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Research article

On the generalized Ramanujan-Nagell equation $x^2 + (2k - 1)^y = k^z$ with $k \equiv 3 \pmod 4$

Yahui Yu[1] **and Jiayuan Hu**[2,*]

[1] Department of Mathematics and Physics, Luoyang Institute of Science and Technology, Luoyang, Henan, 471023, China

[2] Department of Mathematics and Computer, Hetao College, Bayannur, Inner Mongolia, 015000, China

* **Correspondence:** Email: hujiayuan1986@163.com.

Abstract: Let k be a fixed positive integer with $k > 1$. In 2014, N. Terai [6] conjectured that the equation $x^2 + (2k - 1)^y = k^z$ has only the positive integer solution $(x, y, z) = (k - 1, 1, 2)$. This is still an unsolved problem as yet. For any positive integer n, let $Q(n)$ denote the squarefree part of n. In this paper, using some elementary methods, we prove that if $k \equiv 3 \pmod 4$ and $Q(k - 1) \geq 2.11 \log k$, then the equation has only the positive integer solution $(x, y, z) = (k-1, 1, 2)$. It can thus be seen that Terai's conjecture is true for almost all positive integers k with $k \equiv 3 (\mathrm{mod}\ 4)$.

Keywords: polynomial-exponential diophantine equation; generalized Ramanujan-Nagell equation
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1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let k be a fixed positive integer with $k > 1$. In this paper, we deal with an exponential generalized Ramanujan-Nagell equation with the form

$$x^2 + (2k - 1)^y = k^z, x, y, z \in \mathbb{N}. \tag{1.1}$$

In 2014, N. Terai [6] proposed the following conjecture:
Conjecture 1.1. (1.1) has only the solution $(x, y, z) = (k - 1, 1, 2)$.

Obviously, if $k \equiv 2 (\mathrm{mod}\ 4)$, then $2k - 1 \equiv 3 (\mathrm{mod}\ 8)$. Since $z > 1$, by (1.1), we have $2 \nmid x, 2 \nmid y$ and $k^z \equiv x^2 + (2k - 1)^y \equiv 1 + 3 \equiv 4 (\mathrm{mod}\ 8)$. It implies that $z = 2$. Therefore, Conjecture 1.1 is true for $k \equiv 2 (\mathrm{mod}\ 4)$. However, in addition to this case, it is only proved in some special cases (see [1–4, 6, 7]). For example, M. J. Deng, J. Guo and A. J. Xu [3] gave some conditions for (1.1) to have solutions

(x, y, z) with $(x, y, z) \neq (k-1, 1, 2)$. So they proved that if $k \equiv 3 \pmod 4$ and $k < 500$, then Conjecture 1.1 is true.

For any fixed positive r, there exist unique positive integers d and s such that $r = ds^2$ and d is square-free. Such d is call the square-free part of r, and denoted by $Q(r)$. In this paper, using some elementary methods, we prove a general result as follows:

Theorem 1.1. If $k \equiv 3 (\mathrm{mod}\ 4)$ and $Q(k-1) \geq 2.65 \log k$, then (1.1) has only the solution $(x, y, z) = (k-1, 1, 2)$.

By the above theorem, we can deduce the following corollaries:

Corollary 1.1. If $k \equiv 3 (\mathrm{mod}\ 4)$ and $k-1$ is square-free, then Conjecture 1.1 is true.

Corollary 1.2. If $k \equiv 3 (\mathrm{mod}\ 4)$ and $500 < k < 1000$, then Conjecture 1.1 is true.

Corollary 1.3. Conjecture 1.1 is true for almost all positive integers k with $k \equiv 3 (\mathrm{mod}\ 4)$.

2. Preliminaries

Lemma 2.1. ([5]) Let n be an odd integer with $n > 1$, and let X, Y be coprime positive integers. Further, let p be an odd prime with $p \nmid XY$. If $p \mid X + Y$ and $p^r \mid (X^n + Y^n)/(X + Y)$, where r is a positive integer, then $p^r \mid n$.

Here and below, we assume that $k \equiv 3 (\mathrm{mod}\ 4)$ and (1.1) has a solution (x, y, z) with $(x, y, z) \neq (k-1, 1, 2)$.

Lemma 2.2. ((i) of Lemma 2.5 of [3]) $2 \nmid y$ and $2 \mid z$.

Lemma 2.3. ((ii) of Lemma 2.5 of [3]) $y > 3$.

Lemma 2.4. (Lemma 2.6 of [3]) There exist positive integers a and b such that

$$2k - 1 = ab, a > b > 1, gcd(a, b) = 1 \tag{2.1}$$

and

$$a^y + b^y = 2k^{z/2}. \tag{2.2}$$

Lemma 2.5. ((ii) of Theorem 1.1 of [3]) $a \equiv b \equiv 1 \pmod 4$.

Lemma 2.6. (Theorem 1.2 of [3]) k is not an odd prime power.

Lemma 2.7. ((i) of Theorem 1.1 of [3]) $2k - 1$ is not an odd prime power.

Lemma 2.8. (Corollary 1.4 of [3]) $k > 500$.

Lemma 2.9. $y < z < 2y$.

Proof. We see from (1.1) that $k^z > (2k-1)^y > k^y$. So we have $y < z$. On the other hand, by Lemma 2.4, we get from (2.1) and (2.2) that $b \geq 3$ and

$$\begin{aligned} 2k^{z/2} &= a^y + b^y < 2a^y = 2\left(\frac{2k-1}{b}\right)^y \leq 2\left(\frac{2k-1}{3}\right)^y \\ &< 2k^y \end{aligned} \tag{2.3}$$

Therefore, by (2.3), we obtain $z < 2y$. The lemma is proved. □

Lemma 2.10. $(a + b)y \geq 2.3^{z/2}$.

Proof. Notice that $2 \nmid y$, $2 \mid (a+b)$, $a+b>2$ and $(a^y+b^y)/(a+b)$ is an odd positive integer. By (2.2), we have

$$a+b=2fk_1^{z/2}, \frac{a^y+b^y}{a+b}=gk_2^{z/2}, fg=k_0^{z/2},$$

$$k=k_0k_1k_2, \quad f,g,k_0,k_1,k_2 \in \mathbb{N}, \tag{2.4}$$

where k_0 is square free, neither f nor g has $z/2$-th power divisors.

If $k_0=1$, then from (2.4) we get $f=1$, $k_1>1$ and $a+b=2k_1^{z/2} \geq 2 \cdot 3^{z/2}$. Therefore, the lemma holds for this case.

If $k_0>1$, then k_0 has an odd prime divisor p. Since neither f nor g has $z/2$-th power divisors, by (2.4), there exists a positive integer s such that

$$p^s \mid f, p^{z/2-s} \mid g, 1 \leq s < \frac{z}{2}. \tag{2.5}$$

Hence, applying Lemma 2.1 to (2.4) and (2.5), we have

$$p^{z/2-s} \mid y. \tag{2.6}$$

Therefore, by (2.4) and (2.6), we get $(a+b)y \geq 2p^{z/2} \geq 2 \cdot 3^{z/2}$. The lemma is proved. □

Lemma 2.11. $y < 2.65 \log k$, where log is the Napierian logarithm.

Proof. By Lemmas 2.9 and 2.10, we have

$$(a+b)y \geq 2 \cdot 3^{z/2} > 2 \cdot 3^{y/2},$$

whence we get

$$\frac{y}{2}\log 3 < \log\left(\frac{a+b}{2}\right) + \log y. \tag{2.7}$$

Futher, by Lemma 2.5, we see from (2.1) that

$$\frac{a+b}{2} < a = \frac{2k-1}{b} \leq \frac{2k-1}{5}. \tag{2.8}$$

Hence, by (2.7) and (2.8), we get

$$y < \frac{2}{\log 3}\left(\log\left(\frac{2k-1}{5}\right) + \log y\right). \tag{2.9}$$

Let

$$y = t\log k. \tag{2.10}$$

Subtitute (2.10) into (2.9), we have

$$t < \frac{2}{\log 3}\left(\frac{\log((2k-1)/5)}{\log k} + \frac{\log t}{\log k} + \frac{\log\log k}{\log k}\right) \tag{2.11}$$

Since $k \geq 503$ by Lemma 2.8, we get

$$\frac{\log((2k-1)/5)}{\log k} < 1, \quad \frac{\log\log k}{\log k} < 0.2939. \tag{2.12}$$

Therefore, by (2.11) and (2.12), we have

$$t < \frac{2}{\log 3}\left(1.2939 + \frac{\log t}{\log k}\right) \leq \frac{2}{\log 3}\left(1.2939 + \frac{\log t}{\log 503}\right). \tag{2.13}$$

Let

$$f(t) = t - \frac{2}{\log 3}\left(1.2939 + \frac{\log t}{\log 503}\right). \tag{2.14}$$

Then we have

$$f'(t) = 1 - \frac{2}{(\log 3)(\log 503)t}, \tag{2.15}$$

where $f'(t)$ is the derivative of $f(t)$. We see from (2.15) that $f'(t) > 0$ for $t > 2$. Hence, $f(t)$ is an increasing function for $t > 2$. Notice that $f(2.65) > 0$. We have $f(t) > 0$ for $t > 2.65$. Thus, by (2.13) and (2.14), we get that $t < 2.65$, and by (2.10), the lemma is proved. □

3. Proofs

In this section, we assume that $k \equiv 3 \pmod 4$ and (x, y, z) is a solution of (1.1) with $(x, y, z) \neq (k-1, 1, 2)$.

3.1. Proof of Theorem 1.1

Proof. By (1.1), we have

$$\begin{aligned} x^2 &= k^z - (2k-1)^y = (1 + (k-1))^z - (1 + 2(k-1))^y \\ &= \sum_{i=0}^{z} \binom{z}{i} (k-1)^i - \sum_{j=0}^{y} \binom{y}{j} (2(k-1))^j, \end{aligned}$$

whence we get

$$x^2 \equiv (z - 2y)(k-1) \left(\bmod (k-1)^2\right). \tag{3.1}$$

By the definition of the square-free part, we have

$$k - 1 = Q(k-1)m^2, \quad Q(k-1), m \in \mathbb{N}, \tag{3.2}$$

$Q(k-1)$ is square free. Hence, we see from (3.1) and (3.2) that $x \equiv 0 (\bmod\ Q(k-1)m)$. Further, since $x^2 \equiv 0(\bmod (Q(k-1))^2 m^2)$ and $(k-1)^2 \equiv 0(\bmod (Q(k-1))^2 m^2)$, by (3.1) and (3.2), we get

$$z - 2y \equiv 0 \,(\bmod\, Q(k-1)). \tag{3.3}$$

Furthermore, by Lemma 2.9, we find from (3.3) that $z - 2y \neq 0$ and

$$y > 2y - z \geq Q(k-1). \tag{3.4}$$

Therefore, combining (3.4) with Lemma 2.11, we get $2.65 \log k > Q(k-1)$. This implies that if $Q(k-1) \geq 2.65 \log k$, then(1.1) has no solution (x, y, z) with $(x, y, z) \neq (k-1, 1, 2)$. The theorem is proved. □

3.2. Proof of Corollary 1.1

Proof. Since $k-1$ is square free, we have $Q(k-1)=k-1$. Therefore, by Theorem 1.1, if (1.1) has solution $(x,y,z)\neq(k-1,1,2)$, then

$$k-1\leq 2.65\log k. \tag{3.5}$$

But, since $k>500$ by Lemma 2.8, (3.5) is false. The corollary is proved. □

3.3. Proof of Corollary 1.2

Proof. We now assume that $k\equiv 3 \pmod 4$, $500<k<1000$ and (1.1) has solutions (x,y,z) with $(x,y,z)\neq(k-1,1,2)$. By Theorem 1.1, we have

$$Q(k-1)\leq 2.65\log k<2.65\log 1000<18.31. \tag{3.6}$$

Since $Q(k-1)$ is square free with $2\mid Q(k-1)$, by (3.6), we get

$$Q(k-1)\in\{2,6,10,14\}. \tag{3.7}$$

Further, by (3.2), we have

$$m=\sqrt{\frac{k-1}{Q(k-1)}}<\sqrt{\frac{1000}{Q(k-1)}}. \tag{3.8}$$

Therefore, by (3.7) and (3.8), we just have to consider the following cases:

$$\begin{aligned}(Q(k-1),m,k)=&(2,17,579),\ (2,19,723),\ (2,21,883),\\ &(6,11,727),\ (10,9,901),\ (14,7,687).\end{aligned} \tag{3.9}$$

Since 727 and 883 are odd primes, by Lemma 2.6, Conjecture 1.1 is true for $k\in\{727,\ 883\}$. Similarly, since 1373 and 1801 are odd primes, by Lemma 2.7, Conjecture 1.1 is true for $k\in\{687,\ 901\}$.

When $k=579$, we have $2k-1=1157=13\times 89$, where 13 and 89 are odd primes. Hence, by Lemma 2.4, if (1.1) has solutions (x,y,z) with $(x,y,z)\neq(k-1,1,2)$ for $k=579$, then we have

$$89^y+13^y=2.579^{z/2}. \tag{3.10}$$

But, since $2\nmid y$, $89+13=102=2\times 3\times 17$ and $17\nmid 579$, (3.10) is false. Therefore, Conjecture 1.1 is true for $k=579$.

Similarly, when $k=723$, we have $2k-1=1445=5\times 17^2$, $17^2+5=294=2\times 3\times 7^2$ and $7\nmid 723$. Therefore, $17^{2y}+5^y\neq 2.723^{x/2}$ and Conjecture 1.1 is true for $k=723$. Thus, by (3.9), the corollary is proved. □

3.4. Proof of Corollary 1.3

Proof. For any positive integer K, let $F(K)$ denote the number of positive integer k with $k\leq K$ and $k\equiv 3 \pmod 4$. Then we have

$$F(K)=\left[\frac{k+1}{4}\right], \tag{3.11}$$

where $[(k+1)/4]$ is the integer part of $(K+1)/4$. Further, let $G(K)$ denote the number of positive integers k such that $k \le K$, $k \equiv 3 \pmod 4$ and k can make (1.1) has solutions (x, y, z) with $(x, y, z) \neq (k-1, 1, 2)$. By Theorem 1.1, we see from (3.2) that

$$G(K) \le \sum_{d} \sqrt{\frac{K-1}{d}} = \sqrt{K-1} \sum_{d} \frac{1}{\sqrt{d}} < \sqrt{K} \sum_{d} 1 < 2.11 \sqrt{K} \log K, \tag{3.12}$$

where d through all positive integers with $d \le 2.11 \log K$ and d is square free. Therefore, by (3.11) and (3.12), we get

$$\lim_{K\to\infty} \frac{G(K)}{F(K)} = 0.$$

This implies that Conjecture 1.1 is true for almost all positive integers k with $k \equiv 3 \pmod 4$. The corollary is proved. □

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Conflict of interest

The authors declare that they have no competing interests.

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