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Research article

Hamilton type gradient estimates for a general type of nonlinear parabolic equations on Riemannian manifolds

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Abstract: In this paper, we prove Hamilton type gradient estimates for positive solutions to a general type of nonlinear parabolic equation concerning V-Laplacian:

$$(\Delta_V - q(x,t) - \partial_t)u(x,t) = A(u(x,t))$$

on complete Riemannian manifold (with fixed metric). When $V = 0$ and the metric evolves under the geometric flow, we also derive some Hamilton type gradient estimates. Finally, as applications, we obtain some Liouville type theorems of some specific parabolic equations.

Keywords: parabolic equation; gradient estimate; Liouville type theorem; geometric flow; curvature
Mathematics Subject Classification: 35B09, 35B45, 35R01, 53C44

1. Introduction

Gradient estimates are very powerful tools in geometric analysis. In 1970s, Cheng-Yau [3] proved a local version of Yau's gradient estimate (see [25]) for the harmonic function on manifolds. In [16], Li and Yau introduced a gradient estimate for positive solutions of the following parabolic equation,

$$(\Delta - q(x,t) - \partial_t)u(x,t) = 0, \tag{1.1}$$

which was known as the well-known Li-Yau gradient estimate and it is the main ingredient in the proof of Harnack-type inequalities. In [10], Hamilton proved an elliptic type gradient estimate for heat equations on compact Riemannian manifolds, which was known as the Hamilton's gradient estimate and it was later generalized to the noncompact case by Kotschwar [15]. The Hamilton's gradient estimate is useful for proving monotonicity formulas (see [9]). In [22], Souplet and Zhang derived a localized Cheng-Yau type estimate for the heat equation by adding a logarithmic correction term, which is called the Souplet-Zhang's gradient estimate. After the above work, there is a rich literature on extensions of the Li-Yau gradient estimate, Hamilton's gradient estimate and Souplet-Zhang's gradient estimate to diverse settings and evolution equations. We only cite [1, 8, 11, 12, 18, 19, 24, 28, 31] here and one may find more references therein.

An important generalization of the Laplacian is the following diffusion operator

$$\Delta_V \cdot = \Delta + \langle V, \nabla \cdot \rangle$$

on a Riemannian manifold (M, g) of dimension n, where V is a smooth vector field on M. Here ∇ and Δ are the Levi-Civita connection and Laplacian with respect to metric g, respectively. The V-Laplacian can be considered as a special case of V-harmonic maps introduced in [5]. Recall that on a complete Riemannian manifold (M, g), we can define the ∞-Bakry-Émery Ricci curvature and m-Bakry-Émery Ricci curvature as follows [6, 20]

$$\mathrm{Ric_V} = \mathrm{Ric} - \frac{1}{2}\mathcal{L}_V g, \tag{1.2}$$

$$\mathrm{Ric_V^m} = \mathrm{Ric_V} - \frac{1}{m-n} V \otimes V, \tag{1.3}$$

where $m \geq n$ is a constant, Ric is the Ricci curvature of M and $\mathcal{L}_V$ denotes the Lie derivative along the direction V. In particular, we use the convention that $m = n$ if and only if $V \equiv 0$. There have been plenty of gradient estimates obtained not only for the heat equation, but more generally, for other nonlinear equations concerning the V-Laplacian on manifolds, for example, [4, 13, 20, 27, 32].

In [7], Chen and Zhao proved Li-Yau type gradient estimates and Souplet-Zhang type gradient estimates for positive solutions to a general parabolic equation

$$(\Delta_V - q(x,t) - \partial_t) u(x,t) = A(u(x,t)) \tag{1.4}$$

on $M \times [0, T]$ with m-Bakry-Émery Ricci tensor bounded below, where $q(x,t)$ is a function on $M \times [0, T]$ of C^2 in x-variables and C^1 in t-variable, and $A(u)$ is a function of C^2 in u. In the present paper, by studying the evolution of quantity $u^{\frac{1}{3}}$ instead of $\ln u$, we derive localised Hamilton type gradient estimates for $\frac{|\nabla u|}{\sqrt{u}}$. Most previous studies cited in the paper give the gradient estimates for $\frac{|\nabla u|}{u}$. The main theorems are below.

Theorem 1.1. *Let (M^n, g) be a complete Riemannian manifold with*

$$\mathrm{Ric}_V^m \geq -(m-1)K_1$$

for some constant $K_1 > 0$ in $B(\overline{x}, \rho)$, some fixed point $\overline{x}$ in M and some fixed radius ρ. Assume that there exists a constant $D_1 > 0$ such that $u \in (0, D_1]$ is a smooth solution to the general parabolic Eq (1.4) in $Q_{2\rho, T_1 - T_0} = B(\overline{x}, 2\rho) \times [T_0, T_1]$, where $T_1 > T_0$. Then there exists a universal constant $c(n)$ that depends only on n so that

$$\begin{aligned} \frac{|\nabla u|}{\sqrt{u}} \leq & c(n)\sqrt{D_1}\left(\frac{(m-1)\sqrt{K_1}}{\rho} + \frac{2m-1}{\rho^2} + \frac{1}{t-T_0} + \max_{Q_{\rho,T_1-T_0}} |\nabla q|^{\frac{2}{3}}\right)^{\frac{1}{2}} \\ & + 3\sqrt{D_1}\left((m-1)K_1 + \max_{Q_{\rho,T_1-T_0}} \frac{|q|}{2} - \min\left\{0, \min_{Q_{\rho,T_1-T_0}}\left(A'(u) - \frac{A(u)}{2u}\right)\right\}\right)^{\frac{1}{2}} \end{aligned} \tag{1.5}$$

in $Q_{\frac{\rho}{2}, T_1 - T_0}$ with $t \neq T_0$.

Remark 1.1. Hamilton [10] first got this gradient estimate for the heat equation on a compact manifold. We also have Hamilton type estimates if we assume that $\mathrm{Ric}_V \geq -(m-1)K_1$ for some constant $K_1 > 0$, and notice that $\mathrm{Ric}_V \geq -(m-1)K_1$ is weak than $\mathrm{Ric}_V^m \geq -(m-1)K_1$. Since we do not have a good enough V-Laplacian comparison for general smooth vector field V, we need the condition that $|V|$ is bounded in this case. Nevertheless, when $V = \nabla f$, we can use the method given in [23] to obtain all results in this paper, without assuming that $|V|$ is bounded.

If $q = 0$ and $A(u) = au \ln u$, where a is a constant, then following the proof of Theorem 1.1 we have

Corollary 1.2. *Let (M^n, g) be a complete Riemannian manifold with*

$$\mathrm{Ric}_V^m \geq -(m-1)K_1$$

for some constant $K_1 > 0$ in $B(\overline{x}, \rho)$, some fixed point $\overline{x}$ in M and some fixed radius ρ. Assume that u is a positive smooth solution to the equation

$$\partial_t u = \Delta_V u - au \ln u \tag{1.6}$$

in $Q_{2\rho, T_1 - T_0} = B(\overline{x}, 2\rho) \times [T_0, T_1]$, where $T_1 > T_0$.

(1) When $a > 0$, assuming that $1 \leq u \leq D_1$ in $Q_{2\rho, T_1-T_0}$, there exists a constant $c = c(n)$ such that

$$\frac{|\nabla u|}{\sqrt{u}} \leq c\sqrt{D_1}\left(\sqrt[4]{\frac{(m-1)^2 K_1}{\rho^2}} + \frac{\sqrt{2m-1}}{\rho} + \frac{1}{\sqrt{t-T_0}} + \left|2(m-1)K_1 - 2a\right|^{\frac{1}{2}}\right) \tag{1.7}$$

in $Q_{\frac{\rho}{2}, T_1-T_0}$ with $t \neq T_0$.

(2) When $a < 0$, assuming that $0 < u \leq D_1$ in $Q_{2\rho, T_1-T_0}$, there exists a constant $c = c(n)$ such that

$$\frac{|\nabla u|}{\sqrt{u}} \leq c\sqrt{D_1}\left(\sqrt[4]{\frac{(m-1)^2 K_1}{\rho^2}} + \frac{\sqrt{2m-1}}{\rho} + \frac{1}{\sqrt{t-T_0}} + \left|2(m-1)K_1 - a(2+\ln D_1)\right|^{\frac{1}{2}}\right) \tag{1.8}$$

in $Q_{\frac{\rho}{2}, T_1-T_0}$ with $t \neq T_0$.

Using the corollary, we get the following Liouville type result.

Corollary 1.3. *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_V^m \geq -(m-1)K_1$ for some constant $K_1 > 0$. Assume that u is a positive and bounded solution to the Eq (1.6) and u is independent of time.*

(1) When $a > 0$, assume that $1 \leq u \leq D_1$. If $a = (m-1)K_1$, then $u \equiv 1$.

(2) When $a < 0$, assume that $0 < u \leq \exp\{-2 + \frac{2(m-1)K_1}{a}\}$, then u does not exist.

Remark 1.2. When $V = 0$, Δ_V and Ric_V^m become Δ and Ric, respectively. It is clear that Corollary 1.2–1.3 generalize Theorem 1.3 and Corollary 1.1 in [14].

We can obtain a global estimate from Theorem 1.1 by taking $\rho \to 0$.

Corollary 1.4. *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_V^m \geq -(m-1)K_1$ for some constant $K_1 > 0$. u is a positive smooth solution to the general parabolic Eq (1.4) on $M^n \times [T_0, T_1]$. Suppose that $u \leq D_1$ on $M^n \times [T_0, T_1]$. We also suppose that*

$$\left|A'(u) - \frac{A(u)}{2u}\right| \leq D_2, \qquad \frac{|q|}{2} \leq D_3, \qquad |\nabla q|^{\frac{2}{3}} \leq D_4$$

on $M^n \times [T_0, T_1]$. Then there exists a universal constant c that depends only on n so that

$$\frac{|\nabla u|}{\sqrt{u}} \leq c(n)\sqrt{D_1}\left(\frac{1}{t-T_0}+D_4\right)^{\frac{1}{2}} + 3\sqrt{D_1}\Big((m-1)K_1 + D_3 + D_2\Big)^{\frac{1}{2}} \tag{1.9}$$

in $M^n \times [T_0, T_1]$ with $t \neq T_0$.

Let $A(u) = a(u(x,t))^\beta$ in Corollary 1.4, we obtain Hamilton type gradient estimates for bounded positive solutions of the equation

$$(\Delta_V - q(x,t) - \partial_t)u(x,t) = a(u(x,t))^\beta, \qquad a \in \mathbb{R}, \qquad \beta \in (-\infty, 0] \cup [\frac{1}{2}, +\infty). \tag{1.10}$$

Corollary 1.5. *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_V^m \geq -(m-1)K_1$ for some constant $K_1 > 0$. u is a positive smooth solution to (1.10) on $M^n \times [T_0, T_1]$. Suppose that $u \leq D_1$ on $M^n \times [T_0, T_1]$. We also suppose that*

$$\frac{|q|}{2} \leq D_3, \qquad |\nabla q|^{\frac{2}{3}} \leq D_4$$

on $M^n \times [T_0, T_1]$. Then in $M^n \times [T_0, T_1]$ with $t \neq T_0$, there exists a universal constant c that depends only on n so that

$$\frac{|\nabla u|}{\sqrt{u}} \leq c(n)\sqrt{D_1}\left(\frac{1}{t-T_0}+D_4\right)^{\frac{1}{2}} + 3\sqrt{D_1}\Big((m-1)K_1 + D_3 + \Lambda_0\Big)^{\frac{1}{2}}, \tag{1.11}$$

where

$$\Lambda_0 = \begin{cases} 0, & a \geq 0, \beta \geq \frac{1}{2}, \\ -a(\beta - \frac{1}{2})D_1^{\beta-1}, & a \leq 0, \beta \geq \frac{1}{2}, \\ 0, & a \leq 0, \beta \leq 0, \\ a(\frac{1}{2} - \beta)(\min_{M^n \times [T_0,T_1]} u)^{\beta-1}, & a \geq 0, \beta \leq 0. \end{cases}$$

In the next part, our result concerns gradient estimates for positive solutions of

$$(\Delta_t - q(x,t) - \partial_t)u(x,t) = A(u(x,t)) \tag{1.12}$$

on $(M^n, g(t))$ with the metric evolving under the geometric flow:

$$\frac{\partial}{\partial t} g(t) = 2\mathrm{S}(t), \tag{1.13}$$

where Δ_t depends on t and it denotes the Laplacian of $g(t)$, and $\mathrm{S}(t)$ is a symmetric $(0, 2)$-tensor field on $(M^n, g(t))$. In [31], Zhao proved localised Li-Yau type gradient estimates and Souplet-Zhang type gradient estimates for positive solutions of (1.12) under the geometric flow (1.13). In this paper, we have the following localised Hamilton type gradient estimates for positive solutions to the general parabolic Eq (1.12) under the geometric flow (1.13).

Theorem 1.6. *Let $(M^n, g(t))_{t \in [0,T]}$ be a complete solution to the geometric flow (1.13) on M^n with*

$$\mathrm{Ric}_{g(t)} \geq -K_2 g(t), \quad \left|\mathrm{S}_{g(t)}\right|_{g(t)} \leq K_3$$

for some K_2, $K_3 > 0$ on $Q_{\rho,T} = B(\bar{x},\rho) \times [0,T]$. Assume that there exists a constant $L_1 > 0$ such that $u \in (0, L_1]$ is a smooth solution to the general parabolic Eq (1.12) *in $Q_{2\rho,T}$. Then there exists a universal constant $c(n)$ that depends only on n so that*

$$\begin{aligned}\frac{|\nabla u|}{\sqrt{u}} \leq & c(n)\sqrt{L_1}\left(\frac{\sqrt{K_2}}{\rho} + \frac{1}{\rho^2} + \frac{1}{t} + K_3 + \max_{Q_{\rho,T}} |\nabla q|^{\frac{2}{3}}\right)^{\frac{1}{2}} \\ & + 3\sqrt{L_1}\left(K_2 + K_3 + \max_{Q_{\rho,T}} \frac{|q|}{2} - \min\left\{0, \min_{Q_{\rho,T}}\left(A'(u) - \frac{A(u)}{2u}\right)\right\}\right)^{\frac{1}{2}}\end{aligned} \tag{1.14}$$

in $Q_{\frac{\rho}{2},T}$.

Remark 1.3. Recently, some Hamilton type estimates have been achieved to positive solutions of

$$(\Delta_t - q - \partial_t)u = au(\ln u)^\alpha$$

under the Ricci flow in [26], and for

$$(\Delta_t - q - \partial_t)u = pu^{b+1}$$

under the Yamabe flow in [29], where $p, q \in C^{2,1}(M^n \times [0,T])$, b is a positive constant and a, α are real constants. Our results generalize many previous well-known gradient estimate results.

The paper is organized as follows. In Section 2, we provide a proof of Theorem 1.1 and a proof of Corollary 1.3 and Corollary 1.5. In Section 3, we study gradient estimates of (1.12) under the geometric flow (1.13) and give a proof of Theorem 1.6.

2. Gradient estimates for (1.4): Proof of Theorem 1.1

2.1. Basic lemmas

We first give some notations for the convenience of writing throughout the paper. Let $h := u^{\frac{1}{3}}$ and $\widehat{A}(h) := \frac{A(u)}{3u^{\frac{2}{3}}}$. Then $\widehat{A}_h = A'(u) - \frac{2A(u)}{3u}$. To prove Theorem 1.1 we need two basic lemmas. First, we derive the following lemma.

Lemma 2.1. *Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric}_V^m \geq -(m-1)K_1$ for some constant $K_1 > 0$. u is a positive smooth solution to the general parabolic Eq* (1.4) *in $Q_{2\rho,T_1-T_0}$. If $h := u^{\frac{1}{3}}$, and $\mu := h \cdot |\nabla h|^2$, then we have*

$$\begin{aligned}(\Delta_V - \partial_t)\mu \geq & 4h^{-3}\mu^2 - 2h^{-1}\langle \nabla h, \nabla \mu\rangle - 2(m-1)K_1\mu + q\mu \\ & - \frac{2}{3}h^{\frac{3}{2}}|\nabla q|\sqrt{\mu} + 2\widehat{A}_h\mu + h^{-1}\widehat{A}(h)\mu.\end{aligned} \tag{2.1}$$

Proof. Since $h := u^{\frac{1}{3}}$, by a simple computation, we can derive the following equation from (1.4):

$$(\Delta_V - \partial_t)h = -2h^{-1}|\nabla h|^2 + \frac{qh}{3} + \widehat{A}(h). \tag{2.2}$$

By direct computations, we have

$$\nabla_i \mu = |\nabla h|^2 \nabla_i h + 2h\nabla_i\nabla_j h \nabla_j h,$$

and

$$
\begin{aligned}
\Delta_V\mu =&\Delta\mu + \langle V, \nabla\mu\rangle\\
=&2h|\nabla^2 h|^2 + 2h\nabla_i\nabla_i\nabla_j h\nabla_j h + 4\nabla^2 h(\nabla h, \nabla h)\\
&+ |\nabla h|^2\Delta h + 2h\nabla_i\nabla_j h\nabla_j hV_i + \langle V, \nabla h\rangle|\nabla h|^2\\
=&2h|\nabla^2 h|^2 + 2h\langle\nabla\Delta h, \nabla h\rangle + 2h\mathrm{Ric}(\nabla h, \nabla h)\\
&+ 4\nabla^2 h(\nabla h, \nabla h) + |\nabla h|^2\Delta_V h + 2h\nabla_i\nabla_j h\nabla_j hV_i\\
=&2h|\nabla^2 h|^2 + 2h\langle\nabla\Delta h, \nabla h\rangle + 2h\mathrm{Ric}_V(\nabla h, \nabla h) + 2h\nabla_j V_i\nabla_i h\nabla_j h\\
&+ 4\nabla^2 h(\nabla h, \nabla h) + |\nabla h|^2\Delta_V h + 2h\nabla_i\nabla_j h\nabla_j hV_i\\
=&2h|\nabla^2 h|^2 + 2h\langle\nabla\Delta_V h, \nabla h\rangle - 2h(\nabla_j V_i\nabla_i h\nabla_j h + \nabla_i\nabla_j h\nabla_j hV_i)\\
&+ 2h\mathrm{Ric}_V(\nabla h, \nabla h) + 2h\nabla_j V_i\nabla_i h\nabla_j h + 4\nabla^2 h(\nabla h, \nabla h)\\
&+ |\nabla h|^2\Delta_V h + 2h\nabla_i\nabla_j h\nabla_j hV_i\\
=&2h|\nabla^2 h|^2 + 2h\langle\nabla\Delta_V h, \nabla h\rangle + 2h\mathrm{Ric}_V(\nabla h, \nabla h)\\
&+ 4\nabla^2 h(\nabla h, \nabla h) + |\nabla h|^2\Delta_V h.
\end{aligned}
\tag{2.3}
$$

By the following fact:

$$
\begin{aligned}
0 \le&\left(\sqrt{\frac{m-n}{mn}}\Delta h - \sqrt{\frac{n}{m(m-n)}}\langle\nabla h, V\rangle\right)^2\\
=&\left(\frac{1}{n} - \frac{1}{m}\right)(\Delta h)^2 - \frac{2}{m}\langle\nabla h, V\rangle\Delta h + \left(\frac{1}{m-n} - \frac{1}{m}\right)\langle\nabla h, V\rangle^2\\
=&\frac{1}{n}(\Delta h)^2 - \frac{1}{m}\left((\Delta h)^2 + 2\langle\nabla h, V\rangle\Delta h + \langle\nabla h, V\rangle^2\right) + \frac{1}{m-n}\langle\nabla h, V\rangle^2\\
\le&|\nabla^2 h|^2 - \frac{1}{m}(\Delta_V h)^2 + \frac{1}{m-n}\langle\nabla h, V\rangle^2,
\end{aligned}
$$

it yields

$$
|\nabla^2 h|^2 \ge \frac{1}{m}(\Delta_V h)^2 - \frac{1}{m-n}\langle\nabla h, V\rangle^2. \tag{2.4}
$$

Plugging (2.4) into (2.3), we have

$$
\begin{aligned}
\Delta_V\mu \ge&2h\left(\frac{1}{m}(\Delta_V h)^2 - \frac{1}{m-n}\langle\nabla h, V\rangle^2\right) + 2h\langle\nabla\Delta_V h, \nabla h\rangle\\
&+ 2h\mathrm{Ric}_V(\nabla h, \nabla h) + 4\nabla^2 h(\nabla h, \nabla h) + |\nabla h|^2\Delta_V h\\
=&\frac{2}{m}h(\Delta_V h)^2 + 2h\mathrm{Ric}_V^m(\nabla h, \nabla h) + 2h\langle\nabla\Delta_V h, \nabla h\rangle\\
&+ 4\nabla^2 h(\nabla h, \nabla h) + |\nabla h|^2\Delta_V h.
\end{aligned}
\tag{2.5}
$$

The partial derivative of μ with respect to t is given by

$$
\begin{aligned}
\partial_t\mu =&|\nabla h|^2\partial_t h + 2h\nabla_i(\partial_t h)\nabla_i h\\
=&|\nabla h|^2\partial_t h + 2h\nabla_i\left(\Delta_V h + 2h^{-1}|\nabla h|^2 - \frac{qh}{3} - \widehat{A}(h)\right)\nabla_i h\\
=&|\nabla h|^2\partial_t h + 2h\langle\nabla\Delta_V h, \nabla h\rangle + 8\nabla^2 h(\nabla h, \nabla h) - 4h^{-1}|\nabla h|^4\\
&- \frac{2qh|\nabla h|^2}{3} - \frac{2h^2\langle\nabla q, \nabla h\rangle}{3} - 2h\widehat{A}_h|\nabla h|^2.
\end{aligned}
\tag{2.6}
$$

It follows from (2.2), (2.5) and (2.6) that

$$\begin{aligned}(\Delta_V-\partial_t)\mu =&\frac{2}{m}h(\Delta_V h)^2+2h\mathrm{Ric}_V^m(\nabla h,\nabla h)-4\nabla^2 h(\nabla h,\nabla h)\\&+|\nabla h|^2(\Delta_V-\partial_t)h+4h^{-1}|\nabla h|^4+\frac{2qh}{3}|\nabla h|^2\\&+\frac{2h^2\langle\nabla q,\nabla h\rangle}{3}+2h\widehat{A}_h|\nabla h|^2\\=&\frac{2}{m}h(\Delta_V h)^2+2h\mathrm{Ric}_V^m(\nabla h,\nabla h)-4\nabla^2 h(\nabla h,\nabla h)\\&+|\nabla h|^2(-2h^{-1}|\nabla h|^2+\frac{qh}{3}+\widehat{A}(h))+4h^{-1}|\nabla h|^4\\&+\frac{2qh}{3}|\nabla h|^2+\frac{2h^2\langle\nabla q,\nabla h\rangle}{3}+2h\widehat{A}_h|\nabla h|^2\\=&\frac{2}{m}h(\Delta_V h)^2+2h\mathrm{Ric}_V^m(\nabla h,\nabla h)-4\nabla^2 h(\nabla h,\nabla h)\\&+2h^{-1}|\nabla h|^4+qh|\nabla h|^2+\frac{2h^2\langle\nabla q,\nabla h\rangle}{3}\\&+2h\widehat{A}_h|\nabla h|^2+\widehat{A}(h)|\nabla h|^2\\\geq&-2(m-1)K_1h|\nabla h|^2-4\nabla^2 h(\nabla h,\nabla h)+2h^{-1}|\nabla h|^4\\&+qh|\nabla h|^2+\frac{2h^2\langle\nabla q,\nabla h\rangle}{3}+2h\widehat{A}_h|\nabla h|^2+\widehat{A}(h)|\nabla h|^2.\end{aligned}\tag{2.7}$$

Note that

$$\begin{aligned}-4\nabla^2 h(\nabla h,\nabla h)=&2h^{-1}|\nabla h|^4-2h^{-1}\langle\nabla h,\nabla(h|\nabla h|^2)\rangle\\=&2h^{-3}\mu^2-2h^{-1}\langle\nabla h,\nabla\mu\rangle.\end{aligned}\tag{2.8}$$

Therefore,

$$\begin{aligned}(\Delta_V-\partial_t)\mu\geq&-2(m-1)K_1h|\nabla h|^2+4h^{-3}\mu^2-2h^{-1}\langle\nabla h,\nabla\mu\rangle\\&+qh|\nabla h|^2+\frac{2h^2\langle\nabla q,\nabla h\rangle}{3}+2h\widehat{A}_h|\nabla h|^2+\widehat{A}(h)|\nabla h|^2\\\geq&-2(m-1)K_1\mu+4h^{-3}\mu^2-2h^{-1}\langle\nabla h,\nabla\mu\rangle+q\mu\\&-\frac{2}{3}h^{\frac{3}{2}}|\nabla q|\sqrt{\mu}+2\widehat{A}_h\mu+h^{-1}\widehat{A}(h)\mu,\end{aligned}\tag{2.9}$$

which is the desired estimate. □

The following cut-off function will be used in the proof of Theorem 1.1 (see [2, 16, 22, 30]).

Lemma 2.2. *Fix T_0, $T_1\in\mathbb{R}$ and $T_0<T_1$. Given $\tau\in(T_0,T_1]$, there exists a smooth function $\overline{\Psi}:[0,+\infty)\times[T_0,T_1]\to\mathbb{R}$ satisfying the following requirements.*

1). The support of $\overline{\Psi}(s,t)$ is a subset of $[0,\rho]\times[T_0,T_1]$, and $0\leq\overline{\Psi}\leq 1$ in $[0,\rho]\times[T_0,T_1]$.

2). The equalities $\overline{\Psi}(s,t)=1$ in $[0,\frac{\rho}{2}]\times[\tau,T_1]$ and $\frac{\partial\overline{\Psi}}{\partial s}(s,t)=0$ in $[0,\frac{\rho}{2}]\times[T_0,T_1]$.

3). The estimate $|\partial_t\overline{\Psi}|\leq\frac{\overline{C}}{\tau-T_0}\overline{\Psi}^{\frac{1}{2}}$ is satisfied on $[0,+\infty]\times[T_0,T_1]$ for some $\overline{C}>0$, and $\overline{\Psi}(s,T_0)=0$ for all $s\in[0,+\infty)$.

4). The inequalities $-\frac{C_b\overline{\Psi}^b}{\rho}\leq\frac{\partial\overline{\Psi}}{\partial s}\leq 0$ and $|\frac{\partial^2\overline{\Psi}}{\partial s^2}|\leq\frac{C_b\overline{\Psi}^b}{\rho^2}$ hold on $[0,+\infty]\times[T_0,T_1]$ for every $b\in(0,1)$ with some constant C_b that depends on b.

Throughout this section, we employ the cut-off function $\Psi : M^n \times [T_0, T_1] \to \mathbb{R}$ by

$$\Psi(x,t) = \overline{\Psi}(r(x),t),$$

where $r(x) := d(x,\overline{x})$ is the distance function from some fixed point $\overline{x} \in M^n$.

2.2. *Proof of Theorem 1.1*

From Lemma 2.1, we have

$$\begin{aligned}(\Delta_V - \partial_t)(\Psi\mu) =&\mu(\Delta_V - \partial_t)\Psi + \Psi(\Delta_V - \partial_t)\mu + 2\langle\nabla\mu, \nabla\Psi\rangle \\ \geq&\mu(\Delta_V - \partial_t)\Psi + 2\langle\nabla\mu, \nabla\Psi\rangle - 2\Psi h^{-1}\langle\nabla h, \nabla\mu\rangle \\ &+ \Psi\big[-2(m-1)K_1\mu + 4h^{-3}\mu^2 - \frac{2}{3}h^{\frac{3}{2}}|\nabla q|\sqrt{\mu} \\ &+ q\mu + 2\widehat{A}_h\mu + h^{-1}\widehat{A}(h)\mu\big] \\ =&\mu(\Delta_V - \partial_t)\Psi + \frac{2}{\Psi}\langle\nabla\Psi, \nabla(\Psi\mu)\rangle - 2\frac{|\nabla\Psi|^2}{\Psi}\mu \\ &- 2h^{-1}\langle\nabla h, \nabla(\Psi\mu)\rangle + 2h^{-1}\mu\langle\nabla h, \nabla\Psi\rangle \\ &+ \Psi\big[-2(m-1)K_1\mu + 4h^{-3}\mu^2 - \frac{2}{3}h^{\frac{3}{2}}|\nabla q|\sqrt{\mu} \\ &+ q\mu + 2\widehat{A}_h\mu + h^{-1}\widehat{A}(h)\mu\big].\end{aligned} \tag{2.10}$$

For fixed $\tau \in (T_0, T_1]$, let (x_1, t_1) be a maximum point for $\Psi\mu$ in $Q_{\rho,\tau-T_0}$. Obviously at (x_1, t_1), we have the following facts: $\nabla(\Psi\mu) = 0$, $\Delta_V(\Psi\mu) \leq 0$, and $\partial_t(\Psi\mu) \geq 0$. It follows from (2.10) that at such point

$$\begin{aligned}0 \geq& h^3\mu(\Delta_V - \partial_t)\Psi - 2h^3\frac{|\nabla\Psi|^2}{\Psi}\mu - 2h^{\frac{3}{2}}\mu^{\frac{3}{2}}|\nabla\Psi| \\ &+ \Psi\big[-2(m-1)K_1h^3\mu + 4\mu^2 - \frac{2}{3}h^{\frac{9}{2}}|\nabla q|\sqrt{\mu} \\ &+ q\mu h^3 + 2h^3\widehat{A}_h\mu + h^2\widehat{A}(h)\mu\big].\end{aligned} \tag{2.11}$$

In other words, we have just proved that

$$\begin{aligned}4\Psi\mu^2 \leq& -h^3\mu(\Delta_V - \partial_t)\Psi + 2h^{\frac{3}{2}}\mu^{\frac{3}{2}}|\nabla\Psi| + \frac{2}{3}h^{\frac{9}{2}}|\nabla q|\sqrt{\mu}\Psi + 2h^3\frac{|\nabla\Psi|^2}{\Psi}\mu \\ &+ \Psi\mu\big[2(m-1)K_1h^3 - qh^3 - 2h^3\widehat{A}_h - h^2\widehat{A}(h)\big]\end{aligned} \tag{2.12}$$

at (x_1, t_1).

Next, to realize the theorem, it suffices to bound each term on the right-hand side of (2.12). To deal with $\Delta_V\Psi(x_1, t_1)$, we divide the arguments into two cases.

Case 1. If $d(\overline{x}, x_1) < \frac{\rho}{2}$, then it follows from Lemma 2.2 that $\Psi(x,t) = 1$ around (x_1, t_1) in the space direction. Therefore, $\Delta_V\Psi(x_1, t_1) = 0$.

Case 2. Suppose that $d(\overline{x}, x_1) \geq \frac{\rho}{2}$. Since $\mathrm{Ric}_V^m \geq -(m-1)K_1$, we can apply the generalized Laplace comparison theorem (see Corollary 3.2 in [20]) to get

$$\Delta_V r \leq (m-1)\sqrt{K_1}\coth(\sqrt{K_1}r) \leq (m-1)\left(\sqrt{K_1} + \frac{1}{r}\right).$$

Using the generalized Laplace comparison theorem and Lemma 2.2, we have

$$
\begin{aligned}
\Delta_V \Psi =& \frac{\partial \overline{\Psi}}{\partial r}\Delta_V r + \frac{\partial^2 \overline{\Psi}}{\partial r^2}|\nabla r|^2 \\
\geq & -\frac{C_{1/2}\Psi^{\frac{1}{2}}}{\rho}(m-1)\left(\sqrt{K_1}+\frac{2}{\rho}\right) - \frac{C_{1/2}\Psi^{\frac{1}{2}}}{\rho^2}
\end{aligned}
$$

at (x_1, t_1), which agrees with Case 1. Therefore, we have

$$
\begin{aligned}
& - h^3\mu(\Delta_V - \partial_t)\Psi \\
=& - u\mu(\Delta_V - \partial_t)\Psi \\
\leq & \left(\frac{C_{1/2}\Psi^{\frac{1}{2}}}{\rho}(m-1)\left(\sqrt{K_1}+\frac{2}{\rho}\right) + \frac{C_{1/2}\Psi^{\frac{1}{2}}}{\rho^2} + \frac{\overline{C}\Psi^{\frac{1}{2}}}{\tau - T_0}\right)u\mu \\
\leq & cD_1\Psi^{\frac{1}{2}}\mu\left(\frac{(m-1)\sqrt{K_1}}{\rho} + \frac{2m-1}{\rho^2} + \frac{1}{\tau - T_0}\right) \\
\leq & \frac{1}{2}\Psi\mu^2 + cD_1^2\left(\frac{(m-1)\sqrt{K_1}}{\rho} + \frac{2m-1}{\rho^2} + \frac{1}{\tau - T_0}\right)^2
\end{aligned}
\tag{2.13}
$$

for some universal constant $c > 0$. Here we used Lemma 2.2, $0 \leq \Psi \leq 1$ and Cauchy's inequality.

On the other hand, by Young's inequality and Lemma 2.2, we obtain

$$
\begin{aligned}
2h^{\frac{3}{2}}\mu^{\frac{3}{2}}|\nabla\Psi| =& 2\Psi^{\frac{3}{4}}\mu^{\frac{3}{2}}u^{\frac{1}{2}}\frac{|\nabla\Psi|}{\Psi^{\frac{3}{4}}} \\
\leq & \frac{1}{2}\Psi\mu^2 + cD_1^2\frac{|\nabla\Psi|^4}{\Psi^3} \\
\leq & \frac{1}{2}\Psi\mu^2 + cD_1^2\frac{1}{\rho^4},
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
\frac{2}{3}\Psi h^{\frac{9}{2}}|\nabla q|\sqrt{\mu} =& \frac{2}{3}\Psi^{\frac{1}{4}}\mu^{\frac{1}{2}}u^{\frac{3}{2}}\Psi^{\frac{3}{4}}|\nabla q| \\
\leq & \frac{1}{2}\Psi\mu^2 + cu^2\Psi|\nabla q|^{\frac{4}{3}} \\
\leq & \frac{1}{2}\Psi\mu^2 + cD_1^2\Psi|\nabla q|^{\frac{4}{3}},
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
2\frac{|\nabla\Psi|^2}{\Psi}h^3\mu =& 2|\nabla\Psi|^2\Psi^{-\frac{3}{2}}\Psi^{\frac{1}{2}}u\mu \\
\leq & 2D_1|\nabla\Psi|^2\Psi^{-\frac{3}{2}}\Psi^{\frac{1}{2}}\mu \\
\leq & \frac{1}{2}\Psi\mu^2 + cD_1^2\frac{|\nabla\Psi|^4}{\Psi^3} \\
\leq & \frac{1}{2}\Psi\mu^2 + \frac{cD_1^2}{\rho^4}
\end{aligned}
\tag{2.16}
$$

and

$$\begin{aligned}
&\Psi\mu[2(m-1)K_1h^3 - qh^3 - 2h^3\widehat{A}_h - h^2\widehat{A}(h)] \\
=&\Psi^{\frac{1}{2}}\mu\Psi^{\frac{1}{2}}u[2(m-1)K_1 - q - 2\widehat{A}_h - h^{-1}\widehat{A}(h)] \\
\leq&\frac{1}{2}\Psi\mu^2 + \frac{1}{2}\Psi u^2[2(m-1)K_1 - q - 2\widehat{A}_h - h^{-1}\widehat{A}(h)]^2 \\
\leq&\frac{1}{2}\Psi\mu^2 + \frac{D_1^2}{2}\Psi[2(m-1)K_1 - q - 2\widehat{A}_h - h^{-1}\widehat{A}(h)]^2.
\end{aligned} \tag{2.17}$$

Now, we plug (2.13)–(2.17) into (2.12) to get

$$\begin{aligned}
&(\Psi\mu)^2(x_1,t_1) \leq (\Psi\mu^2)(x_1,t_1) \\
\leq&cD_1^2\left(\frac{(m-1)\sqrt{K_1}}{\rho} + \frac{2m-1}{\rho^2} + \frac{1}{\tau - T_0} + |\nabla q|^{\frac{2}{3}}\right)^2 + D_1^2\Big((m-1)K_1 \\
&+ \frac{|q|}{2} - \min\left\{0, \min_{Q_{\rho,T_1-T_0}}(\widehat{A}_h + \frac{1}{2}h^{-1}\widehat{A}(h))\right\}\Big)^2 \\
\leq&cD_1^2\left(\frac{(m-1)\sqrt{K_1}}{\rho} + \frac{2m-1}{\rho^2} + \frac{1}{\tau - T_0} + \max_{Q_{\rho,T_1-T_0}}|\nabla q|^{\frac{2}{3}}\right)^2 + D_1^2\Big((m-1)K_1 \\
&+ \max_{Q_{\rho,T_1-T_0}}\frac{|q|}{2} - \min\left\{0, \min_{Q_{\rho,T_1-T_0}}\left(A'(u) - \frac{A(u)}{2u}\right)\right\}\Big)^2.
\end{aligned} \tag{2.18}$$

The finally, since $\Psi(x,\tau) = 1$ in $B(\overline{x}, \frac{\rho}{2})$, it follows from (2.18) that

$$\begin{aligned}
&\mu(x,\tau) \leq \Psi\mu(x_1,t_1) \\
\leq&cD_1\left(\frac{(m-1)\sqrt{K_1}}{\rho} + \frac{2m-1}{\rho^2} + \frac{1}{\tau - T_0} + \max_{Q_{\rho,T_1-T_0}}|\nabla q|^{\frac{2}{3}}\right) + D_1\Big((m-1)K_1 \\
&+ \max_{Q_{\rho,T_1-T_0}}\frac{|q|}{2} - \min\left\{0, \min_{Q_{\rho,T_1-T_0}}\left(A'(u) - \frac{A(u)}{2u}\right)\right\}\Big).
\end{aligned} \tag{2.19}$$

Since $\tau \in (T_0, T_1]$ is arbitrary and $\mu = \frac{|\nabla u|^2}{9u}$, we have

$$\begin{aligned}
\frac{|\nabla u|}{\sqrt{u}} \leq& c\sqrt{D_1}\left(\frac{(m-1)\sqrt{K_1}}{\rho} + \frac{2m-1}{\rho^2} + \frac{1}{t - T_0} + \max_{Q_{\rho,T_1-T_0}}|\nabla q|^{\frac{2}{3}}\right)^{\frac{1}{2}} \\
&+ 3\sqrt{D_1}\Big((m-1)K_1 + \max_{Q_{\rho,T_1-T_0}}\frac{|q|}{2} - \min\left\{0, \min_{Q_{\rho,T_1-T_0}}\left(A'(u) - \frac{A(u)}{2u}\right)\right\}\Big)^{\frac{1}{2}}
\end{aligned} \tag{2.20}$$

in $Q_{\frac{\rho}{2},T_1-T_0}$. We complete the proof.

2.3. Proof of Corollary 1.3 and Corollary 1.5

Proof of Corollary 1.3.

(1) When $a > 0$, for $a = (m-1)K_1$, using the inequality (1.7), we have

$$\frac{|\nabla u|}{\sqrt{u}} \leq c\sqrt{D_1}\left(\sqrt[4]{\frac{(m-1)^2K_1}{\rho^2}} + \frac{\sqrt{2m-1}}{\rho} + \frac{1}{\sqrt{t - T_0}}\right). \tag{2.21}$$

Letting $\rho \to +\infty$, $t \to +\infty$ in (2.21), we get u is a constant. Using $\Delta_V u - au\ln u = 0$, we get $u = 1$.

(2) When $a < 0$, for $D_1 = \exp\{-2 + \frac{2(m-1)K_1}{a}\}$, using the inequality (1.8), we have

$$\frac{|\nabla u|}{\sqrt{u}} \leq c\sqrt{D_1}\left(\sqrt[4]{\frac{(m-1)^2K_1}{\rho^2}} + \frac{\sqrt{2m-1}}{\rho} + \frac{1}{\sqrt{t-T_0}}\right). \tag{2.22}$$

Letting $\rho \to +\infty$, $t \to +\infty$ in (2.22), we get u is a constant. Using $\Delta_V u - au\ln u = 0$, we get $u = 1$, but $0 < u \leq D_1 = \exp\{-2 + \frac{2(m-1)K_1}{a}\} < 1$. So u does not exist.

Proof of Corollary 1.5.

Let

$$\Lambda := -\min\left\{0, \min_{M^n\times[T_0,T_1]}\left(a(\beta - \frac{1}{2})u^{\beta-1}\right)\right\}.$$

From Corollary 1.4, we just have to compute Λ. By the definition, we have

$$\Lambda_0 = \begin{cases} 0, & a \geq 0, \beta \geq \frac{1}{2}, \\ -a(\beta - \frac{1}{2})D_1^{\beta-1}, & a \leq 0, \beta \geq \frac{1}{2}, \\ 0, & a \leq 0, \beta \leq 0, \\ a(\frac{1}{2} - \beta)(\min_{M^n\times[T_0,T_1]} u)^{\beta-1}, & a \geq 0, \beta \leq 0. \end{cases}$$

3. Gradient estimates for (1.12) under geometric flow: Proof of Theorem 1.6

In this section, we consider positive solutions of the nonlinear parabolic Eq (1.12) on (M^n, g) with the metric evolving under the geometric flow (1.13). To prove Theorem 1.6, we follow the procedure used in the proof of Theorem 1.1.

3.1. Basic lemmas

We first derive a general evolution equation under the geometric flow.

Lemma 3.1. *([21]) Suppose the metric evolves by* (1.13). *Then for any smooth function f, we have*

$$(|\nabla f|^2)_t = -2\mathrm{S}(\nabla f, \nabla f) + 2\langle \nabla f, \nabla(f_t)\rangle.$$

Next, we derive the following lemma in the same fashion of Lemma 2.1.

Lemma 3.2. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the geometric flow* (1.13) *and u be a smooth positive solution to the nonlinear parabolic Eq* (1.12) *in $Q_{2\rho,T}$. Suppose that there exists positive constants K_2 and K_3, such that*

$$\mathrm{Ric}_{g(t)} \geq -K_2 g(t), \quad \left|\mathrm{S}_{g(t)}\right|_{g(t)} \leq K_3$$

in $Q_{\rho,T}$. If $h := u^{\frac{1}{3}}$, and $\mu := h \cdot |\nabla h|^2$, then in $Q_{\rho,T}$, we have

$$\begin{aligned}(\Delta_t - \partial_t)\mu \geq & 4h^{-3}\mu^2 - 4h^{-1}\langle \nabla h, \nabla\mu\rangle - \frac{2}{3}h^{\frac{3}{2}}|\nabla q|\sqrt{\mu} \\ & - 2(K_2 + K_3)\mu + q\mu + h^{-1}\widehat{A}(h)\mu + 2\widehat{A}_h\mu.\end{aligned} \tag{3.1}$$

Proof. Since u is a solution to the nonlinear parabolic Eq (1.12), the function $h = u^{\frac{1}{3}}$ satisfies

$$(\Delta_t - \partial_t)h = -2h^{-1}|\nabla h|^2 + \frac{qh}{3} + \widehat{A}(h). \tag{3.2}$$

As in the proof of Lemma 2.1, we have that

$$\begin{aligned}\Delta_t\mu =&2h|\nabla^2 h|^2 + 2h\langle\nabla\Delta_t h, \nabla h\rangle + 2h\mathrm{Ric}(\nabla h, \nabla h)\\ &+ 4\nabla^2 h(\nabla h, \nabla h) + |\nabla h|^2\Delta_t h.\end{aligned}$$

On the other hand, by the equation $\partial_t g(t) = 2\mathrm{S}(t)$, we have

$$\begin{aligned}\partial_t\mu =&|\nabla h|^2\partial_t h + 2h\nabla_i(\partial_t h)\nabla_i h - 2h\mathrm{S}(\nabla f, \nabla f)\\ =&|\nabla h|^2\partial_t h + 2h\nabla_i\left(\Delta_t h + 2h^{-1}|\nabla h|^2 - \frac{qh}{3} - \widehat{A}(h)\right)\nabla_i h\\ &- 2h\mathrm{S}(\nabla f, \nabla f)\\ =&|\nabla h|^2\partial_t h + 2h\langle\nabla\Delta_t h, \nabla h\rangle + 8\nabla^2 h(\nabla h, \nabla h) - 4h^{-1}|\nabla h|^4\\ &- \frac{2qh|\nabla h|^2}{3} - \frac{2h^2\langle\nabla q, \nabla h\rangle}{3} - 2h\widehat{A_h}|\nabla h|^2 - 2h\mathrm{S}(\nabla f, \nabla f).\end{aligned} \tag{3.3}$$

Therefore,

$$\begin{aligned}(\Delta_t - \partial_t)\mu =&2h|\nabla^2 h|^2 - 4\nabla^2 h(\nabla h, \nabla h) + 2h^{-1}|\nabla h|^4\\ &+ \frac{2h^2\langle\nabla q, \nabla h\rangle}{3} + 2h\mathrm{Ric}(\nabla h, \nabla h) + 2h\mathrm{S}(\nabla h, \nabla h)\\ &+ qh|\nabla h|^2 + 2h\widehat{A_h}|\nabla h|^2 + \widehat{A}(h)|\nabla h|^2\\ \geq& - 8\nabla^2 h(\nabla h, \nabla h) - 2h(K_2 + K_3)|\nabla h|^2 - \frac{2}{3}h^2|\nabla h||\nabla q|\\ &+ (qh + 2h\widehat{A_h} + \widehat{A}(h))|\nabla h|^2\\ =&4h^{-3}\mu^2 - 4h^{-1}\langle\nabla h, \nabla\mu\rangle - \frac{2}{3}h^{\frac{3}{2}}|\nabla q|\sqrt{\mu}\\ &- 2(K_2 + K_3)\mu + q\mu + h^{-1}\widehat{A}(h)\mu + 2\widehat{A_h}\mu,\end{aligned} \tag{3.4}$$

where we used (2.8), the assumption on bound of Ric + S and

$$\begin{aligned}&h|\nabla^2 h|^2 + 2\nabla^2 h(\nabla h, \nabla h) + h^{-1}|\nabla h|^4\\ =&|\sqrt{h}\nabla^2 h + \frac{dh\otimes dh}{\sqrt{h}}|^2 \geq 0.\end{aligned}$$

The proof is complete. □

Finally, we employ the cut-off function $\Psi : M^n \times [T_0, T_1] \to \mathbb{R}$, with $T_0 = 0$, $T_1 = T$, by

$$\Psi(x, t) = \overline{\Psi}(r(x, t), t),$$

where $r(x, t) := d_{g(t)}(x, \overline{x})$ is the distance function from some fixed point $\overline{x} \in M^n$ with respect to the metric $g(t)$.

3.2. Proof of Theorem 1.6

To prove Theorem 1.6, we follows the same procedure used previously to prove Theorem 1.1; hence in view of Lemma 3.2,

$$
\begin{aligned}
(\Delta_t - \partial_t)(\Psi\mu) \geq & \mu(\Delta_t - \partial_t)\Psi + \frac{2}{\Psi}\langle \nabla\Psi, \nabla(\Psi\mu)\rangle - 2\frac{|\nabla\Psi|^2}{\Psi}\mu \\
& - 4h^{-1}\langle \nabla h, \nabla(\Psi\mu)\rangle + 4h^{-1}\mu\langle \nabla h, \nabla\Psi\rangle \\
& + \Psi\Big[- 2(K_2 + K_3)\mu + 4h^{-3}\mu^2 - \frac{2}{3}h^{\frac{3}{2}}|\nabla q|\sqrt{\mu} \\
& + q\mu + 2\widehat{A}_h\mu + h^{-1}\widehat{A}(h)\mu\Big].
\end{aligned} \tag{3.5}
$$

For fixed $\tau \in (0, T]$, let (x_2, t_2) be a maximum point for $\Psi\mu$ in $Q_{\rho,\tau} := B(\bar{x}, \rho) \times [0, \tau] \subset Q_{\rho,T}$. It follows from (3.5) that at such point

$$
\begin{aligned}
0 \geq & h^3\mu(\Delta_t - \partial_t)\Psi - 2h^3\frac{|\nabla\Psi|^2}{\Psi}\mu - 4h^{\frac{3}{2}}\mu^{\frac{3}{2}}|\nabla\Psi| \\
& + \Psi\Big[- 2(K_2 + K_3)\mu h^3 + 4\mu^2 - \frac{2}{3}h^{\frac{9}{2}}|\nabla q|\sqrt{\mu} \\
& + q\mu h^3 + 2h^3\widehat{A}_h\mu + h^2\widehat{A}(h)\mu\Big].
\end{aligned} \tag{3.6}
$$

At (x_2, t_2), making use of (3.6), we further obtain

$$
\begin{aligned}
4\Psi\mu^2 \leq & - h^3\mu(\Delta_t - \partial_t)\Psi + 4h^{\frac{3}{2}}\mu^{\frac{3}{2}}|\nabla\Psi| + \frac{2}{3}h^{\frac{9}{2}}|\nabla q|\sqrt{\mu}\Psi + 2h^3\frac{|\nabla\Psi|^2}{\Psi}\mu \\
& + \Psi\mu[2(K_2 + K_3)h^3 - qh^3 - 2h^3\widehat{A}_h - h^2\widehat{A}(h)].
\end{aligned} \tag{3.7}
$$

Next, we need to bound each term on the right-hand side of (3.7). To deal with $\Delta_t\Psi(x_2, t_2)$, we divide the arguments into two cases:

Case 1: $r(x_2, t_2) < \frac{\rho}{2}$. In this case, from Lemma 2.2, it follows that $\Psi(x, t) = 1$ around (x_2, t_2) in the space direction. Therefore, $\Delta_{t_2}\Psi(x_2, t_2) = 0$.

Case 2: $r(x_2, t_2) \geq \frac{\rho}{2}$. Since $\mathrm{Ric}_{g(t)} \geq -(n-1)K_2$, the Laplace comparison theorem (see [17]) implies that

$$
\Delta_t r \leq (n-1)\sqrt{K_2}\coth(\sqrt{K_2}r) \leq (n-1)\left(\sqrt{K_2} + \frac{1}{r}\right).
$$

Then, it follows that

$$
\begin{aligned}
\Delta_t\Psi = & \frac{\partial\overline{\Psi}}{\partial r}\Delta_t r + \frac{\partial^2\overline{\Psi}}{\partial r^2}|\nabla r|^2 \\
\geq & -\frac{C_{1/2}\Psi^{\frac{1}{2}}}{\rho}(n-1)\left(\sqrt{K_2} + \frac{2}{\rho}\right) - \frac{C_{1/2}\Psi^{\frac{1}{2}}}{\rho^2}
\end{aligned}
$$

at (x_2, t_2), which agrees with Case 1.

Next, we estimate $\partial_t\Psi$. $\forall x \in B(\overline{x},\rho)$, let $\gamma : [0,a] \to M^n$ be a minimal geodesic connecting x and $\overline{x}$ at time $t \in [0,T]$. Then, we have

$$\begin{aligned}
|\partial_t r(x,t)|_{g(t)} =& \left| \partial_t \int_0^a |\dot{\gamma}(s)|_{g(t)}\, ds \right|_{g(t)} \\
\le& \frac{1}{2}\int_0^a \left| |\dot{\gamma}(s)|^{-1}_{g(t)} \partial_t g(\dot{\gamma}(s),\dot{\gamma}(s)) \right|_{g(t)} ds \\
\le& K_3 r(x,t) \le K_3\rho.
\end{aligned}$$

Thus, together with Lemma 2.2, we find that

$$\begin{aligned}
\partial_t\Psi \le& |\partial_t\overline{\Psi}|_{g(t)} + |\nabla\overline{\Psi}|_{g(t)}|\partial_t r|_{g(t)} \\
\le& \frac{\overline{C}\Psi^{1/2}}{\tau} + C_{1/2}K_3\Psi^{1/2}.
\end{aligned} \tag{3.8}$$

Therefore, we have

$$\begin{aligned}
& -h^3\mu(\Delta_t - \partial_t)\Psi \\
=& -u\mu(\Delta_t - \partial_t)\Psi \\
\le& \left(\frac{C_{1/2}\Psi^{\frac{1}{2}}}{\rho}(n-1)\left(\sqrt{K_2} + \frac{2}{\rho}\right) + \frac{C_{1/2}\Psi^{\frac{1}{2}}}{\rho^2} + \frac{\overline{C}\Psi^{1/2}}{\tau} + C_{1/2}K_3\Psi^{1/2} \right) u\mu \\
\le& cL_1\Psi^{\frac{1}{2}}\mu\left(\frac{\sqrt{K_2}}{\rho} + \frac{1}{\rho^2} + \frac{1}{\tau} + K_3 \right) \\
\le& \frac{1}{2}\Psi\mu^2 + cL_1^2\left(\frac{\sqrt{K_2}}{\rho} + \frac{1}{\rho^2} + \frac{1}{\tau} + K_3 \right)^2
\end{aligned} \tag{3.9}$$

at (x_2,t_2) for some universal constant $c > 0$ that depends only on n. By similar computations as in the proof of Theorem 1.1, we arrive at

$$4h^{\frac{3}{2}}\mu^{\frac{3}{2}}|\nabla\Psi| \le \Psi\mu^2 + 2cL_1^2\frac{1}{\rho^4}, \tag{3.10}$$

$$\frac{2}{3}\Psi h^{\frac{9}{2}}|\nabla q|\sqrt{\mu} \le \frac{1}{2}\Psi\mu^2 + cL_1^2\Psi|\nabla q|^{\frac{4}{3}}, \tag{3.11}$$

$$2\frac{|\nabla\Psi|^2}{\Psi}h^3\mu \le \frac{1}{2}\Psi\mu^2 + \frac{cL_1^2}{\rho^4} \tag{3.12}$$

and

$$\begin{aligned}
& \Psi\mu[2(K_2+K_3)h^3 - qh^3 - 2h^3\widehat{A}_h - h^2\widehat{A}(h)] \\
\le& \frac{1}{2}\Psi\mu^2 + \frac{L_1^2}{2}\Psi\left[2(K_2+K_3) + |q| - \min\left\{ 0, \min_{Q_{\rho,T}}(2\widehat{A}_h + h^{-1}\widehat{A}(h)) \right\} \right]^2.
\end{aligned} \tag{3.13}$$

Plugging (3.9)–(3.13) into (3.6), we get

$$\begin{aligned}
&(\Psi\mu)^2(x_2,t_2) \le (\Psi\mu^2)(x_2,t_2) \\
\le& cL_1^2\left(\frac{\sqrt{K_2}}{\rho}+\frac{1}{\rho^2}+\frac{1}{\tau}+K_3+|\nabla q|^{\frac{2}{3}}\right)^2 + L_1^2\Big(K_2+K_3 \\
&+\frac{|q|}{2}-\min\left\{0,\min_{Q_{\rho,T}}(\widehat{A}_h+\frac{1}{2}h^{-1}\widehat{A}(h))\right\}\Big)^2 \\
\le& cL_1^2\left(\frac{\sqrt{K_2}}{\rho}+\frac{1}{\rho^2}+\frac{1}{\tau}+K_3+\max_{Q_{\rho,T}}|\nabla q|^{\frac{2}{3}}\right)^2 + L_1^2\Big(K_2+K_3 \\
&+\max_{Q_{\rho,T}}\frac{|q|}{2}-\min\left\{0,\min_{Q_{\rho,T}}\left(A'(u)-\frac{A(u)}{2u}\right)\right\}\Big)^2.
\end{aligned} \tag{3.14}$$

Note that $\Psi(x,\tau)=1$ when $d(x,\overline{x})<\frac{\rho}{2}$, it follows from (2.18) that

$$\begin{aligned}
&\mu(x,\tau) \le \Psi\mu(x_2,t_2) \\
\le& cL_1\left(\frac{\sqrt{K_2}}{\rho}+\frac{1}{\rho^2}+\frac{1}{\tau}+K_3+\max_{Q_{\rho,T}}|\nabla q|^{\frac{2}{3}}\right) + L_1\Big(K_2+K_3 \\
&+\max_{Q_{\rho,T}}\frac{|q|}{2}-\min\left\{0,\min_{Q_{\rho,T}}\left(A'(u)-\frac{A(u)}{2u}\right)\right\}\Big).
\end{aligned} \tag{3.15}$$

Since $\tau\in(0,T]$ is arbitrary and $\mu=\frac{|\nabla u|^2}{9u}$, we have

$$\begin{aligned}
\frac{|\nabla u|}{\sqrt{u}} \le& c\sqrt{L_1}\left(\frac{\sqrt{K_2}}{\rho}+\frac{1}{\rho^2}+\frac{1}{\tau}+K_3+\max_{Q_{\rho,T}}|\nabla q|^{\frac{2}{3}}\right)^{\frac{1}{2}} \\
&+3\sqrt{L_1}\Big(K_2+K_3+\max_{Q_{\rho,T}}\frac{|q|}{2}-\min\left\{0,\min_{Q_{\rho,T}}\left(A'(u)-\frac{A(u)}{2u}\right)\right\}\Big)^{\frac{1}{2}}
\end{aligned} \tag{3.16}$$

in $Q_{\frac{\rho}{2},T}$.

We complete the proof.

Remark 3.1. We also obtain the corresponding applications similar to Corollary 1.4 and Corollary 1.5, which we will not write them down here.

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Conflict of interest

The author declares no conflict of interest.

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