AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(10): 10602–10619.
DOI:10.3934/math.2021616
Received: 29 April 2021
Accepted: 13 July 2021
Published: 22 July 2021

Research article

Analysis of two components parallel repairable degenerate system with vacation

YanLing Li[1,2], **GenQi Xu**[1,*] **and Hao Chen**[3]

[1] School of Mathematics, Tianjin University, Tianjin 300350, China
[2] School of Mathematics and Statistics, Qinghai Nationalities University, Xining 810007, China
[3] School of Mechatronical Engineering, Beijing Institute of Technology, Beijing, 100081, China

* **Correspondence:** Email: gqxu@tju.edu.cn.

Abstract: This article studies a parallel repairable degradation system with two similar components and a repairman who can take a single vacation. Suppose that the system consists of two components that cannot be repaired "as good as new" after failures; when the repairman has a single vacation, the fault component of system may not be repaired immediately, namely, if a component fails and the repairman is on vacation, the repair of the component will be delayed, if a component fails and the repairman is on duty, the fault component can be repaired immediately. Under these assumptions, a replacement policy N based on the failed times of component 1 is studied. The explicit expression of the system average cost rate $C(N)$ and the optimal replacement policy N^* by minimizing the $C(N)$ are obtained, which means the two components of the system will be replaced at the same time if the failures of component 1 reach N^*. To show the advantage of a parallel system, a replacement policy N of the cold standby system consisting of the two similar components is also considered. The numerical results of both systems are given by the numerical analysis. The optimal replacement policy N^* for both systems are obtained. Finally, the comparison of numerical results shows the advantages of the parallel system.

Keywords: geometric process; parallel system; optimal replacement policy; single vacation; imperfect delayed repair
Mathematics Subject Classification: TP391

1. Introduction

With a rapid development of economy, higher requirements are put forward for the reliability of the system because some system failures may cause very serious consequences, for example, the unreliable power supply system of the hospital operating room will lead to the suspension of the operation; the unreliable navigation system will lead to the crash of the aircraft, etc. In order to improve reliability of the system, the maintenance strategy of a system has become an important research content. Many scholars have done a lot of research in this area. see e.g., [1–4, 6, 7, 9, 15, 16]. In recent years, scholars have studied repairable systems from the following two aspects: one is that the components of a system can be repaired "as good as new" after failures; another is that the components cannot be repaired "as good as new" after failures, which is called a degenerate repairable system. In general, the geometric process is used to describe a degenerate repairable system (see [11–13]), such a repair model was proposed by Lam in [5]. For a degenerate repairable system, Lam mainly considered two kinds of replacement policies: N policy and T policy, where T policy is based on the working age of the system and N policy is based on the number of system failures. Under the two policies, the explicit expression of the system average cost rate $C(N)$ is given, the optimal replacement policy N^* is obtained by minimizing the average cost rate $C(N^*)$. In addition, under some conditions, Lam also proved the optimal replacement policy N^* is better than the optimal replacement policy T^*. Based on this result, Zhang and Yam (see [17]) proposed a bivariate replacement policy (T, N) by combining the two policies. Similarly, Zhang et al. shows that under some conditions, the optimal policy $(T, N)^*$ is better than the optimal policy T^* and N^*, herein we refer to some related works such as [10], [14] and [17].

Also, there are some works on the maintenance strategy of two-component cold standby system, such as [11], [14] and [16]. Y. L. Zhang and G. J. Wang [14] studied a repair replacement problem for a cold standby system consisting of two dissimilar components. By the extended geometric process and numerical example the authors obtained an optimal replacement policy N^* that minimises the average cost rate of the system. However, there is little research on the maintenance strategy of a degenerate system with two components in parallel. Y. L. Li and G. Q. Xu [8] introduced a parallel repairable system with two similar components and a repairman who can take a single vacation. The authors obtained the existence of positive solution, the non-negative steady-state solution and the exponential stability of the system by using the functional analysis method and C_0-semigroup theory. Different from Li's study, in this paper, we mainly study a parallel repairable system with degeneration, namely, two components of the system cannot be repaired "as good as new" after failures, herein the repairman still has a single vacation. Since there is an essential difference between degeneration system and non-degeneration system, we mainly consider a replacement policy N of a parallel repairable degeneration system based on the failed times of component 1, because component 1 and component 2 are two similar components, and the number of failures of component 1 is equivalent to the number of failures of component 2 in a renewal cycle. At the same time, we propose a replacement policy N of the cold standby system consisting of the above two similar components. As a comparison, some numerical analyses are used to illustrate the theoretical results and the optimal replacement policy of the two models. The results show that, under the same assumptions, the parallel system with two similar components is better than the cold standby system, and the parallel system can produce much more profits.

The rest is organized as follows. In section 2, some assumptions on the system are introduced, and the explicit expression of the system average cost rate $C(N)$ is obtained according to the renewal reward theorem and the possible course of the system. At the same time, the expression of average cost rate $C_1(N)$ of the cold standby system is also discussed. In section 3, the numerical analysis is carried out to illustrate the existence and uniqueness of the optimal replacement policy of both models. The comparison of numerical results shows the advantages of the parallel system. The last section gives a conclusion of this paper.

2. Model description and average cost rate

In this section, the definition of the geometric process can be referred ([11]), we omit the definition details. Here, we consider a replacement policy N of the repairable degeneration system with two similar components in parallel. At the same time, we also study a replacement policy N of the cold standby repairable system consisting of the above two similar components.

The following is a list of notations used in this paper.

$X_n^{(i)}$	: working time of the component i in the n-th cycle($i = 1, 2.n = 1, 2, \cdots$);
$Y_n^{(i)}$	: repair time of the component i in the n-th cycle($i = 1, 2.n = 1, 2, \cdots$);
$Z_n^{(i)}$	: delayed repair time of the component i in the n-th cycle($i = 1, 2.n = 1, 2, \cdots$);
$S_n^{(i)}$	: standby time of the component i in the n-th cycle($i = 1, 2.n = 1, 2, \cdots$);
a, b	: positive cinstant,where $a > 0, 0 < b < 1$ are called the ratio of geometric process;
$\lambda > 0$	: failure rate;
$\mu > 0$	: repair rate;
$v > 0$	: delayed repair rate;
c_r	: repair cost rate of component $i(i = 1, 2)$;
c_w	: working reward rate of component $i(i = 1, 2)$;
c	: replacement cost of the system;
$W(W_1)$	: the length incurred of a renewal cycle;
$D(D_1)$	: the cost incurred of a renewal cycle;
$U(U_1)$	: total working time of the system;
V_1, V_2	: repair times of component i in a cycle($i = 1, 2$);
$C(N)$	: average cost rate of the parallel system under the policy N;
$C_1(N)$	: average cost rate of the cold standby system under the policy N.

2.1. Parallel repairable system with two similar components

Firstly, we give the following basic assumptions on the system with two similar components in parallel:

Assumption 1. The system consists of two similar components in parallel and one repairman with a single vacation. Initially, the system is new and in a working state. The repairman is taking his vacation.

Assumption 2. Suppose that the two components in the system cannot be repaired "as good as new" after failures. The time interval between the completion of the $(n-1)$-th repair and the completion of the n-th repair of component 1 is called the n-th cycle of the system.

$X_n^{(i)}(i = 1, 2)$ and $Y_n^{(i)}(i = 1, 2)$ respectively show the working time and the repair time of the two components in the n-th cycle, $n = 1, 2, \cdots$. Denote the distribution functions of $X_n^{(i)}(i = 1, 2)$ and $Y_n^{(i)}(i = 1, 2)$ by $F_n^{(i)}(t)$ and $G_n^{(i)}(t)$ respectively given as follows:

$$\begin{cases} F_n^{(i)}(t) = F(a^{n-1}t) = 1 - \exp(-a^{n-1}\lambda t), & i = 1, 2, \\ G_n^{(i)}(t) = G(b^{n-1}t) = 1 - \exp(-b^{n-1}\mu t), & i = 1, 2, \end{cases}$$

where $t \geq 0$, and $a > 0, 0 < b < 1, \lambda > 0, \mu > 0$.

Assumption 3. The repairman takes a single vacation. Single vacation means that the repairman will start to work when he returns from a vacation and finds that at least one failed component in the system waiting to be repaired, otherwise, he will wait idly in the system for the first failed component arrival, upon which he starts repairing it immediately. But when the failed component has been repaired, the repairman will go for a vacation again.

Assume that the two components of the system may not be repaired immediately after failure since the repairman has a single vacation. When a component fails and the repairman is on vacation, the repair of the component is delayed. When a component fails and the repairman is back from vacation and in the system, the faulty component can be repaired immediately. Herein, the vacation time of the repairman does not completely refer to the delay repair time of the component, the delay repair time here refers to the period from the component failure to the repairman's return to the system. It is not a fixed value, but a random variable. Let $Z_n^{(i)}(i = 1, 2)$ be the delayed repair time of component $i(i = 1, 2)$ in the n-th cycle and they have the same exponential distribution function $H(t) = 1 - \exp(-\nu t), (\nu > 0)$.

Assumption 4. A replacement policy N is based on the failure times of component 1. Namely, the system will be replaced by a new one if the failures of component 1 reach N.

Assumption 5. Random variables $X_n^{(i)}, Y_n^{(i)}, Z_n^{(i)}(i = 1, 2), n = 1, 2, \cdots$ are independent of each other.

Assumption 6. The repair cost rate of component i is $c_r(i = 1, 2)$, the working reward of these two components is c_w, and the replacement cost of the system is c.

Based on the above assumptions, a possible course of the repairable system is shown in Figure 1.

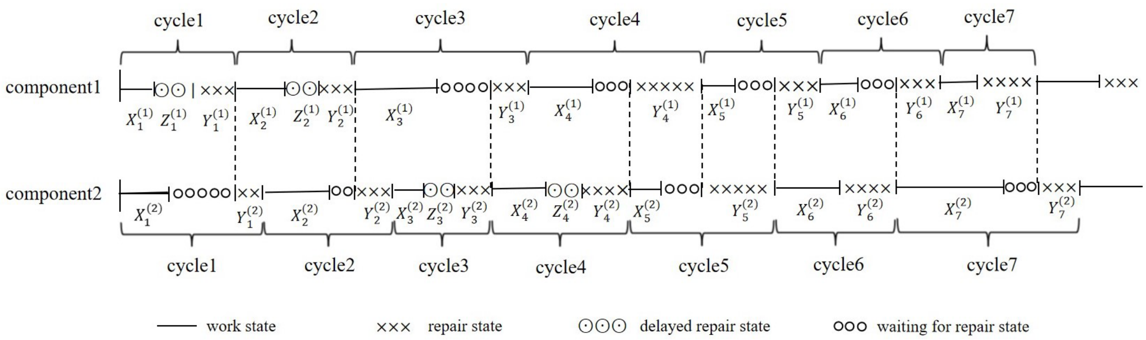

Figure 1. A possible course of the parallel repairable system with two components.

Remark 1. Assumption 4 shows that the system will be replaced if the failures of component 1 reach N-th, herein the system will be replaced, which means that component 1 and component 2 are replaced at the same time. It can be seen from Figure 1 that N-th failure of component 1 is equivalent to N-th failure of component 2. Therefore, it is reasonable to replace the system when the failures of component 1 reaches N-th.

Assumed that τ_1 be the time before the first replacement, and τ_n be the time between the $(n-1)$-th and n-th replacement of the system. And then, $\{\tau_n.n = 1, 2, \cdots\}$ forms a renewal process, from the renewal reward theorem we can get

$$C(N) = \frac{E[D]}{E[W]} \tag{2.1}$$

where W and D denote the length and the cost incurred of a renewal cycle, respectively.

In the system, since the repairman has a single vacation, the repair of component 1 is not always delayed after failures. If component 1 fails while component 2 is in the repair state or delayed repair state, component 1 must wait for the repair of component 2 to be finished. Therefore, Figure 1 is only the possible operation process of the system, from it we can see that component 1 has four states: working state, the delayed repair state, the repair state and the waiting for repair state in a renewal cycle. So we have

$$\begin{aligned} W \;=\;& \sum_{n=1}^{N} X_n^{(1)} + \sum_{n=1}^{N-1} Y_n^{(1)} + \sum_{n=1}^{N-1} Z_n^{(1)} \chi_{\{X_n^{(2)}-X_n^{(1)}>0\}} + \sum_{n=2}^{N-1} Z_n^{(1)} \chi_{\{X_n^{(2)}+Y_{n-1}^{(2)}-X_n^{(1)}>0\}} \\ &+ \sum_{n=2}^{N-1} (X_n^{(2)} + Z_n^{(2)} + Y_n^{(2)} + Y_{n-1}^{(2)} - X_n^{(1)}) \chi_{\{Y_{n-1}^{(2)}+X_n^{(2)}+Z_n^{(2)}+Y_n^{(2)}-X_n^{(1)}>0\}} \chi_{\{Y_{n-1}^{(1)}-X_{n-1}^{(2)}>0\}} \\ &+ \sum_{n=2}^{N-1} (Y_n^{(2)} - X_n^{(1)}) \chi_{\{Y_n^2-X_n^{(1)}>0\}} \chi_{\{Y_{n-1}^{(1)}-X_n^{(2)}>0\}} \\ &+ \sum_{n=2}^{N-1} (Y_n^{(2)} + X_n^{(2)} + Z_n^{(2)} - Y_{n-1}^{(1)} - X_n^{(1)}) \chi_{\{Y_n^{(2)}+X_n^{(2)}+Z_n^{(2)}-Y_{n-1}^{(1)}-X_n^{(1)}>0\}} \chi_{\{Y_{n-1}^{(2)}-X_{n-1}^{(1)}>0\}} \\ &+ \sum_{n=2}^{N-1} (X_n^{(2)} + Y_n^{(2)} - Y_{n-1}^{(1)} - X_n^{(1)}) \chi_{\{Y_n^{(2)}+X_n^{(2)}-Y_{n-1}^{(1)}-X_n^{(1)}>0\}} \chi_{\{Y_{n-1}^{(2)}-X_{n-1}^{(1)}>0\}} \end{aligned} \tag{2.2}$$

where χ is an indicator function, which is defined by

$$\chi_A = \begin{cases} 1, & \text{if event A occur,} \\ 0, & \text{if event A does not occur.} \end{cases}$$

When component 1 is in the delayed repair state, there are two kinds of cases: (i) During the operation of two components, component 1 first fails and the repairman is not in the system, which is denoted by $\chi_{\{X_n^{(2)}-X_n^{(1)}\}}$; (ii) During the repair of component 1, component 2 is in the waiting repair state, and then the repair of component 2 has been completed before component 1 fails again, which is denoted by $\chi_{\{X_n^{(2)}+Y_{n-1}^{(2)}-X_n^{(1)}>0\}}$. The third and fourth terms in Eq (2.2) represent these two cases respectively. As component 1 is in the waiting for repair state, there are four types of cases, they are respectively expressed as the fifth, sixth, seventh and eighth term in Eq (2.2). The fifth term means that when the repair time of component 1 is longer than the working time of component 2, the component 2 will fail during the repair of component 1, which is indicated by $\chi_{\{Y_{n-1}^{(1)}-X_{n-1}^{(2)}\}}$. Moreover the component 1 is in the waiting for repair state if and only if $Y_{n-1}^{(2)} + X_n^{(2)} + Z_n^{(2)} + Y_n^{(2)} - X_n^{(1)} > 0$ is true, that is indicated by $\chi_{\{Y_{n-1}^{(2)}+X_n^{(2)}+Z_n^{(2)}+Y_n^{(2)}-X_n^{(1)}>0\}}$. Other items can be obtained similarly,the details are omitted here.

According to the system assumptions, we have

$$E(X_n^{(i)}) = \frac{1}{a^{n-1}\lambda}, \;\; E(Y_n^{(i)}) = \frac{1}{b^{n-1}\mu}, \;\; E(Z_n^{(i)}) = \frac{1}{\nu} \quad (i = 1, 2). \tag{2.3}$$

Furthermore, according to the definition of convolution, the random variables as well as their distribution functions can be listed as follows

$$\begin{aligned}
X_n^{(2)} - X_n^{(1)}, &\quad \Phi_n(t) = F(a^{n-1}t) * [1 - F(-a^{n-1}t)];\\
Y_n^{(2)} - X_n^{(1)}, &\quad \Psi_n(t) = G(b^{n-1}t) * [1 - F(-a^{n-1}t)];\\
Y_{n-1}^{(1)} - X_n^{(2)}, &\quad \Omega_n(t) = G(b^{n-2}t) * [1 - F(-a^{n-1}t)];\\
X_{n-1}^{(1)} - Y_{n-1}^{(2)}, &\quad M_n(t) = F(a^{n-2}t) * [1 - G(-b^{n-2}t)];\\
X_n^{(2)} + Y_{n-1}^{(2)} - X_n^{(1)}, &\quad O_n(t) = F(a^{n-1}t) * G(b^{n-2}t) * [1 - F(-a^{n-1}t)];\\
X_n^{(2)} + Y_n^{(2)} + Z_n^{(2)} + Y_{n-1}^{(2)} - X_n^{(1)}, &\quad K_n(t) = F(a^{n-1}t) * G(b^{n-1}t) * H_n(t) * G(b^{n-2}t) * [1 - F(-a^{n-1}t)];\\
X_n^{(2)} + Y_n^{(2)} + Z_n^{(2)} - X_n^{(1)} - Y_{n-1}^{(1)}, &\quad P_n(t) = F(a^{n-1}t) * G(b^{n-1}t) * H_n(t) * [1 - F(-a^{n-1}t)]\\
&\qquad * [1 - G(-b^{n-2}t)];\\
X_n^{(2)} + Y_n^{(2)} - X_n^{(1)} - Y_{n-1}^{(1)}, &\quad \Theta_n(t) = F(a^{n-1}t) * G(b^{n-1}t) * [1 - F(-a^{n-1}t)] * [1 - G(-b^{n-2}t)].
\end{aligned}$$

Thus, we can calculate the expectation as following

$$\begin{aligned}
&E\left(\sum_{n=1}^{N-1} Z_n^{(1)} \chi_{\{X_n^{(2)}-X_n^{(1)}>0\}} + \sum_{n=2}^{N-1} Z_n^{(1)} \chi_{\{X_n^{(2)}+Y_{n-1}^{(2)}-X_n^{(1)}>0\}}\right) = \tfrac{1}{v}\left(\sum_{n=1}^{N-1}\Phi_n(0) + \sum_{n=2}^{N-1}(1 - O_n(0)\right);\\
&E\left(\sum_{n=2}^{N-1}(Y_n^{(2)} - X_n^{(1)})\chi_{\{Y_n^{(2)}-X_n^{(1)}>0\}}\chi_{\{Y_{n-1}^{(1)}-X_n^{(2)}>0\}} = \sum_{n=2}^{N-1}[1 - \Omega_n(0)]\int_0^\infty td\Psi_n(t);\right);\\
&E\left(\sum_{n=2}^{N-1}(Y_{n-1}^{(2)} + X_n^{(2)} + Z_n^{(2)} + Y_n^{(2)} - X_n^{(1)})\chi_{\{Y_{n-1}^{(2)}+X_n^{(2)}+Z_n^{(2)}+Y_n^{(2)}-X_n^{(1)}>0\}}\chi_{\{Y_{n-1}^{(1)}-X_{n-1}^{(2)}>0\}}\right) = \sum_{n=2}^{N-1} M_n(0)\int_0^\infty tdK_n(t);\\
&E\left(\sum_{n=2}^{N-1}(Y_n^{(2)} + X_n^{(2)} + Z_n^{(2)} - Y_{n-1}^{(1)} - X_n^{(1)})\chi_{\{Y_n^{(2)}+X_n^{(2)}+Z_n^{(2)}-Y_{n-1}^{(1)}-X_n^{(1)}>0\}}\chi_{\{Y_{n-1}^{(2)}-X_{n-1}^{(1)}>0\}}\right) = \sum_{n=2}^{N-1} M_n(0)\int_0^\infty tdP_n(t);\\
&E\left(\sum_{n=2}^{N-1}(X_n^{(2)} + Y_n^{(2)} - Y_{n-1}^{(1)} - X_n^{(1)})\chi_{\{Y_n^{(2)}+X_n^{(2)}-Y_{n-1}^{(1)}-X_n^{(1)}>0\}}\chi_{\{Y_{n-1}^{(2)}-X_{n-1}^{(1)}>0\}}\right) = \sum_{n=2}^{N-1} M_n(0)\int_0^\infty td\Theta_n(t).
\end{aligned}$$

Therefore, substituting all the above equations into (2.2), we can get the expected length of component 1 in a renewal cycle:

$$\begin{aligned}
E(W) &= \sum_{n=1}^{N}\frac{1}{a^{n-1}\lambda} + \sum_{n=1}^{N-1}\frac{1}{b^{n-1}\mu} + \frac{1}{v}[\sum_{n=1}^{N-1}\Phi_n(0) + \sum_{n=2}^{N-1}(1 - O_n(0)] + \sum_{n=2}^{N-1}[1 - \Omega_n(0)]\int_0^\infty td\Psi_n(t)\\
&\quad + \sum_{n=2}^{N-1} M_n(0)\int_0^\infty tdK_n(t) + \sum_{n=2}^{N-1} M_n(0)\int_0^\infty tdP_n(t) + \sum_{n=2}^{N-1} M_n(0)\int_0^\infty td\Theta_n(t) \qquad (2.4)
\end{aligned}$$

Since the system is a parallel system of two components, the total working time of the system is the largest of the working time of the two components, so the total working time U of the system in a cycle is

$$U = \sum_{n=1}^{N}\max(X_n^{(1)}, X_n^{(2)}).$$

Thus,

$$\begin{aligned}
E(U) &= \sum_{n=1}^{N} E\left[\max(X_n^{(1)}, X_n^{(2)})\right] = \sum_{n=1}^{N}\left\{E(X_n^{(1)}) + \int_0^\infty F(x)[1 - F(x)]dx\right\}\\
&= \sum_{n=1}^{N}\left\{\frac{1}{a^{n-1}\lambda} + \int_0^\infty (1 - e^{-a^{n-1}\lambda t})e^{-a^{n-1}\lambda t}dt\right\} = \frac{3}{2}\sum_{n=1}^{N}\frac{1}{a^{n-1}\lambda}. \qquad (2.5)
\end{aligned}$$

The total repair times of component 1 and component 2 in a cycle are $V_1 = \sum_{n=1}^{N-1} Y_n^{(1)}$ and $V_2 = \sum_{n=1}^{N-1} Y_n^{(2)}$, respectively. So

$$E(V_1) = E(V_2) = \sum_{n=1}^{N-1} \frac{1}{b^{n-1}\mu} \tag{2.6}$$

Finally, substituting (2.4),(2.5) and (2.6) into (2.1), the explicit expression of $C(N)$ under the policy N is given by

$$C(N) = \frac{2c_r \sum\limits_{n=1}^{N-1} \frac{1}{b^{n-1}\mu} + c - \frac{3}{2}c_w \sum\limits_{n=1}^{N} \frac{1}{a^{n-1}\lambda}}{\sum\limits_{n=1}^{N} \frac{1}{a^{n-1}\lambda} + \sum\limits_{n=1}^{N-1} \frac{1}{b^{n-1}\mu} + f_1 + f_2} \tag{2.7}$$

where

$$f_1 = \frac{1}{v}\left[\sum_{n=1}^{N-1} \Phi_n(0) + \sum_{n=2}^{N-1}(1 - O_n(0)] + \sum_{n=2}^{N-1}[1 - \Omega_n(0)\right] \int_0^\infty td\Psi_n(t),$$

$$f_2 = \sum_{n=2}^{N-1} M_n(0)\left[\int_0^\infty tdK_n(t) + \int_0^\infty tdP_n(t) + \int_0^\infty td\Theta_n(t)\right].$$

Moreover, by complicated calculation (see Appendix A.1), the Eq (2.7) is rewritten as follows:

$$C(N) = \frac{2c_r \sum\limits_{n=1}^{N-1} \frac{1}{b^{n-1}\mu} + c - \frac{3}{2}c_w \sum\limits_{n=1}^{N} \frac{1}{a^{n-1}\lambda}}{\sum\limits_{n=1}^{N} \frac{1}{a^{n-1}\lambda} + \sum\limits_{n=1}^{N-1} \frac{1}{b^{n-1}\mu} + \frac{1}{2v} + \frac{1}{v}\sum\limits_{n=2}^{N-1} \frac{b^{n-2}\mu + 2a^{n-1}\lambda}{2(b^{n-1}\mu + a^{n-1}\lambda)} + g_1}, \tag{2.8}$$

where

$$g_1 = \sum_{n=2}^{N-1} \frac{(a^{n-1}\lambda)^2}{b^{n-1}\mu(b^{n-1}\mu + a^{n-1}\lambda)(b^{n-2}\mu + a^{n-1}\lambda)} + \sum_{n=2}^{N-1} \frac{a^{n-2}\lambda[\int_0^\infty tdK_n(t) + \int_0^\infty tdP_n(t) + \int_0^\infty td\Theta_n(t)]}{(a^{n-2}\lambda + b^{n-2}\mu)}.$$

In the section 3, we will further verify the existence and uniqueness of the optimal replacement policy N^* by minimizing the average cost rate $C(N^*)$ through numerical analysis.

2.2. Cold standby repairable system with two similar components

In this subsection, we will consider the average cost rate of the cold standby repairable system consisting of the two similar components.

Firstly, we modify a few conditions.

Assumption 1'. The system is a cold standby repairable system with two similar components and one repairman with a single vacation. Initially, the system is new and the component 1 in a working state, the component 2 is in a cold standby state. The repairman is taking vacation.

Assumption 3'. Let $S_n^{(i)}$ be the standby time of component $i(i = 1, 2)$ in the n-th cycle and the distributions of $S_n^{(i)}$ is the same as the adjacent working time distributions of component $i(i = 1, 2)$.

The other assumptions are the same as that in the parallel system. A possible course of the cold standby repairable system is shown in Figure 2.

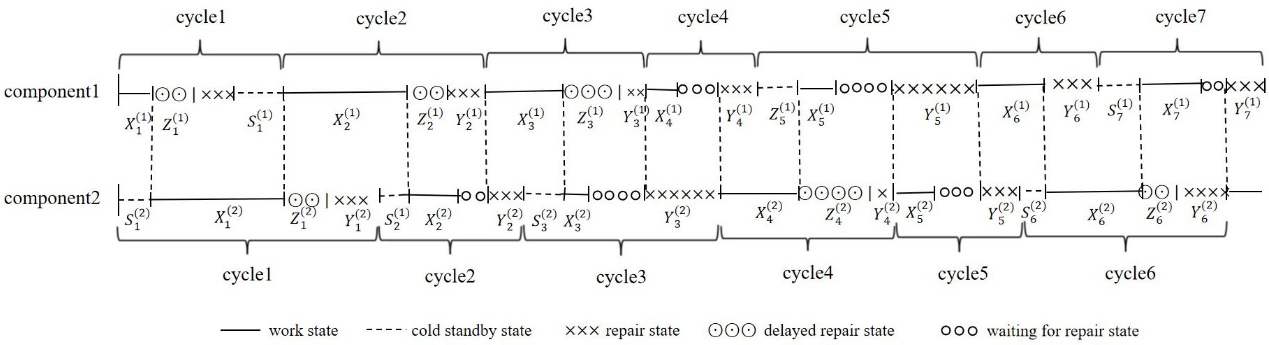

Figure 2. A possible course of the cold standby system with two components.

Let D_1 and W_1 represent the length and the cost incurred of a renewal cycle, respectively. Similarly, the average cost rate $C_1(N)$ of the cold standby system is given by

$$C_1(N) = \frac{E[D_1]}{E[W_1]}. \tag{2.9}$$

Note that, in a renewal cycle, the component 1 has five states: working state, the cold standby state, the delayed repair state, the repair state and the waiting for repair state. From the analysis of a possible course of the system, we see that the length of a renewal cycle is

$$\begin{aligned} W_1 \;=\; & \sum_{n=1}^{N} X_n^{(1)} + \sum_{n=1}^{N-1} Y_n^{(1)} + \sum_{n=1}^{N-1} Z_n^{(1)} \chi_{\{S_n^{(2)}-X_n^{(1)}>0\}} + \sum_{n=2}^{N-1} Z_n^{(2)} \chi_{\{Y_{n-1}^{(2)}+S_n^{(2)}-X_n^{(1)}>0\}} \\ & + \sum_{n=2}^{N-1} Z_n^{(2)} \chi_{\{Z_{n-1}^{(2)}+Y_{n-1}^{(2)}+S_n^{(2)}-X_n^{(1)}>0\}} \\ & + \sum_{n=2}^{N-1} (Y_{n-1}^{(2)} - X_n^{(1)}) \chi_{\{Y_{n-1}^{2}-X_n^{(1)}>0\}} \chi_{\{Y_{n-1}^{(1)}-X_{n-1}^{(2)}>0\}} \\ & + \sum_{n=2}^{N-1} (Z_{n-1}^{(2)} + Y_{n-1}^{(2)} - X_n^{(1)}) \chi_{\{Z_{n-1}^{(2)}+Y_{n-1}^{(2)}-X_n^{(1)}>0\}} \chi_{X_{n-1}^{(2)}-\{Y_{n-1}^{(1)}>0\}} \\ & + \sum_{n=1}^{N-1} (X_n^{(2)} - Z_n^{(1)} - Y_n^{(1)}) \chi_{\{X_n^{(2)}-Z_n^{(1)}-Y_n^{(1)}>0\}} \chi_{\{S_n^{(2)}-X_n^{(1)}>0\}} \\ & + \sum_{n=2}^{N-1} (X_n^{(2)} - Y_n^{(1)}) \chi_{\{X_n^{(2)}-Y_n^{(1)}>0\}} \chi_{\{Y_{n-1}^{(2)}-X_n^{(1)}>0\}} \\ & + \sum_{n=2}^{N-1} (X_n^{(2)} - Y_n^{(1)}) \chi_{\{X_n^{(2)}-Y_n^{(1)}>0\}} \chi_{\{X_n^{(1)}-Y_{n-1}^{(2)}>0\}}. \end{aligned}$$

By complicated calculation (see Appendix A.2) we get the expected length in a renewal cycle of the system

$$E[W_1] \;=\; \sum_{n=1}^{N} \frac{1}{a^{n-1}\lambda} + \sum_{n=1}^{N-1} \frac{1}{b^{n-1}\mu} + \frac{1}{v}[\sum_{n=1}^{N-1} \Phi_n(0) + \sum_{n=2}^{N-1}(1 - \Psi_n(0)) + \sum_{n=2}^{N-1}(1 - \Omega_n(0))]$$

$$+\sum_{n=2}^{N-1} O_n(0)\int_0^{\infty} tdM_n(t) + \sum_{n=2}^{N-1}[1-O_n(0)]\int_0^{\infty} tdK_n(t) + \sum_{n=2}^{N-1}\Phi_n(0)\int_0^{\infty} tdP_n(t)$$
$$+\sum_{n=2}^{N-1}\int_0^{\infty} td\theta_n(t) \tag{2.10}$$

Moreover, the total working time of the system is

$$\begin{aligned} U_1 \;=\; & \sum_{n=1}^{N} X_n^{(1)} + \sum_{n=1}^{N-1} X_n^{(2)} - \sum_{n=1}^{N-1}(X_n^{(2)} - Z_n^{(1)} - Y_n^{(1)} - S_{n+1}^{(1)})\chi_{\{X_n^{(2)}-Z_n^{(1)}-Y_n^{(1)}-S_{n+1}^{(1)}>0\}}\chi_{\{S_n^{(2)}-X_n^{(1)}>0\}} \\ & -\sum_{n=2}^{N-1}(X_n^{(2)} - Y_n^{(1)} - S_{n+1}^{(1)})\chi_{\{X_n^{(2)}-Y_n^{(1)}-S_{n+1}^{(1)}\}}\chi_{\{Y_{n-1}^{(2)}-X_n^{(1)}>0\}} \\ & -\sum_{n=2}^{N-1}(X_n^{(2)} - Y_n^{(1)} - S_{n+1}^{(1)})\chi_{\{X_n^{(2)}-Y_n^{(1)}-S_{n+1}^{(1)}\}}\chi_{\{X_n^{(1)}-Y_{n-1}^{(2)}>0\}}. \end{aligned}$$

Hence the expected total working time in a renewal cycle of the system is

$$E(U_1) = 2\sum_{n=1}^{N}\frac{1}{a^{n-1}\lambda} - \sum_{n=1}^{N-1}\Phi_n(0)\int_0^{\infty} tdR_n(t) - \sum_{n=2}^{N-1}\int_0^{\infty} tdW_n(t), \tag{2.11}$$

the calculation details of (16) are given in Appendix A.2.

Now, substituting Eqs (2.10) and (2.11) into Eq (2.9) and using the calculation results in Appendix A.2, an explicit expression of $C_1(N)$ is given by

$$C_1(N) = \frac{2c_r \sum\limits_{n=1}^{N-1}\frac{1}{b^{n-1}\mu} + c - c_w g_1}{\sum\limits_{n=1}^{N}\frac{1}{a^{n-1}\lambda} + \sum\limits_{n=1}^{N-1}\frac{1}{b^{n-1}\mu} + g_2 + g_3 + g_4} \tag{2.12}$$

where

$$g_1 = 2\sum_{n=1}^{N-1}\Big[\frac{1}{a^{n-1}\lambda} - \frac{b^{n-1}\mu v a^n\lambda}{2a^{n-1}\lambda(a^{n-1}\lambda+b^{n-1}\mu)(a^{n-1}\lambda+v)(a^{n-1}\lambda+a^n\lambda)} - \frac{b^{n-1}\mu a^n\lambda}{a^{n-1}\lambda(a^{n-1}\lambda+b^{n-1}\mu)(a^{n-1}\lambda+a^n\lambda)}\Big];$$
$$g_2 = \frac{1}{v}\Big\{\frac{N-1}{2} + \sum_{n=2}^{N-1}\Big[1 - \frac{b^{n-2}\mu}{2(b^{n-2}\mu+a^{n-1}\lambda)}\Big] + \sum_{n=2}^{N-1}\Big[1 - \frac{b^{n-2}\mu v}{2(a^{n-1}\lambda+v)(b^{n-2}\mu+a^{n-1}\lambda)}\Big]\Big\};$$
$$g_3 = \sum_{n=2}^{N-1}\Big\{\frac{a^{n-1}\lambda a^{n-2}\lambda}{b^{n-2}\mu(a^{n-2}\lambda+b^{n-2}\mu)(a^{n-1}\lambda+b^{n-2}\mu)} + \Big(1 - \frac{a^{n-2}\lambda}{a^{n-2}\lambda+b^{n-2}\mu}\Big)\Big[\frac{va^{n-1}\lambda}{b^{n-2}\mu(v-b^{n-2}\mu)(a^{n-1}\lambda+b^{n-2}\mu)} - \frac{b^{n-2}\mu a^{n-1}\lambda}{v(v-b^{n-2}\mu)(v+a^{n-1}\lambda)}\Big]\Big\};$$
$$g_4 = \sum_{n=2}^{N-1}\Big[\frac{vb^{n-1}\mu}{2a^{n-1}\lambda(a^{n-1}\lambda+b^{n-1}\mu)(a^{n-1}\lambda+v)} + \frac{b^{n-}1\mu}{a^{n-1}\lambda(a^{n-1}\lambda+b^{n-1}\mu)}\Big].$$

Summarizing discussions above, we mainly study a replacement policy N under different models of two similar components. By the renewal reward theorem, we get the explicit expressions of the average cost rate $C(N)$ and $C_1(N)$, respectively. From these expressions we cannot directly assert which model is better. In the coming section, we will give some numerical results for these two models and compare the values of average cost function. Furthermore, we will verify the existence and uniqueness of the optimal replacement policy N^* by the numerical analysis, and analyze and compare the optimal maintenance policy of the two models.

3. Numerical analysis

In this section, we will provide some numerical results of the average cost functions for these two kinds of repairable systems. By the numerical analysis, we illustrate the existence and uniqueness of the optimal replacement policy N^*. Moreover, the optimal replacement policy N^* of these two models will be compared and analyzed.

3.1. Numerical results of average cost functions

In this numerical analysis, the parameter values of the two kinds of systems are taken as follows:

$$a = 1.1,\ \ b = 0.9,\ \ \lambda = 0.01,\ \ \mu = 0.1,\ \ \nu = 0.11,\ \ c_r = 10,\ \ c = 2500,\ \ c_w = 9. \tag{3.1}$$

We will calculate the values of average cost rate $C(N)$ and $C_1(N)$ given by formulas (2.8) and (2.12), respectively. The replacement policy N is taken value in $[1, 30]$.

The curves of the average cost of both systems are pictured as shown in Figure 3.

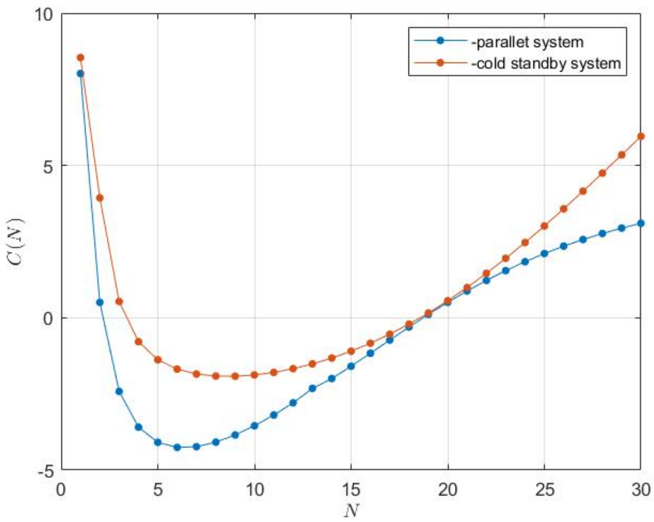

Figure 3. The blue curve presents the average cost of the parallel system and the red curve presents the one of the cold standby system.

Remark 2. The negative values of the average cost means that the system have positive benefits.

3.2. The optimal replacement policy N^*

In this subsection we are mainly concerned with the optimal replacement policy N^* of both systems. From Figure 3 we see that the curves of the average cost of both systems in variable N are similar to a quadratic curve, so the optimal replacement policy N^* may exist uniquely. In what follows we will discuss the optimal replacement policy N^* for the parallel system and the cold standby system, respectively.

Case 1. The optimal replacement policy of the parallel repairable system

Let the parameter values be given as in Eq (3.1) and the average cost rate $C(N)$ be given as in Eq (2.8) where failure times N be regarded as a variable. In the plane, the graph $(N, C(N))$ is pictured by the blue curve (see Figure 3). The change of system average cost rate $C(N)$ with the increasing of failure times N of component 1 is given in Table 1. The $C(N) < 0$ presents the benefits of the system.

Table 1. The systems expected costs.

N	$C(N)$	$C_1(N)$	N	C(N)	$C_1(N)$	N	C(N)	$C_1(N)$
1	8.02	8.54	11	-3.19	-1.79	21	0.88	0.99
2	0.51	3.94	12	-2.79	-1.67	22	1.23	1.46
3	-2.42	0.53	13	-2.32	-1.51	23	1.55	1.95
4	-3.59	-0.78	14	-2.00	-1.32	24	1.84	2.47
5	-4.09	-1.38	15	-1.59	-1.10	25	2.11	3.01
6	-4.25	-1.67	16	-1.17	-0.84	26	2.35	3.58
7	-4.23	-1.85	17	-0.73	-0.54	27	2.57	4.15
8	-4.08	-1.91	18	-0.13	-0.21	28	2.77	4.75
9	-3.85	-1.92	19	0.11	-0.16	29	2.94	5.35
10	-3.55	-1.88	20	0.51	-0.56	30	3.10	5.95

From Table 1 and Figure 3 we can see that the optimal replacement number of component 1 is $N^* = 6$ and the corresponding average cost rate is

$$C(N)_{min} = C(6) = -4.25.$$

That means the system has maximum benefits provided that two similar components of the system are replaced at the same time as the failures of component 1 reaches $N^* = 6$.

Obviously $C(N)$ is decreasing as $N < 6$ and increasing as $N > 6$. So $C(6) = -4.25$ is the minimum value of the expected cost, and the optimal replacement policy of component 1 is $N^* = 6$ that is unique.

Case 2. The optimal replacement policy of the cold standby repairable system

In the case of the cold standby system, we can calculate the values of $C_1(N)$ and picture the graph of the average cost of the system, please see Table 1 and the red curve in Figure 3.

Similarly, we can discuss the optimal replacement policy of the cold standby repairable system. Obviously, the optimal replacement times of component 1 is $N^* = 9$ and the corresponding average cost is

$$C(N)_{min} = C(9) = -1.92.$$

3.3. Comparing and conclusion

Based on the Table 1 and Figure 3 we compare the numerical results of the parallel repairable system and the cold standby system.

Note that, according to the meaning of $C(N)$, it is clear that the $C(N)$ should be negative values, and the smaller the better. For example, for a production system, if the $C(N)$ can take negative values, this means that the system has benefits.

1) When $N = 1, 2$ and $N \geq 21$, both systems have not benefits. In this case, it always holds that $0 < C(N) < C_1(N)$.

2) When $N = 3$, the parallel system has a benefits, but the cold standby system has not; In particular, when $N = 19, 20$, both systems have not benefits and $C(N) = C_1(N) = 0$.

3) When N takes its value in interval $[4, 18]$, it holds that $C(N) < C_1(N) < 0$ and $|C(N)| > |C_1(N)|$. This means that the benefits of the parallel system are larger than that of the cold standby system.

From the above results and analysis, we find out that, under the same assumptions, although the optimal replacement times of the parallel repairable system ($N^* = 6$) is lower than one of the cold standby system ($N^* = 9$), the minimum value of the expected cost rate $C(N^*) = -4.25$ is far smaller than $C_1(N^*) = -1.92$, that is, the profit of the parallel system is far larger than one of the cold standby system. Moreover, figure 3 shows that the blue curve is lower than the red one, indicating that the profit of the parallel system is larger than that of the cold standby system in the same renewal cycle, and then we can conclude that the parallel system with two similar components is better than the cold standby system. So we can assert that the parallel system can produce more profits.

In practice, for many factories, the warm standby system is better than the cold standby system. Generally speaking, the warm standby system means that one of the components is working and the other components are in the warm storage state, the components in the warm storage state are in the start-up state but do not work during the storage period. When the working component fails, the components in the storage state can immediately enter the working state, the components in the storage state have the possibility of failure during the storage period; The cold standby system means that one of the components is working and the other components are in the cold storage state. The components in the cold storage state are closed during the storage period and will not fail. When the working component fails, the components in the storage state will have a period of start-up time before work. In addition, for some large factories, the cost of restarting the components and starting to work will be relatively high. From these points of view, the warm standby system is better than the cold standby system. Besides, in the warm standby system, although the components in the storage state do not work , there is also the possibility of failure during the storage period, so from the perspective of plant benefits, both components are in working condition, which will maximize the benefit of factories. Therefore, parallel devices will be implemented in the actual operation of factories, which can create more profits.

4. Conclusions

This paper discusses the maintenance strategy of a parallel repairable degradation system with two similar components and a repairman who can take a single vacation. Under some assumptions, a replacement policy N based on the failed times of component 1 is studied. Using the renewal reward theorem, an explicit expression of the system's average cost rate $C(N)$ is deduced. To show the advantage of the parallel system, we considered, at the same time, the average cost rate $C_1(N)$ of the cold standby repairable system consisting of the above two similar components. The optimal replacement policy $N^*(N_1^*)$ by minimizing the $C(N)(C_1(N))$ are obtained by the numerical results of both models. In particular, the profits of both systems during N in $[3, 20]$ are compared.From the economic point of view, we can assert that, under the same assumptions, the parallel system with two similar components is better than the cold standby system, and the parallel system can produce more profits provided that N falls in the interval $[4, 18]$. In addition, in future study, in order to improve the reliability of the system, we can continue to discuss the inspection policy for parallel repairable degradation system with periodic and random inspections. It is possible for the system to fail after a period of time, and if the failure possibility can be detected, it can be repaired or replaced immediately, if the possibility fails to be detected, the system continues to run.

Acknowledgments

This research is supported by the Natural Science Foundation of China grant NSFC-61773277 and by the Natural Science Foundation of Qinghai Province (2018-ZJ-717) and by the Key project of Qinghai University for nationalities (2019XJZ10).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. J. Cheng, Y. H. Tang, M. M. Yu, Reliability of unidirectional closed three-part series and parallel cold storage repairable system, *Mathematica Applicata*, **30** (2017), 27–39.
2. W. L. Chen, System reliability analysis of retrial machine repair systems with warm standbys and a single server of working breakdown and recovery policy, *Systems Eng.*, **21** (2018), 59–69.
3. L. M. Hu, D. Q. Yue, Z. Y. Ma, Availability Analysis of a Repairable Series-Parallel System with Redundant Dependency∗, *J. Syst. Sci. Complex.*, **33** (2020), 446–460.
4. J. C. Ke, T. H. Liu, D. Y. Yang, Modelling of Machine Interference Problem with Unreliable Repairman and Standbys Imperfect Switchover, *Reliab. Eng. Syst. Safe.*, **174** (2018), 12–18.
5. Y. Lam, Geometric processes and replacement problem, *Acta Math. Appl. Sin-E*, **4** (1988), 366–377.
6. B. Liu, L. Cui, Y. Wen, F. Guo, A cold standby repairable system with the repairman having multiple vacations and operational, repair, and vacation times following phase-type distributions, *Commun. Stat-Theor M.*, **45** (2016), 850–858.
7. X. D. Li, C. Cheng, Reliability Analysis of Three Components with Priority in Repairable Cold Standby System, *Journal of Lanzhou Institute of Technology*, **25** (2018), 84–87.
8. Y. L. Li, G. Q. Xu, Analysis of two components parallel repairable system with vacation, *Commun. Stat-Theor M.*, **32** (2019), 21–26.
9. M. A. El-Damcese, N. S. Temraz, Analysis for a parallel repairable system with different failure modes, *Journal of Reliability and Statistical Studies*, **5** (2012), 95–106.
10. W. F. Wu, J. S. Song, K. X. Ling, Maintenance and replacement policy for a repairable multi-state system with regular preventive repairs, *Systems Engineering and Electronics*, **37** (2015), 1319–1324.
11. G. J. Wang, Y. L. Zhang, Optimal replacement policy for a two-dissimilar-component cold standby system with different repair actions, *Int. J. Syst. Sci.*, **47** (2016), 1021–1031.

12. W. Z. Yuan, G. Q. Xu, Modeling of a deteriorating system with repair satisfying general distribution, *Appl. Math. Comput.*, **218** (2012), 6340–6350.
13. D. Q. Yue, Q. Q. Gao, A geometric process model for deteriorating operating system with periodic and random inspections, *Qual. Technol. Quant. M.*, **14** (2016), 269–279.
14. Y. L. Zhang, G. J. Wang, An extended geometric process repair model for a cold standby repairable system with imperfect delayed repair, *International Journal of Systems Science: Operations and Logistics*, **3** (2016), 163–175.
15. Y. L. Zhang, G. J. Wang, An optimal age-replacement policy for a simple repairable system with delayed repair, *Commun. Stat-Theor M.*, **46** (2017), 2837–2850.
16. Y. G. Zhang, Y. C. Sun, L. B. Li, M. Zhao, Copula-based reliability analysis for a parallel system with a cold standby, *Commun. Stat-Theor M.*, **47** (2018), 562–582.
17. Y. L. Zhang, R. C. Yam, M. J. Zuo, Optimal replacement policy for multistate repairable system, *J. Oper. Res. Soc.*, **53** (2002), 1123–1127.

A.

A.1. Random variables and density functions

Let probability density functions of $X_n^{(i)}$, $Y_n^{(i)}$, $Z_n^{(i)}$ $(i = 1, 2)$ be $f_n(t), g_n(t), h(t)$, $n = 1, 2, \cdots$, respectively. Then the probability density functions of random variables $X_n^{(2)} - X_n^{(1)}$, $Y_n^{(2)} - X_n^{(1)}$, $Y_{n-1}^{(1)} - X_n^{(2)}, X_{n-1}^{(1)} - Y_{n-1}^{(2)}$, $X_n^{(2)} + Y_{n-1}^{(2)} - X_n^{(1)}$, $X_n^{(2)} + Y_n^{(2)} + Z_n^{(2)} + Y_{n-1}^{(2)} - X_n^{(1)}$ and $X_n^{(2)} + Y_n^{(2)} + Z_n^{(2)} - X_n^{(1)} - Y_{n-1}^{(1)}, X_n^{(2)} + Y_n^{(2)} - Y_{n-1}^{(1)} - X_n^{(1)}$ are given by

$$\phi_n(t) = f_n(t) * f_n(-t) = \begin{cases} \frac{1}{2}a^{n-1}\lambda e^{-a^{n-1}\lambda t}, & t > 0, \\ \frac{1}{2}a^{n-1}\lambda e^{a^{n-1}\lambda t}, & t < 0; \end{cases}$$

$$\psi_n(t) = g_n(t) * f_n(-t) = \begin{cases} \frac{b^{n-1}\mu a^{n-1}\lambda}{b^{n-1}\mu + a^{n-1}\lambda} e^{-b^{n-1}\mu t}, & t > 0, \\ \frac{b^{n-1}\mu a^{n-1}\lambda}{b^{n-1}\mu + a^{n-1}\lambda} e^{a^{n-1}\lambda t}, & t < 0; \end{cases}$$

$$\varpi_n(t) = g_{n-1}(t) * f_n(-t) = \begin{cases} \frac{b^{n-2}\mu a^{n-1}\lambda}{b^{n-2}\mu + a^{n-1}\lambda} e^{-b^{n-2}\mu t}, & t > 0, \\ \frac{b^{n-2}\mu a^{n-1}\lambda}{b^{n-2}\mu + a^{n-1}\lambda} e^{a^{n-1}\lambda t}, & t < 0; \end{cases}$$

$$m_n(t) = f_{n-1}(t) * g_{n-1}(-t) = \begin{cases} \frac{a^{n-2}\lambda b^{n-2}\mu}{a^{n-2}\lambda + b^{n-2}\mu} e^{-a^{n-2}\lambda t}, & t > 0, \\ \frac{a^{n-2}\lambda b^{n-2}\mu}{a^{n-2}\lambda + b^{n-2}\mu} e^{b^{n-2}\mu t}, & t < 0; \end{cases}$$

$$O_n(t) = f_n(t) * g_{n-1}(t) * f_n(-t) = \begin{cases} \frac{b^{n-2}\mu a^{n-1}\lambda}{2(b^{n-2}\mu - a^{n-1}\lambda)} e^{-a^{n-1}\lambda t} - \frac{b^{n-2}\mu (a^{n-1}\lambda)^2}{(b^{n-2}\mu - a^{n-1}\lambda)(b^{n-2}\mu + a^{n-1}\lambda)} e^{-b^{n-2}\mu t}, & t > 0, \\ \frac{b^{n-2}\mu a^{n-1}\lambda}{2(b^{n-2}\mu + a^{n-1}\lambda)} e^{a^{n-1}\lambda t}, & t < 0; \end{cases}$$

$$k_n(t) = f_n(t) * g_n(t) * h_n(t) * g_{n-1}(t) * f_n(-t)$$

$$
= \begin{cases}
\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{(b^{n-1}\mu-v)(v-b^{n-2}\mu)(a^{n-1}\lambda-v)}\left(\frac{1}{a^{n-1}\lambda+b^{n-2}\mu}e^{-b^{n-2}\mu t}-\frac{1}{a^{n-1}\lambda+v}e^{-vt}\right) \\
-\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{(a^{n-1}\lambda-b^{n-1}\mu)(b^{n-1}\mu-v)(b^{n-1}\mu-b^{n-2}\mu)}\left(\frac{1}{b^{n-2}\mu+a^{n-1}\lambda}e^{-b^{n-2}\mu t}-\frac{1}{b^{n-1}\mu+a^{n-1}\lambda}e^{-b^{n-1}\mu t}\right) \\
+\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{(a^{n-1}\lambda-b^{n-1}\mu)(a^{n-1}\lambda-v)(a^{n-1}\lambda-b^{n-2}\mu)}\left(\frac{1}{b^{n-2}\mu+a^{n-1}\lambda}e^{-b^{n-2}\mu t}-\frac{1}{2a^{n-1}\lambda}e^{-a^{n-1}\lambda t}\right), & t>0, \\
\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{(b^{n-1}\mu-v)(v-b^{n-2}\mu)(a^{n-1}\lambda-v)}\left(\frac{1}{a^{n-1}\lambda+b^{n-2}\mu}-\frac{1}{a^{n-1}\lambda+v}\right)e^{a^{n-1}\lambda t} \\
-\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{(a^{n-1}\lambda-b^{n-1}\mu)(b^{n-1}\mu-v)(b^{n-1}\mu-b^{n-2}\mu)}\left(\frac{1}{b^{n-2}\mu+a^{n-1}\lambda}-\frac{1}{b^{n-1}\mu+a^{n-1}\lambda}\right)e^{a^{n-1}\lambda t} \\
+\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{(a^{n-1}\lambda-b^{n-1}\mu)(a^{n-1}\lambda-v)(a^{n-1}\lambda-b^{n-2}\mu)}\left(\frac{1}{b^{n-2}\mu+a^{n-1}\lambda}-\frac{1}{2a^{n-1}\lambda}\right)e^{a^{n-1}\lambda t}, & t<0;
\end{cases}
$$

$$
\begin{aligned}
p_n(t) &= f_n(t) * g_n(t) * h_n(t) * f_n(-t) * g_{n-1}(-t) \\
&= \begin{cases}
\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{(a^{n-1}\lambda-v)(b^{n-1}\mu-v)(a^{n-1}\lambda+v)(b^{n-2}\mu+v)}e^{-vt} \\
-\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{(a^{n-1}\lambda-b^{n-1}\mu)(b^{n-1}\mu-v)(b^{n-1}\mu+a^{n-1}\lambda)(b^{n-1}\mu+b^{n-2}\mu)}e^{-b^{n-1}\mu t} \\
+\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{2a^{n-1}\lambda(a^{n-1}\lambda-b^{n-1}\mu)(a^{n-1}\lambda-v)(a^{n-1}\lambda+b^{n-2}\mu)}e^{-a^{n-1}\lambda t}, & t>0, \\
\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{(a^{n-1}\lambda-v)(b^{n-1}\mu-v)(a^{n-1}\lambda+v)(b^{n-2}\mu+v)}e^{b^{n-2}\mu t} \\
-\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{(a^{n-1}\lambda-b^{n-1}\mu)(b^{n-1}\mu-v)(b^{n-1}\mu+a^{n-1}\lambda)(b^{n-1}\mu+b^{n-2}\mu)}e^{b^{n-2}\mu t} \\
+\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{2a^{n-1}\lambda(a^{n-1}\lambda-b^{n-1}\mu)(a^{n-1}\lambda-v)(a^{n-1}\lambda+b^{n-2}\mu)}e^{b^{n-2}\mu t} \\
+\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v b^{n-2}\mu}{2a^{n-1}\lambda(a^{n-1}\lambda+v)(b^{n-1}\mu+a^{n-1}\lambda)(a^{n-1}\lambda-b^{n-2}\mu)}(e^{b^{n-2}\mu t}-e^{a^{n-1}\lambda}), & t<0;
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\theta_n(t) &= f_n(t) * g_n(t) * f_n(-t) * g_{n-1}(t) \\
&= \begin{cases}
\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu b^{n-2}\mu}{(a^{n-1}\lambda-b^{n-1}\mu)(b^{n-1}\mu+a^{n-1}\lambda)(b^{n-1}\mu+b^{n-2}\mu)}e^{-b^{n-1}\mu t} \\
-\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu b^{n-2}\mu}{2a^{n-1}\lambda(a^{n-1}\lambda-b^{n-1}\mu)(a^{n-1}\lambda+b^{n-2}\mu)}e^{-a^{n-1}\lambda t}, & t>0, \\
\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu b^{n-2}\mu}{(a^{n-1}\lambda-b^{n-1}\mu)(b^{n-1}\mu+a^{n-1}\lambda)(b^{n-1}\mu+b^{n-2}\mu)}e^{b^{n-2}\mu t} \\
-\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu b^{n-2}\mu}{2a^{n-1}\lambda(a^{n-1}\lambda-b^{n-1}\mu)(a^{n-1}\lambda+b^{n-2}\mu)}e^{b^{n-2}\mu t} \\
+\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu b^{n-2}\mu}{2a^{n-1}\lambda(b^{n-1}\mu+a^{n-1}\lambda)(a^{n-1}\lambda-b^{n-2}\mu)}(e^{b^{n-2}\mu t-e^{a^{n-1}\lambda t}}) & t<0.
\end{cases}
\end{aligned}
$$

Using the probability density function above, we can get

$$
\Phi_n(0)=\int_{-\infty}^{0}\phi_n(t)dt=\frac{1}{2}; \quad O_n(0)=\int_{-\infty}^{0}o_n(t)dt=\frac{b^{n-2}\mu}{2(a^{n-1}\lambda+b^{n-2}\mu)};
$$

$$
\Omega_n(0)=\int_{-\infty}^{0}\varpi_n(t)dt=\frac{b^{n-2}\mu}{b^{n-2}\mu+a^{n-1}\lambda}; \quad M_n(0)=\int_{-\infty}^{0}m_n(t)dt=\frac{a^{n-2}\lambda}{b^{n-2}\mu+a^{n-2}\lambda}
$$

$$
\int_{0}^{\infty}td\Psi_n(t)=\frac{a^{n-1}\lambda}{b^{n-1}\mu(b^{n-1}\mu+a^{n-1}\lambda)};
$$

$$
\begin{aligned}
\int_{0}^{\infty}tdk_n(t) &= \frac{(a^{n-1}\lambda)^2 b^{n-1}\mu v}{b^{n-2}\mu(b^{n-1}\mu-v)(v-b^{n-2}\mu)(a^{n-1}\lambda-v)(a^{n-1}\lambda+b^{n-2}\mu)} \\
&\quad -\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu b^{n-2}\mu}{v(b^{n-1}\mu-v)(v-b^{n-2}\mu)(a^{n-1}\lambda-v)(a^{n-1}\lambda+v)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu\nu}{b^{n-2}\mu(a^{n-1}\lambda-b^{n-1}\mu)(b^{n-1}\mu-\nu)(b^{n-1}\mu-b^{n-2}\mu)(b^{n-2}\mu+a^{n-1}\lambda)}\\
&+\frac{(a^{n-1}\lambda)^2 b^{n-2}\mu\nu}{b^{n-1}\mu(a^{n-1}\lambda-b^{n-1}\mu)(b^{n-1}\mu-\nu)(b^{n-1}\mu-b^{n-2}\mu)(b^{n-1}\mu+a^{n-1}\lambda)}\\
&+\frac{(a^{n-1}\lambda)^2 b^{n-1}\mu\nu}{b^{n-2}\mu(a^{n-1}\lambda-b^{n-1}\mu)(a^{n-1}\lambda-\nu)(a^{n-1}\lambda-b^{n-2}\mu)(b^{n-2}\mu+a^{n-1}\lambda)}\\
&-\frac{b^{n-1}\mu\nu b^{n-2}\mu}{2a^{n-1}\lambda(a^{n-1}\lambda-b^{n-1}\mu)(a^{n-1}\lambda-\nu)(a^{n-1}\lambda-b^{n-2}\mu)};
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty t dp_n(t) \quad = \quad & \frac{(a^{n-1}\lambda)^2 b^{n-1}\mu b^{n-2}\mu}{\nu(a^{n-1}\lambda-\nu)(b^{n-1}\mu-\nu)(a^{n-1}\lambda+\nu)(b^{n-2}\mu+\nu)}\\
&-\frac{(a^{n-1}\lambda)^2 \nu b^{n-2}\mu}{b^{n-1}\mu(a^{n-1}\lambda-b^{n-1}\mu)(b^{n-1}\mu-\nu)(b^{n-1}\mu+a^{n-1}\lambda)(b^{n-1}\mu+b^{n-2}\mu)}\\
&+\frac{b^{n-1}\mu\nu b^{n-2}\mu}{2a^{n-1}\lambda(a^{n-1}\lambda-b^{n-1}\mu)(a^{n-1}\lambda-\nu)(a^{n-1}\lambda+b^{n-2}\mu)};
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty t d\theta_n(t) \quad = \quad & \frac{(a^{n-1}\lambda)^2 b^{n-2}\mu}{b^{n-1}\mu(a^{n-1}\lambda-b^{n-1}\mu)(b^{n-1}\mu+a^{n-1}\lambda)(b^{n-1}\mu+b^{n-2}\mu)}\\
&-\frac{b^{n-1}\mu b^{n-2}\mu}{2a^{n-1}\lambda(a^{n-1}\lambda-b^{n-1}\mu)(a^{n-1}\lambda+b^{n-2}\mu)}.
\end{aligned}
$$

A.2. Random variables and distribution functions

According to the definition of convolution, the following distribution functions can be obtained

$$
\begin{array}{lll}
X_n^{(1)}-S_n^{(2)}, & \Phi_n(t)= & F(a^{n-1}t)*[1-F(-a^{n-1}t)];\\
Y_{n-1}^{(2)}+S_n^{(2)}-X_n^{(1)}, & \Psi_n(t)= & G(b^{n-2}t)*F(a^{n-1}t)*[1-F(-a^{n-1}t)]\\
Z_{n-1}^{(2)}+Y_{n-1}^{(2)}+S_n^{(2)}-X_n^{(1)}, & \Omega_n(t)= & H(t)*G(b^{n-2}t)*F(a^{n-1}t)*[1-F(-a^{n-1}t)];\\
Y_{n-1}^{(2)}-X_n^{(1)}, & M_n(t)= & G(b^{n-2}t)*[1-F(-a^{n-1}t)];\\
X_{n-1}^{(2)}-Y_{n-1}^{(1)}, & O_n(t)= & F(a^{n-2}t)*[1-G(-b^{n-2}t)]\\
Z_{n-1}^{(2)}+Y_{n-1}^{(2)}-X_n^{(1)}, & K_n(t)= & H(t)*G(b^{n-2}t)*[1-F(-a^{n-1}t)];\\
X_n^{(2)}-Z_n^{(1)}-Y_n^{(1)}, & P_n(t)= & F(a^{n-1}t)*[1-H(t)]*[1-G(-b^{n-1}t)];\\
X_n^{(2)}-Y_n^{(1)}, & \Theta_n(t)= & F(a^{n-1}t)*[1-G(-b^{n-1}t)]\\
X_n^{(2)}-Z_n^{(1)}-Y_n^{(1)}-S_{n+1}^{(1)}, & R_n(t)= & F(a^{n-1}t)*[1-H(t)]*[1-G(-b^{n-1}t)]*[1-F(-a^nt)];\\
X_n^{(2)}-Y_n^{(1)}-S_{n+1}^{(1)}, & W_n(t)= & F(a^{n-1}t)*[1-G(-b^{n-1}t)]*[1-F(-a^nt)].
\end{array}
$$

Similar to the previous subsection, the probability density function of the above random variables are as follows:

$$
\phi_n(t) \quad = f_n(t)*s_n(-t)=\begin{cases} \frac{1}{2}a^{n-1}\lambda e^{-a^{n-1}\lambda t}, & t>0,\\ \frac{1}{2}a^{n-1}\lambda e^{a^{n-1}\lambda t}, & t<0[\end{cases}
$$

$$
\psi_n(t) \quad = \quad g_{n-1}(t)*s_n(t)*f_n(-t)
$$

$$= \begin{cases} \frac{b^{n-2}\mu a^{n-1}\lambda}{2(b^{n-2}\mu-a^{n-1}\lambda)}e^{-a^{n-1}\lambda t} - \frac{(a^{n-1}\lambda)^2 b^{n-2}\mu}{(b^{n-2}\mu-a^{n-1}\lambda)(b^{n-2}\mu+a^{n-1}\lambda)}e^{-b^{n-2}\mu t}, & t > 0, \\ \frac{b^{n-2}\mu a^{n-1}\lambda}{2(b^{n-2}\mu+a^{n-1}\lambda)}e^{a^{n-1}\lambda t}, & t < 0; \end{cases}$$

$$\begin{aligned} \varpi_n(t) &= g_{n-1}(t) * s_n(t) * h_{n-1}(-t) * f_n(-t) \\ &= \begin{cases} \frac{b^{n-2}\mu(a^{n-1}\lambda)^2 v}{(a^{n-1}\lambda-v)(a^{n-1}\lambda+v)(b^{n-2}\mu-v)}e^{-vt} - \frac{b^{n-2}\mu a^{n-1}\lambda v}{2(b^{n-2}\mu-a^{n-1}\lambda)(a^{n-1}\lambda-v)}e^{-a^{n-1}\lambda t} \\ +\frac{b^{n-2}\mu(a^{n-1}\lambda)^2 v}{(b^{n-2}\mu-a^{n-1}\lambda)(b^{n-2}\mu-v)(a^{n-1}\lambda+b^{n-2}\mu)}e^{-b^{n-2}\mu t}, & t > 0, \\ \frac{b^{n-2}\mu a^{n-1}\lambda v}{2(a^{n-1}\lambda+v)(b^{n-2}\mu+a^{n-1}\lambda)}e^{a^{n-1}\lambda t}, & t < 0; \end{cases} \end{aligned}$$

$$m_n(t) = g_{n-1}(t) * f_n(-t) = \begin{cases} \frac{a^{n-1}\lambda b^{n-2}\mu}{(a^{n-1}\lambda+b^{n-2}\mu)}e^{-b^{n-2}\mu t}, & t > 0, \\ \frac{a^{n-1}\lambda b^{n-2}\mu}{(a^{n-1}\lambda+b^{n-2}\mu)}e^{a^{n-1}\lambda t}, & t < 0; \end{cases}$$

$$O_n(t) = f_{n-1}(t) * g_{n-1}(-t) = \begin{cases} \frac{a^{n-2}\lambda b^{n-2}\mu}{(a^{n-2}\lambda+b^{n-2}\mu)}e^{-a^{n-2}\lambda t}, & t > 0, \\ \frac{a^{n-2}\lambda b^{n-2}\mu}{(a^{n-2}\lambda+b^{n-2}\mu)}e^{b^{n-2}\mu t}, & t < 0; \end{cases}$$

$$\begin{aligned} k_n(t) &= z_{n-1}(t) * g_{n-1}(t) * f_n(-t) \\ &= \begin{cases} \frac{vb^{n-2}\mu a^{n-1}\lambda}{(v-b^{n-2}\mu)(a^{n-1}\lambda+b^{n-2}\mu)}e^{-b^{n-2}\mu t} - \frac{vb^{n-2}\mu a^{n-1}\lambda}{(v-b^{n-2}\mu)(v+a^{n-1}\lambda)}e^{-vt}, & t > 0, \\ \frac{vb^{n-2}\mu a^{n-1}\lambda}{(v+a^{n-1}\lambda)(a^{n-1}\lambda+b^{n-2}\mu)}e^{a^{n-1}\lambda t}, & t < 0; \end{cases} \end{aligned}$$

$$\begin{aligned} p_n(t) &= f_n(t) * g_n(-t) * z_n(-t) \\ &= \begin{cases} \frac{a^{n-1}\lambda b^{n-1}\mu v}{(a^{n-1}\lambda+b^{n-1}\mu)(a^{n-1}\lambda+v)}e^{-a^{n-1}\lambda t}, & t > 0, \\ \frac{a^{n-1}\lambda b^{n-1}\mu v}{(a^{n-1}\lambda+v)(b^{n-1}\mu-v)}e^{vt} - \frac{a^{n-1}\lambda b^{n-1}\mu v}{(a^{n-1}\lambda+b^{n-1}\mu)(b^{n-1}\mu-v)}e^{b^{n-1}\mu t}, & t < 0; \end{cases} \end{aligned}$$

$$\theta_n(t) = f_n(t) * g_n(-t) = \begin{cases} \frac{a^{n-1}\lambda b^{n-1}\mu}{(a^{n-1}\lambda+b^{n-1}\mu)}e^{-a^{n-1}\lambda t}, & t > 0, \\ \frac{a^{n-1}\lambda b^{n-1}\mu}{(a^{n-1}\lambda+b^{n-1}\mu)}e^{b^{n-1}\mu t}, & t < 0; \end{cases}$$

$$\begin{aligned} w_n(t) &= f_n(t) * g_n(-t) * s_{n+1}(-t) \\ &= \begin{cases} \frac{a^{n-1}\lambda b^{n-1}\mu a^n\lambda}{(a^{n-1}\lambda+b^{n-1}\mu)(a^{n-1}\lambda+a^n\lambda)}e^{-a^{n-1}\lambda t}, & t > 0, \\ \frac{a^{n-1}\lambda b^{n-1}\mu a^n\lambda}{(a^{n-1}\lambda+a^n\lambda)(b^{n-1}\mu-a^n\lambda)}e^{a^n\lambda t} - \frac{a^{n-1}\lambda b^{n-1}\mu a^n\lambda}{(a^{n-1}\lambda+b^{n-1}\mu)(b^{n-1}\mu-a^n\lambda)}e^{b^{n-1}\mu t}, & t < 0; \end{cases} \end{aligned}$$

$$\begin{aligned} r_n(t) &= f_n(t) * z_n(-t) * g_n(-t) * s_{n+1}(-t) \\ &= \begin{cases} \frac{a^{n-1}\lambda b^{n-1}\mu v a^n\lambda}{(a^{n-1}\lambda+b^{n-1}\mu)(a^{n-1}\lambda+v)(a^{n-1}\lambda+a^n\lambda)}e^{-a^{n-1}\lambda t}, & t > 0, \\ \frac{a^{n-1}\lambda b^{n-1}\mu v a^n\lambda}{(a^{n-1}\lambda+b^{n-1}\mu)(a^{n-1}\lambda+v)(a^{n-1}\lambda+a^n\lambda)}e^{a^n\lambda t} + \frac{a^{n-1}\lambda b^{n-1}\mu v a^n\lambda}{(b^{n-1}\mu-v)(a^{n-1}\lambda+v)(v-a^n\lambda)}(e^{a^n\lambda t - e^{vt}}) \\ -\frac{a^{n-1}\lambda b^{n-1}\mu v a^n\lambda}{(b^{n-1}\mu-v)(a^{n-1}\lambda+b^{n-1}\mu)(b^{n-1}\mu-a^n\lambda)}(e^{a^n\lambda t - e^{b^{n-1}\mu}}), & t < 0. \end{cases} \end{aligned}$$

Further, we can get

$$\begin{aligned}
&\Phi_n(0) = \tfrac{1}{2}, && \int_0^\infty t dP_n(t) = \tfrac{b^{n-1}\mu v}{a^{n-1}\lambda(a^{n-1}\lambda+b^{n-1}\mu)(a^{n-1}\lambda+v)},\\
&O_n(0) = \tfrac{a^{n-2}\lambda}{a^{n-2}\lambda+b^{n-2}\mu}, && \int_0^\infty t dW_n(t) = \tfrac{b^{n-1}\mu a^n\lambda}{a^{n-1}\lambda(a^{n-1}\lambda+b^{n-1}\mu)(a^{n-1}\lambda+a^n\lambda)},\\
&\Psi_n(0) = \tfrac{b^{n-2}\mu}{2(b^{n-2}\mu+a^{n-1}\lambda)}, && \int_0^\infty t dR_n(t) = \tfrac{b^{n-1}\mu v a^n\lambda}{a^{n-1}\lambda(a^{n-1}\lambda+b^{n-1}\mu)(a^{n-1}\lambda+v)(a^{n-1}\lambda+a^n\lambda)},\\
&\Omega_n(0) = \tfrac{b^{n-2}\mu v}{2(a^{n-1}\lambda+v)(b^{n-2}\mu+a^{n-1}\lambda)}, && \int_0^\infty t dM_n(t) = \tfrac{a^{n-1}\lambda}{b^{n-2}\mu(a^{n-1}\lambda+b^{n-2}\mu)},\\
& && \int_0^\infty t d\Theta_n(t) = \tfrac{b^{n-1}\mu}{a^{n-1}\lambda(a^{n-1}\lambda+b^{n-1}\mu)},
\end{aligned}$$

and

$$\int_0^\infty t dK_n(t) = \frac{a^{n-1}\lambda v}{b^{n-2}\mu(v-b^{n-2}\mu)(a^{n-1}\lambda+b^{n-2}\mu)} - \frac{b^{n-2}\mu a^{n-1}\lambda v}{v(v-b^{n-2}\mu)(v+a^{n-1}\lambda)}.$$