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Research article

New subclass of analytic functions defined by q-analogue of p-valent Noor integral operator

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Abstract: In this paper, we introduce a certain subclass of analytic functions associated with q-analogue of p-valent Noor integral operator in the open unit disc. A variety of useful properties for this subclass are investigated including coefficient estimates and the familiar Fekete-Szegö type inequalities. Several known sequences of the main results are also highlighted.

Keywords: analytic functions; p-valent functions; q-analogue, q-derivative; q-difference operator; Hadamard product (or convolution)
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1. Introduction and definitions

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1.1}$$

which are p-valent and analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. We note that $\mathcal{A}(1) = \mathcal{A}$. For functions $f(z)$ given by (1.1) and $g(z)$ defined by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \quad (p \in \mathbb{N}), \tag{1.2}$$

the convolution of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.3}$$

For $f \in \mathcal{A}(p)$ given by (1.1) and $0 < q < 1$, the q-derivative of a function $f(z)$ is given by (see [1, 6, 7])

$$\mathcal{D}_{q,p} f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & for \quad z \neq 0, \\ f'(0) & for \quad z = 0, \end{cases} \tag{1.4}$$

provided that $f'(z)$ exists. From (1.1) and (1.4), we deduce that

$$\mathcal{D}_{q,p} f(z) = [p]_q z^{p-1} + \sum_{n=p+1}^{\infty} [n]_q a_n z^{n-1}, \tag{1.5}$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}, \;\; [0]_q = 0, \;\; 0 < q < 1. \tag{1.6}$$

We note that

$$\lim_{q \longrightarrow 1^-} \mathcal{D}_{q,p} f(z) = \lim_{q \longrightarrow 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. Further, for $p = 1$, we have $\mathcal{D}_{q,1} f(z) = \mathcal{D}_q f(z)$ (see [20]).

The q-number shift factorial for any non-negative integer n is defined by

$$[n]_q! = \begin{cases} 1 & for \quad n = 0 \\ [1]_q [2]_q \cdots [n]_q & for \quad n \in \mathbb{N}. \end{cases}$$

The Pochhammer q-generalized symbol for $x > 0$ and $n \in \mathbb{N}$ is also

$$[x, q]_n = \begin{cases} 1 & for \quad n = 0 \\ [x]_q [x+1]_q \cdots [x+n-1]_q & for \quad n \in \mathbb{N}, \end{cases}$$

and for $x > 0$, the q-gamma function is defined by

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x) \text{ and } \Gamma_q(1) = 1.$$

For $\lambda > -p$ ($p \in \mathbb{N}$), we define the function $f^{-1}_{\lambda+p-1,q}(z)$ by

$$f_{\lambda+p-1,q}(z) * f^{-1}_{\lambda+p-1,q}(z) = z^p + \sum_{n=p+1}^{\infty} \frac{[p+1, q]_{n-p}}{[1, q]_{n-p}} z^n, \tag{1.7}$$

where the function $f_{\lambda+p-1,q}(z)$ is given by

$$f_{\lambda+p-1,q}(z) = z^p + \sum_{n=p+1}^{\infty} \frac{[\lambda+p, q]_{n-p}}{[1, q]_{n-p}} z^n. \tag{1.8}$$

It is clear that the function defined in (1.8) converges absolutely in $\mathcal{U}$. Using the idea of convolution we define the q-p-valent Noor integral operator $\mathcal{I}_q^{\lambda+p-1} : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ as follows:

$$\mathcal{I}_q^{\lambda+p-1} f(z) = f_{\lambda+p-1,q}^{-1}(z) * f(z) = z^p + \sum_{n=p+1}^{\infty} \Phi_q(\lambda, p, n) a_n z^n, \tag{1.9}$$

where

$$\Phi_q(\lambda, p, n) = \frac{[p+1, q]_{n-p}}{[\lambda + p, q]_{n-p}} \ \ (\lambda > -p, \ p \in \mathbb{N}). \tag{1.10}$$

From (1.9), we can easily get the identity

$$q^\lambda z \mathcal{D}_{q,p}(\mathcal{I}_q^{\lambda+p} f(z)) = [\lambda + p]_q \mathcal{I}_q^{\lambda+p-1} f(z) - [\lambda]_q \mathcal{I}_q^{\lambda+p} f(z). \tag{1.11}$$

We note that:

(i) For $p = 1$, we have the q-Noor integral operator $\mathcal{I}_q^\lambda f(z)$ $(f \in \mathcal{A})$ which was introduced and studied by Arif et al. [4];

(ii) $\lim\limits_{q \longrightarrow 1^-} \mathcal{I}_q^{\lambda+p-1} f(z) = \mathcal{I}^{\lambda+p-1} f(z)$ which is the p-valent Noor integral operator (see [11]);

(iii) Taking $p = 1$ and letting $q \longrightarrow 1^-$ in (1.9), we obtain Noor integral operator for univalent functions (see [13, 14]);

(iv) For $\lambda = 1$, we have $\mathcal{I}_q^p f(z) = f(z)$ and for $\lambda = 0$, we have

$$\mathcal{I}_q^{p-1} f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{[p+1, q]_{n-p}}{[1, q]_{n-p}} a_n z^n = z^p + \sum_{n=p+1}^{\infty} \frac{[n]_q}{[p]_q} a_n z^n = \frac{z \mathcal{D}_{q,p} f(z)}{[p]_q},$$

$$\lim_{q \longrightarrow 1^-} \mathcal{I}_q^{p-1} f(z) = \mathcal{I}^{p-1} f(z) = z + \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right) a_n z^n = \frac{z f'(z)}{p}.$$

By using the operator $\mathcal{I}_q^{\lambda+p-1} f(z)$ we define the subclass $\mathcal{ST}_q(\lambda, p, k, b)$ of $\mathcal{A}(p)$ as follows:

Definition 1.1. Let $k \geq 0$, $\lambda > -p$, $p \in \mathbb{N}$, $b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $0 < q < 1$. A function $f \in \mathcal{A}(p)$ is said to be in the class $\mathcal{ST}_q(\lambda, p, k, b)$ if it satisfies

$$\begin{aligned} &\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{1}{[p]_q} \frac{z \mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1} f(z)\right)}{\mathcal{I}_q^{\lambda+p-1} f(z)} - 1\right)\right\} \\ > \ & k \left| \frac{1}{b}\left(\frac{1}{[p]_q} \frac{z \mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1} f(z)\right)}{\mathcal{I}_q^{\lambda+p-1} f(z)} - 1\right)\right|, \ (z \in \mathcal{U}). \end{aligned} \tag{1.12}$$

We note that: (1) $\lim\limits_{q \longrightarrow 1^-} \mathcal{ST}_q(1, p, k, 1 - \frac{\alpha}{p}) = \mathcal{ST}(p, k, \alpha) =$
$\left\{f \in \mathcal{A}(p) : \mathrm{Re}\left(\frac{zf'(z)}{f(z)} - \alpha\right) > k\left|\frac{zf'(z)}{f(z)} - p\right|, \ 0 \leq \alpha < p, z \in \mathcal{U}\right\}$ (see [19]);
(2) $\lim\limits_{q \longrightarrow 1^-} \mathcal{ST}_q(0, p, k, 1 - \frac{\alpha}{p}) = \mathcal{UST}(p, k, \alpha) =$
$\left\{f \in \mathcal{A}(p) : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > k\left|1 + \frac{zf''(z)}{f'(z)} - p\right|, \ 0 \leq \alpha < p, z \in \mathcal{U}\right\}$ (see [19]).

2. Geometric interpretation

A functions $f \in \mathcal{A}(p)$ is in the class $\mathcal{ST}_q(\lambda, p, k, b)$ if

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\frac{z\mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1} f(z)\right)}{\mathcal{I}_q^{\lambda+p-1} f(z)} - 1\right)$$

takes all the values in the conic domain $\Omega_k = p_k(\mathcal{U})$, where

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\},$$

or, equivalently,

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\frac{z\mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1} f(z)\right)}{\mathcal{I}_q^{\lambda+p-1} f(z)} - 1\right) \prec p_k(z),\ \Omega_k = p_k(\mathcal{U}). \tag{2.1}$$

The boundary $\partial\Omega_k$ of the above set when $k = 0$ becomes the imaginary axis, when $0 < k < 1$ a hyperbola, when $k = 1$ a parabola and an ellipse when $1 < k < \infty$. The functions $p_k(z)$ are defined by

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ 1 + \frac{2}{\pi^2}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \frac{1}{1-k^2}\cos\left(\frac{2}{\pi}(\cos^{-1} k) i \log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) - \frac{k^2}{1-k^2}, & 0 < k < 1, \\ 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int\limits_0^{u(z)/\sqrt{t}}\frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right) + \frac{k^2}{k^2-1}, & 1 < k < \infty, \end{cases} \tag{2.2}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$ $(0 < t < 1, z \in \mathcal{U})$, t is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind, and $R'(t)$ is complementary integral of $R(t)$ (see [9, 10, 18]).

By giving a specific value to the parameters $q, \lambda, p, k,$ and b in the class $\mathcal{ST}_q(\lambda, p, k, b)$, we get a lot of new and known subclasses studied by various others, for example,

(1) $\mathcal{ST}_q(\lambda, 1, k, b) = \mathcal{ST}_q(\lambda, k, b) = \left\{f \in \mathcal{A} : 1 + \frac{1}{b}\left(\frac{z\mathcal{D}_q\left(\mathcal{I}_q^{\lambda} f(z)\right)}{\mathcal{I}_q^{\lambda} f(z)} - 1\right) \prec p_k(z),\ z \in \mathcal{U}\right\}$;

(2) $\mathcal{ST}_q(\lambda, 1, k, 1) = \mathcal{ST}_q(\lambda, k) = \left\{f \in \mathcal{A} : \frac{z\mathcal{D}_q\left(\mathcal{I}_q^{\lambda} f(z)\right)}{\mathcal{I}_q^{\lambda} f(z)} \prec p_k(z),\ z \in \mathcal{U}\right\}$;

(3) $\mathcal{ST}_q(\lambda, p, k, 1 - \frac{\alpha}{[p]_q}) = \mathcal{ST}_q(\lambda, p, k, \alpha) =$
$\left\{f \in \mathcal{A}(p) : \frac{1}{([p]_q - \alpha)}\left(\frac{z\mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1} f(z)\right)}{\mathcal{I}_q^{\lambda+p-1} f(z)} - \alpha\right) \prec p_k(z),\ 0 \le \alpha < [p]_q,\ z \in \mathcal{U}\right\}$;

(4) $\mathcal{ST}_q\left(\lambda, p, k, \left(1 - \frac{\alpha}{[p]_q}\right)\cos\gamma e^{-i\gamma}\right) = \mathcal{ST}_q^{\gamma}(\lambda, p, k, \alpha) =$
$\left\{f \in \mathcal{A}(p) : e^{i\gamma}\frac{z\mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1} f(z)\right)}{\mathcal{I}_q^{\lambda+p-1} f(z)} \prec ([p]_q - \alpha)\cos\gamma p_k(z) + \alpha\cos\gamma + i[p]_q\sin\gamma,\right.$
$\left. 0 \le \alpha < [p]_q, |\gamma| < \frac{\pi}{2}, z \in \mathcal{U}\right\}$;

(5) $\mathcal{ST}_q(1, p, k, b) = \mathcal{ST}_q(p, k, b) = \left\{f \in \mathcal{A}(p) : 1 + \frac{1}{b}\left(\frac{1}{[p]_q}\frac{z\mathcal{D}_{q,p} f(z)}{f(z)} - 1\right) \prec p_k(z),\ z \in \mathcal{U}\right\}$;

(6) $\mathcal{ST}_q(1, p, k, 1 - \frac{\alpha}{[p]_q}) = \mathcal{ST}_q(p, k, \alpha) =$
$\left\{f \in \mathcal{A}(p) : \frac{1}{([p]_q - \alpha)}\left(\frac{z\mathcal{D}_{q,p} f(z)}{f(z)} - \alpha\right) \prec p_k(z),\ 0 \le \alpha < [p]_q,\ z \in \mathcal{U}\right\}$;

(7) $\mathcal{ST}_q\left(1,p,k,\left(1-\frac{\alpha}{[p]_q}\right)\cos\gamma e^{-i\gamma}\right)=\mathcal{ST}_q^{\gamma}(p,k,\alpha)=$
$\Big\{f\in\mathcal{A}(p): e^{i\gamma}\frac{z\mathcal{D}_{q,p}f(z)}{f(z)}\prec([p]_q-\alpha)\cos\gamma p_k(z)+\alpha\cos\gamma+i[p]_q\sin\gamma,$
$0\le\alpha<[p]_q,|\gamma|<\frac{\pi}{2},z\in\mathcal{U}\Big\}.$
Also we note that:
(8) $\mathcal{ST}_q(\lambda,p,0,b)=\mathcal{S}_q(\lambda,p,b)=$
$\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{[p]_q+\frac{1}{b}\left(\frac{z\mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1}f(z)\right)}{\mathcal{I}_q^{\lambda+p-1}f(z)}-[p]_q\right)\right\}>0,\ z\in\mathcal{U}\right\},$
$\mathcal{S}_q(\lambda,p,1-\frac{\alpha}{[p]_q})=\mathcal{S}_q(\lambda,p,\alpha)=$
$\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{\frac{z\mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1}f(z)\right)}{\mathcal{I}_q^{\lambda+p-1}f(z)}\right\}>\alpha,0\le\alpha<[p]_q,z\in\mathcal{U}\right\},$
$\mathcal{S}_q(1,p,\alpha)=\mathcal{S}_q(p,\alpha)=$
$\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{\frac{z\mathcal{D}_{q,p}f(z)}{f(z)}\right\}>\alpha,0\le\alpha<[p]_q,z\in\mathcal{U}\right\},\mathcal{S}_q(1,\alpha)=\mathcal{S}_q(\alpha)$ (see [20]);
(9) $\mathcal{ST}_q\left(\lambda,p,0,\left(1-\frac{\alpha}{[p]_q}\right)\cos\gamma e^{-i\gamma}\right)=\mathcal{S}_q^{\gamma}(\lambda,p,\alpha)=$
$\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{e^{i\gamma}\frac{z\mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1}f(z)\right)}{\mathcal{I}_q^{\lambda+p-1}f(z)}\right\}>\alpha\cos\gamma,0\le\alpha<[p]_q,\ |\gamma|<\frac{\pi}{2},\ z\in\mathcal{U}\right\},$
$\mathcal{S}_q^{\gamma}(1,p,\alpha)=\mathcal{S}_q^{\gamma}(p,\alpha)=$
$\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{e^{i\gamma}\frac{z\mathcal{D}_{q,p}f(z)}{f(z)}\right\}>\alpha\cos\gamma,0\le\alpha<[p]_q,\ |\gamma|<\frac{\pi}{2},\ z\in\mathcal{U}\right\};$
(10) $\lim\limits_{q\longrightarrow 1^-}\mathcal{ST}_q(\lambda,p,0,b)=\mathcal{S}(\lambda,p,b)=$
$\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{p+\frac{1}{b}\left(\frac{z\left(\mathcal{I}_q^{\lambda+p-1}f(z)\right)'}{\mathcal{I}_q^{\lambda+p-1}f(z)}-p\right)\right\}>0,z\in\mathcal{U}\right\},$
$\mathcal{S}\left(\lambda,p,\left(1-\frac{\alpha}{p}\right)\cos\gamma e^{-i\gamma}\right)=\mathcal{S}^{\gamma}(\lambda,p,\alpha)=$
$\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{e^{i\gamma}\frac{z\left(\mathcal{I}_q^{\lambda+p-1}f(z)\right)'}{\mathcal{I}_q^{\lambda+p-1}f(z)}\right\}>\alpha\cos\gamma,\ 0\le\alpha<p,\ \ |\gamma|<\frac{\pi}{2},\ z\in\mathcal{U}\right\},$
$\mathcal{S}^{\gamma}\left(1,p,\left(1-\frac{\alpha}{p}\right)\cos\gamma e^{-i\gamma}\right)=\mathcal{S}^{\gamma}(p,\alpha)=$
$\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\}>\alpha\cos\gamma,0\le\alpha<p,\ |\gamma|<\frac{\pi}{2},\ z\in\mathcal{U}\right\}$ (see [22]),
$\mathcal{S}(1,p,b)=\mathcal{S}(p,b)=\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{p+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-p\right)\right\}>0,\ z\in\mathcal{U}\right\}$ (see [23]),
$\mathcal{S}(0,p,b)=C(p,b)=\left\{f\in\mathcal{A}(p):\operatorname{Re}\left\{p+\frac{1}{b}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right\}>0,\ z\in\mathcal{U}\right\}$ (see [2, 3, 21, 23]), $\mathcal{S}(1,b)=\mathcal{S}(b)$ and $C(1,b)=C(b)$ (see [15–17]);
(11) $\lim\limits_{q\longrightarrow 1^-}\mathcal{ST}_q(1,1,k,1-\alpha)=\mathcal{ST}(k,\alpha)=$
$\left\{f\in\mathcal{A}:\operatorname{Re}\left(\frac{zf'(z)}{f(z)}-\alpha\right)>k\left|\frac{zf'(z)}{f(z)}-1\right|,\ 0\le\alpha<1,z\in\mathcal{U}\right\}$ (see [5]);
(12) $\lim\limits_{q\longrightarrow 1^-}\mathcal{ST}_q\left(1,p,k,\left(1-\frac{\alpha}{p}\right)\cos\gamma e^{-i\gamma}\right)=\mathcal{ST}^{\gamma}(p,k,\alpha)=$
$\left\{f\in\mathcal{A}(p):\operatorname{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}-\alpha\cos\gamma\right)>k\left|\frac{zf'(z)}{f(z)}-p\right|,\ 0\le\alpha<p,\ |\gamma|<\frac{\pi}{2},z\in\mathcal{U}\right\},$
$\lim\limits_{q\longrightarrow 1^-}\mathcal{ST}_q\left(0,p,k,\left(1-\frac{\alpha}{p}\right)\cos\gamma e^{-i\gamma}\right)=\mathcal{UST}^{\gamma}(p,k,\alpha)=$
$\Big\{f\in\mathcal{A}(p):\operatorname{Re}\left\{e^{i\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)-\alpha\cos\gamma\right\}>k\left|1+\frac{zf''(z)}{f'(z)}-p\right|,$
$0\le\alpha<p,\ |\gamma|<\frac{\pi}{2},z\in\mathcal{U}\Big\}.$

We need the following lemmas in order to establish our main results.

Lemma 2.1. *[8] Let* $0 \le k < \infty$ *be fixed and let* p_k *be defined by (2.2). If* $p_k(z) = 1 + Q_1 z + Q_2 z^2 + \cdots$ *, then*

$$Q_1 = \begin{cases} \frac{2A^2}{1-k^2}, & 0 \le k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4\sqrt{t}(k^2-1)R^2(t)(1+t)}, & 1 < k < \infty, \end{cases} \tag{2.3}$$

and

$$Q_2 = \begin{cases} \frac{(A^2+2)}{3} Q_1, & 0 \le k < 1, \\ \frac{2}{3} Q_1, & k = 1, \\ \frac{4R^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t}R^2(t)(1+t)} Q_1, & 1 < k < \infty, \end{cases} \tag{2.4}$$

where $A = \frac{2\cos^{-1} k}{\pi}$*, and* $t \in (0,1)$ *is chosen such that* $k = \cosh\left(\frac{\pi R'(t)}{R(t)}\right)$*, where* $R(t)$ *is the Legendre's complete elliptic integral of the first kind.*

Lemma 2.2. *[12] Let* $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$*, i.e., let h be analytic in* $\mathcal{U}$ *and satisfies* $Re\,(h(z)) > 0$ $(z \in \mathcal{U})$*, then*

$$\left|c_2 - v c_1^2\right| \le 2 \max\{1, |2v - 1|\} \ \ (v \in \mathbb{C}). \tag{2.5}$$

The result is sharp for a function given by

$$g(z) = \frac{1+z^2}{1-z^2} \text{ or } g(z) = \frac{1+z}{1-z}.$$

Lemma 2.3. *[12] If* $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$*, then*

$$\left|c_2 - v c_1^2\right| \le \begin{cases} 2 - 4v & \text{if} \quad v \le 0, \\ 2 & \text{if} \quad 0 \le v \le 1, \\ 4v - 2 & \text{if} \quad v \ge 1, \end{cases} \tag{2.6}$$

where $v < 0$ *or* $v > 1$*, the equality holds iff* $h(z) = \frac{1+z}{1-z}$ *or one of its rotations. If* $0 < v < 1$*, then he equality holds iff* $h(z) = \frac{1+z^2}{1-z^2}$ *or one of its rotations. If* $v = 0$*, then he equality holds iff* $h(z) = \left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z}(0 \le \lambda \le 1)$ *or one of its rotations. If* $v = 1$*, then he equality holds if and only if g is reciprocal of one of the function such that the equality holds in the case of* $v = 0$.

Also the above upper bound is sharp, and it can improved as follows when $0 < v < 1$:

$$\left|c_2 - v c_1^2\right| + v\,|c_1|^2 \le 2 \ \ (0 \le v \le \frac{1}{2}),$$

and

$$\left|c_2 - v c_1^2\right| + (1 - v)\,|c_1|^2 \le 2 \ \ (\frac{1}{2} \le v \le 1).$$

3. Main results

We shall assume throughout this paper, unless otherwise stated, that $0 \leq k < \infty$, $p \in \mathbb{N}$, $\lambda > -p$, $b \in \mathbb{C}^*$, $0 < q < 1$, Q_1 is given by (2.3) and Q_2 is given by (2.4), $\Phi_q(\lambda, p, n)$ is given by (1.10) and $z \in \mathcal{U}$.

Theorem 3.1. *Let $f \in \mathcal{A}(p)$ be given by (1.1). If the inequality*

$$\sum_{n=p+1}^{\infty} \left\{(k+1)\left([n]_q - [p]_q\right) + [p]_q |b|\right\} \Phi_q(\lambda, p, n) |a_n| \leq [p]_q |b|, \tag{3.1}$$

holds, then $f \in \mathcal{ST}_q(\lambda, p, k, b)$.

Proof. Assume the inequality (3.1) holds. Let us assume that

$$H(z) = 1 + \frac{1}{b}\left(\frac{1}{[p]_q} \frac{z\mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1} f(z)\right)}{\mathcal{I}_q^{\lambda+p-1} f(z)} - 1\right).$$

We have

$$\begin{aligned} |H(z) - 1| &= \frac{1}{[p]_q |b|} \left| \frac{\sum\limits_{n=p+1}^{\infty} \left([n]_q - [p]_q\right) \Phi_q(\lambda, p, n) a_n z^{n-p}}{1 + \sum\limits_{n=p+1}^{\infty} \Phi_q(\lambda, p, n) a_n z^{n-p}} \right| \\ &\leq \frac{1}{[p]_q |b|} \frac{\sum\limits_{n=p+1}^{\infty} \left([n]_q - [p]_q\right) \Phi_q(\lambda, p, n) |a_n|}{1 - \sum\limits_{n=p+1}^{\infty} \Phi_q(\lambda, p, n) |a_n|}. \end{aligned}$$

Now consider

$$\begin{aligned} k|H(z) - 1| - \operatorname{Re}(H(z) - 1) &\leq (k+1)|H(z) - 1| \\ &< \frac{(k+1) \sum\limits_{n=p+1}^{\infty} \left([n]_q - [p]_q\right) \Phi_q(\lambda, p, n) |a_n|}{[p]_q |b| \left(1 - \sum\limits_{n=p+1}^{\infty} \Phi_q(\lambda, p, n) |a_n|\right)}. \end{aligned}$$

The last inequality is bounded by 1 if (3.1) holds. □

Corollary 3.2. *If $f \in \mathcal{ST}_q(\lambda, p, k, b)$, then*

$$|a_n| \leq \frac{[p]_q |b|}{\left\{(k+1)\left([n]_q - [p]_q\right) + [p]_q |b|\right\} \Phi_q(\lambda, p, n)} \quad (n \geq p+1). \tag{3.2}$$

The inequality (3.2) is sharp for the function

$$f(z) = z^p + \frac{[p]_q |b|}{\left\{(k+1)\left([n]_q - [p]_q\right) + [p]_q |b|\right\} \Phi_q(\lambda, p, n)} z^n \quad (n \geq p+1). \tag{3.3}$$

Choosing $p = 1$ and $b = 1 - \alpha$, $0 \leq \alpha < 1$, in Theorem 3.1, we obtain the following corollary.

Corollary 3.3. *Let $f \in \mathcal{A}$ be given by (1.1) with $p = 1$ and satisfy*

$$\sum_{n=2}^{\infty} \left\{(k+1)\left([n]_q - 1\right) + (1-\alpha)\right\} \Phi_q(\lambda, 1, n) |a_n| \leq 1 - \alpha.$$

Then $f \in \mathcal{ST}_q(\lambda, k, \alpha)$.

Taking $b = 1 - \frac{\alpha}{[p]_q}$ $(0 \leq \alpha < [p]_q)$ in Theorem 3.1, we obtain the following consequence.

Corollary 3.4. *Let $f \in \mathcal{A}(p)$ be given by (1.1) and satisfy*

$$\sum_{n=p+1}^{\infty} \left\{(k+1)\left([n]_q - [p]_q\right) + ([p]_q - \alpha)\right\} \Phi_q(\lambda, p, n) |a_n| \leq [p]_q - \alpha.$$

Then $f \in \mathcal{ST}_q(\lambda, p, k, \alpha)$.

Letting $q \longrightarrow 1^-$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.5. *Let $f \in \mathcal{A}(p)$ be given by (1.1) and satisfy*

$$\sum_{n=p+1}^{\infty} \{(k+1)(n-p) + p|b|\} \Phi_q(\lambda, p, n) |a_n| \leq p|b|.$$

Then $f \in \mathcal{ST}(\lambda, p, k, b)$.

Putting $b = \left(1 - \frac{\alpha}{[p]_q}\right) \cos\gamma e^{-i\gamma}$ $(0 \leq \alpha < [p]_q, |\gamma| < \frac{\pi}{2})$ in Theorem 3.1, we obtain the following consequence.

Corollary 3.6. *Let $f \in \mathcal{A}(p)$ be given by (1.1) and satisfy*

$$\sum_{n=p+1}^{\infty} \left\{(k+1)\left([n]_q - [p]_q\right) + ([p]_q - \alpha)\cos\gamma\right\} \Phi_q(\lambda, p, n) |a_n| \leq \left([p]_q - \alpha\right) \cos\gamma.$$

Then $f \in \mathcal{ST}_q^{\gamma}(\lambda, p, k, \alpha)$.

Letting $q \longrightarrow 1^-$ and putting $b = 1 - \frac{\alpha}{p}$ $(0 \leq \alpha < p)$ and $\lambda = 1$ in Theorem 3.1, we obtain the following corollary (see also [19], Theorem 1, with $n = 0$).

Corollary 3.7. *Let $f \in \mathcal{A}(p)$ be given by (1.1) and satisfy*

$$\sum_{n=p+1}^{\infty} \{(k+1)(n-p) + (p-\alpha)\} |a_n| \leq p - \alpha.$$

Then $f \in \mathcal{ST}(p, k, \alpha)$.

Letting $q \longrightarrow 1^-$ and putting $b = 1 - \frac{\alpha}{p}$ $(0 \leq \alpha < p)$ and $\lambda = 0$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.8. *Let $f \in \mathcal{A}(p)$ be given by (1.1) and satisfy*

$$\sum_{n=p+1}^{\infty} \binom{n}{p} \{(k+1)(n-p) + (p-\alpha)\} |a_n| \le p - \alpha.$$

Then $f \in \mathcal{UST}(p,k,\alpha)$.

Taking $p = 1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.9. *If a function $f \in \mathcal{A}$ has the form (1.1) (with $p = 1$) and satisfy*

$$\sum_{n=2}^{\infty} \left\{(k+1)\left([n]_q - 1\right) + |b|\right\} \Phi_q(\lambda, n) |a_n| \le |b| .$$

Then $f \in \mathcal{ST}_q(\lambda,k,b)$.

Theorem 3.10. *If $f \in \mathcal{ST}_q(\lambda,p,k,b)$. Then*

$$\left|a_{p+1}\right| \le \frac{[p]_q |b| Q_1}{q^p \Phi_q(\lambda,p,p+1)} = \frac{[p]_q |b| Q_1 [\lambda+p]_q}{q^p [p+1]_q}, \tag{3.4}$$

and for all $n \ge 3$

$$\left|a_{n+p-1}\right| \le \frac{[p]_q |b| Q_1}{q^p [n-1]_q \Phi_q(\lambda,p,n+p-1)} \prod_{j=1}^{n-2} \left(1 + \frac{[p]_q |b| Q_1}{q^p [j]_q}\right), \tag{3.5}$$

where Q_1 is given by (2.3).

Proof. Let

$$p(z) = 1 + \frac{1}{b}\left(\frac{1}{[p]_q} \frac{z\mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1} f(z)\right)}{\mathcal{I}_q^{\lambda+p-1} f(z)} - 1\right),$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ is analytic in $\mathcal{U}$ and it can be written as

$$\sum_{n=p+1}^{\infty} \left([n]_q - [p]_q\right) \Phi_q(\lambda,p,n) a_n z^n \le [p]_q b \left(\mathcal{I}_q^{\lambda+p-1} f(z)\right) \left(\sum_{n=1}^{\infty} c_n z^n\right). \tag{3.6}$$

Comparing the coefficients of z^{n+p-1} on both sides of (3.6), we obtain

$$\begin{aligned} &\left([n+p-1]_q - [p]_q\right) \Phi_q(\lambda,p,n+p-1) a_{n+p-1} \\ = \;& [p]_q b \left\{c_1 \Phi_q(\lambda,p,n+p-1) a_{n+p-2} + \cdots + c_{n-1}\right\}. \end{aligned}$$

Taking the absolute value on both sides and using $|c_n| \le Q_1$ $(n \ge 1)$ (see [18]), we obtain

$$\left|a_{n+p-1}\right| \leq \frac{[p]_q |b| Q_1}{q^p [n-1]_q \Phi_q(\lambda, p, n+p-1)}$$
$$\times \left\{1 + \Phi_q(\lambda, p, p+1)\left|a_{p+1}\right| + \cdots + \Phi_q(\lambda, p, n+p-2)\left|a_{n+p-2}\right|\right\}. \tag{3.7}$$

We apply the mathematical induction on (3.7), so for $n = 2$, we have

$$\left|a_{p+1}\right| \leq \frac{[p]_q |b| Q_1}{q^p \Phi_q(\lambda, p, p+1)} = \frac{[p]_q |b| Q_1 [\lambda + p]_q}{q^p [p+1]_q}, \tag{3.8}$$

this shows that the result is true for $n = 2$. Now for $n = 3$ we have

$$\left|a_{p+2}\right| \leq \frac{[p]_q |b| Q_1}{q^p [2]_q \Phi_q(\lambda, p, p+2)} \left(1 + \Phi_q(\lambda, p, p+1)\left|a_{p+1}\right|\right),$$

using (3.8), we obtain

$$\left|a_{p+2}\right| \leq \frac{[p]_q |b| Q_1}{q^p [2]_q \Phi_q(\lambda, p, p+2)} \left(1 + \frac{[p]_q |b| Q_1}{q^p [1]_q}\right),$$

which is true for $n = 3$. Let us assume that (3.7) is true for $n \leq t$, that is

$$\left|a_{t+p-1}\right| \leq \frac{[p]_q |b| Q_1}{q^p [t-1]_q \Phi_q(\lambda, p, t+p-1)} \prod_{j=1}^{t-2} \left(1 + \frac{[p]_q |b| Q_1}{q^p [j]_q}\right).$$

Consider

$$\begin{aligned}
\left|a_{t+p}\right| &\leq \frac{[p]_q |b| Q_1}{q^p [t]_q \Phi_q(\lambda, p, t+p)} \\
&\quad \times \left\{1 + \Phi_q(\lambda, p, p+1)\left|a_{p+1}\right| + \cdots + \Phi_q(\lambda, p, t+p-1)\left|a_{t+p-1}\right|\right\} \\
&\leq \frac{[p]_q |b| Q_1}{q^p [t]_q \Phi_q(\lambda, p, t+p)} \left\{1 + \frac{[p]_q |b| Q_1}{q^p} + \frac{[p]_q |b| Q_1}{q^p [2]_q}\left(1 + \frac{[p]_q |b| Q_1}{q^p [1]_q}\right)\right. \\
&\quad + \frac{[p]_q |b| Q_1}{q^p [3]_q}\left(1 + \frac{[p]_q |b| Q_1}{q^p [1]_q}\right)\left(1 + \frac{[p]_q |b| Q_1}{q^p [2]_q}\right) + \cdots + \\
&\quad \left. \frac{[p]_q |b| Q_1}{q^p [t-1]_q} \prod_{j=1}^{t-2}\left(1 + \frac{[p]_q |b| Q_1}{q^p [j]_q}\right)\right\} \\
&= \frac{[p]_q |b| Q_1}{q^p [t]_q \Phi_q(\lambda, p, t+p)} \prod_{j=1}^{t-1}\left(1 + \frac{[p]_q |b| Q_1}{q^p [j]_q}\right).
\end{aligned}$$

So, the result is true for $n = t + 1$. Also, we proved that the result true for all $n (n \geq 2)$ using mathematical induction. □

Taking $p = 1$ in Theorem 3.10, we obtain the following corollary.

Corollary 3.11. *Let $f \in \mathcal{A}$ be given by (1.1)* (*with* $p = 1$). *If* $f \in \mathcal{ST}_q(\lambda, k, b)$, *then*

$$|a_2| \leq \frac{[\lambda+1]_q \, |b| \, Q_1}{q[2]_q},$$

and

$$|a_n| \leq \frac{|b| \, Q_1}{q[n-1]_q \Phi_q(\lambda, 1, n)} \prod_{j=1}^{n-2} \left(1 + \frac{|b| \, Q_1}{q[j]_q}\right) \ (n \geq 3).$$

Taking $b = 1 - \alpha$ $(0 \leq \alpha < 1)$ and $p = 1$ in Theorem 3.10, we obtain the following consequence.

Corollary 3.12. *Let $f \in \mathcal{A}$ be given by (1.1)* (*with* $p = 1$). *If* $f \in \mathcal{ST}_q(\lambda, k, \alpha)$, *then*

$$|a_2| \leq \frac{P_1[\lambda+1]_q}{q[2]_q},$$

and

$$|a_n| \leq \frac{[p]_q P_1}{q[n-1]_q \Phi_q(\lambda, n)} \prod_{j=1}^{n-2} \left(1 + \frac{P_1}{q[j]_q}\right) \ (n \geq 3),$$

where $P_1 = (1-\alpha)Q_1$ *and* Q_1 *is given by (2.3).*

Taking $b = 1 - \frac{\alpha}{[p]_q}$ $(0 \leq \alpha < [p]_q)$ in Theorem 3.10, we obtain the following result.

Corollary 3.13. *Let $f \in \mathcal{A}(p)$ be given by (1.1). If $f \in \mathcal{ST}_q(\lambda, p, k, \alpha)$, then*

$$\left|a_{p+1}\right| \leq \frac{\left([p]_q - \alpha\right) Q_1}{q^p \Phi_q(\lambda, p, n+p-1)},$$

and for all $n \geq 3$,

$$\left|a_{n+p-1}\right| \leq \frac{\left([p]_q - \alpha\right) Q_1}{q^p [n-1]_q \Phi_q(\lambda, p, n+p-1)} \prod_{j=1}^{n-2} \left(1 + \frac{\left([p]_q - \alpha\right) Q_1}{q^p [j]_q}\right).$$

Putting $b = \left(1 - \frac{\alpha}{[p]_q}\right) \cos\gamma e^{-i\gamma}$ $(0 \leq \alpha < [p]_q, \ |\gamma| < \frac{\pi}{2})$ in Theorem 3.10, we obtain the following consequence.

Corollary 3.14. *Let $f \in \mathcal{A}(p)$ be given by (1.1). If $f \in \mathcal{ST}_q(\lambda, p, k, \alpha)$, then*

$$\left|a_{p+1}\right| \leq \frac{\left([p]_q - \alpha\right) \cos\gamma Q_1}{q^p \Phi_q(\lambda, p, p+1)},$$

and for all $n \geq 3$,

$$\left|a_{n+p-1}\right| \leq \frac{\left([p]_q - \alpha\right) \cos\gamma Q_1}{q^p [n-1]_q \Phi_q(\lambda, p, n+p-1)} \prod_{j=1}^{n-2} \left(j + \frac{\left([p]_q - \alpha\right) \cos\gamma Q_1}{q^p [j]_q}\right).$$

Theorem 3.15. *Let $f \in \mathcal{ST}_q(\lambda, p, k, b)$. Then $f(\mathcal{U})$ contains an open disc*

$$r = \frac{q^p[p+1]_q}{q^p(p+1)[p+1]_q + [p]_q\,|b|}.$$

Proof. Let $w_0 \in \mathbb{C}$ and $w_0 \neq 0$ such that $f(z) \neq w_0$ for $z \in \mathcal{U}$. Then

$$f_1(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z^p + \left(a_{p+1} + \frac{1}{w_0}\right)z^{p+1} + \cdots .$$

Since f_1 is univalent, so

$$\left|a_{p+1} + \frac{1}{w_0}\right| \leq p+1.$$

Now using Theorem 3.10, we have

$$\left|\frac{1}{w_0}\right| \leq p+1 + \frac{[p]_q\,|b|\,Q_1[\lambda+p]_q}{q^p[p+1]_q},$$

and hence we have

$$|w_0| \geq \frac{q^p[p+1]_q}{q^p(p+1)[p+1]_q + [p]_q\,|b|\,Q_1[\lambda+p]_q}.$$

This completes the proof of Theorem 3.15 □

Theorem 3.16. *Let $0 \leq k < \infty$ be fixed and let $f \in \mathcal{ST}_q(\lambda, p, k, b)$ with the form (1.1). Then for a complex μ, we have*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{[p]_q\,|b|\,Q_1[\lambda+p,q]_2}{2[2]_q q^p[p+1,q]_2} \max\{1, |2v-1|\}, \tag{3.9}$$

where

$$v = \frac{1}{2}\left\{1 - \frac{Q_2}{Q_1} - \frac{[p]_q b Q_1}{q^p}\left(1 - \frac{[2]_q[\lambda+p]_q[p+2]_q}{[\lambda+p+1]_q[p+1]_q}\mu\right)\right\},$$

where Q_1 and Q_2 are given by (2.3) and (2.4), respectively. The result is sharp.

Proof. Let $f \in \mathcal{ST}_q(\lambda, p, k, b)$, then there exist a function w, with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\frac{z\mathcal{D}_{q,p}\left(\mathcal{I}_q^{\lambda+p-1}f(z)\right)}{\mathcal{I}_q^{\lambda+p-1}f(z)} - 1\right) = p_k(w(z)) \quad (z \in \mathcal{U}). \tag{3.10}$$

Let $h \in P$ be a function defined by

$$h(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathcal{U}).$$

This gives

$$w(z) = \frac{c_1}{2}z + \frac{1}{2}(c_2 - \frac{c_1^2}{2})z^2 + \cdots ,$$

and

$$p_k(w(z)) = 1 + \frac{1}{2}c_1Q_1z + \frac{1}{2}\left\{\frac{c_1^2Q_2}{2} + (c_2 - \frac{c_1^2}{2})Q_1\right\}z^2 + \cdots . \tag{3.11}$$

Using (3.11) in (3.10) along with (1.9), we obtain

$$a_{p+1} = \frac{[p]_q bc_1Q_1[\lambda+1]_q}{2q^p[p+1]_q},$$

and

$$a_{p+2} = \frac{[p]_q b[\lambda+p,q]_2}{[2]_q q^p[p+1,q]_2}\left\{\frac{c_1^2Q_2}{4} + \frac{1}{2}(c_2 - \frac{c_1^2}{2})Q_1 + \frac{[p]_q bQ_1^2c_1^2}{4q^p}\right\}.$$

For any complex number μ, we have

$$\begin{aligned} a_{p+2} - \mu a_{p+1}^2 \quad = \quad & \frac{[p]_q b[\lambda+p,q]_2}{2[2]_q q^p[p+1,q]_2}\left\{\frac{c_1^2Q_2}{2} + (c_2 - \frac{c_1^2}{2})Q_1 + \frac{[p]_q bQ_1^2c_1^2}{4q^p}\right\} \\ & -\frac{[p]_q^2b^2c_1^2Q_1^2}{4q^{2p}}\left(\frac{[\lambda+1]_q}{[p+1]_q}\right)^2\mu. \end{aligned} \tag{3.12}$$

Thus (3.12) can be written as

$$a_{p+2} - \mu a_{p+1}^2 = \frac{[p]_q bQ_1[\lambda+p,q]_2}{2[2]_q q^p[p+1,q]_2}\left\{c_2 - vc_1^2\right\}, \tag{3.13}$$

where

$$v = \frac{1}{2}\left\{1 - \frac{Q_2}{Q_1} - \frac{[p]_q bQ_1}{q^p}\left(1 - \frac{[2]_q[\lambda+p]_q[p+2]_q}{[\lambda+p+1]_q[p+1]_q}\mu\right)\right\}. \tag{3.14}$$

Now, taking absolute value and using Lemma 2.2, we obtain the required result. The sharpness of (3.9) follows from the sharpness of (2.5). □

Putting $p = 1$ in Theorem 3.16, we obtain the following consequence.

Corollary 3.17. *Let $0 \le k < \infty$ be fixed and let $f \in \mathcal{ST}_q(\lambda, k, b)$ with the form (1.1) (with $p = 1$). Then for a complex parameter μ, we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|b|\, Q_1[\lambda+1,q]_2}{2[2]_q q[2,q]_2}\max\{1, |2v-1|\},$$

where

$$v = \frac{1}{2}\left\{1 - \frac{Q_2}{Q_1} - \frac{bQ_1}{q}\left(1 - \frac{[\lambda+1]_q[3]_q}{[\lambda+2]_q}\mu\right)\right\},$$

where Q_1 and Q_2 are given by (2.3) and (2.4), respectively. The result is sharp.

Putting $p = 1$ and $b = 1 - \alpha$ $(0 \le \alpha < 1)$ in Theorem 3.16, we get the following corollary.

Corollary 3.18. *Suppose that the function $f(z)$ given by (1.1) (with $p = 1$) is in the class $\mathcal{ST}_q(\lambda, k, \alpha)$. Then for a complex parameter μ, we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{P_1[\lambda+1,q]_2}{2q[2]_q[2,q]_2}\max\left\{1, \frac{P_2}{P_1} - \frac{P_1}{q}\left(1 - \frac{[\lambda+1]_q[3]_q}{[\lambda+2]_q}\mu\right)\right\}, \tag{3.15}$$

where $P_1 = (1-\alpha)Q_1$ and $P_2 = (1-\alpha)Q_2$, Q_1 and Q_2 are given by (2.3) and (2.4), respectively. The result is sharp.

Putting $b = 1 - \frac{\alpha}{[p]_q}$ $(0 \le \alpha < [p]_q)$ in Theorem 3.16, we get the following corollary.

Corollary 3.19. *Let $0 \le k < \infty$ be fixed and let $f \in \mathcal{ST}_q(\lambda, p, k, \alpha)$ with the form (1.1). Then for a complex parameter μ, we have*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{([p]_q - \alpha)Q_1[\lambda + p, q]_2}{2[2]_q q^p[p+1, q]_2} \times \max\left\{1, \left|\frac{Q_2}{Q_1} - \frac{([p]_q - \alpha)Q_1}{q^p}\left(1 - \frac{[2]_q[\lambda + p]_q[p+2]_q}{[\lambda + p + 1]_q[p+1]_q}\mu\right)\right|\right\},$$

where Q_1 and Q_2 are given by (2.3) and (2.4), respectively. The result is sharp.

Putting $b = \left(1 - \frac{\alpha}{[p]_q}\right)\cos\gamma e^{-i\gamma}$ $(0 \le \alpha < [p]_q,\ |\gamma| < \frac{\pi}{2})$ in Theorem 3.16, we get the following corollary.

Corollary 3.20. *Let $0 \le k < \infty$ be fixed and let $f \in \mathcal{ST}_q^{\gamma}(\lambda, p, k, \alpha)$. Then for a complex parameter μ, we have*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{([p]_q - \alpha)\cos\gamma Q_1[\lambda + p, q]_2}{2[2]_q q^p[p+1, q]_2} \times \max\left\{1, \left|\frac{Q_2}{Q_1} - \frac{([p]_q - \alpha)\cos\gamma Q_1}{q^p}\left(1 - \frac{[2]_q[\lambda + p]_q[p+2]_q}{[\lambda + p + 1]_q[p+1]_q}\mu\right)\right|\right\}.$$

The result is sharp.

Theorem 3.21. *Let*

$$\sigma_1 = \frac{\left[[p]_q bQ_1^2 + q^p(Q_2 - Q_1)\right][\lambda + p + 1]_q[p+1]_q}{[2]_q[p]_q bQ_1^2[\lambda + p]_q[p+2]_q},$$

$$\sigma_2 = \frac{\left[[p]_q bQ_1^2 + q^p(Q_2 + Q_1)\right][\lambda + p + 1]_q[p+1]_q}{[2]_q[p]_q bQ_1^2[\lambda + p]_q[p+2]_q},$$

$$\sigma_3 = \frac{\left[[p]_q bQ_1^2 + q^p Q_2\right][\lambda + p + 1]_q[p+1]_q}{[2]_q[p]_q bQ_1^2[\lambda + p]_q[p+2]_q}.$$

If f given by (1.1) belong to the class $\mathcal{ST}_q(\lambda, p, k, b)$ $(b > 0)$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \begin{cases} \frac{[p]_q bQ_1[\lambda+p,q]_2}{q^p[2]_q[p+1,q]_2}\left\{\frac{Q_2}{Q_1} + \frac{[p]_q bQ_1}{q^p}\left(1 - \frac{[2]_q[\lambda+p]_q[p+2]_q}{[\lambda+p+1]_q[p+1]_q}\mu\right)\right\}, & \mu \le \sigma_1, \\ \frac{[p]_q bQ_1[\lambda+p,q]_2}{q^p[2]_q[p+1,q]_2}, & \sigma_1 \le \mu \le \sigma_2, \\ -\frac{[p]_q bQ_1[\lambda+p,q]_2}{q^p[2]_q[p+1,q]_2}\left\{\frac{Q_2}{Q_1} + \frac{[p]_q bQ_1}{q^p}\left(1 - \frac{[2]_q[\lambda+p]_q[p+2]_q}{[\lambda+p+1]_q[p+1]_q}\mu\right)\right\}, & \mu \ge \sigma_2. \end{cases}$$

Further, if $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{q^p\left([p+1]_q\right)^2[\lambda+p,q]_2}{[2]_q[p]_q bQ_1\left([\lambda+p]_q\right)^2[p+1,q]_2}$$
$$\times\left\{1-\frac{Q_2}{Q_1}-\frac{bQ_1}{q^p}\left(1-\frac{[2]_q[\lambda+p]_q[p+2]_q}{[\lambda+p+1]_q[p+1]_q}\mu\right)\right\}\left|a_{p+1}\right|^2 \le \frac{[p]_q bQ_1[\lambda+p,q]_2}{q^p[2]_q[p+1,q]_2}.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{q^p\left([p+1]_q\right)^2[\lambda+p,q]_2}{[2]_q bQ_1\left([\lambda+p]_q\right)^2[p+1,q]_2}$$
$$\times\left\{1+\frac{Q_2}{Q_1}+\frac{[p]_q bQ_1}{q^p}\left(1-\frac{[2]_q[\lambda+p]_q[p+2]_q}{[\lambda+p+1]_q[p+1]_q}\mu\right)\right\}\left|a_{p+1}\right|^2 \le \frac{[p]_q bQ_1[\lambda+p,q]_2}{q^p[2]_q[p+1,q]_2}.$$

The result is sharp.

Proof. Applying Lemma 2.3 to (3.12) and (3.13), we can obtain our results asserted by Theorem 3.21. □

Putting $p = 1$ in Theorem 3.21, we obtain the following corollary.

Corollary 3.22. *Let*

$$\varsigma_1 = \frac{\left[bQ_1^2+q(Q_2-Q_1)\right][\lambda+2]_q}{bQ_1^2[\lambda+1]_q[3]_q},$$
$$\varsigma_2 = \frac{\left[bQ_1^2+q(Q_2+Q_1)\right][\lambda+2]_q}{bQ_1^2[\lambda+1]_q[3]_q},$$
$$\varsigma_3 = \frac{\left[bQ_1^2+qQ_2\right][\lambda+2]_q}{bQ_1^2[\lambda+1]_q[3]_q}.$$

If f given by (1.1) (with $p = 1$) belong to the class $\mathcal{ST}_q(\lambda,k,b)$ with $b > 0$, then

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{bQ_1[\lambda+1,q]_2}{q[2]_q[2,q]_2}\left\{\frac{Q_2}{Q_1}+\frac{bQ_1}{q}\left(1-\frac{[\lambda+1]_q[3]_q}{[\lambda+2]_q}\mu\right)\right\}, & \mu \le \varsigma_1, \\ \frac{bQ_1[\lambda+1,q]_2}{q[2]_q[2,q]_2}, & \varsigma_1 \le \mu \le \varsigma_2, \\ -\frac{bQ_1[\lambda+1,q]_2}{q[2]_q[2,q]_2}\left\{\frac{Q_2}{Q_1}+\frac{bQ_1}{q}\left(1-\frac{[\lambda+1]_q[3]_q}{[\lambda+2]_q}\mu\right)\right\}, & \mu \ge \varsigma_2. \end{cases}$$

Further, if $\varsigma_1 \le \mu \le \varsigma_3$, then

$$\left|a_3-\mu a_2^2\right| + \frac{q[2]_q[\lambda+1,q]_2}{bQ_1\left([\lambda+1]_q\right)^2[2,q]_2}$$
$$\times\left\{1-\frac{Q_2}{Q_1}-\frac{bQ_1}{q}\left(1-\frac{[\lambda+1]_q[3]_q}{[\lambda+2]_q}\mu\right)\right\}|a_2|^2 \le \frac{bQ_1[\lambda+1,q]_2}{q[2]_q[2,q]_2}.$$

If $\varsigma_3 \leq \mu \leq \varsigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{q[2]_q[\lambda+1,q]_2}{qbQ_1\left([\lambda+1]_q\right)^2[2,q]_2}$$
$$\times\left\{1 + \frac{Q_2}{Q_1} + \frac{bQ_1}{q}\left(1 - \frac{[\lambda+1]_q[3]_q}{[\lambda+2]_q}\mu\right)\right\}|a_2|^2 \leq \frac{bQ_1[\lambda+1,q]_2}{q[2]_q[2,q]_2}.$$

The result is sharp.

Putting $p = 1$ and $b = 1 - \alpha$ $(0 \leq \alpha < 1)$ in Theorem 3.21, we obtain the following corollary.

Corollary 3.23. *Let*

$$\vartheta_1 = \frac{\left[P_1^2 + q(P_2 - P_1)\right][\lambda+2]_q}{P_1^2[\lambda+1]_q[3]_q},$$
$$\vartheta_2 = \frac{\left[P_1^2 + q(P_2 + P_1)\right][\lambda+2]_q}{P_1^2[\lambda+1]_q[3]_q},$$
$$\vartheta_3 = \frac{\left[P_1^2 + qP_2\right][\lambda+2]_q}{P_1^2[\lambda+1]_q[3]_q}.$$

If f *given by (1.1) (with* $p = 1$*) belong to the class* $\mathcal{ST}_q(\lambda, k, \alpha)$, *then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{P_1[\lambda+1,q]_2}{q[2]_q[2,q]_2}\left\{\frac{P_2}{P_1} + \frac{P_1}{q}\left(1 - \frac{[\lambda+1]_q[3]_q}{[\lambda+2]_q}\mu\right)\right\}, & \mu \leq \vartheta_1, \\ \frac{P_1[\lambda+1,q]_2}{q[2]_q[2,q]_2}, & \vartheta_1 \leq \mu \leq \vartheta_2, \\ -\frac{P_1[\lambda+1,q]_2}{q[2]_q[2,q]_2}\left\{\frac{P_2}{P_1} + \frac{P_1}{q}\left(1 - \frac{[\lambda+1]_q[3]_q}{[\lambda+2]_q}\mu\right)\right\}, & \mu \geq \vartheta_2. \end{cases}$$

Further, if $\vartheta_1 \leq \mu \leq \vartheta_3$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{q[2]_q[\lambda+1,q]_2}{P_1\left([\lambda+1]_q\right)^2[2,q]_2}$$
$$\times\left\{1 - \frac{P_2}{P_1} - \frac{P_1}{q}\left(1 - \frac{[\lambda+1]_q[3]_q}{[\lambda+2]_q}\mu\right)\right\}|a_2|^2 \leq \frac{P_1[\lambda+1,q]_2}{q[2]_q[2,q]_2}.$$

If $\vartheta_3 \leq \mu \leq \vartheta_2$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{q[2]_q[\lambda+1,q]_2}{qP_1\left([\lambda+1]_q\right)^2[2,q]_2}$$
$$\times\left\{1 + \frac{P_2}{P_1} + \frac{P_1}{q}\left(1 - \frac{[\lambda+1]_q[3]_q}{[\lambda+2]_q}\mu\right)\right\}|a_2|^2 \leq \frac{P_1[\lambda+1,q]_2}{q[2]_q[2,q]_2}.$$

The result is sharp.

Putting $b = \left(1 - \frac{\alpha}{[p]_q}\right)$ $(0 \leq \alpha < [p]_q)$ in Theorem 3.21, we obtain the following corollary.

Corollary 3.24. *Let*

$$\begin{aligned}
\epsilon_1 &= \frac{\left[\left([p]_q - \alpha\right) Q_1^2 + q^P(Q_2 - Q_1)\right][\lambda + P + 1]_q[P + 1]_q}{[2]_q\left([p]_q - \alpha\right) Q_1^2[\lambda + P]_q[P + 2]_q},\\
\epsilon_2 &= \frac{\left[\left([p]_q - \alpha\right) Q_1^2 + q^P(Q_2 + Q_1)\right][\lambda + P + 1]_q[P + 1]_q}{[2]_q\left([p]_q - \alpha\right) Q_1^2[\lambda + P]_q[P + 2]_q},\\
\epsilon_3 &= \frac{\left[\left([p]_q - \alpha\right) Q_1^2 + q^P Q_2\right][\lambda + P + 1]_q[P + 1]_q}{[2]_q\left([p]_q - \alpha\right) Q_1^2[\lambda + P]_q[P + 2]_q}.
\end{aligned}$$

If f given by (1.1) belong to the class $\mathcal{ST}_q(\lambda, P, k, b)$ with $b > 0$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} \frac{\left([p]_q-\alpha\right)Q_1[\lambda+p,q]_2}{q^p[2]_q[p+1,q]_2}\left\{\frac{Q_2}{Q_1} + \frac{\left([p]_q-\alpha\right)Q_1}{q}\left(1 - \frac{[2]_q[\lambda+p]_q[p+2]_q}{[\lambda+p+1]_q[p+1]_q}\mu\right)\right\}, & \mu \leq \epsilon_1,\\[2ex] \frac{\left([p]_q-\alpha\right)Q_1[\lambda+p,q]_2}{q^p[2]_q[p+1,q]_2}, & \epsilon_1 \leq \mu \leq \epsilon_2,\\[2ex] -\frac{\left([p]_q-\alpha\right)Q_1[\lambda+p,q]_2}{q^p[2]_q[p+1,q]_2}\left\{\frac{Q_2}{Q_1} + \frac{\left([p]_q-\alpha\right)Q_1}{q}\left(1 - \frac{[2]_q[\lambda+p]_q[p+2]_q}{[\lambda+p+1]_q[p+1]_q}\mu\right)\right\}, & \mu \geq \epsilon_2. \end{cases}$$

Further, if $\epsilon_1 \leq \mu \leq \epsilon_3$, then

$$\begin{aligned}
&\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{q^p\left([p + 1]_q\right)^2[\lambda + p, q]_2}{[2]_q\left([p]_q - \alpha\right) Q_1\left([\lambda + p]_q\right)^2[p + 1, q]_2}\\
&\times\left\{1 - \frac{Q_2}{Q_1} - \frac{\left([p]_q - \alpha\right) Q_1}{q^p}\left(1 - \frac{[2]_q[\lambda + p]_q[p + 2]_q}{[\lambda + p + 1]_q[p + 1]_q}\mu\right)\right\}\left|a_{p+1}\right|^2\\
\leq\ &\frac{\left([p]_q - \alpha\right) Q_1[\lambda + p, q]_2}{q^p[2]_q[p + 1, q]_2}.
\end{aligned}$$

If $\epsilon_3 \leq \mu \leq \epsilon_2$, then

$$\begin{aligned}
&\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{q^p\left([p + 1]_q\right)^2[\lambda + p, q]_2}{[2]_q\left([p]_q - \alpha\right)\left([\lambda + p]_q\right)^2[p + 1, q]_2}\\
&\times\left\{1 + \frac{Q_2}{Q_1} + \frac{\left([p]_q - \alpha\right) Q_1}{q^p}\left(1 - \frac{[2]_q[\lambda + p]_q[p + 2]_q}{[\lambda + p + 1]_q[p + 1]_q}\mu\right)\right\}\left|a_{p+1}\right|^2\\
\leq\ &\frac{\left([p]_q - \alpha\right) Q_1[\lambda + p, q]_2}{q^p[2]_q[p + 1, q]_2}.
\end{aligned}$$

The result is sharp.

4. Conclusions

Studies of the coefficient problems including the Fekete-Szegö problems continue to motivate researchers in Geometric Function Theory of Complex Analysis. In our present investigation, we have introduced and studied a new class $\mathcal{ST}_q(\lambda, p, k, b)$ of analytic functions associated with q-analogue of p-valent Noor integral operator in the open unit disc $\mathcal{U}$. For functions in this class, we have derived the coefficient estimates of the coefficients $\left|a_{p+1}\right|$ and $\left|a_{n+p+1}\right|$ for $n \geq 3$, and Fekete-Szegö functional problems for functions belonging to this new class. Several of new results are shown to follow upon specializing the parameters involved in our main results.

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Conflict of interest

The authors declare that they have no competing interests.

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