Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(10): 10495–10505.
DOI: 10.3934/math.2021609
Received: 23 February 2021
Accepted: 15 July 2021
Published: 20 July 2021

Research article

Sensitivity for topologically double ergodic dynamical systems

Risong Li[1,3], Tianxiu Lu[2,*], Xiaofang Yang[2] and Yongxi Jiang[2]

[1] School of Mathematic and Computer Science, Guangdong Ocean University, Zhanjiang 524025, China

[2] College of Mathematics and Statistics, Sichuan University of Science and Engineering; The Key Laboratory of Higher Education of Sichuan Province for Enterprise Informationalization and Internet of Things, Zigong, Sichuan 643000, China

[3] Artificial Intelligence Key Laboratory of Sichuan Province; Bridge Non-destruction Detecting and Engineering Computing Key Laboratory of Sichuan Province, Zigong, Sichuan 643000, China

* **Correspondence:** Email: lubeeltx@163.com; Tel: +8613408138464.

Abstract: As a stronger form of multi-sensitivity, the notion of ergodic multi-sensitivity (resp. strongly ergodically multi-sensitivity) is introduced. In particularly, it is proved that every topologically double ergodic continuous selfmap (resp. topologically double strongly ergodic selfmap) on a compact metric space is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive). And for any given integer $m \geq 2$, f is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) if and only if so is f^m. Also, it is shown that if f is a continuous surjection, then f is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) if and only if so is σ_f, where σ_f is the shift selfmap on the inverse limit space $\varprojlim(X, f)$. Moreover, it is proved that if $f : X \to X$ (resp. $g : Y \to Y$) is a map on a nontrivial metric space (X, d) (resp. (Y, d')), and π is a semiopen factor map between (X, f) and (Y, g), then the ergodic multi-sensitivity (resp. the strongly ergodic multi-sensitivity) of g implies the same property of f.

Keywords: sensitivity; multi-sensitivity; syndetic sensitivity; ergodic (resp. strongly ergodic) multi-sensitivity
Mathematics Subject Classification: 54H20, 37B45, 37C20, 37C50

1. Introduction

It is well known that chaos characterizes the unpredictability of complex systems (see [1–9], for example). Sensitive dependence on initial conditions (sensitivity for short) is the essential component of various definitions of chaos. It is widely used in control theory, chaotic cryptography, Chemistry and so on (see [10–14]). And ergodicity is an important part of Markov chain theory. While, what conditions imply that a system is sensitive? This question has gained some attention in [1,2,4–10] and others.

For continuous self-maps on compact metric spaces, Moothathu [6] initiated a preliminary study of stronger forms of sensitivity formulated in terms of large subsets of $Z^+ = \{0, 1, \cdots\}$, named syndetic sensitivity and cofinite sensitivity. Moreover, he constructed a transitive, sensitive map which is not syndetically sensitive and established the following. (1) Any syndetically transitive, non-minimal map is syndetically sensitive (this improves the result that sensitivity is redundant in Devaney's definition of chaos). (2) Any sensitive map of $[0, 1]$ is cofinitely sensitive. (3) Any sensitive subshift of finite type is cofinitely sensitive. (4) Any syndetically transitive, infinite subshift is syndetically sensitive. (5) No Sturmian subshift is cofinitely sensitive. Also, Moothathu [6] tells us that every topologically mixing (resp. topologically weakly mixing) selfmap on a compact metric space is cofinitely sensitive (resp. multi-sensitive). By the definitions, any topologically double ergodic (topologically double strongly ergodic) selfmap of a compact metric space is topologically weakly mixing. So, any topologically double ergodic selfmap (resp. topologically double strongly ergodic selfmap) of a compact metric space is multi-sensitive.

This paper introduces the notion of ergodic (resp. strongly ergodic) multi-sensitivity which is a stronger form of multi-sensitivity. Particularly, if a continuous map of a compact metric space is topologically double ergodic (topologically double strongly ergodic), then it is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive). In Section 3, some necessary and sufficient conditions for ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) are given. These results improve and extend some existing ones.

2. Preliminaries

Let $|A|$ denote the cardinality of A. An upper density of a set $A \subset Z^+$ is the number

$$d^*(A) = \limsup_{k\to\infty} \frac{1}{k+1} \mid \{0 \leq j \leq k : j \in A\} \mid .$$

An lower density of a set $A \subset Z^+$ is the number

$$d_*(A) = \liminf_{k\to\infty} \frac{1}{k+1} \mid \{0 \leq j \leq k : j \in A\} \mid .$$

For a dynamical system (X, f) (i.e., X is a compact metric space and $f : X \to X$ is a continuous map) with an admissible metric d on X, according to the classical definition, f is sensitive if there is $\delta > 0$ such that for each $x \in X$ and any open neighborhood V_x of x, there exists $n \in Z^+$ with $\sup\{d(f^n(x), f^n(y)) : y \in V_x\} > \delta$. One can write this in a slightly different way. For $V \subset X$ and $\delta > 0$, write $S_f(V, \delta) = \{n \in Z^+ :$ there exist $x, y \in V$ with $d(f^n(x), f^n(y)) > \delta\}$. Now, the following conclusions is obtained.

(1) f is sensitive if there is $\delta > 0$ such that for any nonempty open set $V \subset X$, the set $S_f(V, \delta)$ is nonempty.

(2) f is syndetically sensitive if there is $\delta > 0$ such that for every nonempty open subset $V \subset X$, the set $S_f(V, \delta)$ is syndetic (that is, there is an integer $L > 0$ such that $S_f(V, \delta) \cap \{n, n+1, \cdots, n+L-1\} \neq \emptyset$ for any integer $n \geq 0$).

(3) f is cofinitely sensitive if there is $\delta > 0$ such that for every nonempty open subset $V \subset X$, the set $S_f(V, \delta)$ is cofinite.

(4) f is ergodically sensitive if there is $\delta > 0$ such that for every nonempty open subset $V \subset X$, the set $S_f(V, \delta)$ has positive upper density.

(5) f is multi-sensitive if there is $\delta > 0$ such that for every $k \geq 1$ and any nonempty open subset $V_1, V_2, \cdots, V_k \subset X$, the set $\bigcap_{1 \leq i \leq k} S_f(V_i, \delta)$ is nonempty.

Definition 2.1. For a dynamical system (X, f), f is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) if there is $\delta > 0$ such that for every $k \geq 1$ and any nonempty open subset $V_1, V_2, \cdots, V_k \subset X$, the set $\bigcap_{1 \leq i \leq k} S_f(V_i, \delta)$ has positive upper density (resp. is syndetic).

Here $\delta > 0$ will be referred as a constant of sensitivity. Clearly, syndetic sensitivity implies ergodic sensitivity. It is known from the definition of the ergodic sensitivity and Theorem 7 in [6] that ergodic sensitivity implies sensitivity and the converse does not hold. By Theorem 5 and Corollary 3 in [6], one can conclude that both syndetic sensitivity and ergodic sensitivity are weaker than cofinite sensitivity. It is easy to show that,

(i) Cofinite sensitivity $\Rightarrow$ ergodic (resp. strongly ergodic) multi-sensitivity.

(ii) Ergodic (resp. strongly ergodic) multi-sensitivity implies multi-sensitivity and ergodic sensitivity (resp. syndetic sensitivity).

For a dynamical system (X, f) and subsets $U, V \subset X$, let

$$N_f(U, V) = \{n \in Z^+ : f^n(U) \cap V \neq \emptyset\}.$$

One can say that

(1) f is topologically transitive if for every pair of nonempty open sets $U, V \subset X$, the set $N_f(U, V)$ is nonempty.

(2) f is topologically mixing if for every pair of nonempty open sets $U, V \subset X$, the set $N_f(U, V)$ is cofinite.

(3) f is topologically ergodic (resp. topologically strongly ergodic or syndetically transitive) if for every pair of nonempty open sets $U, V \subset X$, the set $N_f(U, V)$ has positive upper density (resp. is syndetic).

(4) f is topologically double ergodic (resp. topologically double strongly ergodic) if for every pair of nonempty open sets $U, V \subset X$, the map $f \times f$ is topologically ergodic (resp. topologically strongly ergodic).

Obviously, topological ergodicity implies topological transitivity, and syndetic transitivity (i.e., topologically strong ergodicity) implies topological ergodicity, and that topologically double ergodicity (resp. topologically double ergodicity) implies topologically weak mixing.

A continuous map f from a compact metric space (X, d) to itself is chaotic in the sense of Devaney if:

(1) f is topologically transitive,

(2) the set of all periodic points of f is dense in X, and,

(3) f has sensitive dependence on initial conditions.

Let (X, d) be a metric space and let $f : X \to X$ be a continuous map. Let $\kappa(X)$ denote the collection of all nonempty compact subsets of X. The Hausdorff metric d_H on $K(X)$ is defined by

$$d_H(C, D) = \max\{\rho(C, D), \rho(D, C)\}$$

for any $C, D \in \kappa(X)$, where $\rho(C, D) = \inf\{\varepsilon > 0 : d(y, C) < \varepsilon, y \in D\}$. It is known that for any compact metric space (X, d), the topology on $\kappa(X)$ induced by $d_H$is same as the Vietoris or finite topology, which is generated by a basis consisting of all sets of the form,

$$\{V_1, V_2, \cdots, V_n\} = \left\{A \in \kappa(X) : A \subset \bigcup_{1\leq i\leq n} V_i, A \cap V_i \neq \emptyset, 1 \leq i \leq n\right\},$$

where $V_1, V_2, \cdots, V_n$ are nonempty and open subsets of X. It is known that this topology is admissible in the sense that the map $i : X \to \kappa(X)$ defined as $i(x) = \{x\}$ is continuous, and $\kappa(X)$ is compact if and only if X is compact. Let $\mathcal{F}(X)$ denote the set of all finite subsets of X. Under this topology, $\mathcal{F}(X)$ is dense in $\kappa(X)$ (see [15, 16]).

For any continuous selfmap f of X, a continuous map $\overline{f} : \kappa(X) \to\to \kappa(X)$ is defined by $\overline{f}(K) = f(K)$ for any $K \in \kappa(X)$. When a point $x \in X$ is identified as a subset $\{x\}$ of X, the system (X, f) is a subsystem of the induced system $(\kappa(X), \overline{f})$ (see [17–23]).

3. Main results

Motivated by Theorem 31 in [24], the following result can be proved.

Theorem 3.1. Let (X, d) be a nontrivial compact metric space and (X, f) be a dynamical system. Then, for any given integer $m \geq 2$, f is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) if and only if so is f^m.

Proof. Suppose $m \geq 2$ and $k \geq 1$ are given integers. Then, for any given integer $i \in \{1, 2, \cdots, k\}$, any nonempty open set V_i and for any constant $\theta > 0$, $\{mn : n \in S_{f^m}(V_i, \theta)\} \subset S_f(V_i, \theta)$, which implies that if f^m is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive), then so is f by the related definitions.

Now, suppose that f is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) with sensitivity constant $\delta > 0$, and that $m \geq 2$ and $k \geq 1$ are given integers. As f is uniformly continuous, f^i is uniformly continuous for each $i \in \{0, 1, \cdots, m\}$. By the definition, there exists a constant $\varepsilon \in (0, \delta)$ such that $d(x, y) \leq \varepsilon$ $(x, y \in X)$ implies $d(f^i(x), f^i(y)) \leq \delta$ for any $i \in \{0, 1, \cdots, m\}$. By the definition, for any k nonempty open sets V_i, $1 \leq i \leq k$, the set

$$\bigcap_{1\leq i\leq k} S_f(V_i, \delta)$$

has positive upper density (resp. is syndetic). Let

$$n \in \bigcap_{1\leq i\leq k} S_f(V_i, \delta)$$

and $n = lm + r$ with $0 \leq r \leq m - 1$ and $l \in Z^+$. Then

$$l \in \bigcap_{1\leq i\leq k} S_{f^m}(V_i, \varepsilon).$$

This implies the set $\bigcap_{1\leq i\leq k} S_{f^m}(V_i, \varepsilon)$ has positive upper density (resp. is syndetic). □

Let (X, d) be a compact metric space and (X, f) be a dynamical system. The inverse limit space $\varprojlim(X, f)$ of the system (X, f) or the map f is the metric space $\{(x_0, x_1, x_2, \cdots) : x_i = f(x_{i+1}), x_i \in X, i = 0, 1, 2, \cdots\}$ with the metric $\widetilde{d}$ defined by $\widetilde{d}(\widetilde{x}, \widetilde{y}) = \sum_{i=0}^{\infty} \frac{1}{2^i} d(x_i, y_i)$, where $\widetilde{x} = (x_0, x_1, x_2, \cdots) \in \varprojlim(X, f)$ and $\widetilde{y} = (y_0, y_1, y_2, \cdots) \in \varprojlim(X, f)$. Clearly, The inverse limit space $\varprojlim(X, f)$ is a compact subspace of the product space $\prod_{i=0}^{\infty} X_i$ where $X_i = X$ for every $i \in \{0, 1, 2, \cdots\}$. The shift selfmap σ_f on the inverse limit space $\varprojlim(X, f)$ is defined as $\sigma_f(x_0, x_1, x_2, \cdots) = (f(x_0), x_0, x_1, \cdots)$ for any $(x_0, x_1, x_2, \cdots) \in \varprojlim(X, f)$. Then the inverse limit dynamical system is denoted by $\left(\varprojlim(X, f), \sigma_f\right)$. The projection map $\pi_i : \varprojlim(X, f) \to X$ is defined as $\pi_i((x_0, x_1, x_2, \cdots)) = x_i$ for any $(x_0, x_1, x_2, \cdots) \in \varprojlim(X, f)$ and each $i \in \{0, 1, 2, \cdots\}$. Obviously, π_i is a continuous open map, and $f \circ \pi_i = \pi_i \circ \sigma_f$ for each $i \in \{0, 1, 2, \cdots\}$. If f is a surjective map, then π_i is an open surjective mapping for each $i \in \{0, 1, 2, \cdots\}$. The inverse limit topology induced by $\widetilde{d}$ has the following basis:

$$\mathscr{T} = \{V : V = \pi_i^{-1}(U) \text{ for some } i \geq 0 \text{ and some open subset } U \subset X\}.$$

Now, one can get the following result.

Theorem 3.2. Let (X, f) be a dynamical system and f be a onto map. Then f is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) if and only if so is σ_f.

Proof. Suppose that f is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) with sensitivity constant $\delta > 0$. For any integer $k \geq 1$, let $\widetilde{V_i} \subset \varprojlim(X, f)$ be any nonempty open subset for each $i = 1, 2, \cdots, k$. Since π_0 is an open map, $\pi_0(\widetilde{V_i})$ is nonempty and open. By the definitions, the set

$$\bigcap_{1 \leq i \leq k} S_f(V_i, \delta)$$

has positive upper density (resp. is syndetic), where $V_i = \pi_0(\widetilde{V_i})$. For any given $n \in \bigcap_{1 \leq i \leq k} S_f(V_i, \delta)$, by the definition there are $x_{i0}, y_{i0} \in V_i$ with $d\left(f^n(x_{i0}), f^n(y_{i0})\right) > \delta$ for each $i = 1, 2, \cdots, k$.

Take

$$\widetilde{x_i} = (x_{i0}, x_{i1}, \cdots) \in \pi_0^{-1}(x_{i0}) \cap \widetilde{V_i} \quad \text{and} \quad \widetilde{y_i} = (y_{i0}, y_{i1}, \cdots) \in \pi_0^{-1}(y_{i0}) \cap \widetilde{V_i}$$

for each $i = 1, 2, \cdots, k$. Then, by the definitions we have

$$\widetilde{d}\left(\sigma_f^n(\widetilde{x_i}), \sigma_f^n(\widetilde{y_i})\right) \geq d(f^n(x_{i0}), f^n(y_{i0})) > \delta$$

for each $i = 1, 2, \cdots, k$. This implies that

$$\bigcap_{1 \leq i \leq k} S_{\sigma_f}(\widetilde{V_i}, \delta) \supset \bigcap_{1 \leq i \leq k} S_f(V_i, \delta).$$

So, the set

$$\bigcap_{1 \leq i \leq k} S_{\sigma_f}(\widetilde{V_i}, \delta)$$

has positive upper density (resp. is syndetic).

Assume that σ_f is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) with sensitivity constant $\delta > 0$. For any integer $k \geq 1$, let $V_i \subset X$ be any nonempty open subset for each

$i = 1, 2, \cdots, k$. As π_0 is continuous, $\widetilde{V_i} = \pi_0^{-1}(V_i)$ is nonempty and open for each $i = 1, 2, \cdots, k$. Take $\widetilde{x_i} \in \widetilde{V_i}$ for each $i = 1, 2, \cdots, k$. Then there is an integer $m > 8$ with $B\left(\widetilde{x_i}, \frac{\delta}{m}\right) \subset \widetilde{V_i}$ for each $i = 1, 2, \cdots, k$, where

$$B\left(\widetilde{x_i}, \frac{\delta}{m}\right) = \{\widetilde{y} \in \lim_{\leftarrow}(X, f) : \widetilde{d}(\widetilde{y}, \widetilde{x_i}) < \frac{\delta}{m}\}$$

for each $i = 1, 2, \cdots, k$.

By the definitions, the set

$$\bigcap_{1 \le i \le k} S_{\sigma_f}(B\left(\widetilde{x_i}, \frac{\delta}{m}\right), \delta)$$

has positive upper density (resp. is syndetic). For any given $n \in \bigcap_{1 \le i \le k} S_{\sigma_f}(B\left(\widetilde{x_i}, \frac{\delta}{m}\right), \delta)$, there are $\widetilde{x_i}', \widetilde{y_i}' \in B\left(\widetilde{x_i}, \frac{\delta}{m}\right)$ with $\widetilde{d}\left(\sigma_f^n(\widetilde{x_i}'), \sigma_f^n(\widetilde{y_i}')\right) > \delta$ for each $i = 1, 2, \cdots, k$. Since σ_f^{n-1} is uniformly continuous, for the above $\widetilde{x_i}$, there exists $\delta' < \frac{\delta}{8}$ such that $\widetilde{y_i}' \in B(\widetilde{x_i}, \delta')$ implies $\widetilde{d}(\sigma_f^{n-1}(\widetilde{y_i}'), \sigma_f^{n-1}(\widetilde{x_i})) < \frac{\delta}{8}$ for each $i = 1, 2, \cdots, k$.

Let $\widetilde{x_i}' = (x'_{i0}, x'_{i1}, \cdots)$ and $\widetilde{y_i}' = (y'_{i0}, y'_{i1}, \cdots)$ for each $i = 1, 2, \cdots, k$. Clearly, $x'_{i0}, y'_{i0} \in V_i$ for each $i = 1, 2, \cdots, k$. Then, by the definition, one has

$$\widetilde{d}\left(\sigma_f^n(\widetilde{x_i}'), \sigma_f^n(\widetilde{y_i}')\right) = d(f^n(x'_{i0}), f^n(y'_{i0}) + \frac{1}{2}\widetilde{d}\left(\sigma_f^{n-1}(\widetilde{x_i}'), \sigma_f^{n-1}(\widetilde{y_i}')\right) \le d(f^n(x'_{i0}), f^n(y'_{i0}) + \frac{1}{8}\delta.$$

for each $i = 1, 2, \cdots, k$. So,

$$d(f^n(x'_{i0}), f^n(y'_{i0}) > \frac{1}{2}\delta$$

for each $i = 1, 2, \cdots, k$. This means that

$$\bigcap_{1 \le i \le k} S_{\sigma_f}(\widetilde{V_i}, \delta) \subset \bigcap_{1 \le i \le k} S_f\left(V_i, \frac{1}{2}\delta\right).$$

So, the set $\bigcap_{1 \le i \le k} S_f\left(V_i, \frac{1}{2}\delta\right)$ has positive upper density (resp. is syndetic). □

Inspired by Lemma 10 in [24], the following result can be obtained.

Theorem 3.3. Let (X, d) be a nontrivial compact metric space, (X, f) be a dynamical system and f be topologically double ergodic (resp. topologically double strongly ergodic), then f is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive).

Proof. Write $f^{(k)} = f_1 \times f_2 \times \cdots \times f_k$ for any integer $k > 0$ where $f_i = f$ for every $i \in \{1, 2, \cdots, k\}$. By Lemma 2 in [25] and Lemma 2 in [26] or the definitions, f is topologically double ergodic (resp. topologically double strongly ergodic) if and only if so is $f^{(k)}$ for any integer $k \ge 1$. Since X is not reduced to a single point, there is $\delta > 0$ such that for every $x \in X$, there is $y \in Y$ satisfying $d(x, y) > 3\delta$. Fix any integer $k > 0$ and let $V_i \subset X, 1 \le i \le k$, be any bounded and nonempty open sets with $\mathrm{diam}(V_i) < \delta$ where the diameter $\mathrm{diam}(V_i)$ of V_i is defined by $\mathrm{diam}(V_i) = \sup_{x,y \in V_i}\{d(x, y)\}$. Then, for each $i \in \{1, 2, \cdots, k\}$ there is a nonempty open subset U_i with $d(U_i, V_i) > \delta$. Since $f^{(2k)}$ is topologically double ergodic (resp. topologically double strongly ergodic), by the definitions we get the set

$$N_{f^{(2k)}}((V_1 \times V_1) \times (V_2 \times V_2) \times \cdots \times (V_k \times V_k), (V_1 \times U_1) \times (V_2 \times U_2) \times \cdots \times (V_k \times U_k))$$

has positive upper density (resp. is syndetic). Fix

$$n \in N_{f^{(2k)}}((V_1 \times V_1) \times (V_2 \times V_2) \times \cdots \times (V_k \times V_k), (V_1 \times U_1) \times (V_2 \times U_2) \times \cdots \times (V_k \times U_k)),$$

$f^n(V_i) \cap V_i \neq \emptyset$ and $f^n(V_i) \cap U_i \neq \emptyset$ for $1 \leq i \leq k$. Consequently, there are $x_i, x_i' \in V_i$ such that $f^n(x_i) \in V_i$ and $f^n(x_i') \in U_i$ for $1 \leq i \leq k$. So, we have $d(f^n(x_i), f^n(x_i')) > \delta$ for $1 \leq i \leq k$. This implies that the set

$$\bigcap_{i=1}^{k} S_f(V_i, \delta) \supset N_{f^{(2k)}}((V_1 \times V_1) \times (V_2 \times V_2) \times \cdots \times (V_k \times V_k), (V_1 \times U_1)$$

for any integer $k \geq 1$. Hence,

$$\bigcap_{i=1}^{k} S_f(V_i, \delta)$$

has positive upper density (resp. is syndetic) for any integer $k \geq 1$. □

In [13], the authors studied the relations between the various forms of sensitivity of the systems (X, f) and $(\kappa(X), f)$, and proved that all forms of sensitivity of $(\kappa(X), f)$ partly imply the same for (X, f), and the converse holds in some cases. In particular, they proved that (X, f) is cofinitely sensitive if and only so is $(\kappa(X), \overline{f})$. In [27] we proved that f is syndetically sensitive or multi-sensitive if and only if so does $\overline{f}$. For topologically double ergodic (resp. topologically double strongly ergodic) continuous selfmap f of a compact metric space, the following result is right.

Theorem 3.4. Assume that $f : X \to X$ is a topologically double ergodic (resp. topologically double strongly ergodic) continuous map on a nontrivial compact metric space (X, d). Then $\overline{f}$ is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive).

Proof. From Theorem 2 in [26], f is topologically double ergodic (resp. topologically double strongly ergodic) if and only if so is $\overline{f}$. By hypothesis and Theorem 3.3, $\overline{f}$ is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive). □

Let $\mathcal{B}(X)$ denotes the $\sigma-$algebra of Borel subsets of a compact metric space X. Let $M(X)$ be the collection of all probability measures defined on the measurable space $(X, \mathcal{B}(X))$. The members of $M(X)$ are called Borel probability measures on X. Each $x \in X$ determines a member δ_x (i.e., point measure) of $M(X)$ defined by $\delta_x(A) = 1$ if $x \in A$; $\delta_x(A) = 0$ if $x \notin A$. So, the map $x \to \delta_x$ imbeds X inside $M(X)$. For a given dynamical system (X, f), it is well known that the map defined by $f_M(\mu)(B) = \mu(f^{-1}(B))$ for any $\mu \in M(X)$ and any $B \in \mathcal{B}(X)$ and the map $x \to \delta_x$ from X into $M(X)$ are continuous, and $M(X)$ is a nonempty convex set which is compact in the weak topology (see [28, 29]). Clearly, the map $x \to \delta_x$ imbeds X inside $M(X)$. It is well known that the convex combinations of point measures (i. e. the measures with finite support) are dense in $M(X)$ (see [28, 29]).

Suppose that X is a compact metric space with metric d and $M(X)$ is the space of Borel probability measures on X provided with the Prohorov metric p defined by $p(\lambda, \mu) = \inf\{\varepsilon : \lambda(A) \leq \mu(A^{\varepsilon}) + \varepsilon$ and $\mu(A) \leq \lambda(A^{\varepsilon}) + \varepsilon$ for all Borel sets $A \in \mathcal{B}(X)\}$ for $\lambda, \mu \in M(X)$, where $A^{\varepsilon} = \{xeX : d(x, A) < \varepsilon\}$. As V. Stassen showed in [30], one has $p(\lambda, \mu) = \inf\{\varepsilon : \lambda(A) \leq \mu(A^{\varepsilon}) + \varepsilon$ for all Borel sets $A \in \mathcal{B}(X)\}$. The induced topology is just the weak topology [28, 29] for measures. It turns $M(X)$ into a compact space [28, 31].

Theorem 3.5. Assume that $f : X \to X$ is a topologically double ergodic (resp. topologically double strongly ergodic) continuous map on a nontrivial compact metric space (X, d). Then f_M is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive).

Proof. From Theorem 3.5 in [32], if f is topologically double ergodic, then so is f_M. By the proof of Theorem 3.5 in [32], one can easily prove that if f is topologically double strongly ergodic, then so is f_M. By hypothesis and Theorem 3.1, f_M is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive). □

Remark 3.1. Theorem 3.5 extends and improves Theorem 3.10 in [32].

Theorem 3.6. Assume that $f : X \to X$ (resp. $g : Y \to Y$) is a continuous map on a nontrivial compact metric space (X, d) (resp. (Y, d')). Then $f \times g$ is ergodically multi-sensitive if and only if f or g is ergodically multi-sensitive.

Proof. The proof is easily obtained by Theorem 3.1 in [33] and Theorem 10 in [34] and is omitted. □

Remark 3.2. It is not known whether the following conclusion holds: $f \times g$ is strongly ergodically multi-sensitive if and only if f or g is strongly ergodically multi-sensitive.

Theorem 3.7. Assume that $f : X \to X$ is a continuous map on a nontrivial compact metric space (X, d) (resp. (Y, d')). Then f is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) if and only if so is $\overline{f}$.

Proof. By the definition and the proofs of Theorems 3.2 and 3.3 in [27], the proof is easily obtained and is omitted. □

Assume that $f : X \to X$ (resp. $g : Y \to Y$) is a map on a nontrivial metric space (X, d) (resp. (Y, d')). If there exists a continuous and surjective map $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$, then (Y, g) is said to be a factor of the system (X, f), and (X, f) is said to be a extension of (Y, g), while π is said to be a factor map between (X, f) and (Y, g).

Theorem 3.8. Assume that $f : X \to X$ (resp. $g : Y \to Y$) is a map on a nontrivial metric space (X, d) (resp. (Y, d')), and let π be a semiopen factor map between (X, f) and (Y, g). If g is ergodically multi-sensitive (resp. strongly ergodically multi-sensitive) then so is f.

Proof. The proof is similar to that of Proposition 9 in [24] and is omitted. □

4. An example

Let I be the compact interval [0,1] and f be defined by

$$f(x) = \begin{cases} 2x + \frac{1}{2} & for \quad x \in [0, \frac{1}{4}] \\ -2x - \frac{3}{2} & for \quad x \in [\frac{1}{4}, \frac{3}{4}] \\ 2x - \frac{3}{2} & for \quad x \in [\frac{3}{4}, 1] \end{cases}$$

For arbitrarily $x_1, x_2 \in [0, \frac{1}{4}]$, $x_1 < x_2$, one has

$$| f(x_1) - f(x_2) |=| 2x_1 + \frac{1}{2} - (-2x_2 + \frac{3}{2}) |= 2 | x_1 - x_2 |>| x_1 - x_2 | .$$

For $x_1 \in [0, \frac{1}{4}], x_2 \in [\frac{1}{4}, \frac{3}{4}]$, one has

$$| f(x_1) - f(x_2) |=| 2x_1 + \frac{1}{2} - (-2x_2 + \frac{3}{2}) |= 2 | x_2 + x_1 - \frac{1}{2} |>| x_2 + x_1 - \frac{1}{2} | .$$

For $x_1, x_2 \in [\frac{1}{4}, \frac{3}{4}]$, one has

$$| f(x_1) - f(x_2) |=| -2x_1 + \frac{3}{2} - (-2x_2 + \frac{3}{2}) |= 2 | x_2 - x_1 |>| x_2 - x_1 | .$$

For $x_1 \in [\frac{1}{4}, \frac{3}{4}], x_2 \in [\frac{3}{4}, 1]$, one has

$$| f(x_1) - f(x_2) |=| -2x_1 + \frac{3}{2} - (-2x_2 - \frac{3}{2}) |= 2 | -x_2 - x_1 + \frac{3}{2} |>| -x_2 - x_1 + \frac{3}{2} | .$$

And for $x_1, x_2 \in [\frac{3}{4}, 1]$, one has

$$| f(x_1) - f(x_2) |=| 2x_1 - \frac{3}{2} - (2x_2 - \frac{3}{2}) |= 2 | x_1 - x_2 |>| x_1 - x_2 | .$$

If $x_1 \in [0, \frac{1}{4}], x_2 \in [\frac{3}{4}, 1]$, one has

$$| f(x_1) - f(x_2) |=| 2x_1 + \frac{1}{2} - 2x_2 + \frac{3}{2} |= 2 | x_1 - x_2 + 1 |>| x_1 - x_2 + 1 | .$$

Then, for any $x_1, x_2 \in [0, 1]$, let $\delta_1 =| x_1 - x_2 |, \delta_2 =| x_2 + x_1 - \frac{1}{2} |, \delta_3 =| -x_2 - x_1 + \frac{3}{2} |$ and $\delta_4 =| x_1 - x_2 + 1 |$. Taking $\delta = min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. For any $n \in \mathbb{N}$, one has $| f^n(x_1) - f^n(x_2) |\geq \delta$. So, for every nonempty open subset $V \subset X$, the set $S_f(V, \delta) = \{n \in Z^+$: There exist $x, y \in V$ with $d(f^n(x), f^n(y)) > \delta\}$ has positive upper density. Thus, f is ergodically sensitive. Similarly, it can be proved that for every nonempty open subsets $V_1, V_2, \cdots, V_k \subset X$, the set $\bigcap_{1\leq i\leq k} S_f(V_i, \delta)$ is nonempty. So, f is multi-sensitive.

5. Conclusions

Two kinds of sensitivities associated with ergodic (i.e. ergodic multi-sensitivity and strongly ergodically multi-sensitivity) are preserved in the composite case and in inverse limit system. Moreover, for two systems (X, d) and (Y, d'), under the condition of that there is a semiopen factor map between them, the above sensitivities of X and Y are consistent.

Acknowledgments

This work was supported by the Opening Project of Artificial Intelligence Key Laboratory of Sichuan Province (2018RZJ03), the Opening Project of Bridge Non-destruction Detecting and Engineering Computing Key Laboratory of Sichuan Province (2018QZJ03), the Opening Project of Key Laboratory of Higher Education of Sichuan Province for Enterprise Informationalization and Internet of Things (No. 2020WZJ01), Ministry of Education Science and Technology Development center (No. 2020QT13) and Scientific Research Project of Sichuan University of Science and Engineering (No. 2020RC24).

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, On Devaney's definition of chaos, *Am. Math. Mon.*, **99** (1992), 332–334.
2. E. Glasner, B. Weiss, Sensitive dependence on initial conditions, *Nonlinearity*, **6** (1993), 1067–1075.
3. C. Abraham, G. Biau, B. Cadre, Chaotic properties of mapping on a probbility space, *J. Math. Anal. Appl.*, **266** (2002), 420–431.
4. L. He, X. Yan, L. Wang, Weak-mixing implies sensitive dependence, *J. Math. Anal. Appl.*, **299** (2004), 300–304.
5. R. Li, T. Lu, A. Waseem, Sensitivity and transitivity of systems satisfying the large deviations theorem in a sequence, *Int. J. Bifurcation Chaos.*, **29** (2019), 1950125.
6. T. K. S. Moothathu, Stronger forms of sensitivity for dynamical systems, *Nonlinearity*, **20** (2007), 2115–2126.
7. W. Huang, J. Li, X. Ye, X. Zhou, Positive topological entropy and $\triangle$-weakly mixing sets, *Adv. Math.*, **306** (2017), 653–683.
8. X. Wu, S. D. Liang, X. Ma, T. X. Lu, The mean sensitivity and mean equicontinuity in uniform spaces, *Int. J. Bifurcation Chaos.*, **30** (2020), 2050122.
9. H. Liu, E. Shi, G. Liao, Sensitivity of set-valuted discrete systems, *Nonlinear Anal.: Theory Methods Appl.*, **71** (2009), 6122–6125.
10. S. A. Eisa, P. Stechlinski, Sensitivity analysis of nonsmooth power control systems with an example of wind turbines, *Commun. Nonlinear Sci. Numer. Simul.*, **95** (2020), 105633.
11. G. Sakai, N. Matsunaga, K. Shimanoe, N. Yamazoe. Theory of gas-diffusion controlled sensitivity for thin film semiconductor gas sensor, *Sensor. Actuat. B: Chem.*, **80** (2001), 125–131.
12. M. A. Midoun, X. Wang, M. Z. Talhaoui, A sensitive dynamic mutual encryption system based on a new 1D chaotic map, *Opt. Lasers Eng.*, **139** (2020), 106485.
13. C. Caginalp, A dynamical systems approach to cryptocurrency stability, *AIMS Math.*, **4** (2019), 1065–1077.
14. B. Chaboki, A. Shakiba, An image encryption algorithm with a novel chaotic coupled mapped lattice and chaotic image scrambling technique, *J. Electr. Eng. Comput. Sci.*, **21** (2021), 1103–1124.
15. G. Beer, *Topologies on closed and closed convex sets*, Kluwer Academic Publishers, 1993.
16. E. Michael, Topologies on spaces of subsets, *Trans. Am. Math. Soc.*, **71** (1951), 152–182.
17. J. Banks, Chaos for induced hyperspace maps, *Chaos, Solitons Fractals*, **25** (2005), 681–685.

18. R. Gu, Kato's chaos in set valued discrete systems, *Chaos, Solitons Fractals*, **31** (2007), 765–771.

19. Z. Yin, Y. Chen, Q. Xiang, Dynamics of operator-weighted shifts, *Int. J. Bifurcation Chaos*, **29** (2019), 1950110.

20. H. Wang, F. C. Lei, L. D. Wang, DC3 and Li-Yorke chaos, *Appl. Math. Lett.*, **31** (2014), 29–33.

21. R. Hunter, B. E. Raines, Omega chaos and the specification property, *J. Math. Anal. Appl.*, **448** (2017), 908–913.

22. D. Kwietniak, P. Oprocha, Topological entropy and chaos for maps induced on hyperspaces, *Chaos, Solitons Fractals*, **33** (2007), 76–86.

23. Y. Wang, G. Wei, W. H. Campbell, Sensitive dependence on initial conditions between dynamical systems and their induced hyperspace dynamical systems, *Topol. Appl.*, **156** (2009), 803–811.

24. R. Li, Y. Shi, Stronger forms of sensitivity for measure-preserving maps and semiflows on probability spaces, *Abstr. Appl. Anal.*, (2014), 769523.

25. R. Yang, Topological ergodicity and topological double ergodicity, *Acta Math. Sin.*, **46** (2003), 555–560.

26. R. S. Li, Topological ergodicity, transitivity and chaos of the set-valued maps, *J. Nanjing Univ. Math. Biquarterly*, **25** (2008), 114–121.

27. R. S. Li, A note on stronger forms of sensitivity for dynamical systems, *Chaos, Solitons Fractals*, **45** (2012), 753–758.

28. P. Walter, *An introduction to ergodic theory*, New York: Spring-Verlag, 1982.

29. W. Bauer, K. Sigmund, Topological dynamics of transformations induced on the space of probability measures, *Monatsh. Math.*, **79** (1975), 81–92.

30. V. Strassen, The existence of probability measures with given marginals, *Ann. Math. Stat.*, **36** (1965), 423–439.

31. K. R. Parthasarathy, *Probability measures on metric spaces*, AMS Chelsea Publishing, 1967.

32. R. Li, A note on shadowing with chain transitivity, *Commun. Nonlinear Sci. Numer. Simul.*, **17** (2012), 2815–2823.

33. X. Wu, R. Li, Y. Zhang, The multi-$\mathcal{F}$-sensitivity and $(\mathcal{F}_1, \mathcal{F}_2)$-sensitivity for product systems, *J. Nonlinear Sci. Appl.*, **9** (2016), 4364–4370.

34. X. Wu, J. Wang, G. Chen, $\mathcal{F}$-sensitivity and multi-sensitivity of hyperspatial dynamical systems, *J. Math. Anal. Appl.*, **429** (2015), 16–26.