{2\epsilon}}(dt)^\epsilon\\
=&\beta^{-2\epsilon}\Big[\frac{1}{\Gamma(1+\epsilon)}\int_0^1(1-u)^\epsilon\Big(1-\frac{u}{2}(1-\frac{\alpha}{\beta})\Big)^{-2\epsilon}(du)^\epsilon\\
&+\frac{2^\epsilon}{\Gamma(1+\epsilon)}\int_0^1 t^\epsilon(1-t)^\epsilon\Big(1-(1-\frac{\alpha}{\beta})t\Big)^{-2\epsilon}(dt)^\epsilon\\
&-\frac{1}{\Gamma(1+\epsilon)}\int_0^1(1-t)^\epsilon\Big(1-(1-\frac{\alpha}{\beta})t\Big)^{-2\epsilon}(dt)^\epsilon\Big]\\
=&\beta^{-2\epsilon}\Big[{}_2F_1^\epsilon\Big(2,1;3;\frac{1}{2}(1-\frac{\alpha}{\beta})\Big)\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\\
&+2^\epsilon{}_2F_1^\epsilon\Big(2,2;4;1-\frac{\alpha}{\beta}\Big)\Big(\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}-\frac{\Gamma(1+2\epsilon)}{\Gamma(1+3\epsilon)}\Big)\\
&-{}_2F_1^\epsilon\Big(2,1;3;1-\frac{\alpha}{\beta}\Big)\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\Big]=K_3^\epsilon(\alpha,\beta).
\end{aligned}
\tag{3.9}
$$

From (3.4)–(3.9), we get inequality (3.2). This completes the proof. □

Theorem 3. Let $I \subset (0,\infty)$ be an interval, $\phi : I^\circ \to \mathbb{R}^\epsilon$ is an increasing function on I° such that $\phi \in D_\epsilon(I^\circ)$ and $\phi^{(\epsilon)} \in C_\epsilon[\alpha,\beta]$ for $\alpha,\beta \in I^\circ$ with $\alpha < \beta$. If $|\phi^{(\epsilon)}|^q$ is generalized harmonically convex on $[\alpha,\beta]$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for all $x \in [\alpha,\beta]$, the following local fractional integrals inequality holds.

$$\begin{aligned}&\left|\phi\Big(\frac{2\alpha\beta}{\alpha+\beta}\Big) - \Gamma(1+\epsilon)\frac{\alpha^\epsilon\beta^\epsilon}{(\beta-\alpha)^\epsilon}{}_\alpha I_\beta^{(\epsilon)}\frac{\phi(x)}{x^{2\epsilon}}\right| \\ \le&\frac{\phi(\beta)-\phi(\alpha)}{2^\epsilon} + \frac{\alpha^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon\beta^\epsilon}\left[\frac{\Gamma(1+p\epsilon)}{\Gamma(1+(p+1)\epsilon)}\right]^{\frac{1}{p}}\left(\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\right)^{\frac{1}{q}} \\ &\times\left[{}_2F_1^\epsilon\Big(2q,2;3;1-\frac{\alpha}{\beta}\Big)\left|\phi^{(\epsilon)}(\alpha)\right|^q + {}_2F_1^\epsilon\Big(2q,1;3;1-\frac{\alpha}{\beta}\Big)\left|\phi^{(\epsilon)}(\beta)\right|^q\right]^{\frac{1}{q}}. \end{aligned}\tag{3.10}$$

Proof. From inequality (3.4), we have

$$\left|\phi\Big(\frac{2\alpha\beta}{\alpha+\beta}\Big) - \Gamma(1+\epsilon)\frac{\alpha^\epsilon\beta^\epsilon}{(\beta-\alpha)^\epsilon}{}_\alpha I_\beta^{(\epsilon)}\frac{\phi(x)}{x^{2\epsilon}}\right| \le \frac{\phi(\beta)-\phi(\alpha)}{2^\epsilon} + |I_3|. \tag{3.11}$$

From Lemma 6, using the property of the modulus and the generalized Hölder's inequality, we have

$$\begin{aligned}\left|I_3\right| \le&\frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 |1-2t|^\epsilon\left|\frac{1}{A_t^{2\epsilon}}\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|(dt)^\epsilon \\ \le&\frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\left[\frac{1}{\Gamma(1+\epsilon)}\int_0^1 |1-2t|^{\epsilon p}(dt)^\epsilon\right]^{\frac{1}{p}}\left[\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{1}{A_t^{2\epsilon q}}\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|^q(dt)^\epsilon\right]^{\frac{1}{q}}. \end{aligned}\tag{3.12}$$

Since $|\phi^{(\epsilon)}|^q$ is generalized harmonically convex on $[\alpha,\beta]$, we can get

$$\begin{aligned}&\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{1}{A_t^{2\epsilon q}}\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|^q(dt)^\epsilon \\ \le&\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{1}{A_t^{2q\epsilon}}\Big(t^\epsilon\left|\phi^{(\epsilon)}(\beta)\right|^q + (1-t)^\epsilon\left|\phi^{(\epsilon)}(\alpha)\right|^q\Big)(dt)^\epsilon \\ =&\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{t^\epsilon}{A_t^{2q\epsilon}}(dt)^\epsilon\right)\left|\phi^{(\epsilon)}(\beta)\right|^q + \left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{(1-t)^\epsilon}{A_t^{2q\epsilon}}(dt)^\epsilon\right)\left|\phi^{(\epsilon)}(\alpha)\right|^q. \end{aligned}\tag{3.13}$$

By calculating, we have

$$\begin{aligned}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{t^\epsilon}{A_t^{2q\epsilon}}(dt)^\epsilon &= \beta^{-2q\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 t^\epsilon\Big(1-(1-\frac{\alpha}{\beta})t\Big)^{-2q\epsilon}(dt)^\epsilon \\ &= \beta^{-2q\epsilon}{}_2F_1^\epsilon\Big(2q,2;3;1-\frac{\alpha}{\beta}\Big)B_\epsilon(2,1), \\ &= \frac{\Gamma(1+\epsilon)}{\beta^{2q\epsilon}\Gamma(1+2\epsilon)}{}_2F_1^\epsilon\Big(2q,2;3;1-\frac{\alpha}{\beta}\Big), \end{aligned}\tag{3.14}$$

$$\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{(1-t)^\epsilon}{A_t^{2q\epsilon}}(dt)^\epsilon = \beta^{-2q\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 (1-t)^\epsilon\Big(1-(1-\frac{\alpha}{\beta})t\Big)^{-2q\epsilon}(dt)^\epsilon$$
$$= \beta^{-2q\epsilon}{}_2F_1^\epsilon\Big(2q,1;3;1-\frac{\alpha}{\beta}\Big)B_\epsilon(1,2),$$
$$= \frac{\Gamma(1+\epsilon)}{\beta^{2q\epsilon}\Gamma(1+2\epsilon)}{}_2F_1^\epsilon\Big(2q,1;3;1-\frac{\alpha}{\beta}\Big), \tag{3.15}$$

and

$$\frac{1}{\Gamma(1+\epsilon)}\int_0^1 |1-2t|^{\epsilon p}(dt)^\epsilon = \frac{\Gamma(1+p\epsilon)}{\Gamma(1+(p+1)\epsilon)}. \tag{3.16}$$

Thus, combining (3.11)–(3.16), we obtain the required inequality. The proof is completed. □

Theorem 4. Let $I \subset (0,\infty)$ be an interval, $\phi : I^\circ \to \mathbb{R}^\epsilon$ is an increasing function on I° such that $\phi \in D_\epsilon(I^\circ)$ and $\phi^{(\epsilon)} \in C_\epsilon[\alpha,\beta]$ for $\alpha,\beta \in I^\circ$ with $\alpha < \beta$. If $|\phi^{(\epsilon)}|^q$ is generalized harmonically convex on $[\alpha,\beta]$, $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then for all $x \in [\alpha,\beta]$, the following local fractional integrals inequality holds.

$$\left|\phi\Big(\frac{2\alpha\beta}{\alpha+\beta}\Big) - \Gamma(1+\epsilon)\frac{\alpha^\epsilon\beta^\epsilon}{(\beta-\alpha)^\epsilon}{}_\alpha I_\beta^{(\epsilon)}\frac{\phi(x)}{x^{2\epsilon}}\right|$$
$$\le \frac{\phi(\beta)-\phi(\alpha)}{2^\epsilon} + \frac{\alpha^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon\beta^\epsilon}\left[\frac{{}_2F_1^\epsilon\Big(2p,1;2;1-\frac{\alpha}{\beta}\Big)}{\Gamma(1+\epsilon)}\right]^{\frac{1}{p}}\left(\frac{\Gamma(1+q\epsilon)}{2^\epsilon\Gamma(1+(q+1)\epsilon)}\right)^{\frac{1}{q}}$$
$$\times\left[\left|\phi^{(\epsilon)}(\alpha)\right|^q + \left|\phi^{(\epsilon)}(\beta)\right|^q\right]^{\frac{1}{q}}. \tag{3.17}$$

Proof. From Lemma 6, using the generalized Hölder's inequality and the generalized harmonically convexity of $|\phi^{(\epsilon)}|^q$, we have

$$\left|I_3\right| \le \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 |1-2t|^\epsilon\frac{1}{A_t^{2\epsilon}}\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|(dt)^\epsilon$$
$$\le \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\left[\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{1}{A_t^{2\epsilon p}}(dt)^\epsilon\right]^{\frac{1}{p}}\left[\frac{1}{\Gamma(1+\epsilon)}\int_0^1 |1-2t|^{\epsilon q}\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|^q(dt)^\epsilon\right]^{\frac{1}{q}}$$
$$\le \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\left[\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{1}{A_t^{2\epsilon p}}(dt)^\epsilon\right]^{\frac{1}{p}}$$
$$\times\left[\frac{1}{\Gamma(1+\epsilon)}\int_0^1 |1-2t|^{\epsilon q}\Big(t^\epsilon\left|\phi^{(\epsilon)}(\beta)\right|^q + (1-t)^\epsilon\left|\phi^{(\epsilon)}(\alpha)\right|^q\Big)(dt)^\epsilon\right]^{\frac{1}{q}}. \tag{3.18}$$

By calculating, we have

$$\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{1}{A_t^{2p\epsilon}}(dt)^\epsilon = \beta^{-2p\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \Big(1-(1-\frac{\alpha}{\beta})t\Big)^{-2p\epsilon}(dt)^\epsilon$$
$$= \beta^{-2p\epsilon}{}_2F_1^\epsilon\Big(2p,1;2;1-\frac{\alpha}{\beta}\Big)B_\epsilon(1,1),$$
$$= \frac{{}_2F_1^\epsilon\Big(2p,1;2;1-\frac{\alpha}{\beta}\Big)}{\beta^{2p\epsilon}\Gamma(1+\epsilon)}, \tag{3.19}$$

$$\frac{1}{\Gamma(1+\epsilon)}\int_0^1 |1-2t|^{\epsilon q}t^\epsilon(dt)^\epsilon = \frac{1}{\Gamma(1+\epsilon)}\int_0^{1/2}(1-2t)^{\epsilon q}t^\epsilon(dt)^\epsilon$$
$$+\frac{1}{\Gamma(1+\epsilon)}\int_{1/2}^1(2t-1)^{\epsilon q}t^\epsilon(dt)^\epsilon$$
$$= \frac{\Gamma(1+q\epsilon)}{2^\epsilon\Gamma(1+(q+1)\epsilon)}, \tag{3.20}$$

and

$$\frac{1}{\Gamma(1+\epsilon)}\int_0^1 |1-2t|^{\epsilon q}(1-t)^\epsilon(dt)^\epsilon$$
$$=\frac{1}{\Gamma(1+\epsilon)}\int_0^{1/2}(1-2t)^{\epsilon q}(1-t)^\epsilon(dt)^\epsilon$$
$$+\frac{1}{\Gamma(1+\epsilon)}\int_{1/2}^1(2t-1)^{\epsilon q}(1-t)^\epsilon(dt)^\epsilon$$
$$=\frac{\Gamma(1+q\epsilon)}{2^\epsilon\Gamma(1+(q+1)\epsilon)}. \tag{3.21}$$

From (3.11) in Theorem 3, combining (3.18)–(3.21), we obtain the required inequality. The proof is completed. □

Theorem 5. Let $I\subset(0,\infty)$ be an interval, $\phi: I^\circ\to\mathbb{R}^\epsilon$ is an increasing function on I° such that $\phi\in D_\epsilon(I^\circ)$ and $\phi^{(\epsilon)}\in C_\epsilon[\alpha,\beta]$ for $\alpha,\beta\in I^\circ$ with $\alpha<\beta$. If $|\phi^{(\epsilon)}|^q$ is generalized harmonically convex on $[\alpha,\beta]$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then for all $x\in[\alpha,\beta]$, the following local fractional integrals inequality holds.

$$\left|\phi\Big(\frac{2\alpha\beta}{\alpha+\beta}\Big)-\Gamma(1+\epsilon)\frac{\alpha^\epsilon\beta^\epsilon}{(\beta-\alpha)^\epsilon}{}_\alpha I_\beta^{(\epsilon)}\frac{\phi(x)}{x^{2\epsilon}}\right|$$
$$\le\frac{\phi(\beta)-\phi(\alpha)}{2^\epsilon}+\frac{\alpha^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon\beta^\epsilon}\left[\frac{\Gamma(1+p\epsilon)}{\Gamma(1+(p+1)\epsilon)}\right]^{\frac{1}{p}}\left[\left({}_2F_1^\epsilon\Big(2p,1;p+2;1-\frac{\alpha}{\beta}\Big)\right)^{\frac{1}{p}}\right.$$
$$\left.+\left({}_2F_1^\epsilon\Big(2p,p+1;p+2;1-\frac{\alpha}{\beta}\Big)\right)^{\frac{1}{p}}\right]\left(\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\right)^{\frac{1}{q}}\left[\left|\phi^{(\epsilon)}(\alpha)\right|^q+\left|\phi^{(\epsilon)}(\beta)\right|^q\right]^{\frac{1}{q}}. \tag{3.22}$$

Proof. Note that $(\alpha-\beta)^\epsilon=\alpha^\epsilon-\beta^\epsilon$. From Lemma 6, using the generalized Hölder's inequality and the generalized harmonically convexity of $|\phi^{(\epsilon)}|^q$, we have

$$
\begin{aligned}
\left|I_3\right| \leq & \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{|1-2t|^\epsilon}{A_t^{2\epsilon}}\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|(dt)^\epsilon \\
= & \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{|(1-t)^\epsilon-t^\epsilon|}{A_t^{2\epsilon}}\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|(dt)^\epsilon \\
\leq & \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\left[\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{(1-t)^\epsilon}{A_t^{2\epsilon}}\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|(dt)^\epsilon\right. \\
& \left.+\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{t^\epsilon}{A_t^{2\epsilon}}\left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|(dt)^\epsilon\right] \\
\leq & \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\left[\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{(1-t)^{\epsilon p}}{A_t^{2\epsilon p}}(dt)^\epsilon\right)^{1/p}\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|^q(dt)^\epsilon\right)^{1/q}\right. \\
& \left.+\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{t^{\epsilon p}}{A_t^{2\epsilon p}}(dt)^\epsilon\right)^{1/p}\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|^q(dt)^\epsilon\right)^{1/q}\right] \\
= & \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\left[\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{(1-t)^{\epsilon p}}{A_t^{2\epsilon p}}(dt)^\epsilon\right)^{1/p}\right. \\
& \left.+\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{t^{\epsilon p}}{A_t^{2\epsilon p}}(dt)^\epsilon\right)^{1/p}\right] \\
& \times\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left|\phi^{(\epsilon)}\left(\frac{\alpha\beta}{A_t}\right)\right|^q(dt)^\epsilon\right)^{1/q} \\
\leq & \frac{\alpha^\epsilon\beta^\epsilon(\beta-\alpha)^\epsilon}{2^\epsilon}\left[\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{(1-t)^{\epsilon p}}{A_t^{2\epsilon p}}(dt)^\epsilon\right)^{1/p}\right. \\
& \left.+\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{t^{\epsilon p}}{A_t^{2\epsilon p}}(dt)^\epsilon\right)^{1/p}\right] \\
& \times\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \left[t^\epsilon\left|\phi^{(\epsilon)}(\beta)\right|^q+(1-t)^\epsilon\left|\phi^{(\epsilon)}(\alpha)\right|^q\right](dt)^\epsilon\right)^{1/q}.
\end{aligned} \tag{3.23}
$$

By calculating, we have

$$
\begin{aligned}
\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{(1-t)^{\epsilon p}}{A_t^{2\epsilon p}}(dt)^\epsilon &= \beta^{-2p\epsilon}\frac{1}{\Gamma(1+\epsilon)}\int_0^1 (1-t)^{\epsilon p}\left[1-(1-\frac{\alpha}{\beta})t\right]^{-2p\epsilon}(dt)^\epsilon \\
&= \beta^{-2p\epsilon}{}_2F_1^\epsilon\left(2p,1;p+2;1-\frac{\alpha}{\beta}\right)B_\epsilon(1,p+1) \\
&= \frac{\beta^{-2p\epsilon}\Gamma(1+p\epsilon)}{\Gamma(1+(p+1)\epsilon)}{}_2F_1^\epsilon\left(2p,1;p+2;1-\frac{\alpha}{\beta}\right).
\end{aligned} \tag{3.24}
$$

Similarly,

$$\frac{1}{\Gamma(1+\epsilon)}\int_0^1 \frac{t^{\epsilon p}}{A_t^{2\epsilon p}}(dt)^\epsilon = \beta^{-2p\epsilon}{}_2F_1^\epsilon\Big(2p, p+1; p+2; 1-\frac{\alpha}{\beta}\Big)B_\epsilon(p+1,1)$$
$$= \frac{\beta^{-2p\epsilon}\Gamma(1+p\epsilon)}{\Gamma(1+(p+1)\epsilon)}{}_2F_1^\epsilon\Big(2p, p+1; p+2; 1-\frac{\alpha}{\beta}\Big). \tag{3.25}$$

And

$$\frac{1}{\Gamma(1+\epsilon)}\int_0^1 t^\epsilon (dt)^\epsilon = \frac{1}{\Gamma(1+\epsilon)}\int_0^1 (1-t)^\epsilon (dt)^\epsilon = \frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}. \tag{3.26}$$

From (3.11) in Theorem 3, combining (3.23)–(3.26), we obtain the required inequality. The proof is completed. □

4. Applications to special means

We consider the following ϵ-type generalized special means of the real line numbers $\alpha^\epsilon, \beta^\epsilon$ with $\alpha < \beta$ on Yang's fractal sets.

(1) The generalized arithmetic mean

$$A_\epsilon(\alpha, \beta) = \frac{\alpha^\epsilon + \beta^\epsilon}{2^\epsilon};$$

(2) The generalized p-logarithmic mean

$$L_{p\epsilon}(\alpha, \beta) = \left[\frac{\Gamma(1+p\epsilon)}{\Gamma(1+(p+1)\epsilon)}\frac{\beta^{(p+1)\epsilon} - \alpha^{(p+1)\epsilon}}{(\beta-\alpha)^\epsilon}\right]^{1/p}, p \in \mathbb{R}\backslash\{-1, 0\};$$

(3) The generalized geometric mean

$$G_\epsilon(\alpha, \beta) = (\alpha^\epsilon \beta^\epsilon)^{\frac{1}{2}};$$

(4) The generalized harmonic mean

$$H_\epsilon(\alpha, \beta) = \frac{(2\alpha\beta)^\epsilon}{\alpha^\epsilon + \beta^\epsilon}.$$

Consider the function $\phi : (0, \infty) \to \mathbb{R}^\epsilon$, $\phi(x) = \frac{\Gamma(1+k\epsilon)}{\Gamma(1+(k+1)\epsilon)}x^{(k+1)\epsilon}$, $x > 0, k \geq 1$ and $q \geq 1$. Because the function $\varphi(x) = |\phi^{(\epsilon)}(x)|^q = x^{kq\epsilon}$ is generalized convex and nondecreasing on $(0, \infty)$, by Proposition 3.3 in [22], the function $\varphi(x)$ is generalized harmonically convex on $(0, \infty)$.

Let $\phi(x) = \frac{\Gamma(1+k\epsilon)}{\Gamma(1+(k+1)\epsilon)}x^{(k+1)\epsilon}$, $x > 0, k > 1$ and $q > 1$. Then

$$\phi\Big(\frac{2\alpha\beta}{\alpha+\beta}\Big) = \frac{\Gamma(1+k\epsilon)}{\Gamma(1+(k+1)\epsilon)}H_\epsilon^{k+1}(\alpha, \beta),$$

$$\frac{\alpha^\epsilon\beta^\epsilon}{(\beta-\alpha)^\epsilon}{}_\alpha I_\beta^{(\epsilon)}\frac{\phi(x)}{x^{2\epsilon}} = \frac{\Gamma(1+k\epsilon)}{\Gamma(1+(k+1)\epsilon)}L_{(k-1)\epsilon}^{k-1}(\alpha, \beta)G_\epsilon^2(\alpha, \beta),$$

$$\frac{\phi(\beta)-\phi(\alpha)}{2^{\epsilon}}=\frac{(\beta-\alpha)^{\epsilon}}{2^{\epsilon}}L^{k}_{k\epsilon}(\alpha,\beta).$$

Proposition 1. From Theorem 2, we obtain the following inequality

$$\begin{aligned}&\left|H^{k+1}_{\epsilon}(\alpha,\beta)-\Gamma(1+\epsilon)L^{k-1}_{(k-1)\epsilon}(\alpha,\beta)G^{2}_{\epsilon}(\alpha,\beta)\right|\\ \leq&\frac{(\beta-\alpha)^{\epsilon}\Gamma(1+(k+1)\epsilon)}{2^{\epsilon}\Gamma(1+k\epsilon)}\left[L^{k}_{k\epsilon}(\alpha,\beta)+\alpha^{\epsilon}\beta^{\epsilon}\Big(K^{\epsilon}_{1}(\alpha,\beta)\Big)^{1-\frac{1}{q}}\Big(K^{\epsilon}_{2}(\alpha,\beta)\alpha^{kq\epsilon}+K^{\epsilon}_{3}(\alpha,\beta)\beta^{kq\epsilon}\Big)^{\frac{1}{q}}\right],\end{aligned}$$

where $K^{\epsilon}_{1}(\alpha,\beta), K^{\epsilon}_{2}(\alpha,\beta)$ and $K^{\epsilon}_{3}(\alpha,\beta)$ as in Theorem 2.

Proposition 2. From Theorem 3, we obtain the following inequality

$$\begin{aligned}&\left|H^{k+1}_{\epsilon}(\alpha,\beta)-\Gamma(1+\epsilon)L^{k-1}_{(k-1)\epsilon}(\alpha,\beta)G^{2}_{\epsilon}(\alpha,\beta)\right|\\ \leq&\frac{(\beta-\alpha)^{\epsilon}\Gamma(1+(k+1)\epsilon)}{2^{\epsilon}\Gamma(1+k\epsilon)}\left[L^{k}_{k\epsilon}(\alpha,\beta)+\frac{\alpha^{\epsilon}}{\beta^{\epsilon}}\left(\frac{\Gamma(1+p\epsilon)}{\Gamma(1+(p+1)\epsilon)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\right)^{\frac{1}{q}}\right.\\ &\left.\times\left({}_{2}F^{\epsilon}_{1}\Big(2q,2;3;1-\frac{\alpha}{\beta}\Big)\alpha^{kq\epsilon}+{}_{2}F^{\epsilon}_{1}\Big(2q,1;3;1-\frac{\alpha}{\beta}\Big)\beta^{kq\epsilon}\right)^{\frac{1}{q}}\right],\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1, q>1$.

Proposition 3. From Theorem 4, we obtain the following inequality

$$\begin{aligned}&\left|H^{k+1}_{\epsilon}(\alpha,\beta)-\Gamma(1+\epsilon)L^{k-1}_{(k-1)\epsilon}(\alpha,\beta)G^{2}_{\epsilon}(\alpha,\beta)\right|\\ \leq&\frac{(\beta-\alpha)^{\epsilon}\Gamma(1+(k+1)\epsilon)}{2^{\epsilon}\Gamma(1+k\epsilon)}\left[L^{k}_{k\epsilon}(\alpha,\beta)+\frac{\alpha^{\epsilon}}{\beta^{\epsilon}}\left(\frac{{}_{2}F^{\epsilon}_{1}\big(2p,1;2;1-\frac{\alpha}{\beta}\big)}{\Gamma(1+\epsilon)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+q\epsilon)}{2^{\epsilon}\Gamma(1+(q+1)\epsilon)}\right)^{\frac{1}{q}}\right.\\ &\left.\times\left(\alpha^{kq\epsilon}+\beta^{kq\epsilon}\right)^{\frac{1}{q}}\right],\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1, q>1$.

Proposition 4. From Theorem 5, we obtain the following inequality

$$\begin{aligned}&\left|H^{k+1}_{\alpha}(\alpha,\beta)-\Gamma(1+\epsilon)L^{k-1}_{(k-1)\epsilon}(\alpha,\beta)G^{2}_{\epsilon}(\alpha,\beta)\right|\\ \leq&\frac{(\beta-\alpha)^{\epsilon}\Gamma(1+(k+1)\epsilon)}{2^{\epsilon}\Gamma(1+k\epsilon)}\left\{L^{k}_{k\epsilon}(\alpha,\beta)+\frac{\alpha^{\epsilon}}{\beta^{\epsilon}}\left(\frac{\Gamma(1+p\epsilon)}{\Gamma(1+(p+1)\epsilon)}\right)^{\frac{1}{p}}\left[\left({}_{2}F^{\epsilon}_{1}\Big(2p,1;p+2;1-\frac{\alpha}{\beta}\Big)\right)^{\frac{1}{p}}\right.\right.\\ &\left.\left.+\left({}_{2}F^{\epsilon}_{1}\Big(2p,p+1;p+2;1-\frac{\alpha}{\beta}\Big)\right)^{\frac{1}{p}}\right]\left(\frac{\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}\right)^{\frac{1}{q}}\left(\alpha^{kq\epsilon}+\beta^{kq\epsilon}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1, q>1$.

5. Conclusions

In this paper, the research on Hermite-Hadamard type inequalities is extended to Yang's fractal space. By using the definitions of generalized harmonically convex function and the theory of local fractional calculus, we construct some new Hermite-Hadamard type integral inequalities for monotonically increasing functions with generalized harmonically convexity. Some applications related to the special mean are established by using the obtained inequalities, which shows that our results have certain application significance. Our research may inspire more scholars to further explore Hermite-Hadamard type integral inequalities on Yang's fractal sets.

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Conflict of interest

This work does not have any conflicts of interest.

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