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Research article

Explicit characteristic equations for integral operators arising from well-posed boundary value problems of finite beam deflection on elastic foundation

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Abstract: Characteristic equations for the whole class of integral operators arising from arbitrary well-posed two-point boundary value problems of finite beam deflection resting on elastic foundation are obtained in terms of 4×4 matrices in block-diagonal form with explicit 2×2 blocks.

Keywords: beam; block-diagonal; characteristic equation; deflection; eigenfunction; eigenvalue; elastic foundation; integral operator; spectrum
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1. Introduction

We consider characteristic equations, *i.e.*, equations for eigenvalues and eigenfunctions of the class of integral operators on the Hilbert space $L^2[-l, l]$ of the form

$$\mathcal{K}_{\mathbf{M}}[w](x) = \int_{-l}^{l} G_{\mathbf{M}}(x, \xi) w(\xi)\, d\xi, \qquad x \in [-l, l],\ w \in L^2[-l, l], \tag{1.1}$$

where $G_{\mathbf{M}}$ is the Green function [1, 2] for the boundary value problem consisting of the fourth-order linear differential equation

$$EI \cdot u^{(4)}(x) + k \cdot u(x) = w(x), \qquad x \in [-l, l] \tag{1.2}$$

and a well-posed two-point boundary condition

$$\mathbf{M} \cdot \begin{pmatrix} u(-l) & u'(-l) & u''(-l) & u'''(-l) & u(l) & u'(l) & u''(l) & u'''(l) \end{pmatrix}^T = \mathbf{0}. \tag{1.3}$$

Here, $\mathbf{M} \in \mathrm{gl}(4, 8, \mathbb{C})$ is called a *boundary matrix*, where $\mathrm{gl}(4, 8, \mathbb{C})$ is the set of 4×8 matrices with complex entries. For example, the two-point boundary condition $u(-l) = u'(-l) = u(l) = u'(l) = 0$ can be expressed by (1.3) with

$$\mathbf{M} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

The differential equation (1.2) is the classical Euler–Bernoulli beam equation [3] which governs the vertical downward deflection $u(x)$ of a linear-shaped beam with finite length $2l$ resting horizontally on an elastic foundation with spring constant density k. The constants E and I are the Young's modulus and the mass moment of inertia of the beam respectively, and $w(x)$ is the downward load density applied vertically on the beam. The beam deflection problem has been one of the central topics in mechanical engineering with diverse and important applications [3–12].

Throughout this paper, we assume that l, E, I, k in (1.2) are positive constants and put $\alpha = \sqrt[4]{k/(EI)} > 0$. When the boundary value problem consisting of (1.2) and (1.3) is well-posed or, equivalently, when (1.2) and (1.3) has a unique solution, we call the boundary matrix $\mathbf{M}$ *well-posed*. The set of well-posed boundary matrices is denoted by $\mathrm{wp}(4, 8, \mathbb{C})$. It was shown in [2] that, up to a natural equivalence relation, $\mathrm{wp}(4, 8, \mathbb{C})$ is in one-to-one correspondence with the 16-dimensional algebra $\mathrm{gl}(4, \mathbb{C})$ of 4×4 matrices with complex entries.

For $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$, we denote by $\mathrm{Spec}\,\mathcal{K}_{\mathbf{M}}$ the *spectrum* or, the set of eigenvalues, of the integral operator $\mathcal{K}_{\mathbf{M}}$ in (1.1). Since $\mathcal{K}_{\mathbf{M}}[w]$ is the unique solution of the boundary value problem (1.2) and (1.3) for every $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$, analyzing the behavior of the integral operators $\mathcal{K}_{\mathbf{M}}$ is important in understanding the beam deflection problem. In general, spectral analysis for integral operators arising from various differential equations is crucial in many applications such as inverse problem [13] and nonlinear problem [5, 6]. In contrast to this importance, there are few explicit spectral analyses for the integral operators $\mathcal{K}_{\mathbf{M}}$ which arise from a most fundamental and basic differential equation (1.2) in the history of mechanical engineering.

Choi [14] analyzed $\mathrm{Spec}\,\mathcal{K}_{\mathbf{Q}}$ of a special integral operator $\mathcal{K}_{\mathbf{Q}}$ in detail, where

$$\mathbf{Q} = \left(\begin{array}{cccc|cccc} 0 & \alpha^2 & -\sqrt{2}\alpha & 1 & 0 & 0 & 0 & 0 \\ \sqrt{2}\alpha^3 & -\alpha^2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^2 & \sqrt{2}\alpha & 1 \\ 0 & 0 & 0 & 0 & -\sqrt{2}\alpha^3 & -\alpha^2 & 0 & 1 \end{array}\right), \tag{1.4}$$

which is in $\mathrm{wp}(4, 8, \mathbb{C})$ [2]. The Green function $G_{\mathbf{Q}}(x, \xi)$ corresponding to $\mathbf{Q}$ is the restriction in $[-l, l] \times [-l, l]$ of the Green function for the boundary value problem consisting of the *infinite* version $EI \cdot u^{(4)}(x) + k \cdot u(x) = w(x)$, $x \in (-\infty, \infty)$ of (1.2) and the boundary condition $\lim_{x \to \pm\infty} u(x) = 0$.

For two positive sequences a_n, b_n, we denote $a_n \sim b_n$ if there exists $N > 0$ such that $m \le a_n/b_n \le M$ for every $n > N$ for some constants $0 < m \le M < \infty$.

Proposition 1.1 ([14]). *For every $l > 0$, the spectrum* $\mathrm{Spec}\,\mathcal{K}_{\mathbf{Q}}$ *of the operator $\mathcal{K}_{\mathbf{Q}}$ is of the form* $\{\mu_n/k \,|\, n = 1, 2, 3, \cdots\} \cup \{\nu_n/k \,|\, n = 1, 2, 3, \cdots\} \subset (0, 1/k)$, *where* $1 > \mu_1 > \nu_1 > \mu_2 > \nu_2 > \cdots \searrow 0$. *Each of μ_n and ν_n for $n = 1, 2, 3, \ldots$ is determined only by the intrinsic length $L = 2l\alpha$ of the beam. $\mu_n \sim \nu_n \sim n^{-4}$, and*

$$\frac{1}{1 + \left\{h^{-1}\left(2\pi n + \frac{\pi}{2}\right)\right\}^4} < \nu_n < \frac{1}{1 + \left\{h^{-1}\left(2\pi n\right)\right\}^4} < \mu_n < \frac{1}{1 + \left\{h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right\}^4}, \quad n = 1, 2, 3, \ldots,$$

$$\frac{1}{1+\left\{h^{-1}\left(2\pi n-\frac{\pi}{2}\right)\right\}^4}-\mu_n\sim\nu_n-\frac{1}{1+\left\{h^{-1}\left(2\pi n+\frac{\pi}{2}\right)\right\}^4}\sim n^{-5}e^{-2\pi n},$$
$$\frac{1}{1+\frac{1}{L^4}\left(2\pi(n-1)-\frac{\pi}{2}\right)^4}-\mu_n\sim\frac{1}{1+\frac{1}{L^4}\left(2\pi(n-1)+\frac{\pi}{2}\right)^4}-\nu_n\sim n^{-6}.$$

Here, $h : [0,\infty) \to [0,\infty)$ is the strictly increasing real-analytic function defined in Supplementary D, with the properties $h(0) = 0$ and $h^{-1}(a_n) \sim a_n/L$ for any positive sequence a_n such that $a_n \to \infty$. See [14] for numerical computations of μ_n, ν_n with arbitrary precision.

Recently, Choi [2] derived explicit characteristic equations for the integral operator $\mathcal{K}_{\mathbf{M}}$ in (1.1) for *arbitrary* well-posed $\mathbf{M} \in \mathrm{wp}(4,8,\mathbb{C})$, which are stated in more detail in Section 2. Although these characteristic equations are expressed in terms of the explicit 4×4 matrices $\mathcal{G}(\mathbf{M})$, $\mathbf{X}_\lambda$, $\mathbf{Y}_\lambda$, they still involve determinants of *full* 4×4 matrices, which makes it hard to analyze the structure of $\operatorname{Spec}\mathcal{K}_{\mathbf{M}}$ for general well-posed boundary matrix $\mathbf{M}$.

In this paper, we utilize some of the symmetries in the 4×4 matrices $\mathbf{X}_\lambda$, $\mathbf{Y}_\lambda$ to block-diagonalize them with explicit 2×2 blocks $\mathbf{X}^\pm_\lambda$, $\mathbf{Y}^\pm_\lambda$, which enables us to obtain new and simpler forms of characteristic equations for the integral operator $\mathcal{K}_{\mathbf{M}}$ for *arbitrary* well-posed boundary matrix $\mathbf{M} \in \mathrm{wp}(4,8,\mathbb{C})$. In particular, the entries of the 2×2 blocks $\mathbf{X}^\pm_\lambda$ and $\mathbf{Y}^\pm_\lambda$ are represented explicitly with the concrete holomorphic functions $\delta^\pm(z,\kappa)$ and $p^\pm(z)$ introduced in Section 3.

Our results significantly reduce the complexity of dealing with determinants of 4×4 matrices and facilitate to represent $\operatorname{Spec}\mathcal{K}_{\mathbf{M}}$ for arbitrary $\mathbf{M} \in \mathrm{wp}(4,8,\mathbb{C})$ essentially as the *zero set* of *one* explicit holomorphic function composed with the concrete functions $\delta^\pm(z,\kappa)$. For example, Corollary 1 in Section 3 states that $0, 1/k \neq \lambda \in \operatorname{Spec}\mathcal{K}_{\mathbf{Q}}$ if and only if λ is a zero of the holomorphic function $\delta^+(\alpha l,\chi(\lambda))\cdot\delta^-(\alpha l,\chi(\lambda))$, where χ is a 4th root transformation introduced in Section 2. In particular, the holomorphic functions $\delta^\pm(z,\kappa)$ unify the real-analytic functions which were analyzed in detail in [14, 15] to obtain concrete results on $\operatorname{Spec}\mathcal{K}_{\mathbf{Q}}$ such as Proposition 1.1. The fact that $\delta^\pm(z,\kappa)$ encapsulate condensed information on $\operatorname{Spec}\mathcal{K}_{\mathbf{Q}}$, and hence on $\operatorname{Spec}\mathcal{K}_{\mathbf{M}}$ in general, is demonstrated in Supplementary D by showing that the seemingly complex-looking conditions $\varphi^\pm(\kappa) = p(\kappa)$, which were derived in [14] with the help of computer algebra systems, can be directly and elegantly recovered from $\delta^\pm(z,\kappa)$.

Our results open up practical ways to direct and concrete spectral analysis for the whole 16-dimensional class of the integral operators $\mathcal{K}_{\mathbf{M}}$ arising from arbitrary well-posed boundary value problem of finite beam deflection on elastic foundation.

After introducing basic notations, definitions, and previous results relevant to our analysis in Section 2, we state our main results Theorems 1, 2 and 3 in Section 3, which are proved in Sections 4, 5 and 6 respectively. Some remarks and future directions are given in Section 7. In Supplementary D, the conditions $\varphi^\pm(\kappa) = p(\kappa)$ on $\operatorname{Spec}\mathcal{K}_{\mathbf{Q}}$ in [14] are derived from our holomorphic functions $\delta^\pm(z,\kappa)$.

2. Preliminaries

2.1. Basic notations and definitions

We denote $\mathbb{i} = \sqrt{-1}$. Denote by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, the set of integers, the set of real numbers, and the set of complex numbers respectively. The set of $m\times n$ matrices with entries in $\mathbb{C}$ is denoted by $\mathrm{gl}(m,n,\mathbb{C})$.

When $m = n$, we also denote $\mathrm{gl}(m, n, \mathbb{C}) = \mathrm{gl}(n, \mathbb{C})$. We write $\mathbf{A} = \left(a_{i,j}\right)_{1\le i\le m,\, 1\le j\le n}$ when the (i, j)th entry of $\mathbf{A} \in \mathrm{gl}(m, n, \mathbb{C})$ is $a_{i,j}$. When $m = n$, we also write $\mathbf{A} = \left(a_{i,j}\right)_{1\le i,j\le n}$. For $\mathbf{A} \in \mathrm{gl}(m, n, \mathbb{C})$, we denote the (i, j)th entry of $\mathbf{A}$ by $\mathbf{A}_{i,j}$. The complex conjugate, the transpose, and the conjugate transpose of $\mathbf{A} \in \mathrm{gl}(m, n, \mathbb{C})$ are denoted by $\overline{\mathbf{A}}$, $\mathbf{A}^T$, and $\mathbf{A}^*$ respectively. For $\mathbf{A} \in \mathrm{gl}(n, \mathbb{C})$, $\operatorname{adj}\mathbf{A}$ is the classical adjoint of $\mathbf{A}$, so that, if $\mathbf{A}$ is invertible then $\mathbf{A}^{-1} = \operatorname{adj}\mathbf{A}/\det\mathbf{A}$.

Regardless of size, the identity matrix and the zero matrix are denoted by $\mathbf{I}$ and $\mathbf{O}$ respectively. The zero column vector with any size is denoted by $\mathbf{0}$. The diagonal matrix with diagonal entries $c_1, c_2, \cdots, c_n$ is denoted by $\operatorname{diag}(c_1, c_2, \cdots, c_n)$.

Definition 2.1. Denote $\omega = e^{\mathbb{i}\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \mathbb{i}\frac{1}{\sqrt{2}}$ and $\omega_n = \mathbb{i}^{n-1}\omega$ for $n \in \mathbb{Z}$. Denote $\mathbf{\Omega} = \operatorname{diag}(\omega_1, \omega_2, \omega_3, \omega_4)$ and $\mathbf{W}_0 = \left(\omega_j^{i-1}\right)_{1\le i,j\le 4}$.

$\omega_1 = \omega$, ω_2, ω_3, ω_4 are the primitive 4th roots of -1 and satisfy

$$\begin{aligned} &\overline{\omega} = \omega_4 = -\omega_2 = -\mathbb{i}\omega, \qquad \omega_3 = -\omega, \qquad \overline{\omega_n} = \omega_n^{-1},\ n \in \mathbb{Z}, \\ &\omega + \overline{\omega} = \sqrt{2}, \qquad \omega - \overline{\omega} = \mathbb{i}\sqrt{2}, \qquad \omega^2 = \mathbb{i}, \qquad \omega\overline{\omega} = 1. \end{aligned} \tag{2.1}$$

Definition 2.2. Denote $\epsilon_1 = \epsilon_4 = 1$, $\epsilon_2 = \epsilon_3 = -1$, and $\epsilon_{n+4} = \epsilon_n$ for $n \in \mathbb{Z}$. Denote $\mathcal{E} = \operatorname{diag}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = \operatorname{diag}(1, -1, -1, 1)$.

By Definitions 2.1, 2.2 and (2.1), we have

$$\begin{aligned} e^{-\mathcal{E}\mathbf{\Omega}z} &= \operatorname{diag}(e^{-\omega_1 z}, e^{\omega_2 z}, e^{\omega_3 z}, e^{-\omega_4 z}) = \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega}z}, e^{-\omega z}, e^{-\overline{\omega}z}\right) \\ &= \begin{pmatrix} \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega}z}\right) & \mathbf{O} \\ \mathbf{O} & \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega}z}\right) \end{pmatrix}, \qquad z \in \mathbb{C}. \end{aligned} \tag{2.2}$$

Definition 2.3. Denote

$$\mathbf{V} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \qquad \widehat{\mathbf{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $\mathbf{V}$ and $\widehat{\mathbf{V}}$ are orthogonal and

$$\mathbf{V}^{-1} = \mathbf{V}^T, \quad \widehat{\mathbf{V}}^{-1} = \widehat{\mathbf{V}}^T = \widehat{\mathbf{V}}, \quad \det\mathbf{V} = 1, \quad \det\widehat{\mathbf{V}} = -1. \tag{2.3}$$

Lemma 2.1. $\mathbf{V}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}\mathbf{V}^T = \begin{pmatrix} \mathbf{A}+\mathbf{B} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}-\mathbf{B} \end{pmatrix}$ *for* $\mathbf{A}, \mathbf{B} \in \mathrm{gl}(2, \mathbb{C})$.

Proof. By Definition 2.3,

$$\begin{aligned} \mathbf{V}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}\mathbf{V}^T &= \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{A}+\mathbf{B} & \mathbf{A}+\mathbf{B} \\ -\mathbf{A}+\mathbf{B} & \mathbf{A}-\mathbf{B} \end{pmatrix}\begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A}+\mathbf{B} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}-\mathbf{B} \end{pmatrix}. \end{aligned}$$

□

By (2.2) and Lemma 2.1,

$$\mathbf{V}e^{-\mathcal{E}\mathbf{\Omega}z}\mathbf{V}^T = \begin{pmatrix} \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega}z}\right) + \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega}z}\right) - \mathbf{O} \end{pmatrix}$$
$$= \begin{pmatrix} \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega}z}\right) & \mathbf{O} \\ \mathbf{O} & \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega}z}\right) \end{pmatrix} = e^{-\mathcal{E}\mathbf{\Omega}z}, \qquad z \in \mathbb{C}. \tag{2.4}$$

By (2.1),

$$\det \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega}z}\right) = e^{-\omega z} \cdot e^{-\overline{\omega}z} = e^{-(\omega+\overline{\omega})z} = e^{-\sqrt{2}z}, \quad z \in \mathbb{C}. \tag{2.5}$$

2.2. Previous results

Definition 2.4. For $\lambda \in \mathbb{C} \setminus \{0, 1/k\}$, define $\chi(\lambda)$ to be the unique complex number satisfying $\chi(\lambda)^4 = 1 - 1/(\lambda k)$ and $0 \le \operatorname{Arg}\chi(\lambda) < \pi/2$.

Note that χ is a one-to-one correspondence from $\mathbb{C} \setminus \{0, 1/k\}$ to the set $\{\kappa \in \mathbb{C} \,|\, 0 \le \operatorname{Arg}\kappa < \pi/2\} \setminus \{0, 1\}$.

Definition 2.5. Let $0 \ne \lambda \in \mathbb{C}$ and $x \in \mathbb{R}$. For $\lambda \ne 1/k$, let $\kappa = \chi(\lambda)$. Denote

$$\mathbf{W}(x) = \begin{pmatrix} \mathbf{y}(x) & \mathbf{y}'(x) & \mathbf{y}''(x) & \mathbf{y}'''(x) \end{pmatrix}^T, \qquad \mathbf{W}_\lambda(x) = \left(y_{\lambda,j}^{(i-1)}(x)\right)_{1\le i,j\le 4},$$

where $\mathbf{y}(x) = \begin{pmatrix} e^{\omega_1\alpha x} & e^{\omega_2\alpha x} & e^{\omega_3\alpha x} & e^{\omega_4\alpha x} \end{pmatrix}^T$ and $y_{\lambda,j}(x) = \begin{cases} \frac{1}{(j-1)!} \cdot x^{j-1}, & \text{if } \lambda = 1/k, \\ e^{\omega_j\kappa\alpha x}, & \text{if } \lambda \ne 1/k, \end{cases}$ $j = 1, 2, 3, 4$. Denote $\mathbf{X}_\lambda(x) = \operatorname{diag}(0, 1, 1, 0) \cdot \mathbf{W}(-x)^{-1}\mathbf{W}_\lambda(-x) + \operatorname{diag}(1, 0, 0, 1) \cdot \mathbf{W}(x)^{-1}\mathbf{W}_\lambda(x)$. When $\det \mathbf{X}_\lambda(x) \ne 0$, denote $\mathbf{Y}_\lambda(x) = \mathbf{X}_\lambda(-x)\mathbf{X}_\lambda(x)^{-1} - \mathbf{I}$.

Definition 2.6. Define $\mathcal{G} : \mathrm{wp}(4, 8, \mathbb{C}) \to \mathrm{gl}(4, \mathbb{C})$ by

$$\mathcal{G}(\mathbf{M}) = \{\mathbf{M}^-\mathbf{W}(-l) + \mathbf{M}^+\mathbf{W}(l)\}^{-1}\,\mathbf{M}^+\mathbf{W}(l)\mathcal{E} - \operatorname{diag}(1, 0, 0, 1),$$

where $\mathbf{M}^-, \mathbf{M}^+ \in \mathrm{gl}(4, \mathbb{C})$ are the 4×4 minors of $\mathbf{M}$ such that $\mathbf{M} = \left(\mathbf{M}^- \,\middle|\, \mathbf{M}^+ \right)$. Define $\psi : \mathrm{gl}(4, \mathbb{C}) \to \mathrm{gl}(4, 8, \mathbb{C})$ by

$$\psi(\mathbf{G}) = \left(\{\operatorname{diag}(0, 1, 1, 0) - \mathbf{G}\mathcal{E}\}\,\mathbf{W}(-l)^{-1} \,\middle|\, \{\operatorname{diag}(1, 0, 0, 1) + \mathbf{G}\mathcal{E}\}\,\mathbf{W}(l)^{-1} \right).$$

The map $\mathcal{G}$ in Definition 2.6 is well defined since, for $\mathbf{M} = \left(\mathbf{M}^- \,\middle|\, \mathbf{M}^+ \right) \in \mathrm{gl}(4, 8, \mathbb{C})$, $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$ if and only of $\det\{\mathbf{M}^-\mathbf{W}(-l) + \mathbf{M}^+\mathbf{W}(l)\} \ne 0$ [2, Lemma 3.1]. $\mathcal{G}(\mathbf{M})$ is denoted by $\mathbf{G_M}$ in [2]. Define the equivalence relation $\approx$ on $\mathrm{wp}(4, 8, \mathbb{C})$ by $\mathbf{M} \approx \mathbf{N}$ if and only if $\mathbf{M} = \mathbf{AN}$ for some invertible $\mathbf{A} \in \mathrm{gl}(4, \mathbb{C})$.

Proposition 2.1. (a) ([2, Lemma 6.1]) *For* $\mathbf{M}, \mathbf{N} \in \mathrm{wp}(4, 8, \mathbb{C})$*, the following* (i), (ii), (iii) *are equivalent:* (i) $\mathbf{M} \approx \mathbf{N}$, (ii) $\mathcal{G}(\mathbf{M}) = \mathcal{G}(\mathbf{N})$, (iii) $\mathcal{K}_\mathbf{M} = \mathcal{K}_\mathbf{N}$.
(b) ([2, Eq 6.4]) *For* $\mathbf{G} \in \mathrm{gl}(4, \mathbb{C})$, $\psi(\mathbf{G}) \in \mathrm{wp}(4, 8, \mathbb{C})$ *and* $\mathcal{G}(\psi(\mathbf{G})) = \mathbf{G}$.

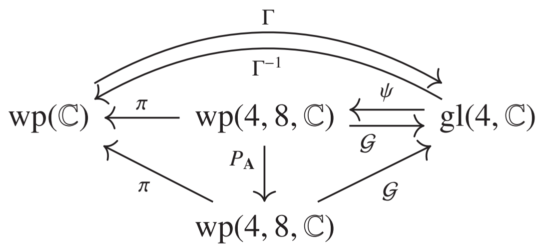

Figure 1. The commutative diagram showing the one-to-one correspondence Γ between $\mathrm{gl}(4,\mathbb{C})$ and the set $\mathrm{wp}(\mathbb{C})$ of all *equivalent* well-posed boundary matrices. $\mathrm{wp}(\mathbb{C})$ is also in one-to-one correspondence with the set of all integral operators $\mathcal{K}_{\mathbf{M}}$ in (1.1). This commutative diagram holds for any invertible $\mathbf{A} \in \mathrm{gl}(4,\mathbb{C})$, where $P_{\mathbf{A}}(\mathbf{M}) = \mathbf{AM}$. π is the canonical projection which maps a well-posed boundary matrix $\mathbf{M}$ to its equivalence class $[\mathbf{M}]$ with respect to $\approx$. The maps $\mathcal{G}$ and ψ defined in Definition 2.6 are explicitly computable.

Denote by $\mathrm{wp}(\mathbb{C})$ the quotient set $\mathrm{wp}(4,8,\mathbb{C})/\approx$ of $\mathrm{wp}(4,8,\mathbb{C})$ with respect to the relation $\approx$. For $\mathbf{M} \in \mathrm{wp}(4,8,\mathbb{C})$, denote by $[\mathbf{M}]$ the equivalence class in $\mathrm{wp}(4,8,\mathbb{C})/\approx$ which contains $\mathbf{M}$. Then we have the canonical projection $\pi : \mathrm{wp}(4,8,\mathbb{C}) \to \mathrm{wp}(\mathbb{C})$ defined by $\pi(\mathbf{M}) = [\mathbf{M}]$. By Proposition 2.1, the map $\pi \circ \psi : \mathrm{gl}(4,\mathbb{C}) \to \mathrm{wp}(\mathbb{C})$ is a one-to-one correspondence, and we denote its inverse by $\Gamma : \mathrm{wp}(\mathbb{C}) \to \mathrm{gl}(4,\mathbb{C})$. Thus we have the commutative diagram in Figure 1 which holds for any invertible $\mathbf{A} \in \mathrm{gl}(4,\mathbb{C})$. Here, the map $P_{\mathbf{A}} : \mathrm{wp}(4,8,\mathbb{C}) \to \mathrm{wp}(4,8,\mathbb{C})$ is defined by $P_{\mathbf{A}}(\mathbf{M}) = \mathbf{AM}$.

By Proposition 2.1, the set of integral operators $\mathcal{K}_{\mathbf{M}}$ in (1.1) is in one-to-one correspondence with the set $\mathrm{wp}(\mathbb{C})$ of *equivalent* well-posed boundary matrices, and hence is also in one-to-one correspondence with $\mathrm{gl}(4,\mathbb{C})$. Note that both of the maps $\mathcal{G}$ and ψ in Definition 2.6 are explicitly computable, hence Γ and its inverse Γ^{-1} are explicitly computable. For the special boundary matrix $\mathbf{Q}$ in (1.4), we have [2, Eq 6.2]

$$\mathcal{G}(\mathbf{Q}) = \mathbf{O}. \tag{2.6}$$

Proposition 2.2. *For* $\mathbf{M} \in \mathrm{wp}(4,8,\mathbb{C})$ *and* $\lambda \in \mathbb{C}$*, the following* (a) *and* (b) *hold.*

(a) ([2, Theorem 1 and Corollary 1]) $\mathcal{K}_{\mathbf{M}}[u] = \lambda \cdot u$ *for some* $0 \neq u \in L^2[-l,l]$ *if and only if* $\lambda \neq 0$ *and* $u = \mathbf{c}^T \mathbf{y}_\lambda$ *for some* $\mathbf{0} \neq \mathbf{c} \in \mathrm{gl}(4,1,\mathbb{C})$ *such that* $[\mathcal{G}(\mathbf{M})\{\mathbf{X}_\lambda(l) - \mathbf{X}_\lambda(-l)\} + \mathbf{X}_\lambda(l)]\,\mathbf{c} = \mathbf{0}$. $\mathcal{K}_{\mathbf{Q}}[u] = \lambda \cdot u$ *for some* $0 \neq u \in L^2[-l,l]$ *if and only if* $\lambda \neq 0$ *and* $u = \mathbf{c}^T \mathbf{y}_\lambda$ *for some* $\mathbf{0} \neq \mathbf{c} \in \mathrm{gl}(4,1,\mathbb{C})$ *such that* $\mathbf{X}_\lambda(l)\mathbf{c} = \mathbf{0}$. *In particular,* $0 \neq \lambda \in \mathrm{Spec}\,\mathcal{K}_{\mathbf{Q}}$ *if and only if* $\det \mathbf{X}_\lambda(l) = 0$.

(b) ([2, Corollary 2]) *Let* $0 \neq \lambda \in \mathbb{C}\backslash\mathrm{Spec}\,\mathcal{K}_{\mathbf{Q}}$. *Then* $\lambda \in \mathrm{Spec}\,\mathcal{K}_{\mathbf{M}}$ *if and only if* $\det\{\mathcal{G}(\mathbf{M})\mathbf{Y}_\lambda(l) - \mathbf{I}\} = 0$.

3. Main results

3.1. Block-diagonalization of $\mathbf{X}_\lambda(x)$ *for* $\lambda \neq 1/k$

The following is well defined since the range $\chi\,(\mathbb{C} \setminus \{0, 1/k\})$ of χ in Definition 2.4 does not contain $1, -1, \mathrm{i}, -\mathrm{i}$.

Definition 3.1. For $\lambda \in \mathbb{C} \setminus \{0, 1/k\}$ and $x \in \mathbb{R}$, denote

$$\mathbf{X}_\lambda^\pm(x) = \frac{1-\kappa^4}{4} \cdot \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega} z}\right) \begin{pmatrix} \frac{e^{\omega\kappa z}}{1-\kappa} \pm \frac{e^{-\omega\kappa z}}{1+\kappa} & \frac{e^{-\overline{\omega}\kappa z}}{1-\mathbb{i}\kappa} \pm \frac{e^{\overline{\omega}\kappa z}}{1+\mathbb{i}\kappa} \\ \frac{e^{-\omega\kappa z}}{1+\mathbb{i}\kappa} \pm \frac{e^{\omega\kappa z}}{1-\mathbb{i}\kappa} & \frac{e^{\overline{\omega}\kappa z}}{1-\kappa} \pm \frac{e^{-\overline{\omega}\kappa z}}{1+\kappa} \end{pmatrix},$$

where $z = \alpha x$ and $\kappa = \chi(\lambda)$.

The following is well defined, since

$$\left(\frac{1+\kappa^2}{1-\kappa^2}\right)^2 - \left(\frac{2\kappa}{1-\kappa^2}\right)^2 = 1, \qquad \kappa \in \mathbb{C} \setminus \{-1, 1\},$$
$$\left(\frac{1-\kappa^2}{1+\kappa^2}\right)^2 + \left(\frac{2\kappa}{1+\kappa^2}\right)^2 = 1, \qquad \kappa \in \mathbb{C} \setminus \{-\mathbb{i}, \mathbb{i}\}.$$

Definition 3.2. Denote by $\beta(\kappa)$ any holomorphic branch in $\mathbb{C} \setminus \{-1, 1\}$ satisfying

$$\cosh\beta(\kappa) = \frac{1+\kappa^2}{1-\kappa^2}, \qquad \sinh\beta(\kappa) = \frac{2\kappa}{1-\kappa^2},$$

and denote by $\gamma(\kappa)$ any holomorphic branch in $\mathbb{C} \setminus \{-\mathbb{i}, \mathbb{i}\}$ satisfying

$$\cos\gamma(\kappa) = \frac{1-\kappa^2}{1+\kappa^2}, \qquad \sin\gamma(\kappa) = \frac{2\kappa}{1+\kappa^2}.$$

For $z \in \mathbb{C}$ and $\kappa \in \mathbb{C} \setminus \{1, -1, \mathbb{i}, -\mathbb{i}\}$, define

$$\delta^\pm(z, \kappa) = \sinh\left(\sqrt{2}\kappa z + \beta(\kappa)\right) \pm \sin\left(\sqrt{2}\kappa z + \gamma(\kappa)\right).$$

$\beta(\kappa)$ and $\gamma(\kappa)$ are holomorphic branches of $2\operatorname{arctanh}\kappa$ and $2\arctan\kappa$ respectively, which, in turn, are anti-derivatives of $2/\left(1-\kappa^2\right)$ and $2/\left(1+\kappa^2\right)$ respectively.

Definition 3.3. Define $\mathcal{F} : \mathrm{wp}(4, 8, \mathbb{C}) \to \mathrm{gl}(4, \mathbb{C})$ by $\mathcal{F}(\mathbf{M}) = \mathbf{V}\mathcal{G}(\mathbf{M})\mathbf{V}^T$ and $\phi : \mathrm{gl}(4, \mathbb{C}) \to \mathrm{wp}(4, 8, \mathbb{C})$ by $\phi(\mathbf{G}) = \psi\left(\mathbf{V}^T\mathbf{G}\mathbf{V}\right)$. $\mathcal{F}(\mathbf{M})$ is called the *fundamental boundary matrix* corresponding to the well-posed boundary matrix $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$.

Denote by $\mathrm{Sim}_{\mathbf{V}^T}, \mathrm{Sim}_{\mathbf{V}} : \mathrm{gl}(4, \mathbb{C}) \to \mathrm{gl}(4, \mathbb{C})$ the similarity transforms defined by $\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{G} = \mathbf{V}\mathbf{G}\mathbf{V}^T$ and $\mathrm{Sim}_{\mathbf{V}}\,\mathbf{G} = \mathbf{V}^T\mathbf{G}\mathbf{V}$ respectively, so that $\mathcal{F} = \mathrm{Sim}_{\mathbf{V}^T} \circ \mathcal{G}$ and $\phi = \psi \circ \mathrm{Sim}_{\mathbf{V}}$ by Definition 3.3. By (2.3), $\mathrm{Sim}_{\mathbf{V}^T}^{-1} = \mathrm{Sim}_{\mathbf{V}}$, hence, by Proposition 2.1 (b), $\mathcal{F}(\phi(\mathbf{G})) = \mathrm{Sim}_{\mathbf{V}^T}\,\mathcal{G}(\psi(\mathrm{Sim}_{\mathbf{V}}\,\mathbf{G})) = \mathrm{Sim}_{\mathbf{V}^T}\,\mathrm{Sim}_{\mathbf{V}}\,\mathbf{G} = \mathbf{G}$ for $\mathbf{G} \in \mathrm{gl}(4, \mathbb{C})$. Thus Definition 3.3 gives a new one-to-one correspondence $\Phi : \mathrm{wp}(\mathbb{C}) \to \mathrm{gl}(4, \mathbb{C})$ defined by $\Phi = \mathrm{Sim}_{\mathbf{V}^T} \circ \Gamma$. See Figure 2 for a commutative diagram which expands the one in Figure 1 to incorporate Φ.

By Proposition 2.1 and Definition 3.3, the set of integral operators $\mathcal{K}_{\mathbf{M}}$ in (1.1) is in one-to-one correspondence with the 16-dimensional algebra $\mathrm{gl}(4, \mathbb{C})$. Both of Φ and its inverse Φ^{-1} are explicitly computable by using the maps $\mathcal{F}$ and ϕ in Definition 3.3.

Theorem 1. *For $\lambda \in \mathbb{C} \setminus \{0, 1/k\}$, the following* (a) *and* (b) *hold.*

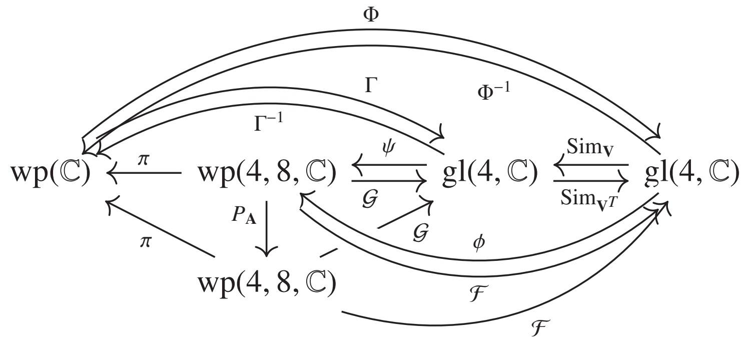

Figure 2. Commutative diagram showing the one-to-one correspondence Φ between $\mathrm{gl}(4,\mathbb{C})$ and the set $\mathrm{wp}(\mathbb{C})$ of all *equivalent* well-posed boundary matrices, which is also in one-to-one correspondence with the set of all integral operators $\mathcal{K}_{\mathbf{M}}$ in (1.1). This commutative diagram holds for any invertible $\mathbf{A} \in \mathrm{gl}(4,\mathbb{C})$, and extends the one for the map Γ in Figure 1 to incorporate Φ. $\mathrm{Sim}_{\mathbf{V}^T}$ and $\mathrm{Sim}_{\mathbf{V}}$ are the similarity transforms defined by $\mathrm{Sim}_{\mathbf{V}^T}\,\mathbf{G} = \mathbf{V}\mathbf{G}\mathbf{V}^T$ and $\mathrm{Sim}_{\mathbf{V}}\,\mathbf{G} = \mathbf{V}^T\mathbf{G}\mathbf{V}$ respectively. The maps $\mathcal{F}$ and ϕ defined in Definition 3.3 are explicitly computable.

(a) *For* $\mathbf{M} \in \mathrm{wp}(4,8,\mathbb{C})$*,* $\mathcal{K}_{\mathbf{M}}[u] = \lambda \cdot u$ *for some* $0 \neq u \in L^2[-l,l]$ *if and only if* $u = \mathbf{c}^T\mathbf{y}_\lambda$ *for some* $\mathbf{0} \neq \mathbf{c} \in \mathrm{gl}(4,1,\mathbb{C})$ *such that*

$$\left\{\mathcal{F}(\mathbf{M})\begin{pmatrix}\mathbf{X}_\lambda^+(-l) - \mathbf{X}_\lambda^+(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(-l) - \mathbf{X}_\lambda^-(l)\end{pmatrix} - \begin{pmatrix}\mathbf{X}_\lambda^+(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(l)\end{pmatrix}\right\}\mathbf{V}\mathbf{c} = \mathbf{0}.$$

$\mathcal{K}_{\mathbf{Q}}[u] = \lambda \cdot u$ *for some* $0 \neq u \in L^2[-l,l]$ *if and only if* $u = \mathbf{c}^T\mathbf{y}_\lambda$ *for some* $\mathbf{0} \neq \mathbf{c} \in \mathrm{gl}(4,1,\mathbb{C})$ *such that* $\begin{pmatrix}\mathbf{X}_\lambda^+(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(l)\end{pmatrix}\mathbf{V}\mathbf{c} = \mathbf{0}$.

(b) *Let* $\kappa = \chi(\lambda)$ *and* $z = \alpha x$*. Then, for* $x \in \mathbb{R}$*,*

$$\det \mathbf{X}_\lambda^\pm(x) = \frac{e^{-\sqrt{2}z}\kappa\left(1-\kappa^4\right)}{4} \cdot \delta^\pm(z,\kappa),$$

$$\det \mathbf{X}_\lambda(x) = \det \mathbf{X}_\lambda^+(x)\det \mathbf{X}_\lambda^-(x) = \frac{e^{-2\sqrt{2}z}\kappa^2\left(1-\kappa^4\right)^2}{16} \cdot \delta^+(z,\kappa)\delta^-(z,\kappa).$$

The proof of Theorem 1 will be given at the end of Section 4.

By Proposition 1.1, $0, 1/k \notin \mathrm{Spec}\,\mathcal{K}_{\mathbf{Q}}$ for every $l > 0$. Note that $\kappa \neq 0$ and $\kappa^4 \neq 1$ when $\kappa = \chi(\lambda)$ and $\lambda \in \mathbb{C} \setminus \{0, 1/k\}$. Thus, by Proposition 2.2 (a) and Theorem 1, the zero sets of the holomorphic functions $\delta^\pm(z,\kappa)$ in Definition 3.2 completely describe $\mathrm{Spec}\,\mathcal{K}_{\mathbf{Q}}$ in Proposition 1.1.

Corollary 1. *For every* $l > 0$*,* $\lambda \in \mathbb{C}$ *is in* $\mathrm{Spec}\,\mathcal{K}_{\mathbf{Q}}$ *if and only if* $\lambda \neq 0$*,* $\lambda \neq 1/k$*, and* $\delta^+(\alpha l,\chi(\lambda)) \cdot \delta^-(\alpha l,\chi(\lambda)) = 0$.

3.2. Block-diagonalization of $\mathbf{X}_{1/k}(x)$

Definition 3.4. For $z \in \mathbb{C}$, denote $p_n(z) = \sum_{r=0}^{n} \frac{\omega^{n-r}}{r!} z^r$, $n = 0, 1, 2, 3$, where it is understood that $0^0 = 1$, and denote

$$\mathbf{P}^{+}(z) = \begin{pmatrix} \overline{p_0(\overline{z})} & \overline{p_2(\overline{z})} \\ p_0(z) & p_2(z) \end{pmatrix}, \qquad \mathbf{P}^{-}(z) = \begin{pmatrix} -\overline{p_1(\overline{z})} & -\overline{p_3(\overline{z})} \\ p_1(z) & p_3(z) \end{pmatrix}.$$

For $x \in \mathbb{R}$, denote

$$\mathbf{X}^{+}_{1/k}(x) = \frac{1}{2\sqrt{2}} \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega} z}\right) \cdot \mathbf{P}^{+}(z) \cdot \operatorname{diag}\left(1, \alpha^{-2}\right),$$
$$\mathbf{X}^{-}_{1/k}(x) = \frac{1}{2\sqrt{2}} \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega} z}\right) \cdot \mathbf{P}^{-}(z) \cdot \operatorname{diag}\left(\alpha^{-1}, \alpha^{-3}\right),$$

where $z = \alpha x$.

Definition 3.5. For $z \in \mathbb{C}$, denote

$$p^{+}(z) = 1 + \frac{z}{\sqrt{2}}, \qquad p^{-}(z) = 1 + \sqrt{2} z + z^2 + \frac{z^3}{3\sqrt{2}}.$$

Theorem 2. *The following* (a) *and* (b) *hold.*

(a) *For* $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$*,* $\mathcal{K}_{\mathbf{M}}[u] = \frac{1}{k} \cdot u$ *for some* $0 \neq u \in L^2[-l, l]$ *if and only if* $u = \mathbf{c}^T \mathbf{y}_{1/k}$ *for some* $\mathbf{0} \neq \mathbf{c} \in \mathrm{gl}(4, 1, \mathbb{C})$ *such that*

$$\left\{ \mathcal{F}(\mathbf{M}) \begin{pmatrix} \mathbf{X}^{+}_{1/k}(-l) - \mathbf{X}^{+}_{1/k}(l) & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{-}_{1/k}(-l) - \mathbf{X}^{-}_{1/k}(l) \end{pmatrix} - \begin{pmatrix} \mathbf{X}^{+}_{1/k}(l) & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{-}_{1/k}(l) \end{pmatrix} \right\} \widehat{\mathbf{V}} \mathbf{c} = \mathbf{0}.$$

(b) *For* $x \in \mathbb{R}$*,*

$$\det \mathbf{X}^{+}_{1/k}(x) = \frac{\mathbb{i} e^{-\sqrt{2} z}}{4\alpha^2} \cdot p^{+}(z), \qquad \det \mathbf{X}^{-}_{1/k}(x) = -\frac{\mathbb{i} e^{-\sqrt{2} z}}{4\alpha^4} \cdot p^{-}(z),$$
$$\det \mathbf{X}_{1/k}(x) = -\det \mathbf{X}^{+}_{1/k}(x) \det \mathbf{X}^{-}_{1/k}(x) = -\frac{e^{-2\sqrt{2} z}}{16\alpha^6} \cdot p^{+}(z) p^{-}(z),$$

where $z = \alpha x$. $\det \mathbf{X}^{\pm}_{1/k}(x) \neq 0$ *and* $\det \mathbf{X}_{1/k}(x) \neq 0$ *for* $x > 0$.

The proof of Theorem 2 will be given at the end of Section 5.

3.3. Block-diagonalization of $\mathbf{Y}_{\lambda}(x)$

Definition 3.6. For $0 \neq \lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ such that $\det \mathbf{X}^{\pm}_{\lambda}(x) \neq 0$, denote $\mathbf{Y}^{\pm}_{\lambda}(x) = \mathbf{X}^{\pm}_{\lambda}(-x) \cdot \mathbf{X}^{\pm}_{\lambda}(x)^{-1} - \mathbf{I}$.

Theorem 3. *The following* (a) *and* (b) *hold.*

(a) *For* $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$ *and* $0 \neq \lambda \in \mathbb{C} \setminus \operatorname{Spec} \mathcal{K}_{\mathbf{Q}}$*,* $\lambda \in \operatorname{Spec} \mathcal{K}_{\mathbf{M}}$ *if and only if*

$$\det \left\{ \mathcal{F}(\mathbf{M}) \begin{pmatrix} \mathbf{Y}^{+}_{\lambda}(l) & \mathbf{O} \\ \mathbf{O} & \mathbf{Y}^{-}_{\lambda}(l) \end{pmatrix} - \mathbf{I} \right\} = 0.$$

(b) *Let* $0 \neq \lambda \in \mathbb{C}$*,* $x \in \mathbb{R}$*, and* $z = \alpha x$*. Suppose that* $\det \mathbf{X}_\lambda^\pm(x) \neq 0$*. If* $\lambda \neq 1/k$*, then*

$$\mathbf{Y}_\lambda^\pm(x) = \frac{1}{\delta^\pm(z,\kappa)} \begin{pmatrix} e^{2\omega z}\delta^\pm(-\mathbb{i}z,\kappa) - \delta^\pm(z,\kappa) & \sqrt{2}\omega e^{\sqrt{2}z}\,\mathrm{s}^\pm(z\kappa) \\ \sqrt{2}\overline{\omega} e^{\sqrt{2}z}\,\mathrm{s}^\pm(z\kappa) & e^{2\overline{\omega}z}\delta^\pm(\mathbb{i}z,\kappa) - \delta^\pm(z,\kappa) \end{pmatrix},$$

where $\kappa = \chi(\lambda)$ *and* $\mathrm{s}^\pm(\zeta) = \sinh\left(\sqrt{2}\zeta\right) \pm \sin\left(\sqrt{2}\zeta\right)$ *for* $\zeta \in \mathbb{C}$*. Also,*

$$\mathbf{Y}_{1/k}^\pm(x) = \frac{1}{p^\pm(z)} \begin{pmatrix} e^{2\omega z}p^\pm(-\mathbb{i}z) - p^\pm(z) & \frac{1}{2\mp 1}\omega e^{\sqrt{2}z} z^{2\mp 1} \\ \frac{1}{2\mp 1}\overline{\omega} e^{\sqrt{2}z} z^{2\mp 1} & e^{2\overline{\omega}z}p^\pm(\mathbb{i}z) - p^\pm(z) \end{pmatrix}.$$

The proof of Theorem 3 will be given at the end of Section 6.

4. Block-diagonalization of $\mathbf{X}_\lambda(x)$ for $\lambda \neq 1/k$: proof of Theorem 1

Definition 4.1. For $z, \kappa \in \mathbb{C}$, denote

$$\mathbf{X}(z,\kappa) = \frac{1}{4} e^{-\mathcal{E}\mathbf{\Omega} z} \Big\{ \mathrm{diag}(0,1,1,0) \cdot \mathbf{W}_0^* \cdot \mathrm{diag}\left(1,\kappa,\kappa^2,\kappa^3\right) \cdot \mathbf{W}_0 e^{-\mathbf{\Omega}\kappa z} + \mathrm{diag}(1,0,0,1) \cdot \mathbf{W}_0^* \cdot \mathrm{diag}\left(1,\kappa,\kappa^2,\kappa^3\right) \cdot \mathbf{W}_0 e^{\mathbf{\Omega}\kappa z} \Big\}.$$

Proposition 4.1. ([2, Eq 7.9]) *For* $\lambda \in \mathbb{C} \setminus \{0, 1/k\}$ *and* $x \in \mathbb{R}$*,* $\mathbf{X}_\lambda(x) = \mathbf{X}(z,\kappa)$*, where* $z = \alpha x$ *and* $\kappa = \chi(\lambda)$*.*

Definition 4.2. Denote $\mathbb{D} = \mathbb{C} \setminus \{0, 1, -1, \mathbb{i}, -\mathbb{i}\}$. For $z \in \mathbb{C}$ and $\kappa \in \mathbb{D}$, denote

$$\widehat{\mathbf{X}}(z,\kappa) = \frac{1}{1-\kappa^4} \Big\{ \mathrm{diag}(0,1,1,0) \cdot \mathbf{W}_0^* \cdot \mathrm{diag}\left(1,\kappa,\kappa^2,\kappa^3\right) \cdot \mathbf{W}_0 e^{-\mathbf{\Omega}\kappa z} + \mathrm{diag}(1,0,0,1) \cdot \mathbf{W}_0^* \cdot \mathrm{diag}\left(1,\kappa,\kappa^2,\kappa^3\right) \cdot \mathbf{W}_0 e^{\mathbf{\Omega}\kappa z} \Big\}.$$

By Definitions 4.1 and 4.2, we have

$$\mathbf{X}(z,\kappa) = \frac{1-\kappa^4}{4} \cdot e^{-\mathcal{E}\mathbf{\Omega} z} \cdot \widehat{\mathbf{X}}(z,\kappa), \qquad z \in \mathbb{C},\ \kappa \in \mathbb{D}. \tag{4.1}$$

Lemma 4.1. *For* $z \in \mathbb{C}$ *and* $\kappa \in \mathbb{D}$*,* $\widehat{\mathbf{X}}(z,\kappa) = \left(\frac{e^{\epsilon_i \omega_j \kappa z}}{1-\frac{\omega_j}{\omega_i}\kappa}\right)_{1\le i,j\le 4}$.

Proof. By Definition 2.1 and (2.1), $\mathbf{W}_0^* = \left(\overline{\omega_i}^{j-1}\right)_{1\le i,j\le 4} = \left(\omega_i^{1-j}\right)_{1\le i,j\le 4}$, hence

$$\left\{\mathbf{W}_0^* \cdot \mathrm{diag}\left(1,\kappa,\kappa^2,\kappa^3\right) \cdot \mathbf{W}_0\right\}_{i,j} = \sum_{r=1}^4 \omega_i^{1-r} \cdot \kappa^{r-1} \cdot \omega_j^{r-1} = \sum_{r=1}^4 \left(\frac{\omega_j}{\omega_i} \cdot \kappa\right)^{r-1} = \frac{1 - \frac{\omega_j^4}{\omega_i^4} \cdot \kappa^4}{1 - \frac{\omega_j}{\omega_i} \cdot \kappa} = \frac{1-\kappa^4}{1 - \frac{\omega_j}{\omega_i}\kappa}$$

for $1 \le i, j \le 4$. So by Definition 4.2, we have

$$\begin{aligned} \widehat{\mathbf{X}}(z,\kappa) &= \mathrm{diag}(0,1,1,0) \cdot \left(\frac{1}{1-\frac{\omega_j}{\omega_i}\kappa}\right)_{1\le i,j\le 4} \cdot e^{-\mathbf{\Omega}\kappa z} + \mathrm{diag}(1,0,0,1) \cdot \left(\frac{1}{1-\frac{\omega_j}{\omega_i}\kappa}\right)_{1\le i,j\le 4} \cdot e^{\mathbf{\Omega}\kappa z} \\ &= \mathrm{diag}(0,1,1,0) \cdot \left(\frac{e^{-\omega_j \kappa z}}{1-\frac{\omega_j}{\omega_i}\kappa}\right)_{1\le i,j\le 4} + \mathrm{diag}(1,0,0,1) \cdot \left(\frac{e^{\omega_j \kappa z}}{1-\frac{\omega_j}{\omega_i}\kappa}\right)_{1\le i,j\le 4}. \end{aligned}$$

Thus the result follows by Definition 2.2. □

Definition 4.3. For $z \in \mathbb{C}$ and $\kappa \in \mathbb{D}$, denote

$$\widehat{\mathbf{X}}^{\pm}(z,\kappa) = \begin{pmatrix} \frac{e^{\omega\kappa z}}{1-\kappa} \pm \frac{e^{-\omega\kappa z}}{1+\kappa} & \frac{e^{-\overline{\omega}\kappa z}}{1-\mathbb{i}\kappa} \pm \frac{e^{\overline{\omega}\kappa z}}{1+\mathbb{i}\kappa} \\ \frac{e^{-\omega\kappa z}}{1+\mathbb{i}\kappa} \pm \frac{e^{\omega\kappa z}}{1-\mathbb{i}\kappa} & \frac{e^{\overline{\omega}\kappa z}}{1-\kappa} \pm \frac{e^{-\overline{\omega}\kappa z}}{1+\kappa} \end{pmatrix}, \qquad \mathbf{X}^{\pm}(z,\kappa) = \frac{1-\kappa^4}{4} \cdot \mathrm{diag}\left(e^{-\omega z}, e^{-\overline{\omega} z}\right) \cdot \widehat{\mathbf{X}}^{\pm}(z,\kappa).$$

Note from Definitions 3.1 and 4.3 that

$$\mathbf{X}^{\pm}_{\lambda}(x) = \mathbf{X}^{\pm}(z,\kappa), \qquad \lambda \in \mathbb{C} \setminus \{0, 1/k\},\ x \in \mathbb{R}, \tag{4.2}$$

where $z = \alpha x$ and $\kappa = \chi(\lambda)$.

Lemma 4.2. *For $z \in \mathbb{C}$ and $\kappa \in \mathbb{D}$,* $\mathbf{V}\widehat{\mathbf{X}}(z,\kappa)\mathbf{V}^T = \begin{pmatrix} \widehat{\mathbf{X}}^{+}(z,\kappa) & \mathbf{O} \\ \mathbf{O} & \widehat{\mathbf{X}}^{-}(z,\kappa) \end{pmatrix}$.

Proof. By (2.1), Definition 2.2 and Lemma 4.1,

$$\widehat{\mathbf{X}}(z,\kappa)_{i+2,j+2} = \frac{e^{\epsilon_{i+2}\omega_{j+2}\kappa z}}{1 - \frac{\omega_{j+2}}{\omega_{i+2}}\kappa} = \frac{e^{(-\epsilon_i)(-\omega_j)\kappa z}}{1 - \frac{(-\omega_j)}{(-\omega_i)}\kappa} = \frac{e^{\epsilon_i\omega_j\kappa z}}{1 - \frac{\omega_j}{\omega_i}\kappa} = \widehat{\mathbf{X}}(z,\kappa)_{i,j},$$
$$\widehat{\mathbf{X}}(z,\kappa)_{i+2,j} = \frac{e^{\epsilon_{i+2}\omega_{j}\kappa z}}{1 - \frac{\omega_{j}}{\omega_{i+2}}\kappa} = \frac{e^{(-\epsilon_i)(-\omega_{j+2})\kappa z}}{1 - \frac{(-\omega_{j+2})}{(-\omega_i)}\kappa} = \frac{e^{\epsilon_i\omega_{j+2}\kappa z}}{1 - \frac{\omega_{j+2}}{\omega_i}\kappa} = \widehat{\mathbf{X}}(z,\kappa)_{i,j+2}$$

for $1 \le i,j \le 2$, which implies that $\widehat{\mathbf{X}}(z,\kappa) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}$, where we put $\mathbf{A} = \left\{\widehat{\mathbf{X}}(z,\kappa)_{i,j}\right\}_{1\le i,j\le 2}$, $\mathbf{B} = \left\{\widehat{\mathbf{X}}(z,\kappa)_{i,j+2}\right\}_{1\le i,j\le 2} \in \mathrm{gl}(2,\mathbb{C})$. So by Lemma 2.1, we have

$$\mathbf{V}\widehat{\mathbf{X}}(z,\kappa)\mathbf{V}^T = \begin{pmatrix} \mathbf{A}+\mathbf{B} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}-\mathbf{B} \end{pmatrix}. \tag{4.3}$$

By Lemma 4.1, we have

$$\begin{aligned} \mathbf{A} \pm \mathbf{B} &= \left\{\widehat{\mathbf{X}}(z,\kappa)_{i,j}\right\}_{1\le i,j\le 2} \pm \left\{\widehat{\mathbf{X}}(z,\kappa)_{i,j+2}\right\}_{1\le i,j\le 2} \\ &= \left(\frac{e^{\epsilon_i\omega_j\kappa z}}{1 - \frac{\omega_j}{\omega_i}\kappa} \pm \frac{e^{\epsilon_i\omega_{j+2}\kappa z}}{1 - \frac{\omega_{j+2}}{\omega_i}\kappa}\right)_{1\le i,j\le 2} = \begin{pmatrix} \frac{e^{\epsilon_1\omega_1\kappa z}}{1-\frac{\omega_1}{\omega_1}\kappa} \pm \frac{e^{\epsilon_1\omega_3\kappa z}}{1-\frac{\omega_3}{\omega_1}\kappa} & \frac{e^{\epsilon_1\omega_2\kappa z}}{1-\frac{\omega_2}{\omega_1}\kappa} \pm \frac{e^{\epsilon_1\omega_4\kappa z}}{1-\frac{\omega_4}{\omega_1}\kappa} \\ \frac{e^{\epsilon_2\omega_1\kappa z}}{1-\frac{\omega_1}{\omega_2}\kappa} \pm \frac{e^{\epsilon_2\omega_3\kappa z}}{1-\frac{\omega_3}{\omega_2}\kappa} & \frac{e^{\epsilon_2\omega_2\kappa z}}{1-\frac{\omega_2}{\omega_2}\kappa} \pm \frac{e^{\epsilon_2\omega_4\kappa z}}{1-\frac{\omega_4}{\omega_2}\kappa} \end{pmatrix}, \end{aligned}$$

hence, by (2.1) and Definitions 2.2, 4.3,

$$\mathbf{A} \pm \mathbf{B} = \begin{pmatrix} \frac{e^{\omega\kappa z}}{1-\kappa} \pm \frac{e^{-\omega\kappa z}}{1+\kappa} & \frac{e^{-\overline{\omega}\kappa z}}{1-\mathbb{i}\kappa} \pm \frac{e^{\overline{\omega}\kappa z}}{1+\mathbb{i}\kappa} \\ \frac{e^{-\omega\kappa z}}{1+\mathbb{i}\kappa} \pm \frac{e^{\omega\kappa z}}{1-\mathbb{i}\kappa} & \frac{e^{\overline{\omega}\kappa z}}{1-\kappa} \pm \frac{e^{-\overline{\omega}\kappa z}}{1+\kappa} \end{pmatrix} = \widehat{\mathbf{X}}^{\pm}(z,\kappa).$$

Thus the lemma follows by (4.3). □

Lemma 4.3. *For $z \in \mathbb{C}$ and $\kappa \in \mathbb{D}$,* $\mathbf{V}\mathbf{X}(z,\kappa)\mathbf{V}^T = \begin{pmatrix} \mathbf{X}^{+}(z,\kappa) & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{-}(z,\kappa) \end{pmatrix}$.

Proof. By (2.3), (2.4), (4.1) and Lemma 4.2,

$$\begin{aligned}\mathbf{V}\mathbf{X}(z,\kappa)\mathbf{V}^T &= \mathbf{V}\left\{\frac{1-\kappa^4}{4}\cdot e^{-\mathcal{E}\mathbf{\Omega} z}\cdot\widehat{\mathbf{X}}(z,\kappa)\right\}\mathbf{V}^T = \frac{1-\kappa^4}{4}\cdot\mathbf{V}e^{-\mathcal{E}\mathbf{\Omega} z}\mathbf{V}^T\cdot\mathbf{V}\widehat{\mathbf{X}}(z,\kappa)\mathbf{V}^T\\ &= \frac{1-\kappa^4}{4}\cdot\begin{pmatrix}\operatorname{diag}\left(e^{-\omega z},e^{-\overline{\omega}z}\right) & \mathbf{O}\\ \mathbf{O} & \operatorname{diag}\left(e^{-\omega z},e^{-\overline{\omega}z}\right)\end{pmatrix}\begin{pmatrix}\widehat{\mathbf{X}}^+(z,\kappa) & \mathbf{O}\\ \mathbf{O} & \widehat{\mathbf{X}}^-(z,\kappa)\end{pmatrix}\\ &= \frac{1-\kappa^4}{4}\cdot\begin{pmatrix}\operatorname{diag}\left(e^{-\omega z},e^{-\overline{\omega}z}\right)\widehat{\mathbf{X}}^+(z,\kappa) & \mathbf{O}\\ \mathbf{O} & \operatorname{diag}\left(e^{-\omega z},e^{-\overline{\omega}z}\right)\widehat{\mathbf{X}}^-(z,\kappa)\end{pmatrix}.\end{aligned}$$

Thus the lemma follows by Definition 4.3. □

By Proposition 4.1, (4.2) and Lemma 4.3, we have

$$\mathbf{X}_\lambda(x) = \mathbf{V}^T\begin{pmatrix}\mathbf{X}_\lambda^+(x) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(x)\end{pmatrix}\mathbf{V},\qquad \lambda\in\mathbb{C}\setminus\{0,1/k\},\ x\in\mathbb{R}. \tag{4.4}$$

Lemma 4.4. *For $z\in\mathbb{C}$ and $\kappa\in\mathbb{D}$, $\det\widehat{\mathbf{X}}^\pm(z,\kappa) = \frac{4\kappa}{1-\kappa^4}\cdot\delta^\pm(z,\kappa)$.*

See Supplementary A for proof of Lemma 4.4.

Proof of Theorem 1. Let $\lambda\in\mathbb{C}\setminus\{0,1/k\}$ and $\mathbf{M}\in\mathrm{wp}(4,8,\mathbb{C})$. By Proposition 2.2 (a), $\mathcal{K}_{\mathbf{M}}[u] = \lambda\cdot u$ for some $0\neq u\in L^2[-l,l]$ if and only if $u = \mathbf{c}^T\mathbf{y}_\lambda$ for some $\mathbf{0}\neq\mathbf{c}\in\mathrm{gl}(4,1,\mathbb{C})$ such that

$$\mathbf{0} = \mathbf{V}\left[\mathcal{G}(\mathbf{M})\left\{\mathbf{X}_\lambda(-l)-\mathbf{X}_\lambda(l)\right\}-\mathbf{X}_\lambda(l)\right]\mathbf{c}, \tag{4.5}$$

since $\mathbf{V}$ is invertible by (2.3). Thus the first assertion in (a) follows, since (4.5) is equivalent to

$$\begin{aligned}\mathbf{0} &= \left[\mathbf{V}\mathcal{G}(\mathbf{M})\left\{\mathbf{V}^T\begin{pmatrix}\mathbf{X}_\lambda^+(-l) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(-l)\end{pmatrix}\mathbf{V}-\mathbf{V}^T\begin{pmatrix}\mathbf{X}_\lambda^+(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(l)\end{pmatrix}\mathbf{V}\right\}-\mathbf{V}\cdot\mathbf{V}^T\begin{pmatrix}\mathbf{X}_\lambda^+(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(l)\end{pmatrix}\mathbf{V}\right]\mathbf{c}\\ &= \left[\mathcal{F}(\mathbf{M})\begin{pmatrix}\mathbf{X}_\lambda^+(-l)-\mathbf{X}_\lambda^+(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(-l)-\mathbf{X}_\lambda^-(l)\end{pmatrix}-\begin{pmatrix}\mathbf{X}_\lambda^+(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(l)\end{pmatrix}\right]\mathbf{V}\mathbf{c}\end{aligned}$$

by (4.4) and Definition 3.3. The second assertion in (a) follows from the first one, since $\mathcal{F}(\mathbf{Q}) = \mathbf{V}\mathcal{G}(\mathbf{Q})\mathbf{V}^T = \mathbf{O}$ by (2.6) and Definition 3.3.

Let $\kappa = \chi(\lambda)$, $x\in\mathbb{R}$, and $z = \alpha x$. By (2.3) and (4.4), we have

$$\begin{aligned}\det\mathbf{X}_\lambda(x) &= \det\left\{\mathbf{V}^T\cdot\begin{pmatrix}\mathbf{X}_\lambda^+(x) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}_\lambda^-(x)\end{pmatrix}\cdot\mathbf{V}\right\} = \det\mathbf{V}^T\cdot\left\{\det\mathbf{X}_\lambda^+(x)\cdot\det\mathbf{X}_\lambda^-(x)\right\}\cdot\det\mathbf{V}\\ &= \det\mathbf{X}_\lambda^+(x)\cdot\det\mathbf{X}_\lambda^-(x).\end{aligned} \tag{4.6}$$

By (4.2) and Definition 4.3,

$$\begin{aligned}\det\mathbf{X}_\lambda^\pm(x) = \det\mathbf{X}^\pm(z,\kappa) &= \det\left\{\frac{1-\kappa^4}{4}\cdot\operatorname{diag}\left(e^{-\omega z},e^{-\overline{\omega}z}\right)\widehat{\mathbf{X}}^\pm(z,\kappa)\right\}\\ &= \left(\frac{1-\kappa^4}{4}\right)^2\cdot\det\operatorname{diag}\left(e^{-\omega z},e^{-\overline{\omega}z}\right)\cdot\det\widehat{\mathbf{X}}^\pm(z,\kappa),\end{aligned}$$

hence, by (2.5) and Lemma 4.4,

$$\det \mathbf{X}_\lambda^\pm(x) = \frac{\left(1-\kappa^4\right)^2}{16} \cdot e^{-\sqrt{2}z} \cdot \frac{4\kappa}{1-\kappa^4}\delta^\pm(z,\kappa) = \frac{e^{-\sqrt{2}z}\kappa\left(1-\kappa^4\right)}{4} \cdot \delta^\pm(z,\kappa).$$

So by (4.6), we have

$$\det \mathbf{X}_\lambda(x) = \frac{e^{-\sqrt{2}z}\kappa\left(1-\kappa^4\right)}{4} \cdot \delta^+(z,\kappa) \cdot \frac{e^{-\sqrt{2}z}\kappa\left(1-\kappa^4\right)}{4} \cdot \delta^-(z,\kappa) = \frac{e^{-2\sqrt{2}z}\kappa^2\left(1-\kappa^4\right)^2}{16} \cdot \delta^+(z,\kappa)\delta^-(z,\kappa).$$

Thus we showed (b), and the proof is complete. □

5. Block-diagonalization of $\mathbf{X}_{1/k}(x)$: proof of Theorem 2

Definition 5.1. For $z \in \mathbb{C}$, denote

$$\mathbf{P}(z) = \begin{pmatrix} \overline{p_0(\overline{z})} & \overline{p_1(\overline{z})} & \overline{p_2(\overline{z})} & \overline{p_3(\overline{z})} \\ p_0(z) & -p_1(z) & p_2(z) & -p_3(z) \\ \overline{p_0(\overline{z})} & -\overline{p_1(\overline{z})} & \overline{p_2(\overline{z})} & -\overline{p_3(\overline{z})} \\ p_0(z) & p_1(z) & p_2(z) & p_3(z) \end{pmatrix}.$$

Proposition 5.1. (a) ([2, Eq 7.13]) $\mathbf{X}_{1/k}(x) = \frac{1}{4}e^{-\mathcal{E}\mathbf{\Omega}z}\mathbf{P}(z) \cdot \operatorname{diag}\left(1,\alpha,\alpha^2,\alpha^3\right)^{-1}$ *for* $x \in \mathbb{R}$, *where* $z = \alpha x$.

(b) ([2, Lemma B1]) *For* $z \in \mathbb{C}$, $\mathbf{V}\mathbf{P}(z)\widehat{\mathbf{V}} = \sqrt{2}\begin{pmatrix} \mathbf{P}^+(z) & \mathbf{O} \\ \mathbf{O} & \mathbf{P}^-(z) \end{pmatrix}$.

The result in Proposition 5.1 (b) was for $z \in \mathbb{R}$ in [2] originally, but it can immediately be extended to $z \in \mathbb{C}$.

By (2.3), we have

$$\begin{aligned} &\widehat{\mathbf{V}}^T \operatorname{diag}\left(1,\alpha^{-1},\alpha^{-2},\alpha^{-3}\right)\widehat{\mathbf{V}} = \widehat{\mathbf{V}} \operatorname{diag}\left(1,\alpha^{-1},\alpha^{-2},\alpha^{-3}\right)\widehat{\mathbf{V}} \\ &= \begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&\alpha^{-1}&0&0\\0&0&\alpha^{-2}&0\\0&0&0&\alpha^{-3} \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\0&0&\alpha^{-2}&0\\0&\alpha^{-1}&0&0\\0&0&0&\alpha^{-3} \end{pmatrix}\begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{pmatrix} \\ &= \operatorname{diag}\left(1,\alpha^{-2},\alpha^{-1},\alpha^{-3}\right) = \begin{pmatrix} \operatorname{diag}\left(1,\alpha^{-2}\right) & \mathbf{O} \\ \mathbf{O} & \operatorname{diag}\left(\alpha^{-1},\alpha^{-3}\right) \end{pmatrix}. \end{aligned} \tag{5.1}$$

By Proposition 5.1 (a) and (2.3),

$$\begin{aligned} \mathbf{V}\mathbf{X}_{1/k}(x)\widehat{\mathbf{V}} &= \mathbf{V}\left\{\frac{1}{4}e^{-\mathcal{E}\mathbf{\Omega}z}\mathbf{P}(z) \cdot \operatorname{diag}\left(1,\alpha,\alpha^2,\alpha^3\right)^{-1}\right\}\widehat{\mathbf{V}} \\ &= \frac{1}{4}\mathbf{V}e^{-\mathcal{E}\mathbf{\Omega}z}\mathbf{V}^T \cdot \mathbf{V}\mathbf{P}(z)\widehat{\mathbf{V}} \cdot \widehat{\mathbf{V}}^T \operatorname{diag}\left(1,\alpha^{-1},\alpha^{-2},\alpha^{-3}\right)\widehat{\mathbf{V}}, \end{aligned}$$

hence, by (2.4), (5.1) and Proposition 5.1 (b),

$$\mathbf{V}\mathbf{X}_{1/k}(x)\widehat{\mathbf{V}} = \frac{1}{4}\begin{pmatrix} \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega} z}\right) & \mathbf{O} \\ \mathbf{O} & \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega} z}\right) \end{pmatrix}$$
$$\cdot \sqrt{2}\begin{pmatrix} \mathbf{P}^{+}(z) & \mathbf{O} \\ \mathbf{O} & \mathbf{P}^{-}(z) \end{pmatrix} \cdot \begin{pmatrix} \operatorname{diag}\left(1, \alpha^{-2}\right) & \mathbf{O} \\ \mathbf{O} & \operatorname{diag}\left(\alpha^{-1}, \alpha^{-3}\right) \end{pmatrix}.$$

Thus, by (2.3) and Definition 3.4, we have

$$\mathbf{X}_{1/k}(x) = \mathbf{V}^{T}\begin{pmatrix} \mathbf{X}^{+}_{1/k}(x) & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{-}_{1/k}(x) \end{pmatrix}\widehat{\mathbf{V}}, \qquad x \in \mathbb{R}. \tag{5.2}$$

By Definition 3.4 and (2.1), we have

$$\begin{aligned} p_0(z) &= 1, \\ p_1(z) &= \omega + z, \\ p_2(z) &= \omega^2 + \omega z + \frac{1}{2}z^2 = \mathbb{i} + \omega z + \frac{1}{2}z^2, \\ p_3(z) &= \omega^3 + \omega^2 z + \frac{1}{2}\omega z^2 + \frac{1}{6}z^3 = -\overline{\omega} + \mathbb{i}z + \frac{1}{2}\omega z^2 + \frac{1}{6}z^3. \end{aligned} \tag{5.3}$$

Lemma 5.1. *For* $z \in \mathbb{C}$, $\det \mathbf{P}^{+}(z) = 2\mathbb{i} \cdot p^{+}(z)$ *and* $\det \mathbf{P}^{-}(z) = -2\mathbb{i} \cdot p^{-}(z)$.

Proof. By Definitions 3.4, 3.5, (2.1) and (5.3),

$$\begin{aligned} \det \mathbf{P}^{+}(z) &= \overline{p_0(\overline{z})} \cdot p_2(z) - p_0(z) \cdot \overline{p_2(\overline{z})} \\ &= 1 \cdot \left(\mathbb{i} + \omega z + \frac{1}{2}z^2\right) - 1 \cdot \left(-\mathbb{i} + \overline{\omega} z + \frac{1}{2}z^2\right) = 2\mathbb{i} + \sqrt{2}\mathbb{i}z = 2\mathbb{i} \cdot p^{+}(z), \\ \det \mathbf{P}^{-}(z) &= -\overline{p_1(\overline{z})} \cdot p_3(z) + p_1(z) \cdot \overline{p_3(\overline{z})} \\ &= -(\overline{\omega} + z)\left(-\overline{\omega} + \mathbb{i}z + \frac{1}{2}\omega z^2 + \frac{1}{6}z^3\right) + (\omega + z)\left(-\omega - \mathbb{i}z + \frac{1}{2}\overline{\omega} z^2 + \frac{1}{6}z^3\right) \\ &= \left\{-\mathbb{i} - \sqrt{2}\mathbb{i}z - \left(\frac{1}{2} + \mathbb{i}\right)z^2 - \left(\frac{\omega}{2} + \frac{\overline{\omega}}{6}\right)z^3 - \frac{1}{6}z^4\right\} \\ &\quad + \left\{-\mathbb{i} - \sqrt{2}\mathbb{i}z + \left(\frac{1}{2} - \mathbb{i}\right)z^2 + \left(\frac{\overline{\omega}}{2} + \frac{\omega}{6}\right)z^3 + \frac{1}{6}z^4\right\} \\ &= -2\mathbb{i} - 2\sqrt{2}\mathbb{i}z - 2\mathbb{i}z^2 - \frac{\sqrt{2}\mathbb{i}}{3}z^3 = -2\mathbb{i} \cdot p^{-}(z). \end{aligned}$$

□

Proof of Theorem 2. Let $\mathbf{M} \in \mathrm{wp}(4, 8, \mathbb{C})$. By Proposition 2.2 (a), $\mathcal{K}_{\mathbf{M}}[u] = \frac{1}{k} \cdot u$ for some $0 \neq u \in L^2[-l, l]$ if and only if $u = \mathbf{c}^T \mathbf{y}_{1/k}$ for some $\mathbf{c} \in \mathrm{gl}(4, 1, \mathbb{C})$ such that

$$\mathbf{0} = \mathbf{V}\left[\mathcal{G}(\mathbf{M})\left\{\mathbf{X}_{1/k}(-l) - \mathbf{X}_{1/k}(l)\right\} - \mathbf{X}_{1/k}(l)\right]\mathbf{c}, \tag{5.4}$$

since $\mathbf{V}$ is invertible by (2.3). Thus (a) follows, since (5.4) is equivalent to

$$\mathbf{0} = \left[\mathbf{V}\mathcal{G}(\mathbf{M})\left\{\mathbf{V}^{T}\begin{pmatrix} \mathbf{X}^{+}_{1/k}(-l) & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{-}_{1/k}(-l) \end{pmatrix}\widehat{\mathbf{V}} - \mathbf{V}^{T}\begin{pmatrix} \mathbf{X}^{+}_{1/k}(l) & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{-}_{1/k}(l) \end{pmatrix}\widehat{\mathbf{V}}\right\} - \mathbf{V} \cdot \mathbf{V}^{T}\begin{pmatrix} \mathbf{X}^{+}_{\lambda}(l) & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^{-}_{\lambda}(l) \end{pmatrix}\widehat{\mathbf{V}}\right]\mathbf{c}$$

$$= \left[\mathcal{F}(\mathbf{M}) \begin{pmatrix} \mathbf{X}^+_{1/k}(-l) - \mathbf{X}^+_{1/k}(l) & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^-_{1/k}(-l) - \mathbf{X}^-_{1/k}(l) \end{pmatrix} - \begin{pmatrix} \mathbf{X}^+_{1/k}(l) & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^-_{1/k}(l) \end{pmatrix} \right] \widehat{\mathbf{V}}\mathbf{c}$$

by (5.2) and Definition 3.3.

Let $x \in \mathbb{R}$ and $z = \alpha x$. By (2.3) and (5.2),

$$\det \mathbf{X}_{1/k}(x) = \det \mathbf{V}^T \cdot \det \begin{pmatrix} \mathbf{X}^+_{1/k}(x) & \mathbf{O} \\ \mathbf{O} & \mathbf{X}^-_{1/k}(x) \end{pmatrix} \cdot \det \widehat{\mathbf{V}} = -\det \mathbf{X}^+_{1/k}(x) \cdot \det \mathbf{X}^-_{1/k}(x). \tag{5.5}$$

By (2.5), Definition 3.4 and Lemma 5.1,

$$\begin{aligned} \det \mathbf{X}^+_{1/k}(x) &= \left(\frac{1}{2\sqrt{2}}\right)^2 \det \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega} z}\right) \cdot \det \mathbf{P}^+(z) \cdot \det \operatorname{diag}\left(1, \alpha^{-2}\right) \\ &= \frac{1}{8} e^{-\sqrt{2}z} \cdot \{2\mathbb{i} \cdot p^+(z)\} \cdot \alpha^{-2} = \frac{\mathbb{i} e^{-\sqrt{2}z}}{4\alpha^2} \cdot p^+(z), \end{aligned} \tag{5.6}$$

$$\begin{aligned} \det \mathbf{X}^-_{1/k}(x) &= \left(\frac{1}{2\sqrt{2}}\right)^2 \det \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega} z}\right) \cdot \det \mathbf{P}^-(z) \cdot \det \operatorname{diag}\left(\alpha^{-1}, \alpha^{-3}\right) \\ &= \frac{1}{8} e^{-\sqrt{2}z} \cdot \{-2\mathbb{i} \cdot p^-(z)\} \cdot \alpha^{-4} = -\frac{\mathbb{i} e^{-\sqrt{2}z}}{4\alpha^4} \cdot p^-(z). \end{aligned} \tag{5.7}$$

By (5.5), (5.6), (5.7),

$$\det \mathbf{X}_{1/k}(x) = -\frac{\mathbb{i} e^{-\sqrt{2}z}}{4\alpha^2} \cdot p^+(z) \left\{ -\frac{\mathbb{i} e^{-\sqrt{2}z}}{4\alpha^4} \cdot p^-(z) \right\} = -\frac{e^{-2\sqrt{2}z}}{16\alpha^6} \cdot p^+(z) p^-(z).$$

It follows that $\det \mathbf{X}^\pm_{1/k}(x) \neq 0$ and $\det \mathbf{X}_{1/k}(x) \neq 0$ for $x > 0$, since $p^\pm(z) > 0$ for $z > 0$ by Definition 3.5. Thus we showed (b), and the proof is complete. □

6. Block-diagonalization of $\mathbf{Y}_\lambda(x)$: proof of Theorem 3

6.1. The case $\lambda \neq 1/k$

Denote $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For $a, b, c, d \in \mathbb{C}$, we have

$$\mathbf{R} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{6.1}$$

By Definition 4.3,

$$\operatorname{adj} \widehat{\mathbf{X}}^\pm(z, \kappa) = \begin{pmatrix} \frac{e^{\overline{\omega}\kappa z}}{1-\kappa} \pm \frac{e^{-\overline{\omega}\kappa z}}{1+\kappa} & -\left(\frac{e^{-\overline{\omega}\kappa z}}{1-\mathbb{i}\kappa} \pm \frac{e^{\overline{\omega}\kappa z}}{1+\mathbb{i}\kappa}\right) \\ -\left(\frac{e^{-\omega\kappa z}}{1+\mathbb{i}\kappa} \pm \frac{e^{\omega\kappa z}}{1-\mathbb{i}\kappa}\right) & \frac{e^{\omega\kappa z}}{1-\kappa} \pm \frac{e^{-\omega\kappa z}}{1+\kappa} \end{pmatrix} \tag{6.2}$$

for $z \in \mathbb{C}$ and $\kappa \in \mathbb{D}$. Note from Definition 4.2 that $\overline{\kappa} \in \mathbb{D}$ if and only if $\kappa \in \mathbb{D}$.

Lemma 6.1. *For $z \in \mathbb{C}$ and $\kappa \in \mathbb{D}$,*

$$\begin{aligned} \left\{\widehat{\mathbf{X}}^\pm(-z, \kappa) \cdot \operatorname{adj} \widehat{\mathbf{X}}^\pm(z, \kappa)\right\}_{2,1} &= \overline{\left\{\widehat{\mathbf{X}}^\pm(-\overline{z}, \overline{\kappa}) \cdot \operatorname{adj} \widehat{\mathbf{X}}^\pm(\overline{z}, \overline{\kappa})\right\}_{1,2}}, \\ \left\{\widehat{\mathbf{X}}^\pm(-z, \kappa) \cdot \operatorname{adj} \widehat{\mathbf{X}}^\pm(z, \kappa)\right\}_{2,2} &= \overline{\left\{\widehat{\mathbf{X}}^\pm(-\overline{z}, \overline{\kappa}) \cdot \operatorname{adj} \widehat{\mathbf{X}}^\pm(\overline{z}, \overline{\kappa})\right\}_{1,1}}. \end{aligned}$$

Proof. Let $z \in \mathbb{C}$ and $\kappa \in \mathbb{D}$. It can be checked from Definition 4.3 and (6.2) that $\widehat{\mathbf{X}}^{\pm}(z,\kappa)_{2,1} = \overline{\widehat{\mathbf{X}}^{\pm}(\overline{z},\overline{\kappa})_{1,2}}$, $\widehat{\mathbf{X}}^{\pm}(z,\kappa)_{2,2} = \overline{\widehat{\mathbf{X}}^{\pm}(\overline{z},\overline{\kappa})_{1,1}}$, and $\left\{\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{2,1} = \overline{\left\{\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(\overline{z},\overline{\kappa})\right\}_{1,2}}$, $\left\{\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{2,2} = \overline{\left\{\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(\overline{z},\overline{\kappa})\right\}_{1,1}}$, which, by (6.1), are equivalent to $\mathbf{R}\cdot\widehat{\mathbf{X}}^{\pm}(z,\kappa)\cdot\mathbf{R} = \overline{\widehat{\mathbf{X}}^{\pm}(\overline{z},\overline{\kappa})}$, $\mathbf{R}\cdot\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\cdot\mathbf{R} = \overline{\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(\overline{z},\overline{\kappa})}$. So we have

$$\begin{aligned}\mathbf{R}\left\{\widehat{\mathbf{X}}^{\pm}(-z,\kappa)\cdot\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}\mathbf{R} &= \left\{\mathbf{R}\cdot\widehat{\mathbf{X}}^{\pm}(-z,\kappa)\cdot\mathbf{R}\right\}\left\{\mathbf{R}\cdot\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\cdot\mathbf{R}\right\}\\ &= \overline{\widehat{\mathbf{X}}^{\pm}(-\overline{z},\overline{\kappa})}\cdot\overline{\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(\overline{z},\overline{\kappa})} = \overline{\left\{\widehat{\mathbf{X}}^{\pm}(-\overline{z},\overline{\kappa})\cdot\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(\overline{z},\overline{\kappa})\right\}},\end{aligned}$$

since $\mathbf{R}^2 = \mathbf{I}$. Thus the result follows by (6.1). □

Lemma 6.2. *For $z \in \mathbb{R}$ and $\kappa \in \mathbb{D}$,*

$$\widehat{\mathbf{X}}^{\pm}(-z,\kappa)\cdot\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa) = \frac{4\kappa}{1-\kappa^4}\begin{pmatrix}\delta^{\pm}(-\mathrm{i}z,\kappa) & \sqrt{2}\omega\,\mathrm{s}^{\pm}(z\kappa)\\ \sqrt{2}\overline{\omega}\,\mathrm{s}^{\pm}(z\kappa) & \delta^{\pm}(\mathrm{i}z,\kappa)\end{pmatrix},$$

where $\mathrm{s}^{\pm}(\zeta) = \sinh\left(\sqrt{2}\zeta\right) \pm \sin\left(\sqrt{2}\zeta\right)$ *for* $\zeta \in \mathbb{C}$.

See Supplementary B for proof of Lemma 6.2.

Definition 6.1. For $z \in \mathbb{C}$ and $\kappa \in \mathbb{D}$ such that $\det\mathbf{X}^{\pm}(z,\kappa) \neq 0$, denote $\mathbf{Y}^{\pm}(z,\kappa) = \mathbf{X}^{\pm}(-z,\kappa)\cdot\mathbf{X}^{\pm}(z,\kappa)^{-1} - \mathbf{I}$.

By Definitions 3.6, 6.1 and (4.2),

$$\mathbf{Y}^{\pm}_{\lambda}(x) = \mathbf{Y}^{\pm}(z,\kappa), \qquad \lambda \in \mathbb{C}\setminus\{0,1/k\},\ x \in \mathbb{R},\ \det\mathbf{X}^{\pm}_{\lambda}(x) \neq 0, \tag{6.3}$$

where $z = \alpha x$ and $\kappa = \chi(\lambda)$. Note from (2.1) that, for $a,b,c,d,\delta \in \mathbb{C}$, $\delta \neq 0$,

$$\frac{1}{\delta}\operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega}z}\right)\begin{pmatrix}a & b\\ c & d\end{pmatrix}\operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega}z}\right) - \mathbf{I} = \frac{1}{\delta}\begin{pmatrix}e^{2\omega z}a & e^{\sqrt{2}z}b\\ e^{\sqrt{2}z}c & e^{2\overline{\omega}z}d\end{pmatrix} - \mathbf{I} = \frac{1}{\delta}\begin{pmatrix}e^{2\omega z}a-\delta & e^{\sqrt{2}z}b\\ e^{\sqrt{2}z}c & e^{2\overline{\omega}z}d-\delta\end{pmatrix}. \tag{6.4}$$

Lemma 6.3. *For $z \in \mathbb{C}$ and $\kappa \in \mathbb{D}$ such that* $\det\mathbf{X}^{\pm}(z,\kappa) \neq 0$,

$$\mathbf{Y}^{\pm}(z,\kappa) = \frac{1}{\delta^{\pm}(z,\kappa)}\begin{pmatrix}e^{2\omega z}\delta^{\pm}(-\mathrm{i}z,\kappa)-\delta^{\pm}(z,\kappa) & \sqrt{2}\omega e^{\sqrt{2}z}\,\mathrm{s}^{\pm}(z\kappa)\\ \sqrt{2}\overline{\omega}e^{\sqrt{2}z}\,\mathrm{s}^{\pm}(z\kappa) & e^{2\overline{\omega}z}\delta^{\pm}(\mathrm{i}z,\kappa)-\delta^{\pm}(z,\kappa)\end{pmatrix},$$

where $\mathrm{s}^{\pm}(\zeta) = \sinh\left(\sqrt{2}\zeta\right) \pm \sin\left(\sqrt{2}\zeta\right)$ *for* $\zeta \in \mathbb{C}$.

Proof. Let $z \in \mathbb{C}$, $\kappa \in \mathbb{D}$, and suppose that $\det\mathbf{X}^{\pm}(z,\kappa) \neq 0$. By Definition 4.3,

$$\mathbf{X}^{\pm}(z,\kappa)^{-1} = \left\{\frac{1-\kappa^4}{4}\cdot\operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega}z}\right)\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}^{-1} = \frac{4}{1-\kappa^4}\cdot\widehat{\mathbf{X}}^{\pm}(z,\kappa)^{-1}\operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega}z}\right),$$

hence, by Definition 6.1,

$$\mathbf{Y}^{\pm}(z,\kappa) = \left\{\frac{1-\kappa^4}{4}\cdot\operatorname{diag}\left(e^{-\omega(-z)}, e^{-\overline{\omega}(-z)}\right)\widehat{\mathbf{X}}^{\pm}(-z,\kappa)\right\}\left\{\frac{4}{1-\kappa^4}\cdot\widehat{\mathbf{X}}^{\pm}(z,\kappa)^{-1}\operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega}z}\right)\right\} - \mathbf{I}$$

$$= \operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega} z}\right) \widehat{\mathbf{X}}^{\pm}(-z, \kappa) \cdot \widehat{\mathbf{X}}^{\pm}(z, \kappa)^{-1} \operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega} z}\right) - \mathbf{I}. \tag{6.5}$$

By Lemmas 4.4 and 6.2,

$$\begin{aligned}\widehat{\mathbf{X}}^{\pm}(-z, \kappa) \cdot \widehat{\mathbf{X}}^{\pm}(z, \kappa)^{-1} &= \frac{1}{\det \widehat{\mathbf{X}}^{\pm}(z, \kappa)} \cdot \widehat{\mathbf{X}}^{\pm}(-z, \kappa) \cdot \operatorname{adj} \widehat{\mathbf{X}}^{\pm}(z, \kappa) \\ &= \frac{1}{\frac{4\kappa}{1-\kappa^4}\delta^{\pm}(z, \kappa)} \cdot \frac{4\kappa}{1-\kappa^4} \begin{pmatrix} \delta^{\pm}(-\mathbb{i}z, \kappa) & \sqrt{2}\omega\, \mathrm{s}^{\pm}(z\kappa) \\ \sqrt{2}\overline{\omega}\, \mathrm{s}^{\pm}(z\kappa) & \delta^{\pm}(\mathbb{i}z, \kappa) \end{pmatrix},\end{aligned}$$

hence, by (6.5),

$$\mathbf{Y}^{\pm}(z, \kappa) = \frac{1}{\delta^{\pm}(z, \kappa)} \operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega} z}\right) \begin{pmatrix} \delta^{\pm}(-\mathbb{i}z, \kappa) & \sqrt{2}\omega\, \mathrm{s}^{\pm}(z\kappa) \\ \sqrt{2}\overline{\omega}\, \mathrm{s}^{\pm}(z\kappa) & \delta^{\pm}(\mathbb{i}z, \kappa) \end{pmatrix} \operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega} z}\right) - \mathbf{I}.$$

Thus the lemma follows by (6.4). □

6.2. The case $\lambda = 1/k$ and proof of Theorem 3

By Definition 3.4, we have

$$\operatorname{adj} \mathbf{P}^{+}(z) = \begin{pmatrix} p_2(z) & -\overline{p_2(\overline{z})} \\ -p_0(z) & \overline{p_0(\overline{z})} \end{pmatrix}, \quad \operatorname{adj} \mathbf{P}^{-}(z) = \begin{pmatrix} p_3(z) & \overline{p_3(\overline{z})} \\ -p_1(z) & -\overline{p_1(\overline{z})} \end{pmatrix}, \qquad z \in \mathbb{C}. \tag{6.6}$$

Lemma 6.4. *For $z \in \mathbb{C}$, $\mathbf{P}^{\pm}(-z) \cdot \operatorname{adj} \mathbf{P}^{\pm}(z) = \pm 2\mathbb{i} \begin{pmatrix} p^{\pm}(-\mathbb{i}z) & \frac{1}{2\mp1}\omega z^{2\mp1} \\ \frac{1}{2\mp1}\overline{\omega} z^{2\mp1} & p^{\pm}(\mathbb{i}z) \end{pmatrix}$.*

See Supplementary C for proof of Lemma 6.4.

Lemma 6.5. *Let $x \in \mathbb{R}$, $z = \alpha x$, and suppose that* $\det \mathbf{X}^{\pm}_{1/k}(x) \neq 0$. *Then*

$$\mathbf{Y}^{\pm}_{1/k}(x) = \frac{1}{p^{\pm}(z)} \begin{pmatrix} e^{2\omega z} p^{\pm}(-\mathbb{i}z) - p^{\pm}(z) & \frac{1}{2\mp1}\omega e^{\sqrt{2}z} z^{2\mp1} \\ \frac{1}{2\mp1}\overline{\omega} e^{\sqrt{2}z} z^{2\mp1} & e^{2\overline{\omega} z} p^{\pm}(\mathbb{i}z) - p^{\pm}(z) \end{pmatrix}.$$

Proof. By Definition 3.4,

$$\begin{aligned}\mathbf{X}^{\pm}_{1/k}(x)^{-1} &= \left\{\frac{1}{2\sqrt{2}} \operatorname{diag}\left(e^{-\omega z}, e^{-\overline{\omega} z}\right) \cdot \mathbf{P}^{\pm}(z) \cdot \operatorname{diag}\left(\alpha^{\frac{-1\pm1}{2}}, \alpha^{\frac{-5\pm1}{2}}\right)\right\}^{-1} \\ &= 2\sqrt{2} \cdot \operatorname{diag}\left(\alpha^{\frac{-1\pm1}{2}}, \alpha^{\frac{-5\pm1}{2}}\right)^{-1} \cdot \mathbf{P}^{\pm}(z)^{-1} \operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega} z}\right).\end{aligned}$$

So by Definitions 3.4 and 3.6,

$$\begin{aligned}\mathbf{Y}^{\pm}_{1/k}(x) &= \left\{\frac{1}{2\sqrt{2}} \operatorname{diag}\left(e^{-\omega(-z)}, e^{-\overline{\omega}(-z)}\right) \mathbf{P}^{\pm}(-z) \cdot \operatorname{diag}\left(\alpha^{\frac{-1\pm1}{2}}, \alpha^{\frac{-5\pm1}{2}}\right)\right\} \\ &\quad \cdot \left\{2\sqrt{2} \cdot \operatorname{diag}\left(\alpha^{\frac{-1\pm1}{2}}, \alpha^{\frac{-5\pm1}{2}}\right)^{-1} \cdot \mathbf{P}^{\pm}(z)^{-1} \operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega} z}\right)\right\} - \mathbf{I} \\ &= \operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega} z}\right) \mathbf{P}^{\pm}(-z) \mathbf{P}^{\pm}(z)^{-1} \operatorname{diag}\left(e^{\omega z}, e^{\overline{\omega} z}\right) - \mathbf{I}.\end{aligned} \tag{6.7}$$

By Lemmas 5.1 and 6.4,

$$\mathbf{P}^{\pm}(-z)\cdot\mathbf{P}^{\pm}(z)^{-1}=\frac{1}{\det\mathbf{P}^{\pm}(z)}\cdot\mathbf{P}^{\pm}(-z)\cdot\operatorname{adj}\mathbf{P}^{\pm}(z)=\frac{1}{\pm2\mathbb{i}\cdot p^{\pm}(z)}\cdot\left\{\pm2\mathbb{i}\begin{pmatrix}p^{\pm}(-\mathbb{i}z) & \frac{1}{2\mp1}\omega z^{2\mp1}\\ \frac{1}{2\mp1}\overline{\omega}z^{2\mp1} & p^{\pm}(\mathbb{i}z)\end{pmatrix}\right\},$$

hence, by (6.7),

$$\mathbf{Y}^{\pm}_{1/k}(x)=\frac{1}{p^{\pm}(z)}\operatorname{diag}\left(e^{\omega z},e^{\overline{\omega}z}\right)\begin{pmatrix}p^{\pm}(-\mathbb{i}z) & \frac{1}{2\mp1}\omega z^{2\mp1}\\ \frac{1}{2\mp1}\overline{\omega}z^{2\mp1} & p^{\pm}(\mathbb{i}z)\end{pmatrix}\operatorname{diag}\left(e^{\omega z},e^{\overline{\omega}z}\right)-\mathbf{I}.$$

Thus the lemma follows by (6.4). □

Let $0\neq\lambda\in\mathbb{C}$ and $x\in\mathbb{R}$. Suppose that $\det\mathbf{X}_\lambda(x)\neq0$, which is equivalent to $\det\mathbf{X}^+_\lambda(x)\neq0$ *and* $\det\mathbf{X}^-_\lambda(x)\neq0$ by (4.4) and (5.2). Let $\mathbf{A}=\begin{cases}\mathbf{V}^T, & \text{if } \lambda\neq1/k,\\ \widehat{\mathbf{V}}, & \text{if } \lambda=1/k.\end{cases}$ Then by Definition 2.5 and (2.3),

$$\begin{aligned}\mathbf{V}\mathbf{Y}_\lambda(x)\mathbf{V}^T&=\mathbf{V}\left\{\mathbf{X}_\lambda(-x)\cdot\mathbf{X}_\lambda(x)^{-1}-\mathbf{I}\right\}\mathbf{V}^T=\mathbf{V}\mathbf{X}_\lambda(-x)\mathbf{A}\cdot\mathbf{A}^{-1}\mathbf{X}_\lambda(x)^{-1}\mathbf{V}^T-\mathbf{I}\\&=\mathbf{V}\mathbf{X}_\lambda(-x)\mathbf{A}\cdot\{\mathbf{V}\mathbf{X}_\lambda(x)\mathbf{A}\}^{-1}-\mathbf{I},\end{aligned}$$

hence, by (2.3), (4.4) and (5.2),

$$\begin{aligned}\mathbf{V}\mathbf{Y}_\lambda(x)\mathbf{V}^T&=\begin{pmatrix}\mathbf{X}^+_\lambda(-x) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}^-_\lambda(-x)\end{pmatrix}\begin{pmatrix}\mathbf{X}^+_\lambda(x) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}^-_\lambda(x)\end{pmatrix}^{-1}-\mathbf{I}\\&=\begin{pmatrix}\mathbf{X}^+_\lambda(-x) & \mathbf{O}\\ \mathbf{O} & \mathbf{X}^-_\lambda(-x)\end{pmatrix}\begin{pmatrix}\mathbf{X}^+_\lambda(x)^{-1} & \mathbf{O}\\ \mathbf{O} & \mathbf{X}^-_\lambda(x)^{-1}\end{pmatrix}-\begin{pmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{I}\end{pmatrix}\\&=\begin{pmatrix}\mathbf{X}^+_\lambda(-x)\cdot\mathbf{X}^+_\lambda(x)^{-1}-\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \mathbf{X}^-_\lambda(-x)\cdot\mathbf{X}^-_\lambda(x)^{-1}-\mathbf{I}\end{pmatrix}.\end{aligned}$$

Thus, by (2.3) and Definition 3.6, we have

$$\mathbf{Y}_\lambda(x)=\mathbf{V}^T\begin{pmatrix}\mathbf{Y}^+_\lambda(x) & \mathbf{O}\\ \mathbf{O} & \mathbf{Y}^-_\lambda(x)\end{pmatrix}\mathbf{V},\quad 0\neq\lambda\in\mathbb{C},\ x\in\mathbb{R},\ \det\mathbf{X}^\pm_\lambda(x)\neq0. \tag{6.8}$$

Proof of Theorem 3. Let $\mathbf{M}\in\operatorname{wp}(4,8,\mathbb{C})$ and $0\neq\lambda\in\mathbb{C}\setminus\operatorname{Spec}\mathcal{K}_{\mathbf{Q}}$. By Proposition 2.2 (b), $\lambda\in\operatorname{Spec}\mathcal{K}_{\mathbf{M}}$ if and only if

$$\det\left[\mathbf{V}\left\{\mathcal{G}(\mathbf{M})\mathbf{Y}_\lambda(l)-\mathbf{I}\right\}\mathbf{V}^T\right]=0, \tag{6.9}$$

since $\mathbf{V}$ is invertible by (2.3). Thus (a) follows, since (6.9) is equivalent to

$$0=\det\left\{\mathbf{V}\mathcal{G}(\mathbf{M})\cdot\mathbf{V}^T\begin{pmatrix}\mathbf{Y}^+_\lambda(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{Y}^-_\lambda(l)\end{pmatrix}\mathbf{V}\cdot\mathbf{V}^T-\mathbf{V}\cdot\mathbf{V}^T\right\}=\det\left\{\mathcal{F}(\mathbf{M})\begin{pmatrix}\mathbf{Y}^+_\lambda(l) & \mathbf{O}\\ \mathbf{O} & \mathbf{Y}^-_\lambda(l)\end{pmatrix}-\mathbf{I}\right\}$$

by (6.8) and Definition 3.3.

Let $0\neq\lambda\in\mathbb{C}$, $x\in\mathbb{R}$, and $z=\alpha x$. Suppose that $\det\mathbf{X}^\pm_\lambda(x)\neq0$. (b) follows from (6.3) and Lemma 6.3 when $\lambda\neq1/k$, and from Lemma 6.5 when $\lambda=1/k$. Thus the proof is complete. □

7. Conclusions

The boundary conditions usually considered in practice are only a few in number, including clamped, free, or hinged conditions at each end of the beam. An important aspect of our results is that we have obtained explicit and manageable characteristic equations for the *whole* 16*-dimensional class* of integral operators $\mathcal{K}_{\mathbf{M}}$ arising from *arbitrary* well-posed boundary value problem of the Euler–Bernoulli beam equation.

In our characteristic equations in Theorems 1, 2, and 3, the explicit matrices $\mathbf{X}_\lambda^\pm$ and $\mathbf{Y}_\lambda^\pm$ are *not* affected by specific boundary conditions. The effect of the boundary condition $\mathbf{M}$ is encoded *separately* in the *fundamental boundary matrix* $\mathcal{F}(\mathbf{M})$. The set of equivalent well-posed boundary matrices wp($\mathbb{C}$), and hence the set of integral operators $\mathcal{K}_{\mathbf{M}}$ in (1.1), is in one-to-one correspondence with the 16-dimensional algebra $\mathrm{gl}(4,\mathbb{C})$ via the map Φ. Φ and its inverse Φ^{-1} are explicitly computable using the maps $\mathcal{F}$ and ϕ in Definition 3.3. See Figure 2 in Section 3 for a commutative diagram showing the details.

The 2×2 matrices $\mathbf{X}_\lambda^\pm$ and $\mathbf{Y}_\lambda^\pm$ themselves are pre-calculated in terms of the explicit functions $\delta^\pm(z,\kappa)$ and $p^\pm(z)$. Thus our characteristic equations have simple and manageable expressions with the functions $\delta^\pm(z,\kappa)$ and $p^\pm(z)$, which are amenable to concrete analysis similar to that in [14].

By inverting the 2×2 matrices $\mathbf{Y}_\lambda^\pm(l)$ in Theorem 3, we would have alternate forms of the characteristic equations in Theorem 1 (a) and Theorem 2 (a) with matrix entries also explicitly expressed by $\delta^\pm(z,\kappa)$ and $p^\pm(z)$. However, these forms are suppressed in this paper due to the nontrivial problem of identifying the zeros of $\det \mathbf{Y}_\lambda^\pm(l)$ or $\det\left\{\mathbf{X}_\lambda^\pm(-l) - \mathbf{X}_\lambda^\pm(l)\right\}$, which will be dealt in future works.

Although our results are for boundary matrices with complex entries in general, boundary conditions of practical importance are those represented by boundary matrices with *real* entries. See [2] for the characterization of these real boundary conditions $\mathbf{M}$ in terms of $\mathcal{G}(\mathbf{M})$ by using the $\mathbb{R}$-algebra $\overline{\pi}(4) \subset \mathrm{gl}(4,\mathbb{C})$.

An immediate application of our results would be spectral analysis for a few typical boundary conditions encountered frequently in practice. Specifically, concrete spectral analysis for the following combinations of clamped, free, and hinged boundary conditions at each end of the beam are now possible, which will be performed in future works.

- clamped-clamped or bi-clamped.
- free-free or bi-free.
- hinged-hinged or bi-hinged.
- clamped-free or cantilevered.
- hinged-free.
- clamped-hinged.

In fact, it turns out that the fundamental boundary matrices $\mathcal{F}(\mathbf{M})$ corresponding to the first three symmetric boundary conditions $\mathbf{M}$ above also have the following *block-diagonal* form with 2×2 blocks.

$$\mathcal{F}(\mathbf{M}) = \begin{pmatrix} \mathcal{F}(\mathbf{M})^+ & \mathbf{O} \\ \mathbf{O} & \mathcal{F}(\mathbf{M})^- \end{pmatrix}.$$

In these cases, our characteristic equations in Theorems 1, 2, and 3 are *completely separable* into 2×2 blocks, resulting in further simplified forms which involve determinants of 2×2 matrices only.

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Conflict of interest

The author declares no conflict of interest in this paper.

References

1. I. Stakgold, M. Holst, *Green's functions and boundary value problems*, 3 Eds., Hoboken, NJ: John Wiley & Sons, Inc., 2011.
2. S. W. Choi, Spectral analysis for the class of integral operators arising from well-posed boundary value problems of finite beam deflection on elastic foundation: characteristic equation, *Bull. Korean Math. Soc.*, **58** (2021), 71–111.
3. S. Timoshenko, *History of strength of materials: with a brief account of the history of theory of elasticity and theory of structures*, New York-Toronto-London: McGraw-Hill Book Company, Inc., 1953.
4. F. W. Beaufait, P. W. Hoadley, Analysis of elastic beams on nonlinear foundations, *Comput. Struct.*, **12** (1980), 669–676.
5. S. W. Choi, Existence and uniqueness of finite beam deflection on nonlinear non-uniform elastic foundation with arbitrary well-posed boundary condition, *Bound. Value Probl.*, **2020** (2020), 113.
6. S. W. Choi, T. S. Jang, Existence and uniqueness of nonlinear deflections of an infinite beam resting on a non-uniform nonlinear elastic foundation, *Bound. Value Probl.*, **2012** (2012), 5.
7. M. Galewski, On the nonlinear elastic simply supported beam equation, *An. Ştiinţ Univ. Ovidius Constanţa Ser. Mat.*, **19** (2011), 109–120.
8. M. Hetényi, *Beams on elastic foundation: theory with applications in the fields of civil and mechanical engineering*, Ann Arbor: University of Michigan Press, 1971.
9. Y. Kuo, S. Lee, Deflection of nonuniform beams resting on a nonlinear elastic foundation, *Comput. Struct.*, **51** (1994), 513–519.
10. X. Ma, J. W. Butterworth, G. C. Clifton, Static analysis of an infinite beam resting on a tensionless Pasternak foundation, *Eur. J. Mech. A Solids*, **28** (2009), 697–703.
11. C. Miranda, K. Nair, Finite beams on elastic foundations, *J. Struct. Div.*, **92** (1966), 131–142.
12. B. Y. Ting, Finite beams on elastic foundation with restraints, *J. Struct. Div.*, **108** (1982), 611–621.
13. M. T. Chu, G. H. Golub, *Inverse eigenvalue problems: theory, algorithms, and applications*, Oxford University Press, 2005.
14. S. W. Choi, Spectral analysis of the integral operator arising from the beam deflection problem on elastic foundation II: eigenvalues, *Bound. Value Probl.*, **2015** (2015), 6.
15. S. W. Choi, On positiveness and contractiveness of the integral operator arising from the beam deflection problem on elastic foundation, *Bull. Korean Math. Soc.*, **52** (2015), 1225–1240.

Supplementary

A. Proof of Lemma 4.4

By Definition 4.3 and (2.1),

$$\begin{aligned}
&\det \widehat{\mathbf{X}}^{\pm}(z,\kappa) = \widehat{\mathbf{X}}^{\pm}(z,\kappa)_{1,1}\cdot \widehat{\mathbf{X}}^{\pm}(z,\kappa)_{2,2} - \widehat{\mathbf{X}}^{\pm}(z,\kappa)_{2,1}\cdot \widehat{\mathbf{X}}^{\pm}(z,\kappa)_{1,2}\\
&= \left(\frac{e^{\omega\kappa z}}{1-\kappa} \pm \frac{e^{-\omega\kappa z}}{1+\kappa}\right)\left(\frac{e^{\overline{\omega}\kappa z}}{1-\kappa} \pm \frac{e^{-\overline{\omega}\kappa z}}{1+\kappa}\right) - \left(\frac{e^{-\omega\kappa z}}{1+\mathbb{i}\kappa} \pm \frac{e^{\omega\kappa z}}{1-\mathbb{i}\kappa}\right)\left(\frac{e^{-\overline{\omega}\kappa z}}{1-\mathbb{i}\kappa} \pm \frac{e^{\overline{\omega}\kappa z}}{1+\mathbb{i}\kappa}\right)\\
&= \frac{e^{\sqrt{2}\kappa z}}{(1-\kappa)^2} + \frac{e^{-\sqrt{2}\kappa z}}{(1+\kappa)^2} \pm \frac{e^{\mathbb{i}\sqrt{2}\kappa z}}{1-\kappa^2} \pm \frac{e^{-\mathbb{i}\sqrt{2}\kappa z}}{1-\kappa^2} - \frac{e^{\sqrt{2}\kappa z}}{1+\kappa^2} - \frac{e^{-\sqrt{2}\kappa z}}{1+\kappa^2} \mp \frac{e^{\mathbb{i}\sqrt{2}\kappa z}}{(1-\mathbb{i}\kappa)^2} \mp \frac{e^{-\mathbb{i}\sqrt{2}\kappa z}}{(1+\mathbb{i}\kappa)^2}\\
&= \left\{\frac{1}{(1-\kappa)^2} - \frac{1}{1+\kappa^2}\right\} e^{\sqrt{2}\kappa z} + \left\{\frac{1}{(1+\kappa)^2} - \frac{1}{1+\kappa^2}\right\} e^{-\sqrt{2}\kappa z}\\
&\quad \pm \left\{\frac{1}{1-\kappa^2} - \frac{1}{(1-\mathbb{i}\kappa)^2}\right\} e^{\mathbb{i}\sqrt{2}\kappa z} \pm \left\{\frac{1}{1-\kappa^2} - \frac{1}{(1+\mathbb{i}\kappa)^2}\right\} e^{-\mathbb{i}\sqrt{2}\kappa z}\\
&= \frac{2\kappa}{(1-\kappa)^2(1+\kappa^2)} e^{\sqrt{2}\kappa z} - \frac{2\kappa}{(1+\kappa)^2(1+\kappa^2)} e^{-\sqrt{2}\kappa z} \mp \frac{2\mathbb{i}\kappa}{(1-\kappa^2)(1-\mathbb{i}\kappa)^2} e^{\mathbb{i}\sqrt{2}\kappa z} \pm \frac{2\mathbb{i}\kappa}{(1-\kappa^2)(1+\mathbb{i}\kappa)^2} e^{-\mathbb{i}\sqrt{2}\kappa z}\\
&= \frac{2\kappa}{(1-\kappa^2)^2(1+\kappa^2)} \left\{(1+\kappa)^2 e^{\sqrt{2}\kappa z} - (1-\kappa)^2 e^{-\sqrt{2}\kappa z}\right\}\\
&\quad \mp \frac{2\mathbb{i}\kappa}{(1-\kappa^2)(1+\kappa^2)^2} \left\{(1+\mathbb{i}\kappa)^2 e^{\mathbb{i}\sqrt{2}\kappa z} - (1-\mathbb{i}\kappa)^2 e^{-\mathbb{i}\sqrt{2}\kappa z}\right\}\\
&= \frac{2\kappa}{(1-\kappa^4)(1-\kappa^2)} \left\{2\left(1+\kappa^2\right)\sinh\left(\sqrt{2}\kappa z\right) + 4\kappa\cosh\left(\sqrt{2}\kappa z\right)\right\}\\
&\quad \mp \frac{2\mathbb{i}\kappa}{(1-\kappa^4)(1+\kappa^2)} \left\{2\mathbb{i}\left(1-\kappa^2\right)\sin\left(\sqrt{2}\kappa z\right) + 4\mathbb{i}\kappa\cos\left(\sqrt{2}\kappa z\right)\right\}\\
&= \frac{4\kappa}{1-\kappa^4}\left\{\frac{1+\kappa^2}{1-\kappa^2}\sinh\left(\sqrt{2}\kappa z\right) + \frac{2\kappa}{1-\kappa^2}\cosh\left(\sqrt{2}\kappa z\right)\right\}\\
&\quad \pm \frac{4\kappa}{1-\kappa^4}\left\{\frac{1-\kappa^2}{1+\kappa^2}\sin\left(\sqrt{2}\kappa z\right) + \frac{2\kappa}{1+\kappa^2}\cos\left(\sqrt{2}\kappa z\right)\right\}.
\end{aligned}$$

Thus, by Definition 3.2,

$$\begin{aligned}
\det \widehat{\mathbf{X}}^{\pm}(z,\kappa) &= \frac{4\kappa}{1-\kappa^4}\left\{\sinh\left(\sqrt{2}\kappa z\right)\cosh\beta(\kappa) + \cosh\left(\sqrt{2}\kappa z\right)\sinh\beta(\kappa)\right\}\\
&\quad \pm \frac{4\kappa}{1-\kappa^4}\left\{\sin\left(\sqrt{2}\kappa z\right)\cos\gamma(\kappa) + \cos\left(\sqrt{2}\kappa z\right)\sin\gamma(\kappa)\right\}\\
&= \frac{4\kappa}{1-\kappa^4}\left\{\sinh\left(\sqrt{2}\kappa z + \beta(\kappa)\right) \pm \sin\left(\sqrt{2}\kappa z + \gamma(\kappa)\right)\right\} = \frac{4\kappa}{1-\kappa^4}\cdot\delta^{\pm}(z,\kappa).
\end{aligned}$$

B. Proof of Lemma 6.2

Let $z \in \mathbb{C}$ and $\kappa \in \mathbb{D}$. By Definition 4.3, (2.1) and (6.2),

$$\left\{\widehat{\mathbf{X}}^{\pm}(-z,\kappa)\cdot \operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{1,1} = \widehat{\mathbf{X}}^{\pm}(-z,\kappa)_{1,1}\cdot\left\{\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{1,1} + \widehat{\mathbf{X}}^{\pm}(-z,\kappa)_{1,2}\cdot\left\{\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{2,1}$$

$$
\begin{aligned}
&= \left(\frac{e^{\omega\kappa(-z)}}{1-\kappa} \pm \frac{e^{-\omega\kappa(-z)}}{1+\kappa}\right)\left(\frac{e^{\overline{\omega}\kappa z}}{1-\kappa} \pm \frac{e^{-\overline{\omega}\kappa z}}{1+\kappa}\right) - \left(\frac{e^{-\overline{\omega}\kappa(-z)}}{1-\mathbb{i}\kappa} \pm \frac{e^{\overline{\omega}\kappa(-z)}}{1+\mathbb{i}\kappa}\right)\left(\frac{e^{-\omega\kappa z}}{1+\mathbb{i}\kappa} \pm \frac{e^{\omega\kappa z}}{1-\mathbb{i}\kappa}\right)\\
&= \frac{e^{\mathbb{i}\sqrt{2}\kappa z}}{(1+\kappa)^2} + \frac{e^{-\mathbb{i}\sqrt{2}\kappa z}}{(1-\kappa)^2} \pm \frac{e^{\sqrt{2}\kappa z}}{1-\kappa^2} \pm \frac{e^{-\sqrt{2}\kappa z}}{1-\kappa^2} - \frac{e^{\mathbb{i}\sqrt{2}\kappa z}}{1+\kappa^2} - \frac{e^{-\mathbb{i}\sqrt{2}\kappa z}}{1+\kappa^2} \mp \frac{e^{\sqrt{2}\kappa z}}{(1-\mathbb{i}\kappa)^2} \mp \frac{e^{-\sqrt{2}\kappa z}}{(1+\mathbb{i}\kappa)^2}\\
&= \left\{\frac{1}{(1+\kappa)^2} - \frac{1}{1+\kappa^2}\right\} e^{\mathbb{i}\sqrt{2}\kappa z} + \left\{\frac{1}{(1-\kappa)^2} - \frac{1}{1+\kappa^2}\right\} e^{-\mathbb{i}\sqrt{2}\kappa z}\\
&\quad \mp \left\{\frac{1}{(1-\mathbb{i}\kappa)^2} - \frac{1}{1-\kappa^2}\right\} e^{\sqrt{2}\kappa z} \mp \left\{\frac{1}{(1+\mathbb{i}\kappa)^2} - \frac{1}{1-\kappa^2}\right\} e^{-\sqrt{2}\kappa z}\\
&= -\frac{2\kappa}{(1+\kappa)^2(1+\kappa^2)} e^{\mathbb{i}\sqrt{2}\kappa z} + \frac{2\kappa}{(1-\kappa)^2(1+\kappa^2)} e^{-\mathbb{i}\sqrt{2}\kappa z}\\
&\quad \mp \frac{2\mathbb{i}\kappa}{(1-\mathbb{i}\kappa)^2(1-\kappa^2)} e^{\sqrt{2}\kappa z} \pm \frac{2\mathbb{i}\kappa}{(1+\mathbb{i}\kappa)^2(1-\kappa^2)} e^{-\sqrt{2}\kappa z}\\
&= -\frac{2\kappa}{(1-\kappa^2)^2(1+\kappa^2)} \left\{(1-\kappa)^2 e^{\mathbb{i}\sqrt{2}\kappa z} - (1+\kappa)^2 e^{-\mathbb{i}\sqrt{2}\kappa z}\right\}\\
&\quad \mp \frac{2\mathbb{i}\kappa}{(1+\kappa^2)^2(1-\kappa^2)} \left\{(1+\mathbb{i}\kappa)^2 e^{\sqrt{2}\kappa z} - (1-\mathbb{i}\kappa)^2 e^{-\sqrt{2}\kappa z}\right\}\\
&= -\frac{2\kappa}{(1-\kappa^4)(1-\kappa^2)} \left\{2\mathbb{i}\left(1+\kappa^2\right)\sin\left(\sqrt{2}\kappa z\right) - 4\kappa\cos\left(\sqrt{2}\kappa z\right)\right\}\\
&\quad \mp \frac{2\mathbb{i}\kappa}{(1-\kappa^4)(1+\kappa^2)} \left\{2\left(1-\kappa^2\right)\sinh\left(\sqrt{2}\kappa z\right) + 4\mathbb{i}\kappa\cosh\left(\sqrt{2}\kappa z\right)\right\}\\
&= -\frac{4\kappa}{1-\kappa^4} \left\{\frac{1+\kappa^2}{1-\kappa^2}\sinh\left(\mathbb{i}\sqrt{2}\kappa z\right) - \frac{2\kappa}{1-\kappa^2}\cosh\left(\mathbb{i}\sqrt{2}\kappa z\right)\right\}\\
&\quad \mp \frac{4\kappa}{1-\kappa^4} \left\{\frac{1-\kappa^2}{1+\kappa^2}\sin\left(\mathbb{i}\sqrt{2}\kappa z\right) - \frac{2\kappa}{1+\kappa^2}\cos\left(\mathbb{i}\sqrt{2}\kappa z\right)\right\},
\end{aligned}
$$

hence, by Definition 3.2,

$$
\begin{aligned}
\left\{\widehat{\mathbf{X}}^{\pm}(-z,\kappa)\cdot \operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{1,1} &= -\frac{4\kappa}{1-\kappa^4}\left\{-\sinh\left(-\mathbb{i}\sqrt{2}\kappa z\right)\cosh\beta(\kappa) - \cosh\left(-\mathbb{i}\sqrt{2}\kappa z\right)\sinh\beta(\kappa)\right\}\\
&\quad \mp \frac{4\kappa}{1-\kappa^4}\left\{-\sin\left(-\mathbb{i}\sqrt{2}\kappa z\right)\cos\gamma(\kappa) - \cos\left(-\mathbb{i}\sqrt{2}\kappa z\right)\sin\gamma(\kappa)\right\}\\
&= \frac{4\kappa}{1-\kappa^4}\left\{\sinh\left(-\mathbb{i}\sqrt{2}\kappa z + \beta(\kappa)\right) \pm \sin\left(-\mathbb{i}\sqrt{2}\kappa z + \gamma(\kappa)\right)\right\}\\
&= \frac{4\kappa}{1-\kappa^4}\cdot\delta^{\pm}(-\mathbb{i}z,\kappa). \qquad \text{(B.1)}
\end{aligned}
$$

By Definition 4.3, (2.1) and (6.2),

$$
\begin{aligned}
&\left\{\widehat{\mathbf{X}}^{\pm}(-z,\kappa)\cdot \operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{1,2} = \widehat{\mathbf{X}}^{\pm}(-z,\kappa)_{1,1}\cdot\left\{\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{1,2} + \widehat{\mathbf{X}}^{\pm}(-z,\kappa)_{1,2}\cdot\left\{\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{2,2}\\
&= -\left(\frac{e^{\omega\kappa(-z)}}{1-\kappa} \pm \frac{e^{-\omega\kappa(-z)}}{1+\kappa}\right)\left(\frac{e^{-\overline{\omega}\kappa z}}{1-\mathbb{i}\kappa} \pm \frac{e^{\overline{\omega}\kappa z}}{1+\mathbb{i}\kappa}\right) + \left(\frac{e^{-\overline{\omega}\kappa(-z)}}{1-\mathbb{i}\kappa} \pm \frac{e^{\overline{\omega}\kappa(-z)}}{1+\mathbb{i}\kappa}\right)\left(\frac{e^{\omega\kappa z}}{1-\kappa} \pm \frac{e^{-\omega\kappa z}}{1+\kappa}\right)\\
&= -\frac{e^{\sqrt{2}\kappa z}}{(1+\kappa)(1+\mathbb{i}\kappa)} - \frac{e^{-\sqrt{2}\kappa z}}{(1-\kappa)(1-\mathbb{i}\kappa)} \mp \frac{e^{\mathbb{i}\sqrt{2}\kappa z}}{(1+\kappa)(1-\mathbb{i}\kappa)} \mp \frac{e^{-\mathbb{i}\sqrt{2}\kappa z}}{(1-\kappa)(1+\mathbb{i}\kappa)}
\end{aligned}
$$

$$+\frac{e^{\sqrt{2}\kappa z}}{(1-\kappa)(1-\mathbb{i}\kappa)}+\frac{e^{-\sqrt{2}\kappa z}}{(1+\kappa)(1+\mathbb{i}\kappa)}\pm\frac{e^{\mathbb{i}\sqrt{2}\kappa z}}{(1-\kappa)(1+\mathbb{i}\kappa)}\pm\frac{e^{-\mathbb{i}\sqrt{2}\kappa z}}{(1+\kappa)(1-\mathbb{i}\kappa)}$$
$$=\left\{\frac{1}{(1-\kappa)(1-\mathbb{i}\kappa)}-\frac{1}{(1+\kappa)(1+\mathbb{i}\kappa)}\right\}\left(e^{\sqrt{2}\kappa z}-e^{-\sqrt{2}\kappa z}\right)$$
$$\pm\left\{\frac{1}{(1-\kappa)(1+\mathbb{i}\kappa)}-\frac{1}{(1+\kappa)(1-\mathbb{i}\kappa)}\right\}\left(e^{\mathbb{i}\sqrt{2}\kappa z}-e^{-\mathbb{i}\sqrt{2}\kappa z}\right)$$
$$=\frac{(1+\kappa)(1+\mathbb{i}\kappa)-(1-\kappa)(1-\mathbb{i}\kappa)}{1-\kappa^4}\cdot 2\sinh\left(\sqrt{2}\kappa z\right)$$
$$\pm\frac{(1+\kappa)(1-\mathbb{i}\kappa)-(1-\kappa)(1+\mathbb{i}\kappa)}{1-\kappa^4}\cdot 2\mathbb{i}\sin\left(\sqrt{2}\kappa z\right)$$
$$=\frac{2(1+\mathbb{i})\kappa}{1-\kappa^4}\cdot 2\sinh\left(\sqrt{2}\kappa z\right)\pm\frac{2(1-\mathbb{i})\kappa}{1-\kappa^4}\cdot 2\mathbb{i}\sinh\left(\sqrt{2}\kappa z\right)$$
$$=\frac{\sqrt{2}\omega\cdot 4\kappa}{1-\kappa^4}\left\{\sinh\left(\sqrt{2}\kappa z\right)\pm\sin\left(\sqrt{2}\kappa z\right)\right\}=\frac{4\kappa}{1-\kappa^4}\cdot\sqrt{2}\omega\,\mathrm{s}^{\pm}(z\kappa). \tag{B.2}$$

By Lemma 6.1, (B.1), (B.2) and Definition 3.2,

$$\left\{\widehat{\mathbf{X}}^{\pm}(-z,\kappa)\cdot\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{2,1}=\overline{\left\{\frac{4\overline{\kappa}}{1-\overline{\kappa}^4}\cdot\sqrt{2}\omega\,\mathrm{s}^{\pm}(\overline{z\kappa})\right\}}=\frac{4\kappa}{1-\kappa^4}\cdot\sqrt{2}\overline{\omega}\,\mathrm{s}^{\pm}(z\kappa), \tag{B.3}$$
$$\left\{\widehat{\mathbf{X}}^{\pm}(-z,\kappa)\cdot\operatorname{adj}\widehat{\mathbf{X}}^{\pm}(z,\kappa)\right\}_{2,2}=\overline{\left\{\frac{4\overline{\kappa}}{1-\overline{\kappa}^4}\cdot\delta^{\pm}(-\mathbb{i}\overline{z},\overline{\kappa})\right\}}=\frac{4\kappa}{1-\kappa^4}\cdot\delta^{\pm}(\mathbb{i}z,\kappa). \tag{B.4}$$

Thus the lemma follows from (B.1), (B.2), (B.3), (B.4).

C. Proof of Lemma 6.4

Let $z\in\mathbb{C}$. By Definition 3.4 and (6.6), we have

$$\mathbf{P}^{+}(-z)\cdot\operatorname{adj}\mathbf{P}^{+}(z)=\begin{pmatrix}\overline{p_0(\overline{-z})} & \overline{p_2(\overline{-z})}\\ p_0(-z) & p_2(-z)\end{pmatrix}\begin{pmatrix}p_2(z) & -\overline{p_2(\overline{z})}\\ -p_0(z) & \overline{p_0(\overline{z})}\end{pmatrix}$$
$$=\begin{pmatrix}\overline{p_0(-\overline{z})}p_2(z)-p_0(z)\overline{p_2(-\overline{z})} & -\overline{p_0(-\overline{z})p_2(\overline{z})}+\overline{p_0(\overline{z})p_2(-\overline{z})}\\ p_0(-z)p_2(z)-p_0(z)p_2(-z) & -p_0(-z)\overline{p_2(\overline{z})}+\overline{p_0(\overline{z})}p_2(-z)\end{pmatrix}, \tag{C.1}$$
$$\mathbf{P}^{-}(-z)\cdot\operatorname{adj}\mathbf{P}^{-}(z)=\begin{pmatrix}-\overline{p_1(\overline{-z})} & -\overline{p_3(\overline{-z})}\\ p_1(-z) & p_3(-z)\end{pmatrix}\begin{pmatrix}p_3(z) & \overline{p_3(\overline{z})}\\ -p_1(z) & -\overline{p_1(\overline{z})}\end{pmatrix}$$
$$=\begin{pmatrix}-\overline{p_1(-\overline{z})}p_3(z)+p_1(z)\overline{p_3(-\overline{z})} & -\overline{p_1(-\overline{z})p_3(\overline{z})}+\overline{p_1(\overline{z})p_3(-\overline{z})}\\ p_1(-z)p_3(z)-p_1(z)p_3(-z) & p_1(-z)\overline{p_3(\overline{z})}-\overline{p_1(\overline{z})}p_3(-z)\end{pmatrix}. \tag{C.2}$$

So, by (2.1), (5.3) and Definition 3.5,

$$\{\mathbf{P}^{+}(-z)\cdot\operatorname{adj}\mathbf{P}^{+}(z)\}_{1,1}=\overline{p_0(-\overline{z})}p_2(z)-p_0(z)\overline{p_2(-\overline{z})}=1\cdot\left(\mathbb{i}+\omega z+\frac{1}{2}z^2\right)-1\cdot\overline{\left(\mathbb{i}-\omega\overline{z}+\frac{1}{2}\overline{z}^2\right)}$$
$$=2\mathbb{i}+\sqrt{2}z=2\mathbb{i}\left\{1+\frac{(-\mathbb{i}z)}{\sqrt{2}}\right\}=2\mathbb{i}\cdot p^{+}(-\mathbb{i}z), \tag{C.3}$$

$$\begin{aligned}
\{\mathbf{P}^{+}(-z)\cdot\operatorname{adj}\mathbf{P}^{+}(z)\}_{2,1} &= p_0(-z)p_2(z)-p_0(z)p_2(-z)=1\cdot\left(\mathbb{i}+\omega z+\frac{1}{2}z^2\right)-1\cdot\left(\mathbb{i}-\omega z+\frac{1}{2}z^2\right)\\
&=2\omega z, \qquad (C.4)\\
\{\mathbf{P}^{-}(-z)\cdot\operatorname{adj}\mathbf{P}^{-}(z)\}_{1,1} &= -\overline{p_1(-\overline{z})}p_3(z)+p_1(z)\overline{p_3(-\overline{z})}\\
&=-\overline{(\omega-\overline{z})}\left(-\overline{\omega}+\mathbb{i}z+\frac{1}{2}\omega z^2+\frac{1}{6}z^3\right)+(\omega+z)\overline{\left(-\overline{\omega}-\mathbb{i}\overline{z}+\frac{1}{2}\omega\overline{z}^2-\frac{1}{6}\overline{z}^3\right)}\\
&=\left\{-\mathbb{i}-\sqrt{2}z+\left(-\frac{1}{2}+\mathbb{i}\right)z^2+\left(\frac{\omega}{2}-\frac{\overline{\omega}}{6}\right)z^3+\frac{1}{6}z^4\right\}\\
&\quad+\left\{-\mathbb{i}-\sqrt{2}z+\left(\frac{1}{2}+\mathbb{i}\right)z^2+\left(\frac{\overline{\omega}}{2}-\frac{\omega}{6}\right)z^3-\frac{1}{6}z^4\right\}\\
&=2\left(-\mathbb{i}-\sqrt{2}z+\mathbb{i}z^2+\frac{1}{3\sqrt{2}}z^3\right)=-2\mathbb{i}\left\{1+\sqrt{2}(-\mathbb{i}z)+(-\mathbb{i}z)^2+\frac{1}{3\sqrt{2}}(-\mathbb{i}z)^3\right\}\\
&=-2\mathbb{i}\cdot p^{-}(-\mathbb{i}z), \qquad (C.5)\\
\{\mathbf{P}^{-}(-z)\cdot\operatorname{adj}\mathbf{P}^{-}(z)\}_{2,1} &= p_1(-z)p_3(z)-p_1(z)p_3(-z)\\
&=(\omega-z)\left(-\overline{\omega}+\mathbb{i}z+\frac{1}{2}\omega z^2+\frac{1}{6}z^3\right)-(\omega+z)\left(-\overline{\omega}-\mathbb{i}z+\frac{1}{2}\omega z^2-\frac{1}{6}z^3\right)\\
&=\left(-1-\frac{\mathbb{i}}{2}z^2-\frac{\omega}{3}z^3-\frac{1}{6}z^4\right)+\left(1+\frac{\mathbb{i}}{2}z^2-\frac{\omega}{3}z^3+\frac{1}{6}z^4\right)=-\frac{2\omega}{3}z^3. \qquad (C.6)
\end{aligned}$$

Note from (C.1) and (C.2) that

$$\{\mathbf{P}^{\pm}(-z)\cdot\operatorname{adj}\mathbf{P}^{\pm}(z)\}_{1,2}=-\overline{\{\mathbf{P}^{\pm}(-\overline{z})\cdot\operatorname{adj}\mathbf{P}^{\pm}(\overline{z})\}_{2,1}},\quad \{\mathbf{P}^{\pm}(-z)\cdot\operatorname{adj}\mathbf{P}^{\pm}(z)\}_{2,2}=-\overline{\{\mathbf{P}^{\pm}(-\overline{z})\cdot\operatorname{adj}\mathbf{P}^{\pm}(\overline{z})\}_{1,1}}.$$

So by (C.3), (C.4), (C.5), (C.6),

$$\{\mathbf{P}^{+}(-z)\cdot\operatorname{adj}\mathbf{P}^{+}(z)\}_{1,2}=-\overline{\{\mathbf{P}^{+}(-\overline{z})\cdot\operatorname{adj}\mathbf{P}^{+}(\overline{z})\}_{2,1}}=-\overline{(2\omega\overline{z})}=-2\overline{\omega}z, \qquad (C.7)$$

$$\{\mathbf{P}^{+}(-z)\cdot\operatorname{adj}\mathbf{P}^{+}(z)\}_{2,2}=-\overline{\{\mathbf{P}^{+}(-\overline{z})\cdot\operatorname{adj}\mathbf{P}^{+}(\overline{z})\}_{1,1}}=-\overline{\{2\mathbb{i}\cdot p^{+}(-\mathbb{i}\overline{z})\}}=2\mathbb{i}\cdot p^{+}(\mathbb{i}z), \qquad (C.8)$$

$$\{\mathbf{P}^{-}(-z)\cdot\operatorname{adj}\mathbf{P}^{-}(z)\}_{1,2}=-\overline{\{\mathbf{P}^{-}(-\overline{z})\cdot\operatorname{adj}\mathbf{P}^{-}(\overline{z})\}_{2,1}}=-\overline{\left(-\frac{2\omega}{3}\overline{z}^3\right)}=\frac{2\overline{\omega}}{3}z^3, \qquad (C.9)$$

$$\{\mathbf{P}^{-}(-z)\cdot\operatorname{adj}\mathbf{P}^{-}(z)\}_{2,2}=-\overline{\{\mathbf{P}^{-}(-\overline{z})\cdot\operatorname{adj}\mathbf{P}^{-}(\overline{z})\}_{1,1}}=-\overline{\{-2\mathbb{i}\cdot p^{-}(-\mathbb{i}\overline{z})\}}=-2\mathbb{i}\cdot p^{-}(\mathbb{i}z). \qquad (C.10)$$

Thus, by (C.3), (C.4), (C.5), (C.6), (C.7), (C.8), (C.9), (C.10), we have

$$\begin{aligned}
\mathbf{P}^{+}(-z)\cdot\operatorname{adj}\mathbf{P}^{+}(z)&=\begin{pmatrix}2\mathbb{i}p^{+}(-\mathbb{i}z) & -2\overline{\omega}z\\ 2\omega z & 2\mathbb{i}p^{+}(\mathbb{i}z)\end{pmatrix}=2\mathbb{i}\begin{pmatrix}p^{+}(-\mathbb{i}z) & \omega z\\ \overline{\omega}z & p^{+}(\mathbb{i}z)\end{pmatrix},\\
\mathbf{P}^{-}(-z)\cdot\operatorname{adj}\mathbf{P}^{-}(z)&=\begin{pmatrix}-2\mathbb{i}p^{-}(-\mathbb{i}z) & \frac{2\overline{\omega}}{3}z^3\\ -\frac{2\omega}{3}z^3 & -2\mathbb{i}p^{-}(\mathbb{i}z)\end{pmatrix}=-2\mathbb{i}\begin{pmatrix}p^{-}(-\mathbb{i}z) & \frac{\omega}{3}z^3\\ \frac{\overline{\omega}}{3}z^3 & p^{-}(\mathbb{i}z)\end{pmatrix},
\end{aligned}$$

and the proof is complete.

D. The functions $\delta^{\pm}(z,\kappa)$

We start with some exotic definitions in [14]. For $\kappa \geq 0$, let

$$p(\kappa) = \frac{1 - \sqrt{2}\kappa + \kappa^2}{1 + \sqrt{2}\kappa + \kappa^2}, \qquad \varphi^{\pm}(\kappa) = e^{L\kappa} \cdot \frac{1 \pm \sin h(\kappa)}{\cos h(\kappa)}. \tag{D.1}$$

Here, $L = 2l\alpha$ is the intrinsic length of the beam and

$$h(\kappa) = L\kappa - \hat{h}(\kappa), \tag{D.2}$$

where $\hat{h} : [0,\infty) \to \mathbb{R}$ is defined by

$$\hat{h}(\kappa) = \begin{cases} \arctan\left\{\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4-4\kappa^2+1}\right\}, & \text{if } 0 \leq \kappa < \frac{\sqrt{3}-1}{\sqrt{2}}, \\ -\frac{\pi}{2}, & \text{if } \kappa = \frac{\sqrt{3}-1}{\sqrt{2}}, \\ -\pi + \arctan\left\{\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4-4\kappa^2+1}\right\}, & \text{if } \frac{\sqrt{3}-1}{\sqrt{2}} \leq \kappa \leq \frac{\sqrt{3}+1}{\sqrt{2}}, \\ -\frac{3\pi}{2}, & \text{if } \kappa = \frac{\sqrt{3}+1}{\sqrt{2}}, \\ -2\pi + \arctan\left\{\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4-4\kappa^2+1}\right\}, & \text{if } \kappa > \frac{\sqrt{3}+1}{\sqrt{2}}. \end{cases} \tag{D.3}$$

The branch of arctan here is taken such that $\arctan 0 = 0$. $\hat{h}$ is a strictly decreasing real-analytic function with $\hat{h}(0) = 0$ and $\lim_{\kappa\to\infty} \hat{h}(\kappa) = -2\pi$, hence $h : [0,\infty) \to \mathbb{R}$ is a strictly increasing real-analytic function with $h(0) = 0$ and $\lim_{\kappa\to\infty} h(\kappa) = \infty$.

Proposition D.1. ([14, Eqs 8 and 25]) $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{K}_{\mathbf{Q}} = \mathcal{K}_{l,\alpha,k}$ if and only if $\lambda = \frac{1}{k} \cdot \frac{1}{1+\kappa^4}$ for $\kappa > 0$ such that $\varphi^{+}(\kappa) = p(\kappa)$ or $\varphi^{-}(\kappa) = p(\kappa)$.

Now we demonstrate how the seemingly ad hoc and complex conditions $\varphi^{\pm}(\kappa) = p(\kappa)$ in Proposition D.1, which were practically unobtainable without help of computer algebra systems as indicated in [14], can be derived so naturally and elegantly from our holomorphic functions $\delta^{\pm}(z,\kappa)$.

By Definition 3.2,

$$e^{\mathrm{i}\gamma(\kappa)} = \cos\gamma(\kappa) + \mathrm{i}\sin\gamma(\kappa) = \frac{1-\kappa^2}{1+\kappa^2} + \mathrm{i}\frac{2\kappa}{1+\kappa^2} = \frac{(1+\mathrm{i}\kappa)^2}{1+\kappa^2} = \frac{1+\mathrm{i}\kappa}{1-\mathrm{i}\kappa}, \qquad \kappa \in \mathbb{D}, \tag{D.4}$$

where $\mathbb{D} = \mathbb{C} \setminus \{0, 1, -1, \mathrm{i}, -\mathrm{i}\}$ by Definition 4.2.

Lemma D.1. *For $\kappa \geq 0$, $p(\kappa) = e^{\mathrm{i}\{\gamma(\omega\kappa)-\gamma(\overline{\omega}\kappa)\}}$ and $e^{-\mathrm{i}\hat{h}(\kappa)} = e^{\mathrm{i}\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}}$.*

Proof. By (2.1), (D.1), (D.4),

$$e^{\mathrm{i}\{\gamma(\omega\kappa)-\gamma(\overline{\omega}\kappa)\}} = e^{\mathrm{i}\gamma(\omega\kappa)}e^{-\mathrm{i}\gamma(\overline{\omega}\kappa)} = \frac{1+\mathrm{i}\omega\kappa}{1-\mathrm{i}\omega\kappa} \cdot \frac{1-\mathrm{i}\overline{\omega}\kappa}{1+\mathrm{i}\overline{\omega}\kappa} = \frac{1-\overline{\omega}\kappa}{1+\overline{\omega}\kappa} \cdot \frac{1-\omega\kappa}{1+\omega\kappa} = \frac{1-\sqrt{2}\kappa+\kappa^2}{1+\sqrt{2}\kappa+\kappa^2} = p(\kappa).$$

By (2.1) and (D.4),

$$e^{\mathrm{i}\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}} = e^{\mathrm{i}\gamma(\omega\kappa)}e^{\mathrm{i}\gamma(\overline{\omega}\kappa)} = \frac{1+\mathrm{i}\omega\kappa}{1-\mathrm{i}\omega\kappa} \cdot \frac{1+\mathrm{i}\overline{\omega}\kappa}{1-\mathrm{i}\overline{\omega}\kappa} = \frac{1-\overline{\omega}\kappa}{1+\overline{\omega}\kappa} \cdot \frac{1+\omega\kappa}{1-\omega\kappa} = \frac{1+\mathrm{i}\sqrt{2}\kappa-\kappa^2}{1-\mathrm{i}\sqrt{2}\kappa-\kappa^2}$$

$$= \frac{\left(1+\mathbb{i}\sqrt{2}\kappa-\kappa^2\right)^2}{\left(1-\mathbb{i}\sqrt{2}\kappa-\kappa^2\right)\left(1+\mathbb{i}\sqrt{2}\kappa-\kappa^2\right)} = \frac{\left(1-4\kappa^2+\kappa^4\right)+\mathbb{i}\cdot 2\sqrt{2}\kappa\left(1-\kappa^2\right)}{\left(1-\kappa^2\right)^2+2\kappa^2}.$$

So we have

$$\cos\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\} = \frac{1-4\kappa^2+\kappa^4}{1+\kappa^4}, \quad \sin\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\} = \frac{2\sqrt{2}\kappa\left(1-\kappa^2\right)}{1+\kappa^4},$$

hence

$$\tan\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\} = \frac{2\sqrt{2}\kappa\left(1-\kappa^2\right)}{\kappa^4-4\kappa^2+1}.$$

Thus, by (D.3),

$$\tan\left\{-\hat{h}(\kappa)\right\} = -\tan\hat{h}(\kappa) = \frac{2\sqrt{2}\kappa\left(1-\kappa^2\right)}{\kappa^4-4\kappa^2+1} = \tan\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}.$$

It follows that $e^{-\mathbb{i}\hat{h}(\kappa)} = e^{\mathbb{i}\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}}$, and the proof is complete. □

By (D.2) and Lemma D.1,

$$e^{\mathbb{i}h(\kappa)} = e^{\mathbb{i}\left\{L\kappa-\hat{h}(\kappa)\right\}} = e^{\mathbb{i}L\kappa}e^{-\mathbb{i}\hat{h}(\kappa)} = e^{\mathbb{i}L\kappa}e^{\mathbb{i}\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}} = e^{\mathbb{i}\{L\kappa+\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}}.$$

So we have $\cos h(\kappa) = \cos\{L\kappa+\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}$, $\sin h(\kappa) = \sin\{L\kappa+\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}$, hence, by (D.1),

$$\varphi^{\pm}(\kappa) = e^{L\kappa}\cdot\frac{1\pm\sin\{L\kappa+\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}}{\cos\{L\kappa+\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}}. \tag{D.5}$$

By Definition 3.2,

$$e^{\beta(\kappa)} = \cosh\beta(\kappa)+\sinh\beta(\kappa) = \frac{1+\kappa^2}{1-\kappa^2}+\frac{2\kappa}{1-\kappa^2} = \frac{(1+\kappa)^2}{1-\kappa^2} = \frac{1+\kappa}{1-\kappa}, \qquad \kappa\in\mathbb{D}. \tag{D.6}$$

Comparing (D.4) and (D.6), we have $e^{\mathbb{i}\gamma(\kappa)} = e^{\beta(\mathbb{i}\kappa)}$ for $\kappa\in\mathbb{D}$, hence

$$e^{\beta(\omega\kappa)} = e^{\beta(\mathbb{i}\cdot(-\mathbb{i}\omega\kappa))} = e^{\mathbb{i}\gamma(\overline{\omega}\kappa)}, \qquad \kappa\in\mathbb{D}, \tag{D.7}$$

since $-\mathbb{i}\omega = \overline{\omega}$ by (2.1).

Now let $\lambda = \frac{1}{k}\cdot\frac{1}{1+\kappa^4}$ for $\kappa>0$, and let $z = l\alpha$ so that

$$2\kappa z = L\kappa. \tag{D.8}$$

By Definitions 2.1 and 2.4,

$$\chi(\lambda) = \sqrt[4]{1-\frac{1}{\left(\frac{1}{k}\cdot\frac{1}{1+\kappa^4}\right)\cdot k}} = \sqrt[4]{-\kappa^4} = \omega\kappa,$$

hence $\delta^{\pm}(l\alpha,\chi(\lambda)) = \delta^{\pm}(z,\omega\kappa)$. So by Corollary 1, $\lambda\in\operatorname{Spec}\mathcal{K}_{\mathbf{Q}}$ if and only if $\delta^{+}(z,\omega\kappa) = 0$ or $\delta^{-}(z,\omega\kappa) = 0$. By Definition 2.1, $\sqrt{2}\omega = 1+\mathbb{i}$, hence, by Definition 3.2 and (D.7),

$$2\delta^{\pm}(z,\omega\kappa) = \left\{e^{\sqrt{2}\omega\kappa z}e^{\beta(\omega\kappa)} - e^{-\sqrt{2}\omega\kappa z}e^{-\beta(\omega\kappa)}\right\} \mp \mathbb{i}\left\{e^{\mathbb{i}\sqrt{2}\omega\kappa z}e^{\mathbb{i}\gamma(\omega\kappa)} - e^{-\mathbb{i}\sqrt{2}\omega\kappa z}e^{-\mathbb{i}\gamma(\omega\kappa)}\right\}$$

$$= \left\{e^{\kappa z}e^{\mathrm{i}\kappa z}e^{\mathrm{i}\gamma(\overline{\omega}\kappa)} - e^{-\kappa z}e^{-\mathrm{i}\kappa z}e^{-\mathrm{i}\gamma(\overline{\omega}\kappa)}\right\} \mp \mathrm{i}\left\{e^{-\kappa z}e^{\mathrm{i}\kappa z}e^{\mathrm{i}\gamma(\omega\kappa)} - e^{\kappa z}e^{-\mathrm{i}\kappa z}e^{-\mathrm{i}\gamma(\omega\kappa)}\right\}$$
$$= e^{\kappa z}\left\{e^{\mathrm{i}\kappa z}e^{\mathrm{i}\gamma(\overline{\omega}\kappa)} \pm \mathrm{i}e^{-\mathrm{i}\kappa z}e^{-\mathrm{i}\gamma(\omega\kappa)}\right\} - e^{-\kappa z}\left\{e^{-\mathrm{i}\kappa z}e^{-\mathrm{i}\gamma(\overline{\omega}\kappa)} \pm \mathrm{i}e^{\mathrm{i}\kappa z}e^{\mathrm{i}\gamma(\omega\kappa)}\right\}.$$

So $\delta^{\pm}(z, \omega\kappa) = 0$ if and only if

$$e^{-2\kappa z} = \frac{e^{\mathrm{i}\kappa z}e^{\mathrm{i}\gamma(\overline{\omega}\kappa)} \pm \mathrm{i}e^{-\mathrm{i}\kappa z}e^{-\mathrm{i}\gamma(\omega\kappa)}}{e^{-\mathrm{i}\kappa z}e^{-\mathrm{i}\gamma(\overline{\omega}\kappa)} \pm \mathrm{i}e^{\mathrm{i}\kappa z}e^{\mathrm{i}\gamma(\omega\kappa)}} = \frac{e^{\mathrm{i}\kappa z}e^{\mathrm{i}\gamma(\overline{\omega}\kappa)} \pm \mathrm{i}e^{-\mathrm{i}\kappa z}e^{-\mathrm{i}\gamma(\omega\kappa)}}{e^{-\mathrm{i}\kappa z}e^{-\mathrm{i}\gamma(\overline{\omega}\kappa)} \pm \mathrm{i}e^{\mathrm{i}\kappa z}e^{\mathrm{i}\gamma(\omega\kappa)}} \cdot \frac{e^{-\mathrm{i}\kappa z}e^{-\mathrm{i}\gamma(\overline{\omega}\kappa)} \mp \mathrm{i}e^{\mathrm{i}\kappa z}e^{\mathrm{i}\gamma(\omega\kappa)}}{e^{-\mathrm{i}\kappa z}e^{-\mathrm{i}\gamma(\overline{\omega}\kappa)} \mp \mathrm{i}e^{\mathrm{i}\kappa z}e^{\mathrm{i}\gamma(\omega\kappa)}}$$
$$= \frac{2 \mp \mathrm{i}e^{2\mathrm{i}\kappa z}e^{\mathrm{i}\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}} \pm \mathrm{i}e^{-2\mathrm{i}\kappa z}e^{-\mathrm{i}\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}}}{e^{2\mathrm{i}\kappa z}e^{\mathrm{i}2\gamma(\omega\kappa)} + e^{-2\mathrm{i}\kappa z}e^{-\mathrm{i}2\gamma(\overline{\omega})}}$$
$$= \frac{2 \mp \mathrm{i}\left\{e^{2\mathrm{i}\kappa z}e^{\mathrm{i}\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}} - e^{-2\mathrm{i}\kappa z}e^{-\mathrm{i}\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}}\right\}}{e^{\mathrm{i}\{\gamma(\omega\kappa)-\gamma(\overline{\omega}\kappa)\}}\left\{e^{2\mathrm{i}\kappa z}e^{\mathrm{i}\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}} + e^{-2\mathrm{i}\kappa z}e^{-\mathrm{i}\{\gamma(\omega\kappa)+\gamma(\overline{\omega}\kappa)\}}\right\}}$$
$$= e^{-\mathrm{i}\{\gamma(\omega\kappa)-\gamma(\overline{\omega}\kappa)\}} \cdot \frac{1 \pm \sin\{2\kappa z + \gamma(\omega\kappa) + \gamma(\overline{\omega}\kappa)\}}{\cos\{2\kappa z + \gamma(\omega\kappa) + \gamma(\overline{\omega}\kappa)\}},$$

which is equivalent to $p(\kappa) = \varphi^{\pm}(\kappa)$ by Lemma D.1, (D.5) and (D.8). Thus we conclude that $\lambda \in \operatorname{Spec}\mathcal{K}_{\mathbf{Q}}$ if and only if $p(\kappa) = \varphi^{+}(\kappa)$ or $p(\kappa) = \varphi^{-}(\kappa)$, which is exactly the condition in Proposition D.1.