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Theory article

Uniqueness of difference polynomials

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Abstract: Let $f(z)$ be a transcendental meromorphic function of finite order and $c \in \mathbb{C}$ be a nonzero constant. For any $n \in \mathbb{N}^+$, suppose that $P(z, f)$ is a difference polynomial in $f(z)$ such as $P(z, f) = a_n f(z+nc) + a_{n-1} f(z+(n-1)c) + \cdots + a_1 f(z+c) + a_0 f(z)$, where $a_k (k = 0, 1, 2, \cdots, n)$ are not all zero complex numbers. In this paper, the authors investigate the uniqueness problems of $P(z, f)$.

Keywords: difference polynomial; Borel exceptional values; uniqueness
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1. Introduction

Let $f(z)$ be a function meromorphic in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the general conclussion of the Nevanlinna theory (see [1–3]). The order of $f(z)$ is denoted by $\sigma(f)$. For any $a \in \mathbb{C}$, the exponent of convergence of zeros of $f(z) - a$ is denoted by $\lambda(f, a)$. Especially, we denote $\lambda(f, 0)$ by $\lambda(f)$. Suppose that $f(z)$ is a transcendental meromorphic function of order $\sigma(f)$. If $\lambda(f, a) < \sigma(f)$, then a is said to be a Borel exceptional value of $f(z)$.

Recently, some well-known facts of the Nevanlinna theory of meromorphic function and their applications were extended for the differences of meromorhic functions (see [4–23]).

For any $c \in \mathbb{C} \backslash \{0\}$ and $n \in \mathbb{N}^+$, we define a difference polynomial in $f(z)$ as follows (see [19])

$$P(z, f) = a_n f(z+nc) + a_{n-1} f(z+(n-1)c) + \cdots + a_1 f(z+c) + a_0 f(z), \tag{1.1}$$

where $a_k (k = 0, 1, 2, \cdots, n)$ are not all zero complex numbers. Following [4], we denote the forward difference of f by $\Delta_c^n f(z)$. i.e.

$$\Delta_c f(z) = f(z+c) - f(z), \Delta_c^{n+1} f(z) = \Delta_c^n f(z+c) - \Delta_c^n f(z).$$

Observe that

$$\Delta_c^n f(z) = \sum_{k=0}^{n} (-1)^{n-k} C_n^k f(z+kc),$$

and

$$\sum_{k=0}^{n} (-1)^{n-k} C_n^k = 0,$$

where $C_n^k (k = 0, 1, 2, \cdots, n)$ are the binomial coefficients. If $a_k = C_n^k (-1)^{n-k} (k = 0, 1, 2, \cdots, n)$ in $P(z, f)$, then $P(z, f) = \triangle_c^n f$. Therefore, $P(z, f)$ is a more general difference polynomial than $\triangle_c^n f$. Noting that for $\triangle_c^n f$, $\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (-1)^{n-k} C_n^k = 0$, we assume that $\sum_{k=0}^{n} a_k = 0$ for some a_k of $P(z, f)$ in this paper (see [19]). The main purpose of this paper is to study uniqueness of the difference polynomial $P(z, f)$.

Let $a \in \mathbb{C}$, $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane. If $f - a$ and $g - a$ have the same zeros counting multiplicities, then we say $f(z)$ and $g(z)$ share the value a CM. We say that $f(z)$ and $g(z)$ share the value ∞ CM if $f(z)$ and $g(z)$ have the same poles counting multiplicities (see [24]). For the uniqueness of entire function $f(z)$ and its difference operator $\triangle_c f$, Chen and Yi [15, 16] had proved the following theorems.

Theorem A. [15] Let $f(z)$ be a transcendental entire function of finite order that is of a finite Borel exceptional value β, and let c be a constant such that $f(z+c) \not\equiv f(z)$. If $\Delta_c f(z)$ and $f(z)$ share $a (a \neq \beta)$ CM, then,

$$\frac{\Delta_c f(z) - a}{f(z) - a} = \frac{a}{a - \beta}.$$

Theorem B. [16] Let $f(z)$ be a transcendental entire function of finite order that is of a finite Borel exceptional value β, and let c be a constant such that $f(z+c) \not\equiv f(z)$. If $\Delta_c f(z)$ and $f(z)$ share β CM, then $\beta = 0$ and

$$\frac{f(z+c) - f(z)}{f(z)} = k,$$

for some constant k.

In this paper, the results on the uniqueness of entire function $f(z)$ and its difference operator $\triangle_c f$ established in theorems A and B are extended to meromorphic function $f(z)$ and $P(z, f)$ by using the similar method as that in [15, 16].

Theorem 1.1. *Let f be a transcendental meromorphic function of finite order. Suppose that $\beta \in \mathbb{C}$ and ∞ are Borel exceptional values of f, $P(z, f)$ is defined as that in* (1.1) *and $P(z, f) \not\equiv 0$. If $\beta \neq 0$, then $P(z, f)$ and f can not share the value β CM.*

Under the conditions of Theorem 1.1, there are only two possible scenarios. The first case is $P(z, f)$ and f share the value $a \neq \beta$ CM for any $\beta \in \mathbb{C}$, and the second case is $\beta = 0$, $P(z, f)$ and f share the value 0 CM. For the first case, we shall prove the following Theorem.

Theorem 1.2. *Let f be a transcendental meromorphic function of finite order. Suppose that $\beta \in \mathbb{C}$ and ∞ are Borel exceptional values of f, $P(z, f)$ is defined as that in* (1.1) *and $P(z, f) \not\equiv 0$. If $P(z, f)$ and f share the value $a \neq \beta$ CM. Then*

$$\frac{P(z, f) - a}{f - a} = \frac{a}{a - \beta}.$$

Example 1.3. *Let* $f(z) = e^z, c = \log 3$*,* $P(z, f) = f(z + 2c) - \frac{7}{2} f(z + c) + \frac{5}{2} f(z)$*. Then* $P(z, f)$ *and* $f(z)$ *share the value* 2 *CM and they satisfy*

$$\frac{P(z, f) - 2}{f - 2} = 1,$$

where 1 *satisfies* $\frac{a}{a-\beta}$*,* $a = 2, \beta = 0$.

Corollary 1.4. *Let* f *be a transcendental meromorphic function of finite order. Suppose that* $\beta \in \mathbb{C}$ *and* ∞ *are Borel exceptional values of* f*,* $c \in \mathbb{C}$ *is non-null and* $\Delta_c^n f \not\equiv 0$ *and* $n \in \mathbb{N}^+$*. If* $\Delta_c^n f$ *and* f *share the value* $a \neq \beta$ *CM. Then*

$$\frac{\Delta_c^n f - a}{f - a} = \frac{a}{a - \beta}.$$

For the second case, we shall prove the following Theorem.

Theorem 1.5. *Let* f *be a transcendental meromorphic function of order* $\sigma(f) < 2$*.* $P(z, f)$ *is defined as that in* (1.1) *and* $P(z, f) \not\equiv 0$*. If* $P(z, f)$ *and* f *share the value* 0 *CM. Then*

$$\frac{P(z, f)}{f} = \eta,$$

where η *is a constant.*

2. Proof of Theorems

Lemma 2.1. *[24] Suppose that* $f_1(z), f_2(z), \cdots, f_n(z) (n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \cdots, g_n(z)$ *are entire functions satisfying the following conditions.*

(*i*) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$.

(*ii*) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$.

(*iii*) *For* $1 \leq j \leq n, 1 \leq h < k \leq n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty, r \notin E),$$

where $E \subset (i, +\infty)$ *is of finite linear measure or finite logarithmic measure. Then* $f_j(z) \equiv 0 (j = 1, 2, \cdots, n)$.

Lemma 2.2. *Let* f *be a transcendental meromorphic function of finite order. Suppose that* $\beta \in \mathbb{C}$ *and* ∞ *are Borel exceptional values of* f*, then*

$$f(z) = A(z) e^{P(z)} + \beta,$$

where $P(z)$ *is a polynomial and* $A(z)$ *is a meromorphic function such that* $\lambda(A) = \lambda(\beta, f), \lambda(\frac{1}{A}) = \lambda(\frac{1}{f})$ *and*

$$\sigma(A) \leq \max\{\lambda(\beta, f), \lambda(\frac{1}{f})\} < \sigma(f) = \deg P(z).$$

Proof. Given that β is a Borel exceptional value of f, $f(z)$ can be written as

$$f(z) = z^k \frac{H_1(z)}{H_2(z)} e^{P(z)} + \beta,$$

where $k \in \mathbb{Z}$, $H_1(z)$ and $H_2(z)$ are the canonical products of f formed with the non-null zeros and poles of f, and $P(z)$ is a polynomial with $\sigma(f) = \deg P(z)$.

Put

$$A(z) = z^k \frac{H_1(z)}{H_2(z)}.$$

Since β and ∞ are Borel exceptional values of f, by the Theorem 2.3 in [24], we have

$$\sigma(H_1(z)) = \lambda(\beta, f) < \sigma(f), \sigma(H_1(z)) = \lambda(\frac{1}{f}) < \sigma(f),$$

and

$$\sigma(A) \leq \max\{\lambda(\beta, f), \lambda(\frac{1}{f})\} < \sigma(f) = \deg P(z).$$

□

Lemma 2.3. *[17] Let $A_0(z), A_1(z), \cdots, A_n(z)$ be entire functions of finite order so that among those having the maximal order $\sigma := \max\{\sigma(A_k(z)), 0 \leq k \leq n\}$, exactly one has its type strictly greater than the others. Then for any meromorphic solution of*

$$A_n(z)f(z + \omega_n) + \cdots + A_1(z)f(z + \omega_1) + A_0(z)f(z) = 0,$$

we have $\sigma(f) \geq \sigma + 1$.

2.1. Proof of Theorem 1.1

Suppose that $P(z, f)$ and $f(z)$ share the value β CM, then

$$\frac{P(z, f) - \beta}{f(z) - \beta} = e^{h(z)}, \tag{2.1}$$

where $h(z)$ is a polynomial. Since β and ∞ are Borel exceptional values of f, then by Lemma 2.2, $f(z)$ can be written as

$$f(z) = A(z)e^{P(z)} + \beta, \tag{2.2}$$

where $A(z)$ is a meromorphic function such that

$$\sigma(A) \leq \max\{\lambda(\beta, f), \lambda(\frac{1}{f})\} < \sigma(f) = \deg P(z).$$

It follows from (2.1) and (2.2) that

$$\frac{P(z, A(z)e^{P(z)} + \beta) - \beta}{A(z)e^{P(z)} + \beta - \beta} = e^{h(z)}. \tag{2.3}$$

As $\sum_{i=0}^{n} a_i = 0$, we get

$$P(z, A(z)e^{P(z)} + \beta) = P(z, A(z)e^{P(z)}). \tag{2.4}$$

Next, according to (2.3) and (2.4) ,we infer that

$$\frac{\sum_{i=0}^{n} a_i A(z+ic)e^{P(z+ic)} - \beta}{A(z)e^{P(z)}} = \sum_{i=0}^{n} a_i \frac{A(z+ic)}{A(z)} e^{P(z+ic)-P(z)} - \frac{\beta}{A(z)} e^{-P(z)} = e^{h(z)}. \tag{2.5}$$

As $\sigma(A) < \deg P(z)$ and $\deg(P(z+ic) - P(z)) \leq (\deg P(z)) - 1 = \sigma(f) - 1, i = 0, 1, 2, \cdots, n$, then $\sum_{i=0}^{n} a_i \frac{A(z+ic)}{A(z)} e^{P(z+ic)-P(z)}$ is a small meromorphic function respective to $\frac{\beta}{A(z)} e^{-P(z)}$. Applying the second fundamental theorem to $\frac{\beta}{A(z)} e^{-P(z)}$, we know that

$$\lambda\left(\sum_{i=0}^{n} a_i \frac{A(z+ic)}{A(z)} e^{P(z+ic)-P(z)} - \frac{\beta}{A(z)} e^{-P(z)}\right) = \deg P(z).$$

This contradicts with $e^{h(z)} \neq 0$. Thus, $P(z, f)$ and f can not share the value β CM.

2.2. *Proof of Theorem 1.2*

By the conditions, we can get $a \neq 0$. If $a = 0$, then $\beta \neq 0$. Since β and ∞ are Borel exceptional values of f, then by Lemma 2.2, $f(z)$ can be written as

$$f(z) = A(z)e^{P(z)} + \beta, \tag{2.6}$$

where $P(z)$ is a polynomial and $A(z)$ is a meromorphic function such that

$$\sigma(A) \leq \max\{\lambda(\beta, f), \lambda(\frac{1}{f})\} < \sigma(f) = \deg P(z).$$

Since $P(z, f)$ and $f(z)$ share the value 0 CM, we have

$$\frac{P(z, f)}{f(z)} = e^{h(z)}, \tag{2.7}$$

where $h(z)$ is a polynomial.

It follows from (2.6) and (2.7) that

$$\frac{P(z, A(z)e^{P(z)} + \beta)}{A(z)e^{P(z)} + \beta} = e^{h(z)}. \tag{2.8}$$

Since $\sum_{i=0}^{n} a_i = 0$, there is

$$P(z, A(z)e^{P(z)} + \beta) = P(z, A(z)e^{P(z)}). \tag{2.9}$$

In view of (2.8) and (2.9), it follows that

$$\frac{\sum_{i=0}^{n} a_i A(z+ic)e^{P(z+ic)}}{A(z)e^{P(z)} + \beta} = \frac{\sum_{i=0}^{n} a_i A(z+ic)e^{P(z+ic)-P(z)}}{A(z) + \beta e^{-P(z)}} = e^{h(z)}. \tag{2.10}$$

As $\deg(P(z+ic) - P(z)) \leq (\deg P(z)) - 1 = \sigma(f) - 1, i = 0, 1, 2, \cdots, n$, we see that

$$\lambda\left(\sum_{i=0}^{n} a_i A(z+ic)e^{P(z+ic)-P(z)}\right) \leq \sigma\left(\sum_{i=0}^{n} a_i A(z+ic)e^{P(z+ic)-P(z)}\right) \leq \sigma(f) - 1. \tag{2.11}$$

As $\beta \neq 0$ and $\sigma(A) < \sigma(f)$, applying the second fundamental theorem to $\beta e^{-P(z)}$, we have

$$\lambda(A(z) + \beta e^{-P(z)}) = \sigma(A(z) + \beta e^{-P(z)}) = \sigma(f). \tag{2.12}$$

From (2.10)–(2.12), we can get a contradiction. Thus, $a \neq 0$. Therefore,

$$\frac{P(z,f) - a}{f(z) - a} = e^{q(z)}, \tag{2.13}$$

where $q(z)$ is a polynomial with $\deg q(z) \leq \sigma(f)$. Since $\sum_{i=0}^{n} a_i = 0$, we have

$$P(z,f) = P(z, A(z)e^{P(z)} + \beta) = P(z, A(z)e^{P(z)}). \tag{2.14}$$

Hence, we can derive the following inequality by (2.13) and (2.14)

$$\sum_{i=0}^{n} a_i A(z+ic)e^{P(z+ic)} - a = (\beta - a)e^{q(z)} + e^{q(z)}A(z)e^{p(z)}, \tag{2.15}$$

i.e.

$$\begin{aligned} &a_n A(z+nc)e^{P(z+nc)} + a_{n-1}A(z+(n-1)c)e^{P(z+(n-1)c)} + \cdots \\ &+ a_1 A(z+c)e^{P(z+c)} + (a_0 - e^{q(z)})A(z)e^{P(z)} = (\beta - a)e^{q(z)} + a. \end{aligned} \tag{2.16}$$

Seeing that $q(z)$ is a polynomial with $\deg q(z) \leq \sigma(f)$, then $\deg q(z)$ only satisfies one of the following cases: $1 \leq \deg q(z) < \sigma(f) = \deg P(z)$; $\deg q(z) = \sigma(f) = \deg P(z)$ and $\deg q(z) = 0$.

Case 1. $1 \leq \deg q(z) < \sigma(f) = \deg P(z)$. By (2.16), we have

$$\sum_{i=1}^{n} a_i A(z+ic)e^{P(z+ic)-P(z)} + (a_0 - e^{q(z)})A(z) = \left((\beta - a)e^{q(z)} + a\right)e^{-P(z)}. \tag{2.17}$$

It follows from $\beta - a \neq 0, 1 \leq \deg q(z) < \deg P(z)$ that $(\beta - a)e^{q(z)} + a \not\equiv 0$. Hence, the order of $\left((\beta - a)e^{q(z)} + a\right)e^{-P(z)}$ is equal to $\sigma(f) = \deg P(z)$. As $\deg(P(z+ic) - P(z)) \leq (\deg P(z)) - 1$, $\sigma(A(z)) < \sigma(f) = \deg P(z)$ and $\deg q(z) < \sigma(f) = \deg P(z)$, we see that the order of $\sum_{i=1}^{n} a_i A(z + ic)e^{P(z+ic)-P(z)} + (a_0 - e^{q(z)})A(z)$ is less than $\sigma(f) = \deg P(z)$. We can get a contradiction from (2.17).

Case 2. $\deg q(z) = \sigma(f) = \deg P(z)$. Suppose

$$P(z) = p_k z^k + p_{k-1}z^{k-1} + \cdots + p_1 z + p_0, \quad q(z) = q_k z^k + q_{k-1}z^{k-1} + \cdots + q_1 z + q_0.$$

Thus p_k and q_k only satisfy one of the following cases: $p_k = -q_k$; $p_k = q_k$; $p_k \neq q_k$ and $p_k \neq -q_k$.

Subcase 2.1. $p_k = -q_k$**.** From (2.16), we can get

$$\sum_{i=0}^{n} a_i \frac{A(z+ic)}{A(z)} e^{P(z+ic)-P(z)} - e^{q(z)} = \frac{\beta - a}{A(z)} e^{q(z)-P(z)} + \frac{a}{A(z)} e^{-P(z)}.$$

i.e.

$$B_{11}(z)e^{-P(z)} + B_{12}(z)e^{q(z)-P(z)} + B_{13}(z)e^{r(z)} = 0. \tag{2.18}$$

where

$$\begin{aligned}
&r(z) \equiv 0,\\
&B_{11}(z) = \frac{a}{A(z)} + e^{q(z)+P(z)},\\
&B_{12}(z) = \frac{\beta - a}{A(z)},\\
&B_{13}(z) = -\sum_{i=0}^{n} a_i \frac{A(z+ic)}{A(z)} e^{P(z+ic)-P(z)}.
\end{aligned}$$

Since $p_k = -q_k$, the $\deg(q(z) + P(z)) \leq k - 1$. Note that

$$\deg(P(z + ic) - P(z)) \leq k - 1, i = 1, 2, \cdots,$$

$$\deg(-P(z) - (q(z) - P(z))) = \deg(-P(z) - r(z)) = \deg((q(z) - P(z)) - r(z)) = k.$$

By Lemma 2.1, we can get $\frac{\beta-a}{A(z)} \equiv 0$. Which contradicts with $a \neq \beta$.

Subcase 2.2. $p_k = q_k$**.** From (2.16), we can get

$$\sum_{i=0}^{n} a_i \frac{A(z+ic)}{A(z)} e^{P(z+ic)-P(z)} - e^{q(z)} = \frac{\beta - a}{A(z)} e^{q(z)-P(z)} + \frac{a}{A(z)} e^{-P(z)}.$$

i.e.

$$B_{21}(z)e^{-P(z)} + B_{22}(z)e^{q(z)} + B_{23}(z)e^{r(z)} = 0. \tag{2.19}$$

where

$$\begin{aligned}
&r(z) \equiv 0,\\
&B_{21}(z) = \frac{a}{A(z)},\\
&B_{22}(z) = 1,\\
&B_{23}(z) = \frac{\beta - a}{A(z)} e^{q(z)-P(z)} - \sum_{i=0}^{n} a_i \frac{A(z+ic)}{A(z)} e^{P(z+ic)-P(z)}.
\end{aligned}$$

By Lemma 2.1, we can get a contradiction.

Subcase 2.3. $p_k \neq q_k$ and $p_k \neq -q_k$. From (2.16), we can get

$$\left(\sum_{i=0}^{n} a_i A(z+ic) e^{P(z+ic)-P(z)}\right) e^{P(z)} - a = (\beta - a)e^{q(z)} + A(z)e^{P(z)+q(z)}.$$

i.e.

$$B_{31}(z)e^{P(z)} + B_{32}(z)e^{q(z)+P(z)} + B_{33}(z)e^{q(z)} + B_{34}(z)e^{r(z)} = 0. \tag{2.20}$$

where

$$\begin{aligned}
&r(z) \equiv 0,\\
&B_{31}(z) = \sum_{i=0}^{n} a_i A(z+ic)e^{P(z+ic)-P(z)},\\
&B_{32}(z) = -A(z),\\
&B_{33}(z) = -(\beta - a),\\
&B_{34}(z) = -a.
\end{aligned}$$

By Lemma 2.1, we can get a contradiction.

Case 3. $\deg q(z) = 0$. In this case, $e^{q(z)}$ is a constant. We denote it by C. Suppose that $C \neq \frac{a}{a-\beta}$,by (2.16) we can get

$$\sum_{i=0}^{n} a_i A(z+ic)e^{P(z+ic)} - a = (\beta - a)C + CA(z)e^{p(z)}.$$

i.e.

$$\sum_{i=0}^{n} a_i A(z+ic)e^{P(z+ic)-P(z)} - CA(z) = [(\beta - a)C + a]e^{-p(z)}. \tag{2.21}$$

Since $\deg(P(z+ic) - P(z)) \leq (\deg P(z)) - 1$, $\sigma(A(z)) < \sigma(f) = \deg P(z)$, then

$$\sigma(\sum_{i=0}^{n} a_i A(z+ic)e^{P(z+ic)-P(z)} - CA(z)) < \deg P(z) = \sigma([(\beta - a)C + a]e^{-p(z)}).$$

We can get a contradiction from (2.21). Hence $C = \frac{a}{a-\beta}$.

2.3. Proof of Theorem 1.5

Since $P(z, f)$ and f share the value 0 CM, there holds

$$\frac{P(z, f)}{f} = e^{H(z)},$$

where $H(z)$ is a polynomial. If $H(z) \neq$constant, then

$$a_n f(z+nc) + a_{n-1} f(z+(n-1)c) + \cdots + a_1 f(z+c) + (a_0 - e^{H(z)})f(z) = 0.$$

By Lemma 2.3, we have $\sigma(f) > \deg(H(z)) + 1 > 2$. This contradicts with $\sigma(f) < 2$. Hence $H(z)$ is a constant. Denote $\eta = e^{H(z)}$, then η is a constant and

$$\frac{P(z, f)}{f} = \eta.$$

3. Conclusions

The main result of this paper (Theorem 1.2) shows that $P(z, f)$ is a linear function of f, if the following conditions are satisfied:

(1) f is a transcendental meromorphic function of finite order with two Borel exceptional values β and ∞;

(2) $P(z, f)$ and f share the value $a(\neq \beta)$ CM.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. W. K. Hayman, *Meromorphic functions*, Oxford: Oxford Mathematical Monographs Clarendon Press, 1964.
2. L. Yang, *Value distribution theory*, Berlin: Springer-Verlag, 1993.
3. J. H. Zheng, *Value distribution of meromorphic functions*, Beijing: Tsinghua University Press, 2011.
4. W. Bergweiler, J. K. Langley, Zeros of differences of meromorphic functions, *Math. Proc. Cambridge Philos. Soc.*, **142** (2007), 133–147.
5. Y. M. Chiang, S. J. Feng, On the Nevanlinna characteristic of $f(z + \eta)$ and difference equations in the complex plane, *Ramanujan J.*, **16** (2008), 105–129.
6. Y. M. Chiang, S. J. Feng, On the growth of logarithmic difference, difference equations and logarithmic derivatives of meromorphic functions, *J. Trans. Am. Math. Soc.*, **361** (2009), 3767–3791.
7. R. G. Halburd, R. J. Korhonen, Nevanlinna theory for the difference operator, 2005. Available from: `https://arxiv.org/abs/math/0506011`.
8. R. G. Halburd, R. J. Korhonen, Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations, *J. Phys. A: Math. Theor.*, **40** (2007), 1–38.
9. R. G. Halburd, R. J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, *J. Math. Anal. Appl.*, **314** (2006), 477–487.
10. B. M. Deng, M. L. Fang, D. Liu, Uniqueness of meromorphic functions concerning shared functions with their difference, *Bull. Korean Math. Soc.*, **56** (2019), 1151–1524.
11. Z. B. Huang, R. R. Zhang, Uniqueness of the differences of meromorphic functions, *Anal. Math.*, **44** (2018), 461–473.

12. X. M. Li, C. Y. Kang, H. X. Yi, Uniqueness theorems of entire functions sharing a nonzero complex number with their difference operators, *Arch. Math.*, **96** (2011), 577–587.
13. Z. B. Huang, Value distribution and uniqueness on q-differences of meromorphic functions, *Bull. Korean Math. Soc.*, **50** (2013), 1157–1171.
14. Z. X. Chen, On growth, zeros and poles of meromorphic solutions of linear and nonlinear difference equations, *Sci. China Math.*, **54** (2011), 2123–2133.
15. Z. X. Chen, On the difference counterpart of Brück's conjecture, *Acta Math. Sci.*, **34** (2014), 653–659.
16. Z. X. Chen, H. X. Yi, On sharing values of meromorphic functions and their differences, *Results Math.*, **63** (2013), 557–565.
17. I. Laine, C. C. Yang, Clunie theorems for difference and q-difference polynomials, *J. London Math. Soc.*, **76** (2007), 556–566.
18. K. Liu, H. Z. Cao, T. B. Cao, Entire solutions of Fermat type differential differenceequations, *Arch. Math.*, **99** (2012), 147–155.
19. Z. X. Liu, Q. C. Zhang, Difference uniqueness theorems on meromorphic functions in several variables, *Turk. J. Math.*, **42** (2018), 2481–2505.
20. Z. J. Wu, Value distribution for difference operator of meromorphic functions with maximal deficiency sum, *J. Inequalities Appl.*, **530** (2013), 1–9.
21. H. Y. Xu, T. B. Cao, B. X. Liu, The growth of solutions of systems of complex q-shift difference equations, *Adv. Differ. Equations*, **2012** (2012), 216.
22. J. F. Xu, X. B, Zhang, The zeros of q-shift difference polynomials of meromorphic functions, *Adv. Differ. Equations*, **2012** (2012), 200.
23. R. R. Zhang, Z. X. Chen, Fixed points of meromorphic functions and of their difference, divided differences and shifts, *Acta Math. Sin. English Ser.*, **32** (2016), 1189–1202.
24. C. C. Yang, H. X. Yi, *Uniqueness theory of meromoprhic functions*, Dordrecht: Kluwer Academic Publishers, 2003.

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