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Research article

Nonlinear Fredholm integro-differential equation in two-dimensional and its numerical solutions

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Abstract: This paper proposes a new definition of the nonlinear Fredholm integro-differential equation of the second kind with continuous kernel in two-dimensional (NT-DFIDE). Furthermore, the work is concerned to study this new equation numerically. The existence of a unique solution of the equation is proved. In addition, the approximate solutions of NT-DFIDE are obtained by two powerful methods Adomian Decomposition Method (ADM) and Homotopy Analysis Method (HAM). The given numerical examples showed the efficiency and accuracy of the introduced methods.

Keywords: Fredholm integro-differential equation; Adomian decomposition; homotopy analysis
Mathematics Subject Classification: 34A45, 65C40, 37N30

1. Introduction

The integral equations provide an important tool for modeling the numerous phenomenons and for solving boundary value problems. In addition, the references [1–3], studied the different applications of partial differential equations and analysis in applied mathematics. Kazemi et al. [4] discussed an efficient iterative method based on quadrature formula to solve two-dimensional nonlinear Fredholm integral equations. Fattahzadeh [5] solved two-dimensional linear and nonlinear FIE of the first kind based on Haar wavelet. Torabi and Tari [6] solved T-DIE of the first kind by multi-step method. Atabakan et al. [7] introduced the solving linear FIDEs using the well-known Chebyshev-Gauss-Lobatto collocation points. Rabbani and Zarali [8] discussed the technique of modified decomposition method to solve a system of LIDEs with initial conditions. Arqub et al. [9] discussed the numerical solution of FIDE in a reproducing kernel Hilbert space. Pandey [10] considered a non-standard finite difference method for numerical solution of LFIDEs. Erfanian and Zeidabadi [11] discussed a numerical method for the solutions of the NFIDE in the complex plane is presented. Saadatmandi and Dehghan [12] studied the higher-order LFIDDE with variable coefficients. All previous studies have studied the integro-differential equation in one dimension only. The goal of this paper is to study the N-FIDE of the second kind in two-dimensional.

Consider

$$u''(m,n) + A(m,n)u'(m,n) + B(m,n)u(m,n)$$
$$= Q(m,n) - \lambda \int_a^b \int_c^d L(m,n,t,s)\gamma(t,s,u(t,s))dtds \quad (1)$$

Under the boundary conditions:

$$u(a,c) = q_1 r_1, u(b,d) = q_2 r_2 \quad (2)$$

Where $(m,n) \in J,\ J = [a,b] \times [c,d]$, is a continuous nonlinear in *u* given function, and *u* is the unknown function represents solution of the NT-DIDE (1). Also, λ is a constant. $A\ (m, n)$, $B\ (m, n)$, are known continuous functions in the class C [a, b]×C [a, b] with its derivatives. Integrating (1), twice, then letting m=b, n=d, then, Eq (1) reduce to

$$u(m,n) = f(m,n) + \lambda \int_a^b \int_c^d p(m,n,t,s)\gamma(t,s,u(t,s))dtds \quad (3)$$

Equation (3) represents T-DFIDE in the nonlinear case.

2. Existence of a solution of NT-DFIDE

Theorem 1. Consider a metric space $= (M,d)\ X \neq \Phi$. Suppose that M is complete and let $T : M \to M$ be a contraction on M. Then T has precisely one fixed point. In addition, we can write the formula of Eq (3) in the integral operator form

$$\bar{W}u(m,n) = f(m,n) + Wu(m,n), \quad (4)$$

where

$$Wu(m,n) = \lambda \int_a^b \int_c^d p(m,n,t,s)\gamma(t,s,u(t,s))dtds \quad (5)$$

In addition, we assume the following conditions:

1. The(m,n,t,s), should be satisfies $|p(m,n,t,s)| \leq N$.
2. $f\ (m,n)$ is continuous in$C[a,b] \times C[c,d]$, and its norm is defined as

$$\|f(m,n)\| = \{\int_a^b \int_c^d |f(m,n)|^2 dmdn\}^{\frac{1}{2}} = \delta,\ (\delta \text{ is a constant}).$$

3. The known continuous function $\gamma(m,n,u(m,n))$ satisfies, for the constant $A > A_1$, $A > P$, the following conditions

$$i - \{\int_a^b \int_c^d |\gamma(m,n,u(m,n))|^2 dmdn\}^{\frac{1}{2}} \leq A_1 \|u(m,n)\|$$
$$ii - |\gamma(m,n,u_1(m,n)) - \gamma(m,n,u_2(m,n))| \leq M(m,n)|u_1(m,n) - u_2(m,n)|,$$

where $\|M(m,n)\| = P$.

4. The unknown function $u(x,y)$, behaves in $C[a,b] \times C[c,d]$ as the given function $f(x,y)$ and its

norm is defined as

$$\|u(m,n)\| = \left|\int_a^b \int_c^d |u(m,n)|^2 dmdn\right|^{\frac{1}{2}}.$$

Theorem 2. If the conditions (1)–(3) are verified, then Eq (3) has a unique solution in $C[a,b] \times C[c,d]$.
Lemma 1. Under the conditions (1)–(3-i), the operator $\overline{W}$ defined by (4), maps the space $C[a,b] \times C[c,d]$ into itself.

Proof. In view of the formulas (4) and (5), we get

$$\|\bar{W}u(m,n)\| \leq \|f(m,n)\| + |\lambda| \left\|\int_a^b \int_c^d |p(m,n,t,s)||\gamma(t,s,u(t,s)|dtds\right\| \quad (6)$$

Using the condition (2), we have

$$\|\bar{W}u(m,n)\| \leq \delta + |\lambda|(|p(m,n,t,s)|)(\int_a^b \int_c^d |\gamma(m,n,u(m,n)|^2 dmdn)^{\frac{1}{2}} \quad (7)$$

Using conditions (1) and (3-i):

$$\|\bar{W}u(m,n)\| \leq \delta + \sigma\|u(m,n)\|, (\sigma = |\lambda|NA) \quad (8)$$

Moreover, the inequality (5) involves the roundedness of the operator W of Eq (4), where

$$\|Wu(m,n)\| \leq \sigma\|u(m,n)\|. \quad (9)$$

□

Lemma 2. If the conditions (1) and (3-ii) are satisfied, then the operator $\overline{W}$ is a contractive in the Banach space $C[a,b] \times C[a,b]$.

Proof. For $u_1(m,n)$ and $u_2(m,n)$ in $C[a,b] \times C[c,d]$, the formulas (4) and (5) lead to

$$\|(\bar{W}u_1 - \bar{W}u_2)(m,n)\| \leq |\lambda| \left\|\int_a^b \int_c^d |p(m,n,t,s)||\gamma(t,s,u_1(t,s)) - \gamma(t,s,u_2(t,s))|dtds\right\| (10)$$

From the condition (3-ii), we have

$$\|(\bar{W}u_1 - \bar{W}u_2)(m,n)\| \leq |\lambda|(|p(m,n,t,s)|)(\int_a^b \int_c^d M^2(t,s)\, |u_1(t,s) - u_2(t,s)|^2 dtds)^{\frac{1}{2}} \ (11)$$

Finally, we obtain

$$\|(\bar{W}u_1 - \bar{W}u_2)(m,n)\| \leq \sigma\|u_1(m,n) - u_2(m,n)\| \quad (12)$$

□

3. Numerical methods for solving NT-DFIDE

3.1. ADM

In this section, we will solve the nonlinear T-DFIDE by using the ADM. New algorithms for applying the ADM to nonlinear differential and partial differential equations have been introduced in Behiry et al. [13]. In addition, the error analysis of Adomian series solution for a class of NDIs have been discussed in El-Kalla [14]. A reliable approach for convergence of the ADM when applied to a class of NVIEs have been discussed in El-Kalla [15], also ADM is employed to solve nonlinear

FIEs of the second kind in Atia et al. [16]. EL-Kalla in [17] the proof of convergence of ADM has been applied to a class of NVI-DEs. In [18], Parviz et al. solved systems of FI-DEs by ADM. In [19], Abdou et al. studied the convergence of the series solution to a class of NT-DHIE, and solved it by using ADM and HAM. Zeidan et al. [20] discussed a novel Adomian decomposition method for the solution of Burgers' equation.

In this section, we will discuss and solve the NT-DFIDE of the second kind using ADM. In addition, numerical experiments are prepared to illustrate these considerations, and the estimating error is calculated.

Consider Eq (3), where $f(m,n)$ is assumed to be bounded $\forall m,n \in J=[a,b]\times[c,d]$, and $|p(m,n,t,s)| \le N$. The nonlinear term $\gamma(t,s,u(t,s))$ is Lipchitz continuous with $|\gamma(u)-\gamma(h)| \le L\,|u-h|$. Define $(C[a,b]\times C[c,d],d^*)$, the space of all continuous functions on the rectangle $[a,b]\times[c,d]$ with the distance function $d^*(h,u)$, where

$$d^*(h,u) = \max_{x,y\in J}|h(m,n)-u(m,n)| \tag{13}$$

$u(m,n)$ is assumed of the form:

$$u(m,n)=\sum_{n=0}^{\infty}u_n(m,n) \,. \tag{14}$$

While the nonlinear term $\gamma(t,s,u)$ in Eq (3) is decomposed into an infinite series

$$\gamma(t,s,u)=\sum_{n=0}^{\infty}A_n \,. \tag{15}$$

Where the traditional formula of A_n is:

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}[\gamma(\sum_{i=0}^{\infty}\lambda^i u_i)]_{\lambda=0} \,. \tag{16}$$

Another formula of Adomian polynomials is given by:

$$A_n=\gamma(S_n)-\sum_{i=0}^{n-1}A_i \,. \tag{17}$$

Where,

$$S_n=\sum_{i=0}^{n}u_i(m,n), \tag{18}$$

Then, applying the ADM to Eq (3), yields

$$u(m,n)=\sum_{n=0}^{\infty}u_n(m,n) \,. \tag{19}$$

Where

$$u_0(m,n)=f(m,n) \tag{20}$$

$$u_i(m,n) = \lambda \int_a^b \int_c^d p(m,n,t,s) A_{i-1} dtds \,, i \geq 1. \tag{21}$$

3.2. The HAM

In this section, we will solve the nonlinear T-DFIDE by using the HAM. This study is new, as we have noted that all previous research has been integro differential equation solved in one dimension using this method.

Consider Eq (3), where, $p(m,n,t,s)$ and $f(x,y)$ are known functions, $\gamma(t,s,u(t,s))$ is a known function of u.

For description of the method, we consider:

$$N[u] = u(m,n) - f(m,n) - \int_a^b \int_c^d p(m,n,t,s)\gamma\big(t,s,u(t,s)\big)dtds = 0. \tag{22}$$

Where, N is nonlinear operator. Let $u_0(m,n)$ denote an initial guess of the exact solution $u(m,n)$, $h \neq 0$ an auxiliary parameter, $H(m,n)$ an auxiliary function, $L[g(m,n)] = 0$ when $g(m,n) = 0$. Then using $r \in [0,1]$, we construct such a homotopy.

$$(1-r)L[\varphi(m,n;r) - u_0(m,n)] - rhH(m,n)N[\varphi(m,n;r)]$$
$$= H[\varphi(m,n;r); u_0(m,n), H(m,n), h, r]. \tag{23}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(m,n)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(m,n)$, $\hat{H}$ is the second auxiliary function, enforcing the homotopy (23) to be zero, i.e.,

$$\hat{H}[\varphi(m,nr); u_0(m,n), H(m,n), h, r].$$

Then, we get:

$$(1-r)L[\varphi(m,n;r) - u_0(m,n)] = rhH(m,n)N[\varphi(m,n;r)]. \tag{24}$$

When $r = 0$, then (24) becomes

$$\varphi(m,n;0) = u_0(m,n). \tag{25}$$

Also, when $r = 1$, and $h \neq 0, H(m,n) \neq 0$ then (24) is equivalent to

$$\varphi(m,n;1) = u(m,n). \tag{26}$$

Thus, according to (25) and (26), as the embedding parameter r increases from 0 to 1, $\phi(m,n;r)$ varies continuously from the initial approximation $u_0(m,n)$ to the exact solution $u(m,n)$ such a kind of continuous variation is called deformation in homotopy. $\phi(x,y;r)$, can be represents the power series of r as follows:

$$\phi(m,n;r) = u_0(m,n) + \sum_{l=1}^{\infty} u_l(m,n) r^l \,. \tag{27}$$

Where

$$u_l(m,n) = \frac{1}{l!}\frac{\partial \varphi(m,n;r)}{\partial r^l}\Big|_{r=0}. \quad (28)$$

Then, under these assumptions, we have

$$u(m,n) = \phi(m,n;1) = u_0(m,n) + \sum_{l=1}^{\infty} u_l(m,n) r \quad . \quad (29)$$

Then,

$$\bar{u}_n(m,n) = \{u_0(m,n), u_1(m,n), \ldots, u_n(m,n)\}. \quad (30)$$

According to the Eq (28), the governing equation of $u_m(x,y)$ can be derived from the zero-order deformation equation (24). Differentiating the zero-order deformation equation (24) l times with respective to r and then dividing by $l!$, and setting r=0, we have the so-called mth-order deformation equation:

$$L[u_l(m,n) - \eta_l u_{l-1}(m,n)] = hH(m,n)\Re_l(\bar{u}_{l-1}(m,n))$$
$$u_l(0,0) = 0. \quad (31)$$

Where

$$\Re_l(\bar{u}_{l-1}(m,n)) = \frac{1}{(l-1)!}\frac{\partial^{l-1} N[\varphi(m,n;r)}{\partial r^{l-1}}|_{r=0}. \quad (32)$$

And

$$\eta_l = \begin{cases} 0 & l \le 1 \\ 1 & l > 1 \end{cases}. \quad (33)$$

3.3. The computational procuder for the HAM

In this section, we will use the HAM to solve nonlinear T-DFIDE in two-dimensional (3), which can be written in the form

$$u(m,n) = f(m,n) + \lambda \int_a^b \int_c^d p(m,n,t,s)[u(t,s)]^P dtds \quad (34)$$

p is a positive integer, and $p(m,n,t,s)$. For this, assume:

$$N[u] = u(m,n) - f(m,n) - \lambda \int_a^b \int_c^d p(m,n,t,s)[u(t,s)]^P dtds \quad (35)$$

The corresponding mth-order deformation Eq (31) reads

$$L[u_l(m,n) - \eta_l u_{l-1}(m,n)] = hH(m,n)\Re_{l-1}(\bar{u}_{l-1}(m,n))$$
$$u_l(0,0) = 0. \quad (36)$$

Where

$$\Re_{l-1}(\bar{u}_{l-1}(m,n)) = u_{l-1} - (1-\eta_l)f - \int_a^b \int_c^d p(m,n,t,s)\Re_{l-1}(\varphi^P)dtds \quad (37)$$

$$\Re_l(\phi^P) = \sum_{s_1=0}^{l} u_{l-s_1} \sum_{s_2=0}^{s_1} u_{s_1-s_2} \sum_{s_3=0}^{s_2} u_{s_2-s_3} \cdots \sum_{s_{P-2}=0}^{s_{P-3}} u_{s_{P-3}-s_{P-2}} \sum_{s_{P-1}=0}^{s_{P-2}} u_{s_{P-2}-s_{P-1}} u_{s_{P-1}} \quad . \quad (38)$$

So, to obtain a simple iteration formula for $u_m(x,y)$, choose $Lu = u$, then substituting into (36) to obtain:

$$u_0(m,n) = f(m,n) \quad (39)$$

$$u_l(m,n) = \int_a^b \int_c^d p(m,n,t,s)\Re_{l-1}(\varphi^P)dtds\,, l = 1,2,\ldots \quad (40)$$

In addition, we get:

$$u(m,n) = \sum_{l=0}^{\infty} u_l(m,n) \quad . \quad (41)$$

4. Numerical problems

Example 1. Consider

$$u''(m,n) + Au'(m,n) + Bu(m,n) = Q(m,n) - \int_0^1 \int_0^1 (e^{m,n}.s^2)(u(t,s))^k dtds \quad (42)$$

under the boundary conditions:

$$u(0,0) = 0, u(1,1) = 0. \quad (43)$$

The exact solution is $u\ (m,\ n)=m.n$, if we set $k=1$, in (42), one has

$$u''(m,n) + Au'(m,n) + Bu(m,n) = Q(m,n) - \int_0^1 \int_0^1 (e^{mn}.s^2)(u(t,s))dtds. \quad (44)$$

Which called the LT-DFIDE, and if we set $k \geq 2$ in (42), we obtained the NT-DFIDE, of the second kind, with $\lambda = 1,\ A = (-2/m+n),\ B = 1$. In addition, the corresponding errors for the nonlinear and linear cases are computed. We solve Eq (42) using ADM and HAM. In the following Tables 1 and 2, we present the exact, numerical solutions and the corresponding errors for some points of $m,\ n$, $0 \leq m,n \leq 1$, *at* $N=10$. Maple 10 is used to carry out the computations. In Tables 1 and 2, $u_{Exact} \rightarrow$ the exact solution, u_{ADM} $\rightarrow$approximate solution of *ADM*, $Error_{ADM}\rightarrow$ the absolute error of *ADM*, u_{HAM} $\rightarrow$approximate solution of *HAM*, $Error_{HAM}$ $\rightarrow$ the absolute error of *HAM*.

Table 1. Numerical results and absolute error values by using HAM and ADM, $N=10$, at linear case $k=1$.

m	n	u_{Exact}	*ADM*		*HAM*	
			u_{ADM}	$Error_{ADM}$	u_{HAM}	$Error_{HAM}$
0.0	0.0	0.00	0.00012207	0.000122070	0.00012207	0.000122070
0.1	0.1	0.01	0.00987670	0.000123297	0.00987670	0.000123297
0.2	0.2	0.04	0.03987294	0.000127052	0.03987294	0.000127052
0.3	0.3	0.09	0.08986643	0.000133566	0.08986643	0.000133566
0.4	0.4	0.16	0.15985674	0.000143250	0.15985674	0.000143250
0.5	0.5	0.25	0.24984325	0.000156741	0.24984325	0.000156741
0.6	0.6	0.36	0.35982503	0.000174966	0.35982503	0.000174967
0.7	0.7	0.49	0.48980074	0.000199257	0.48980074	0.000199257
0.8	0.8	0.64	0.63976849	0.000231503	0.63976849	0.000231504
0.9	0.9	0.81	0.80972559	0.000274402	0.80972559	0.000274402
1.0	1.0	1.00	0.99966817	0.000331821	0.99966817	0.000331821

Table 2. Numerical results and absolute error values by using HAM and ADM, *N*=10, at nonlinear case *k*=2.

m	n	u_{Exact}	ADM		HAM	
			u_{ADM}	$Error_{ADM}$	u_{HAM}	$Error_{HAM}$
0.0	0.0	0.00	0.347×10^{-8}	0.347000×10^{-8}	0.347×10^{-8}	0.347×10^{-8}
0.1	0.1	0.01	0.00999999	0.350487×10^{-8}	0.009999996	0.3505×10^{-8}
0.2	0.2	0.04	0.03999999	0.361661×10^{-8}	0.039999996	0.3610×10^{-8}
0.3	0.3	0.09	0.08999999	0.379678×10^{-8}	0.089999996	0.3800×10^{-8}
0.4	0.4	0.16	0.15999999	0.407208×10^{-8}	0.159999995	0.4100×10^{-8}
0.5	0.5	0.25	0.24999999	0.445556×10^{-8}	0.249999995	0.4500×10^{-8}
0.6	0.6	0.36	0.35999999	0.497365×10^{-8}	0.359999995	0.5000×10^{-8}
0.7	0.7	0.49	0.48999999	0.566413×10^{-8}	0.489999994	0.5700×10^{-8}
0.8	0.8	0.64	0.63999999	0.658078×10^{-8}	0.639999993	0.6600×10^{-8}
0.9	0.9	0.81	0.80999999	0.780024×10^{-8}	0.809999992	0.7800×10^{-8}
1.0	1.0	1.00	0.99999999	0.34700×10^{-8}	0.999999999	0.3400×10^{-8}

Example 2. Consider

$$u''(m,n) + Au'(m,n) + Bu(m,n) = Q(m,n) - \int_0^1 \int_0^1 (sin(m.n).s^2)(u(t,s))^k dtds \quad (45)$$

Under the boundary conditions:

$$u(0,0) = 0, u(1,1) = 0 \quad (46)$$

The exact solution is u (m, n)=$m.n$, if we set k=1, in (45), one has

$$u''(m,n) + Au'(m,n) + Bu(m,n) = Q(m,n) - \int_0^1 \int_0^1 (sin(m.n).s^2)(u(t,s))dtds \quad (47)$$

Which called the LT-DFIDE, and if we set $k \geq 2$ in (45), we obtained the NT-DFIDE, of the second kind, with $\lambda = 1$, $A = (-2/m + n)$, $B = 1$, $0 \leq m,n \leq 1$, at N=10. In Tables 3 and 4, $u_{Exact} \rightarrow$ the exact solution, $u_{ADM} \rightarrow$approximate solution of *ADM*, $Error_{ADM} \rightarrow$ the absolute error of *ADM*, $u_{HAM} \rightarrow$approximate solution of *HAM*, $Error_{HAM} \rightarrow$ the absolute error of *HAM*.

Table 3. Numerical results and absolute error values by using HAM and ADM, *N*=10, at linear case *k*=1.

m	n	u_{Exact}	ADM		HAM	
			u_{ADM}	$Error_{ADM}$	u_{HAM}	$Error_{HAM}$
0.0	0.0	0.00	0.0000000	0.00000000	0.0000000	0.000000
0.1	0.1	0.01	0.0099999	0.1999966×10^{-11}	0.00999999	0.2×10^{-12}
0.2	0.2	0.04	0.0399999	0.7997866×10^{-11}	0.03999999	0.1×10^{-10}
0.3	0.3	0.09	0.0899999	0.1797570×10^{-10}	0.08999999	0.2×10^{-10}
0.4	0.4	0.16	0.1600000	0.3186364×10^{-10}	0.16000000	0.00000
0.5	0.5	0.25	0.2500000	0.4948079×10^{-10}	0.25000000	0.00000
0.6	0.6	0.36	0.3599999	0.7045484×10^{-10}	0.35999999	0.1×10^{-9}
0.7	0.7	0.49	0.4899999	0.9412517×10^{-10}	0.48999999	0.1×10^{-9}
0.8	0.8	0.64	0.6399999	0.1194390×10^{-9}	0.63999999	0.1×10^{-9}
0.9	0.9	0.81	0.8099999	0.1448574×10^{-9}	0.80999999	0.1×10^{-9}
1.0	1.0	1.00	0.9999999	0.200×10^{-9}	0.99999999	0.2×10^{-9}

Table 4. Numerical results and absolute error values by using HAM and ADM, *N*=10, at nonlinear case *k*=2.

m	n	u_{Exact}	*ADM*		*HAM*	
			u_{ADM}	$Error_{ADM}$	u_{HAM}	$Error_{HAM}$
0.0	0.0	0.00	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.1	0.01	0.0100000	0.9999833×10^{-13}	0.0100000	0.9999833×10^{-12}
0.2	0.2	0.04	0.0400000	0.3998933×10^{-12}	0.0400000	0.3998933×10^{-12}
0.3	0.3	0.09	0.0900000	0.8987854×10^{-12}	0.0900000	0.8987854×10^{-12}
0.4	0.4	0.16	0.1600000	0.1593182×10^{-11}	0.1600000	0.1593182×10^{-11}
0.5	0.5	0.25	0.2500000	0.2474039×10^{-11}	0.2500000	0.2474039×10^{-11}
0.6	0.6	0.36	0.3600000	0.3522742×10^{-11}	0.3600000	0.3522742×10^{-11}
0.7	0.7	0.49	0.4900000	0.4706258×10^{-11}	0.4900000	0.4706258×10^{-11}
0.8	0.8	0.64	0.6400000	0.5971954×10^{-11}	0.6400000	0.5971954×10^{-11}
0.9	0.9	0.81	0.8100000	0.7242871×10^{-11}	0.8100000	0.7242871×10^{-10}
1.0	1.0	1.00	1.000000	0.1×10^{-10}	1.000000	0.1×10^{-10}

Example 3. Consider

$$u''(m,n) + Au'(m,n) + Bu(m,n) = Q(m,n) - \lambda \int_0^1 \int_0^1 (sin(m.n).s^7/3)(u(t,s))^k dtds \quad (48)$$

Under the boundary conditions:

$$u(0,0) = 0, u(1,1) = 0.01 \quad (49)$$

The exact solution is u (m, n)=$m.n$, if we set k=1, in (48), one has

$$u''(m,n) + Au'(m,n) + Bu(m,n) = Q(m,n) - \lambda \int_0^1 \int_0^1 (sin(m.n).s^7/3)(u(t,s))dtds \quad (50)$$

Which called the LT-DFIDE, and if we set $k \geq 2$ in (48), we obtained the NT-DFIDE, of the second kind, with $\lambda = 0.001$, $A = (-2/m+n)$, $B = 1$, $0 \leq m,n \leq 1$, at N=10. In Tables 5 and 6, u_{Exact} → the exact solution, u_{ADM} →approximate solution of *ADM*, $Error_{ADM}$ → the absolute error of *ADM*, u_{HAM} →approximate solution of *HAM*, $Error_{HAM}$ → the absolute error of *HAM*.

Table 5. Numerical results and absolute error values by using HAM and ADM, *N*=10, at linear case *k*=1.

m	n	u_{Exact}	*ADM*		*HAM*	
			u_{ADM}	$Error_{ADM}$	u_{HAM}	$Error_{HAM}$
0.0	0.0	0.00	0.01000000	0.01000000	0.0100000	0.01000000
0.1	0.1	0.01	0.01981500	0.00981500	0.0198890	0.00988907
0.2	0.2	0.04	0.04926020	0.00926020	0.04955642	0.00955640
0.3	0.3	0.09	0.09833725	0.00833725	0.09900303	0.00900303
0.4	0.4	0.16	0.16705262	0.00705262	0.16823278	0.00823278
0.5	0.5	0.25	0.25542305	0.00542305	0.25725570	0.00725570
0.6	0.6	0.36	0.36348296	0.00348296	0.36609244	0.00609244
0.7	0.7	0.49	0.49129346	0.00129346	0.49477944	0.00477964
0.8	0.8	0.64	0.63895194	0.00104805	0.64337568	0.00337568
0.9	0.9	0.81	0.80660075	0.00339924	0.81196594	0.00196594
1.0	1.0	1.00	0.99443286	0.00556714	1.00066609	0.00066609

Table 6. Numerical results and absolute error values by using HAM and ADM, N=10, at nonlinear case k=2.

m	n	u_{Exact}	ADM		HAM	
			u_{ADM}	$Error_{ADM}$	u_{HAM}	$Error_{HAM}$
0.0	0.0	0.00	0.01000000	0.0100000	0.01000000	0.0100000
0.1	0.1	0.01	0.01988900	0.0098890	0.01988900	0.0098890
0.2	0.2	0.04	0.49556124	0.00955612	0.04955612	0.0095561
0.3	0.3	0.09	0.09900236	0.00900236	0.09900236	0.0090023
0.4	0.4	0.16	0.16823159	0.00823159	0.16823159	0.0082315
0.5	0.5	0.25	0.25725385	0.00725385	0.25725385	0.0072538
0.6	0.6	0.36	0.36608980	0.00608071	0.36608980	0.0060898
0.7	0.7	0.49	0.49477612	0.00477612	0.49477612	0.00477612
0.8	0.8	0.64	0.64337121	0.00337121	0.64337121	0.00337121
0.9	0.9	0.81	0.81196051	0.00196051	0.81196051	0.00196051
1.0	1.0	1.00	1.00065979	0.00065979	1.00065979	0.00065979

5. Conclusions

The goal of this work is studied the NFIDE of the second kind in two-dimensional. This paper proposed an effective two numerical methods to obtain the solution. For this purpose, ADM and HAM has been presented. The given numerical examples showed the efficiency and accuracy of the ADM and HAM. From the previous numerical results we deduce in linear case both ADM and HAM give the same approximate solution. In the nonlinear case, it was found that, ADM converges faster than HAM. Also, the values of absolute errors for linear case larger than the values of errors for nonlinear case. The codes were written in Maple program. The absolute errors of approximate solution in given points are small enough, so it follows that the presentation methods in this article are right. For comparison purpose, many authors have paid the attention to apply, modify and extend the considered methods in this work to tackle a variety type of integro-differential equations, see [21–38].

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Conflict of interest

The author declares no conflict of interest.

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