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Research article

New classes of analytic and bi-univalent functions

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Abstract: Using the (p,q)-derivative operator we introduce new subclasses of analytic and bi-univalent functions, we obtain estimates on coefficients and the Fekete-Szegö functional.

Keywords: Fekete-Szegö problem; (p,q)-derivative operator; univalent functions; bi-univalent functions; analytic functions; coefficient bounds and coefficient estimates
Mathematics Subject Classification: 30C45, 30C50

1. Introduction and Preliminary results

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0, f'(0) = 1$.
Let $S \subset \mathcal{A}$ denote the class of all functions in $\mathcal{A}$ which are univalent in U.
The Koebe One-Quarter Theorem [5] ensures that the image of the unit disk under every $f \in S$ functions contains a disk of radius $1/4$.
It is well known that every functions $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, z \in U$$

and

$$f(f^{-1}(w)) = w, |w| < r_0(f), r_0(f) \geq 1/4$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + ... \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.
Let Σ denote the class of all bi-univalent functions in U given by (1).
The class of bi-univalent functions was first introduced and studied by Lewin [12] and was showed that $|a_2| < 1,51$.
The problem of maximizing the absolute value of the functional $|a_3 - \mu a_2^2|$ is called the Fekete-Szegö problem. Many authors obtained Fekete-Szegö inequalities for different classes of functions [2], [6], [8], [11], [14]. Coefficient estimates of bi-univalent functions are studied in many papers, of which I mention: [1], [3], [7], [9], [10], [12], [17], [18], [19], [20].

For the new results posted in the last section we have to recall the necessary elements of the (p, q)-calculus involving.

There is possibility of extension of the q-calculus to post quantum calculus denoted by the (p, q)-calculus.

When the case $p = 1$ in (p, q)-calculus, the $q-$ calculus may be obtained.

In order to derive our main results, we need to following lemmas:

Lemma 1. *[15] If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in U for which*

$$\mathcal{R}(p(z)) > 0, p(z) = 1 + c_1 z + c_2 z^2 + ...,$$

for $z \in U$.

Lemma 2. *[5] Let $p \in \mathcal{P}$ be of the form $p(z) = 1 + c_1 z + c_2 z^2 + ...$ Then*
$|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}$ *and* $|c_k| \leq 2, k \in \mathbb{N}$.

Lemma 3. *[13] If $p(z) = 1 + c_1 z + c_2 z^2 + ..., z \in U$ is a function with positive real part in U and μ is a complex number, then*
$|c_2 - \mu c_1^2| \leq 2max\{1; |2\mu - 1|\}$.

The result is sharp for the function given by
$p(z) = \frac{1+z^2}{1-z^2}$ and $p(z) = \frac{1+z}{1-z}, z \in U$.

Definition 4. *Let $f \in \mathcal{A}$ given by (1.1) and $0 < q < p \leq 1$. Then the (p, q)-derivative operator or p, q-difference operator for the function f of the form (1.1) is defined by*

$$D_{p,q} f(z) = \frac{f(pz) - f(qz)}{(p - q)z}, z \in U^* = U - \{0\} \tag{1.3}$$

and

$$(D_{p,q} f)(0) = f'(0) \tag{1.4}$$

provided that the function f is differentiable at 0.

From the relation (1.2), we deduce that

$$D_{p,q} f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1} \tag{1.5}$$

where the (p, q)-bracket number or twin-basic is given by

$$[k]_{p,q} = \frac{p^k - q^k}{p - q} = p^{k-1} + p^{k-2}q + p^{k-3}q^2 + ... + pq^{k-2} + q^{k-1}, p \neq q \tag{1.6}$$

which is a natural generalization of the q-number.
Also $\lim_{p\to 1^-}[k]_{p,q} = [k]_q = \frac{1-q^k}{1-q}$
See [4,16].

2. Main results

Definition 5. *A function f given by (1.1) is said to be in the class $H_\Sigma^{p,q,\alpha}(0 < q < p \leq 1, 0 < \alpha \leq 1)$, if the following conditions are satisfied:*

$$\begin{cases} f \in \Sigma \\ |arg(D_{p,q}f(z))| < \frac{\alpha\pi}{2}, (z \in U) \end{cases} \tag{2.1}$$

and

$$|arg(D_{p,q}g(w))| < \frac{\alpha\pi}{2}, (w \in U) \tag{2.2}$$

where the function g is given by (1.2).

Remark 6. *When $p = 1, lim_{q\to 1^-}H_\Sigma^{p,q,\alpha} = H_\Sigma^\alpha$, where H_Σ^α is the class introduced in [19].*

In the next theorem we obtain coefficient bounds for the functions class $H_\Sigma^{p,q,\alpha}$.

Theorem 7. *Let the function f given by (1.1) be in the function class $H_\Sigma^{p,q,\alpha}(0 < q < p \leq 1, 0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2[3]_{p,q}\alpha + (1-\alpha)[2]_{p,q}^2}} \tag{2.3}$$

and

$$|a_3| \leq \frac{4\alpha^2}{[2]_{p,q}^2} + \frac{2\alpha}{[3]_{p,q}} \tag{2.4}$$

Proof. From the relations (2.1) and (2.2) it follows that

$$D_{p,q}f(z) = [P(z)]^\alpha$$

and

$$D_{p,q}g(w) = [Q(w)]^\alpha, (z, w \in U) \tag{2.5}$$

where $P(z) = 1 + p_1z + p_2z^2 + ...$ and $Q(z) = 1 + q_1w + q_2w^2 + ...$ in $\mathcal{P}$.
From the relation (2.5), we obtain the next relations

$$[2]_{p,q}a_2 = a_2\frac{p^2 - q^2}{p - q} = \alpha p_1 \tag{2.6}$$

$$[3]_{p,q}a_3 = a_3\frac{p^3 - q^3}{p - q} = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2}p_1^2 \tag{2.7}$$

$$-[2]_{p,q}a_2 = -a_2\frac{p^2 - q^2}{p - q} = \alpha q_1 \tag{2.8}$$

and

$$[3]_{p,q}(2a_2^2 - a_3) = (2a_2^2 - a_3)\frac{p^3 - q^3}{p - q} = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2}q_1^2 \tag{2.9}$$

It follows that

$$p_1 = -q_1 \tag{2.10}$$

and

$$2[2]_{p,q}^2 a_2^2 = \alpha^2(p_1^2 + q_1^2) \tag{2.11}$$

The relations (2.10) and (2.11) are obtained from the relations (2.6) and (2.8). We obtain that

$$2[3]_{p,q}a_2^2 = 2a_2^2\frac{p^3 - q^3}{p - q} = \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 + q_1^2) =$$

$$= \alpha(p_2 + q_2) + \frac{\alpha - 1}{\alpha}[2]_{p,q}^2 a_2^2. \tag{2.12}$$

This relation is obtained from (2.9) and (2.11). We get from (2.12)

$$a_2^2 = \frac{\alpha^2}{2[3]_{p,q}\alpha + (1 - \alpha)[2]_{p,q}^2}(p_2 + q_2). \tag{2.13}$$

From Lemma 1 for the above equality, we get the estimate on the coefficient $|a_2|$ as asserted in the relation (2.3).
We subtract (2.9) from (2.7) and find the bound on the coefficient $|a_3|$.
We get it that way

$$2[3]_{p,q}a_3 - 2[3]_{p,q}a_2^2 = 2a_3\frac{p^3 - q^3}{p - q} - 2a_2^2\frac{p^3 - q^3}{p - q} = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 - q_1^2). \tag{2.14}$$

It follows that,

$$a_3 = \frac{\alpha^2}{2[2]_{p,q}^2}(p_1^2 + q_1^2) + \frac{\alpha}{2[3]_{p,q}}(p_2 - q_2),$$

from the relations (2.10), (2.11) and (2.14).
And if we apply Lemma 1 for the above equality, we obtain the estimate on the coefficient $|a_3|$ as asserted in (2.4). □

When $p = 1, \quad q \to 1^-$ in Theorem 7, we obtain the following result given in [19].

Corollary 8. *[19] Let $f(z)$ given by (1.1) be in the class $H_\Sigma^\alpha (0 < \alpha \leq 1)$. Then $|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+2}}$ and $|a_3| \leq \frac{\alpha(3\alpha+2)}{3}$.*

Definition 9. *A function f given by (1.1) is said to be in the class $H_\Sigma^{p,q,\beta}(0 < q < p \leq 1, 0 \leq \beta < 1)$ if the following conditions are satisfied:*

$$\begin{cases} f \in \Sigma \\ \mathcal{R}\{D_{p,q}f(z)\} > \beta, z \in U \end{cases} \tag{2.15}$$

$$\mathcal{R}\{D_{p,q}g(w)\} > \beta, w \in U, \tag{2.16}$$

where the function g is defined by (1.2).

Remark 10. *When $p = 1$ and $q \to 1^-$ we obtain the class $H_\Sigma(\beta)$ introduced in [19].*

In the next theorem we obtain coefficient bounds for the functions class $H_\Sigma^{p,q,\beta}$.

Theorem 11. *Let the function f be given by (1.1) be in the function class $H_\Sigma^{p,q,\beta}(0 < q < p \leq 1)$. Then*

$$|a_2| \leq min\{\frac{2(1-\beta)}{[2]_{p,q}}, \sqrt{\frac{2(1-\beta)}{[3]_{p,q}}}\} \tag{2.17}$$

$$|a_3| \leq \frac{2(1-\beta)}{[3]_{p,q}}. \tag{2.18}$$

Proof. It follows from the conditions (2.15) and (2.16) that

$$D_{p,q}f(z) = \beta + (1-\beta)P(z) \quad and \quad D_{p,q}g(w) = \beta + (1-\beta)Q(w), \quad z, w \in U \tag{2.19}$$

respectively, where

$$P(z) = 1 + p_1 z + p_2 z^2 + ...$$

and

$$Q(w) = 1 + q_1 q + q_2 w^2 + ... \quad in \quad \mathcal{P}.$$

We obtain

$$[2]_{p,q}a_2 = a_2\frac{p^2 - q^2}{p - q} = (1-\beta)p_1 \tag{2.20}$$

$$[3]_{p,q}a_3 = a_3\frac{p^3 - q^3}{p - q} = (1-\beta)p_2 \tag{2.21}$$

$$-[2]_{p,q}a_2 = -a_2\frac{p^2 - q^2}{p - q} = (1-\beta)q_1 \tag{2.22}$$

$$[3]_{p,q}(2a_2^2 - a_3) = (2a_2^2 - a_3)\frac{p^3 - q^3}{p - q} = (1-\beta)q_2, \tag{2.23}$$

upon equating the coefficients in (2.19). Now, from the relations (2.20) and (2.22), we obtain

$$p_1 = -q_1 \tag{2.24}$$

and

$$2[2]_{p,q}^2 a_2^2 = (1-\beta)^2(p_1^2 + q_1^2) \tag{2.25}$$

Now, from the relations (2.21) and (2.23) we have

$$2[3]_{p,q}a_2^2 = 2a_2^2\frac{p^3-q^3}{p-q} = (1-\beta)(p_2+q_2) \tag{2.26}$$

From Lemma 1 for the relations (2.26) and (2.25), we obtain the estimate on the coefficient $|a_2|$ as asserted in (2.17).
Now, we subtract (2.23) from (2.21) and we obtain

$$2[3]_{p,q}a_3 - 2[3]_{p,q}a_2^2 = 2a_3\frac{p^3-q^3}{p-q} - 2a_2^2\frac{p^3-q^3}{p-q} = (1-\beta)(p_2-q_2) \tag{2.27}$$

From (2.25) and (2.26) we can obtain a_2^2. From (2.27) we get

$$a_3 = \frac{(1-\beta)^2}{2[2]_{p,q}^2}(p_1^2+q_1^2) + \frac{(1-\beta)}{2[3]_{p,q}}(p_2-q_2) \tag{2.28}$$

By using the relation (2.26) into (2.27) it follows that

$$a_3 = \frac{(1-\beta)}{2[3]_{p,q}}(p_2+q_2) + \frac{1-\beta}{2[3]_{p,q}}(p_2-q_2) = \frac{(1-\beta)p_2(p-q)}{p^3-q^3} = \frac{(1-\beta)p_2}{[3]_{p,q}} \tag{2.29}$$

If we apply Lemma 1 for the relations (2.28) and (2.29) we get the estimate on the coefficient $|a_3|$ as asserted in (2.18). □

We obtain the next corollary.

Corollary 12. *Let $f(z)$ given by (1.1) be in the function class $H_\Sigma(\beta)(0 \le \beta < 1)$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{3}}$$

and

$$|a_3| < \frac{(1-\beta)(5-3\beta)}{3}.$$

Definition 13. *Let $b, t : U \to \mathbb{C}$ be analytic functions and*

$$min\{\mathcal{R}(b(z)), \mathcal{R}(t(z))\} > 0, (z \in U),$$

$$b(0) = t(0) = 1.$$

A function f given by (1.1) is said to be in the class $H_\Sigma^{p,q,b,t}$ if the following conditions are satisfied:

$$D_{p,q}f(z) \in b(U) \tag{2.30}$$

and

$$D_{p,q}g(w) \in t(U), \tag{2.31}$$

where $z, w \in U$ and the function g is given by (1.2).

In the next theorem we obtain coefficient bounds for the functions class $H_\Sigma^{p,q,b,t}$.

Theorem 14. *Let f given by (1.1) be in the class $H_{\Sigma}^{p,q,b,t}$. Then*

$$|a_2| \leq min\{\sqrt{\frac{|b'(0)|^2 + |t'(0)|^2}{2[2]_{p,q}^2}}, \sqrt{\frac{|b''(0)| + |t''(0)|}{2[3]_{p,q}}}\} \tag{2.32}$$

$$|a_3| \leq min\{\frac{|b'(0)|^2 + |t'(0)|^2}{2[2]_{p,q}^2} + \frac{|b''(0)| + |t''(0)|}{4[3]_{p,q}}, \frac{|b''(0)|}{2[3]_{p,q}}\} \tag{2.33}$$

Proof. We will write the equivalent forms of the argument inequalities in the relations (2.32) and (2.33).

$$D_{p,q}f(z) = b(z) \tag{2.34}$$

and

$$D_{p,q}g(w) = t(w), \tag{2.35}$$

where $b(z)$ and $t(w)$ satisfy the conditions from Definition 13 and it have the following Taylor-Maclaurin series expansions:

$$b(z) = 1 + b_1 z + b_2 z^2 + ... \tag{2.36}$$

$$t(w) = 1 + t_1 w + t_2 w^2 + ... \tag{2.37}$$

We find that

$$[2]_{p,q} a_2 = b_1 \tag{2.38}$$

$$[3]_{p,q} a_3 = b_2 \tag{2.39}$$

$$-[2]_{p,q} a_2 = t_1 \tag{2.40}$$

$$[3]_{p,q}(2a_2^2 - a_3) = t_2, \tag{2.41}$$

substituting from (2.36) and (2.37) into (2.34) and (2.35) and equating the coefficients.

We obtain that

$$b_1 = -t_1 \tag{2.42}$$

and

$$2[2]_{p.q}^2 a_2^2 = b_1^2 + t_1^2 \tag{2.43}$$

from the relations (2.38) and (2.40). We obtain that

$$[3]_{p,q} a_3 + [3]_{p,q}(2a_2^2 - a_3) = b_2 + t_2 \tag{2.44}$$

Adding the relation (2.2) and (2.3). From the relations (2.44) and (2.45), we can find

$$a_2^2 = \frac{b_1^2 + t_1^2}{2[2]_{p,q}^2} \tag{2.45}$$

and

$$a_2^2 = \frac{t_2 + b_2}{2[3]_{p,q}}. \tag{2.46}$$

We can calculate from the relations (2.45) and (2.46) that

$$|a_2|^2 \leq \frac{|b'(0)|^2 + |t'(0)|^2}{2[2]_{p,q}^2}$$

and

$$|a_2|^2 \leq \frac{|b''(0)| + |t''(0)|}{2[3]_{p.q}}.$$

So, we obtain the desired estimate on the coefficient $|a_2|$ as asserted in the relation (2.32).
Now, subtracting the relation (2.41) from the relation (2.39), we obtain

$$2[3]_{p,q}a_3 - 2a_2^2[3]_{p,q} = b_2 - t_2 \tag{2.47}$$

Substituting a_2^2 from (2.45) into (2.47) it follows that

$$a_3 = \frac{b_2 - t_2}{2[3]_{p,q}} + \frac{b_1^2 + t_1^2}{2[2]_{p,q}^2}.$$

It follows that

$$|a_3| \leq \frac{|b'(0)|^2 + |t'(0)|^2}{2[2]_{p,q}^2} + \frac{|b''(0)| + |t''(0)|}{4[3]_{p,q}}.$$

Substituting the value of a_2^2 from the relation (2.46) into the relation (2.47), we get

$$a_3 = \frac{b_2}{[3]_{p,q}}.$$

It follows that

$$|a_3| \leq \frac{|b''(0)|}{2[3]_{p,q}}.$$

□

Now, we compute the Fekete-Szegö functional for the class $H_\Sigma^{p,q,\alpha}$.

Theorem 15. *Let f of the form (1.1) be in the class $H_\Sigma^{p,q,\alpha}$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\alpha}{[3]_{p,q}}, |r(\mu)| \leq \frac{1}{2[3]_{p,q}} \\ 2\alpha|r(\mu)|, |r(\mu)| \geq \frac{1}{2[3]_{p,q}} \end{cases} \tag{2.48}$$

Proof. From Theorem 7 we can use the value of the coefficients a_2^2 and a_3 to compute $a_3 - \mu a_2^2$.

$$a_3 - \mu a_2^2 = \alpha[p_2(\frac{1}{2[3]_{p,q}} + \frac{(1-\mu)\alpha}{2[3]_{p,q}\alpha + (1-\alpha)[2]_{p,q}^2}) +$$

$$+ q_2(\frac{(1-\mu)\alpha}{2[3]_{p,q}\alpha + (1-\alpha)[2]_{p,q}^2} - \frac{1}{2[3]_{p,q}})].$$

If we denote by

$$r(\mu) = (1-\mu)\frac{\alpha}{2[3]_{p,q}\alpha + (1-\alpha)[2]_{p,q}^2}.$$

It follows that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\alpha}{[3]_{p,q}}, |r(\mu)| \leq \frac{1}{2[3]_{p,q}} \\ 2\alpha|r(\mu)|, |r(\mu)| \geq \frac{1}{2[3]_{p,q}} \end{cases}$$

□

Now, we compute the Fekete-Szegö functional for the class $H_{\Sigma}^{p,q,\beta}$.

Theorem 16. *Let f of the form (1.1) be in the class $H_{\Sigma}^{p,q,\beta}$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1-\beta}{[3]_{p,q}}, r(\mu) \leq \frac{1}{2[3]_{p,q}} \\ 2(1-\beta)|r(\mu)|, |r(\mu)| \geq \frac{1}{2[3]_{p,q}} \end{cases} . \quad (2.49)$$

Proof. From Theorem 11 we can use the value of the coefficients a_2^2 and a_3 to compute $a_3 - \mu a_2^2$.

$$a_3 - \mu a_2^2 =$$

$$= (1-\beta)[p_2(\frac{1}{2[3]_{p,q}} + \frac{1-\mu}{2[3]_{p,q}}) + q_2(\frac{1-\mu}{2[3]_{p,q}} - \frac{1}{2[3]_{p,q}})].$$

If we denote by

$$r(\mu) = (1-\mu)\frac{1}{2[3]_{p,q}}$$

It follows that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1-\beta}{[3]_{p,q}}, r(\mu) \leq \frac{1}{2[3]_{p,q}} \\ 2(1-\beta)|r(\mu)|, |r(\mu)| \geq \frac{1}{2[3]_{p,q}} \end{cases}$$

□

Conflict of interest

The author declares no conflicts of interest in this paper.

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