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Research article

Global regularity for the tropical climate model with fractional diffusion

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Abstract: In this paper, we investigate the following tropical climate model with fractional diffusion

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p + \Lambda^{2\alpha} u + \mathrm{div}(v\otimes v) = 0, \\ v_t + u\cdot\nabla v + \nabla\theta + \Lambda^{2\beta} v + v\cdot\nabla u = 0, \\ \theta_t + u\cdot\nabla\theta + \Lambda^{2\gamma}\theta + \mathrm{div} v = 0, \\ \mathrm{div} u = 0, \\ (u, v, \theta)(x, 0) = (u_0, v_0, \theta_0), \end{cases}$$

where $(u_0, v_0, \theta_0) \in H^s(R^n)$ with $s \geq 1, n \geq 3$ and $\mathrm{div} u_0 = 0$. When the nonnegative constants α, β and γ satisfy $\alpha \geq \frac{1}{2} + \frac{n}{4}$, $\alpha + \beta \geq 1 + \frac{n}{2}$, $\alpha + \gamma \geq 1 + \frac{n}{2}$, by using the energy methods, we obtain the global existence and uniqueness of solution for the system. In the special case $\theta = 0$, we could obtain the global solution provide that $\alpha \geq \frac{1}{2} + \frac{n}{4}, \alpha + \beta \geq 1 + \frac{n}{2}$ and $(u_0, v_0) \in H^s(s \geq 1)$, which generalizes the existing result.

Keywords: global regularity; tropical climate model; fractional diffusion
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1. Introduction

In this paper, we consider the following tropical climate model with fractional diffusion

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p + \Lambda^{2\alpha}u + \mathrm{div}(v\otimes v) = 0,\\ v_t + u\cdot\nabla v + \nabla\theta + \Lambda^{2\beta}v + v\cdot\nabla u = 0,\\ \theta_t + u\cdot\nabla\theta + \Lambda^{2\gamma}\theta + \mathrm{div}v = 0,\\ \mathrm{div}u = 0,\\ (u,v,\theta)(x,0) = (u_0,v_0,\theta_0), \end{cases} \tag{1.1}$$

where $(x,t)\in\mathbb{R}^n\times\mathbb{R}^+$ with $n\geq 3$, $u = u(x,t)$ and $v = v(x,t)$ denote the barotropic mode and the first baroclinic mode of the velocity, respectively, while $p = p(x,t)$ and $\theta = \theta(x,t)$ denote the scalar pressure and temperature, respectively. $\alpha\geq 0$, $\beta\geq 0$, $\gamma\geq 0$ are real parameters. The fractional Laplacian operator $\Lambda = (-\Delta)^{\frac{1}{2}}$ is defined through the Fourier transform, namely

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha\widehat{f}(\xi),\ \forall\,\alpha\geq 0.$$

By performing a Galerkin trunction to the hydrostatic Boussinesq equations, Frierson, Majda and Pauluis [1] derived the original version of (1.1) without any Laplacian terms. This is because the Laplacian terms are derived from the inviscid primitive equations. In this paper, we consider the viscous counterpart of the system in [1], that is, system (1.1). Before stating our main result, we first summarize the existing results for system (1.1).

For the 2D case, Ma, Jiang and Wan [2] surveyed the local existence and uniqueness of strong solutions with any $\alpha\geq 1,\beta\geq 0,\gamma = 0$ or $0\leq\alpha<1,\beta\geq 1,\gamma = 0$. Li and Titi [3] established the global well-posedness of the strong solution for the case $\alpha=\beta = 1,\gamma = 0$. When $\alpha+\beta = 2, 1<\beta\leq\frac{3}{2},\gamma = 0$, Dong et al. [4] obtained the global regularity of the solution. In the case of $\alpha>0,\beta=\gamma = 1$, Ye [5] obtained the global regularity. Some small initial data results for the system with damp term were studied in [6, 7]. There have been some related works, one can refer to [8–10] and the references therein.

For the 3D case, when $\alpha=\beta=\gamma = 1$, Wu [11] investigated the regularity criterion for the local-in-time smooth solution in the Morrey-Campanato space and Wang, Zhang and Pan [12] proved the global existence. For the case $\alpha\geq\frac{5}{2}$, $\beta=\gamma = 0$, Zhu [13] established the global regularity by using the classical energy methods.

For the n-dimensional case, Ye [14] obtained the global regularity result as long as $\alpha\geq\frac{1}{2}+\frac{n}{4}$, $\alpha+\beta\geq 1+\frac{n}{2},\gamma = 0$. When the initial data satisfy small condition, Li, Deng and Shang [15] established the global well-posedness of solutions with $\frac{1}{2}<\alpha,\beta,\gamma<\frac{1}{2}+\frac{n}{4}$.

Let us mention that when $\theta = 0$, system (1.1) reduces to the generalized MHD system and there have been a lot of results, one can refer to [16–25] and the references therein.

Motivated by the work of [25], we consider the global regularity of the tropical climate model with fractional diffusion in case of $n\geq 3$. The purpose of this paper is to study the global existence and uniqueness of the solution to system (1.1) by fully exploiting its special structure and the energy methods. The main result of this paper can be stated as follows.

Theorem 1.1. *Assume that* $(u_0, v_0, \theta_0) \in H^s(R^n)$ *with* $s \geq 1, n \geq 3$ *and* $\mathrm{div} u_0 = 0$. *Let*

$$\alpha \geq \frac{1}{2} + \frac{n}{4},\ \alpha + \beta \geq 1 + \frac{n}{2},\ \alpha + \gamma \geq 1 + \frac{n}{2}. \tag{1.2}$$

Then system (1.1) *has a unique global solution* (u, v, θ) *satisfying*

$$\begin{aligned} &(u, v, \theta) \in L^\infty(0, T; H^s(R^n)),\ u \in L^2(0, T; H^{s+\alpha}(R^n)), \\ &v \in L^2(0, T; H^{s+\beta}(R^n)),\ \theta \in L^2(0, T; H^{s+\gamma}(R^n)) \end{aligned} \tag{1.3}$$

for any $T > 0$.

Remark 1.1. In the special case $\theta = 0$, system (1.1) reduces to the following GMHD system:

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p + \Lambda^{2\alpha} u + v \cdot \nabla v = 0, \\ v_t + u \cdot \nabla v + \Lambda^{2\beta} v + v \cdot \nabla u = 0, \\ \mathrm{div} u = 0, \mathrm{div} v = 0, \\ (u, v)(x, 0) = (u_0, v_0). \end{cases} \tag{1.4}$$

It is proved in [25] that the solution is globally regular with $\alpha \geq \frac{1}{2} + \frac{n}{4}, \beta \geq \frac{1}{2} + \frac{n}{4}$ and $(u_0, v_0) \in H^s(s \geq \max\{2\alpha, 2\beta\})$. Here, we relax the requirements for the regularity of the initial data, i.e., $(u_0, v_0) \in H^s(s \geq 1)$, and establish the global well-posedness for system (1.4) with $\alpha \geq \frac{1}{2} + \frac{n}{4}, \alpha + \beta \geq 1 + \frac{n}{2}$, which generalizes the result in [25].

Remark 1.2. The global regularity for logarithmically supercritical tropical climate model and MHD system were studied in [14, 23, 24]. Therefore, it may be interesting to improve Theorem 1.1 to the logarithmically supercritical case.

Remark 1.3. When $n = 2$, by examining the proof in section 2 and making a slight adjustment to Theorem 1.1, we can establish the global existance and uniqueness of the solution with $\alpha > 1, \alpha + \beta \geq 2, \alpha + \gamma \geq 2$ or $\alpha = 1, \beta > 1, \gamma > 1$, where $(u_0, v_0, \theta_0) \in H^s(s \geq 1)$.

Some important works related with recent development in fractional calculus and its applications can be referred to [26–28]. For more results of biological fractional diffusion system, one can refer to [29–31].

In next section, we are going to prove Theorem 1.1 and divide the proof into two parts: the global existence part and the uniqueness part.

2. Proof of the main result

We first show the global existence by establishing a priori H^s-bound on solutions to system (1.1), which could be stated as follows.

Proposition 2.1. *Let* (u, v, θ) *be the corresponding solution of system* (1.1). *Then for all* $t \in [0, T]$, *we have*

$$\begin{aligned} &\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s v(t)\|_{L^2}^2 + \|\Lambda^s \theta(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+s} u(\tau)\|_{L^2}^2 + \|\Lambda^{\beta+s} v(\tau)\|_{L^2}^2 \\ &\quad + \|\Lambda^{\gamma+s} \theta(\tau)\|_{L^2}^2) d\tau \leq C. \end{aligned} \tag{2.1}$$

Proof. For the sake of clarity, we divide the proof into three steps.

Step 1 (H^0-estimates). Multiplying $(1.1)_1$ by u, $(1.1)_2$ by v and $(1.1)_3$ by θ, respectively, integrating in $\mathbb{R}^n$ and taking the divergence free property into account, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2)+\|\Lambda^\alpha u\|_{L^2}^2+\|\Lambda^\beta v\|_{L^2}^2+\|\Lambda^\gamma\theta\|_{L^2}^2\\&=-\int_{R^n}\text{div}(v\otimes v)\cdot u\,dx-\int_{R^n}\nabla\theta\cdot v\,dx-\int_{R^n}v\cdot\nabla u\cdot v\,dx-\int_{R^n}\theta\text{div}v\,dx.\end{aligned}\tag{2.2}$$

We now deal with the first and third terms together as

$$\begin{aligned}&-\int_{R^n}\text{div}(v\otimes v)\cdot u\,dx-\int_{R^n}v\cdot\nabla u\cdot v\,dx\\&=-\int_{R^n}\text{div}(v\otimes v)\cdot u\,dx+\int_{R^n}\text{div}(v\otimes v)\cdot u\,dx=0.\end{aligned}\tag{2.3}$$

Note that

$$-\int_{R^n}\nabla\theta\cdot v\,dx-\int_{R^n}\theta\text{div}v\,dx=\int_{R^n}\theta\text{div}v\,dx-\int_{R^n}\theta\text{div}v\,dx=0.\tag{2.4}$$

Substituting (2.3) and (2.4) to (2.2), we know that

$$\begin{aligned}&\|u\|_{L^2}^2+\|v\|_{L^2}^2+\|\theta\|_{L^2}^2+2\int_0^t(\|\Lambda^\alpha u(\tau)\|_{L^2}^2+\|\Lambda^\beta v(\tau)\|_{L^2}^2+\|\Lambda^\gamma\theta(\tau)\|_{L^2}^2)d\tau\\&=\|u_0\|_{L^2}^2+\|v_0\|_{L^2}^2+\|\theta_0\|_{L^2}^2.\end{aligned}\tag{2.5}$$

Step 2 (H^1-estimates). Multiplying $(1.1)_1$–$(1.1)_3$ by Δu, Δv and $\Delta\theta$ respectively, integrating them in R^n, and adding them up, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\beta+1}v\|_{L^2}^2+\|\Lambda^{\gamma+1}\theta\|_{L^2}^2\\&=-\int_{R^n}\partial_k u^i\partial_i u^j\partial_k u^j\,dx+\int_{R^n}\text{div}(v\otimes v)\cdot\Delta u\,dx-\int_{R^n}\partial_k u^i\partial_i v^j\partial_k v^j\,dx\\&\quad+\int_{R^n}\nabla\theta\cdot\Delta v\,dx+\int_{R^n}(v\cdot\nabla u)\cdot\Delta v\,dx-\int_{R^n}\partial_k u^i\partial_i\theta\partial_k\theta\,dx+\int_{R^n}\text{div}v\Delta\theta\,dx\\&:=J_1+J_2+J_3+J_4+J_5+J_6+J_7.\end{aligned}\tag{2.6}$$

Firstly, we have

$$\begin{aligned}J_4+J_7&=\int_{R^n}\nabla\theta\cdot\Delta v\,dx+\int_{R^n}\text{div}v\Delta\theta\,dx\\&=-\int_{R^n}\theta\text{div}\Delta v\,dx+\int_{R^n}\text{div}v\Delta\theta\,dx\\&=-\int_{R^n}\theta\Delta\text{div}v\,dx+\int_{R^n}\text{div}v\Delta\theta\,dx\\&=-\int_{R^n}\Delta\theta\text{div}v\,dx+\int_{R^n}\text{div}v\Delta\theta\,dx=0.\end{aligned}\tag{2.7}$$

For J_1, we recall the following Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^3} \le C\|\nabla u\|_{L^2}^{a}\|\Lambda^{\alpha}u\|_{L^2}^{b}\|\Lambda^{\alpha+1}u\|_{L^2}^{c} \tag{2.8}$$

with

$$a = 1 - \frac{1}{3\alpha}(1 + \frac{n}{2}),\ b = \frac{1}{3},\ c = \frac{1}{3\alpha}(1 - \alpha + \frac{n}{2}).$$

This together with Hölder's inequality and Young's inequality gives rise to

$$\begin{aligned} J_1 &\le \|\nabla u\|_{L^3}^3 \le C\|\nabla u\|_{L^2}^{3a}\|\Lambda^{\alpha}u\|_{L^2}^{3b}\|\Lambda^{\alpha+1}u\|_{L^2}^{3c} \\ &\le \varepsilon\|\Lambda^{\alpha+1}u\|_{L^2}^2 + C\|\Lambda^{\alpha}u\|_{L^2}^{\frac{4\alpha}{6\alpha-2-n}}\|\nabla u\|_{L^2}^2. \end{aligned} \tag{2.9}$$

Next we estimate J_3 in two cases: $\beta < \frac{n}{2}$ and $\beta > \frac{n}{2}$. Note that the remain case of $\beta = \frac{n}{2}$ can be solved by interpolating with the results of $\beta < \frac{n}{2}$ and $\beta > \frac{n}{2}$ (see (2.17) below). The process is used for similar situations later.
If $\beta < \frac{n}{2}$, noting that $\alpha + \beta \ge 1 + \frac{n}{2}$, it holds that

$$\begin{aligned} J_3 &\le C\|\nabla v\|_{L^2}\|\nabla u\|_{L^{\frac{n}{\beta}}}\|\nabla v\|_{L^{\frac{2n}{n-2\beta}}} \\ &\le C\|\nabla v\|_{L^2}\|\Lambda^{\frac{n}{2}-\beta+1}u\|_{L^2}\|\Lambda^{\beta+1}v\|_{L^2} \\ &\le \varepsilon\|\Lambda^{\beta+1}v\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2\|\Lambda^{\frac{n}{2}-\beta+1}u\|_{L^2}^2 \\ &\le \varepsilon\|\Lambda^{\beta+1}v\|_{L^2}^2 + C(\|u\|_{L^2}^2 + \|\Lambda^{\alpha}u\|_{L^2}^2)\|\nabla v\|_{L^2}^2, \end{aligned} \tag{2.10}$$

while if $\beta > \frac{n}{2}$, we derive that

$$\begin{aligned} J_3 &\le C\|\nabla v\|_{L^2}\|\nabla u\|_{L^2}\|\nabla v\|_{L^\infty} \\ &\le C\|\nabla v\|_{L^2}(\|u\|_{L^2} + \|\Lambda^{\alpha}u\|_{L^2})(\|v\|_{L^2} + \|\Lambda^{\beta+1}v\|_{L^2}) \\ &\le \varepsilon\|\Lambda^{\beta+1}v\|_{L^2}^2 + C\|v\|_{L^2}^2 + C(\|u\|_{L^2}^2 + \|\Lambda^{\alpha}u\|_{L^2}^2)\|\nabla v\|_{L^2}^2. \end{aligned} \tag{2.11}$$

For the term J_6, similar to J_3, it follows from $\gamma < \frac{n}{2}$ that

$$J_6 \le \varepsilon\|\Lambda^{\gamma+1}\theta\|_{L^2}^2 + C(\|u\|_{L^2}^2 + \|\Lambda^{\alpha}u\|_{L^2}^2)\|\nabla\theta\|_{L^2}^2, \tag{2.12}$$

on the other hand, if $\gamma > \frac{n}{2}$, it holds that

$$J_6 \le \varepsilon\|\Lambda^{\gamma+1}\theta\|_{L^2}^2 + C\|\theta\|_{L^2}^2 + C(\|u\|_{L^2}^2 + \|\Lambda^{\alpha}u\|_{L^2}^2)\|\nabla\theta\|_{L^2}^2. \tag{2.13}$$

For J_2 and J_5, we compute

$$\begin{aligned} J_2 + J_5 &= \int_{R^n} \mathrm{div}(v \otimes v) \cdot \Delta u\, dx + \int_{R^n} (v \cdot \nabla u) \cdot \Delta v\, dx \\ &= \int_{R^n} \partial_i(v^i v^j)\partial_k\partial_k u^j\, dx + \int_{R^n} v^i \partial_i u^j \partial_k\partial_k v^j\, dx \\ &= \int_{R^n} (\partial_i v^i v^j \partial_k\partial_k u^j + v^i \partial_i v^j \partial_k\partial_k u^j)\, dx - \int_{R^n} (\partial_k v^i \partial_i u^j \partial_k v^j + v^i \partial_k\partial_i u^j \partial_k v^j)\, dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{R^n} (\partial_i v^i v^j \partial_k\partial_k u^j + v^i \partial_i v^j \partial_k\partial_k u^j - v^i \partial_k\partial_i u^j \partial_k v^j)\, dx - \int_{R^n} \partial_k v^i \partial_i u^j \partial_k v^j\, dx \\
&:= N_1 + N_2.
\end{aligned} \tag{2.14}
$$

The estimation of N_2 is the same as that of J_3, which have been handled in (2.10) and (2.11). So we only need to deal with N_1. Using Hölder's inequality, Sobolev embedding theorem, Young's inequality and $\alpha+\beta \geq 1+\frac{n}{2}$, one infers that if $\alpha < 1+\frac{n}{2}$,

$$
\begin{aligned}
N_1 &\leq \|\Delta u\|_{L^{\frac{2n}{2-2\alpha+n}}} \|\nabla v\|_{L^2} \|v\|_{L^{\frac{n}{\alpha-1}}} \\
&\leq C\|\Lambda^{\alpha+1} u\|_{L^2} \|\nabla v\|_{L^2} \|v\|_{L^{\frac{n}{\alpha-1}}} \\
&\leq \varepsilon \|\Lambda^{\alpha+1} u\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2 \|v\|_{L^{\frac{n}{\alpha-1}}}^2 \\
&\leq \varepsilon \|\Lambda^{\alpha+1} u\|_{L^2}^2 + C(\|v\|_{L^2}^2 + \|\Lambda^{\beta} v\|_{L^2}^2)\|\nabla v\|_{L^2}^2,
\end{aligned} \tag{2.15}
$$

and if $\alpha > 1+\frac{n}{2}$,

$$
\begin{aligned}
N_1 &\leq \|\Delta u\|_{L^\infty} \|\nabla v\|_{L^2} \|v\|_{L^2} \\
&\leq C\|\Lambda^{\alpha+1} u\|_{L^2}^{\frac{4+n}{2(\alpha+1)}} \|u\|_{L^2}^{\frac{2\alpha-2-n}{2(\alpha+1)}} \|\nabla v\|_{L^2} \|v\|_{L^2} \\
&\leq \varepsilon \|\Lambda^{\alpha+1} u\|_{L^2}^2 + C\|u\|_{L^2}^{\frac{2(2\alpha-2-n)}{4\alpha-n}} \|\nabla v\|_{L^2}^{\frac{4(\alpha+1)}{4\alpha-n}} \|v\|_{L^2}^{\frac{4(\alpha+1)}{4\alpha-n}} \\
&\leq \varepsilon \|\Lambda^{\alpha+1} u\|_{L^2}^2 + C\|u\|_{L^2}^2 + C\|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2.
\end{aligned} \tag{2.16}
$$

Inserting (2.7), (2.9)–(2.16) into (2.6), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2) + \|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\beta+1} v\|_{L^2}^2 + \|\Lambda^{\gamma+1} \theta\|_{L^2}^2 \\
&\leq C(\|\Lambda^{\alpha} u\|_{L^2}^{\frac{4\alpha}{6\alpha-2-n}} + \|\Lambda^{\alpha} u\|_{L^2}^2 + \|\Lambda^{\beta} v\|_{L^2}^2 + \|u\|_{L^2}^2 + \|v\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \\
&\quad + \|\nabla \theta\|_{L^2}^2) + C(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2),
\end{aligned}
$$

this, together with Gronwall's inequality, $\alpha \geq \frac{1}{2}+\frac{n}{4}$ and (2.5), yields

$$
\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+1} u(\tau)\|_{L^2}^2 + \|\Lambda^{\beta+1} v(\tau)\|_{L^2}^2 \\
&\qquad + \|\Lambda^{\gamma+1} \theta(\tau)\|_{L^2}^2) d\tau \leq C.
\end{aligned} \tag{2.17}
$$

Step 3 (H^s-estimates ($s > 1$)). Applying Λ^s to the first three equations of (1.1), taking the L^2-inner product with $\Lambda^s u$, $\Lambda^s v$, $\Lambda^s \theta$ respectively, and adding them together, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2) + \|\Lambda^{s+\alpha} u\|_{L^2}^2 + \|\Lambda^{s+\beta} v\|_{L^2}^2 + \|\Lambda^{s+\gamma} \theta\|_{L^2}^2 \\
&= -\int_{R^n} \Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\, dx - \int_{R^n} \Lambda^s(\mathrm{div}(v\otimes v))\cdot\Lambda^s u\, dx - \int_{R^n} \Lambda^s(u\cdot\nabla v)\cdot\Lambda^s v\, dx \\
&\quad - \int_{R^n} \Lambda^s(\nabla\theta)\cdot\Lambda^s v\, dx - \int_{R^n} \Lambda^s(v\cdot\nabla u)\cdot\Lambda^s v\, dx - \int_{R^n} \Lambda^s(u\cdot\nabla\theta)\Lambda^s \theta\, dx \\
&\quad - \int_{R^n} \Lambda^s(\mathrm{div} v)\Lambda^s \theta\, dx
\end{aligned}
$$

$$:= K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7. \tag{2.18}$$

It is easy to see that

$$\begin{aligned} K_4 + K_7 &= -\int_{R^n} \Lambda^s(\nabla\theta)\cdot\Lambda^s v\,dx - \int_{R^n} \Lambda^s(\mathrm{div} v)\Lambda^s\theta\,dx \\ &= \int_{R^n} \Lambda^s\theta\Lambda^s(\mathrm{div} v)\,dx - \int_{R^n} \Lambda^s(\mathrm{div} v)\Lambda^s\theta\,dx = 0. \end{aligned} \tag{2.19}$$

To estimate K_1, we need the following Kato-Ponce type commutator estimate [32]:

$$\|[\Lambda^s, f]g\|_{L^r} \le C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{2.20}$$

where $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ and $r, p_i, q_i \in [1,\infty]$, $i = 1, 2$.

Applying Hölder's inequality, (2.20), Young's inequality and Gagliardo-Nirenberg inequality, K_1 can be bounded as

$$\begin{aligned} K_1 &= -\int_{R^n} \Lambda^s(u\cdot\nabla u)\cdot\Lambda^s u\,dx \\ &= -\int_{R^n} [\Lambda^s, u\cdot\nabla]u\cdot\Lambda^s u\,dx \\ &\le C\|\nabla u\|_{L^2}\|\Lambda^s u\|_{L^4}^2 \\ &\le C\|\nabla u\|_{L^2}\|\Lambda^s u\|_{L^2}^{\frac{4\alpha-n}{2\alpha}}\|\Lambda^{s+\alpha}u\|_{L^2}^{\frac{n}{2\alpha}} \\ &\le \varepsilon\|\Lambda^{s+\alpha}u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{\frac{4\alpha}{4\alpha-n}}\|\Lambda^s u\|_{L^2}^2, \end{aligned} \tag{2.21}$$

where we have used $\mathrm{div} u = 0$.

Now we divide the estimate of K_3 into the following two cases. By (2.20), $\mathrm{div} u = 0$ and $\alpha+\beta \ge 1+\frac{n}{2}$, we infer that if $\alpha < \frac{n}{2}$ and $\beta < \frac{n}{2}$,

$$\begin{aligned} K_3 &= -\int_{R^n} \Lambda^s(u\cdot\nabla v)\cdot\Lambda^s v\,dx \\ &= -\int_{R^n} [\Lambda^s, u\cdot\nabla]v\cdot\Lambda^s v\,dx \\ &\le C\|\Lambda^s u\|_{L^{\frac{2n}{n-2\alpha}}}\|\nabla v\|_{L^2}\|\Lambda^s v\|_{L^{\frac{n}{\alpha}}} + C\|\Lambda^s v\|_{L^2}\|\nabla u\|_{L^{\frac{n}{\beta}}}\|\Lambda^s v\|_{L^{\frac{2n}{n-2\beta}}} \\ &\le C\|\Lambda^{s+\alpha}u\|_{L^2}\|\nabla v\|_{L^2}\|\Lambda^s v\|_{L^2}^{\frac{2\alpha+2\beta-n}{2\beta}}\|\Lambda^{s+\beta}v\|_{L^2}^{\frac{n-2\alpha}{2\beta}} \\ &\quad + C\|\Lambda^s v\|_{L^2}\|\Lambda^{\frac{n}{2}-\beta+1}u\|_{L^2}\|\Lambda^{s+\beta}v\|_{L^2} \\ &\le \varepsilon\|\Lambda^{s+\alpha}u\|_{L^2}^2 + \varepsilon\|\Lambda^{s+\beta}v\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\Lambda^{\alpha+1}u\|_{L^2}^2 \\ &\quad + \|\nabla v\|_{L^2}^{\frac{4\beta}{2\alpha+2\beta-n}})\|\Lambda^s v\|_{L^2}^2. \end{aligned} \tag{2.22}$$

If $\alpha > \frac{n}{2}$ and $\beta > \frac{n}{2}$, one deduces

$$\begin{aligned} K_3 &\le C\|\Lambda^s u\|_{L^\infty}\|\nabla v\|_{L^2}\|\Lambda^s v\|_{L^2} + C\|\Lambda^s v\|_{L^2}\|\nabla u\|_{L^2}\|\Lambda^s v\|_{L^\infty} \\ &\le C\|\Lambda^s u\|_{L^2}^{\frac{2\alpha-n}{2\alpha}}\|\Lambda^{s+\alpha}u\|_{L^2}^{\frac{n}{2\alpha}}\|\nabla v\|_{L^2}\|\Lambda^s v\|_{L^2} \end{aligned}$$

$$
\begin{aligned}
&+ C\|\Lambda^s v\|_{L^2}\|\nabla u\|_{L^2}(\|\nabla v\|_{L^2} + \|\Lambda^{s+\beta} v\|_{L^2})\\
&\le \varepsilon\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \varepsilon\|\Lambda^{s+\beta} v\|_{L^2}^2 + C\|\Lambda^s u\|_{L^2}^2\\
&\quad + C\|\nabla v\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2)\|\Lambda^s v\|_{L^2}^2.
\end{aligned}
$$

Summing up the above two cases, one has

$$
\begin{aligned}
K_3 \le\ & \varepsilon\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \varepsilon\|\Lambda^{s+\beta} v\|_{L^2}^2 + C\|\Lambda^s u\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2\\
& + \|\nabla v\|_{L^2}^2 + \|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\nabla v\|_{L^2}^{\frac{4\beta}{2\alpha+2\beta-n}})\|\Lambda^s v\|_{L^2}^2. \qquad (2.23)
\end{aligned}
$$

Similarly, for K_6, we can get

$$
\begin{aligned}
K_6 \le\ & \varepsilon\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \varepsilon\|\Lambda^{s+\gamma} \theta\|_{L^2}^2 + C\|\Lambda^s u\|_{L^2}^2 + C\|\nabla \theta\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2\\
& + \|\nabla \theta\|_{L^2}^2 + \|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^{\frac{4\gamma}{2\alpha+2\gamma-n}})\|\Lambda^s \theta\|_{L^2}^2. \qquad (2.24)
\end{aligned}
$$

Finally, we shall deal with K_2 and K_5 together.

$$
\begin{aligned}
K_2 + K_5 &= -\int_{R^n} \Lambda^s(\mathrm{div}(v\otimes v))\cdot\Lambda^s u\,dx - \int_{R^n}\Lambda^s(v\cdot\nabla u)\cdot\Lambda^s v\,dx\\
&= \int_{R^n} \Lambda^s(v\cdot v)\cdot\Lambda^s\nabla u\,dx - \int_{R^n}\Lambda^s(v\cdot\nabla u)\cdot\Lambda^s v\,dx.
\end{aligned}
$$

Noting that $\alpha+\beta \ge 1+\frac{n}{2}$, if $\alpha < 1+\frac{n}{2}$ and $\beta < \frac{n}{2}$, we know that

$$
\begin{aligned}
K_2 + K_5 &\le C\|v\|_{L^{\frac{n}{\alpha-1}}}\|\Lambda^s\nabla u\|_{L^{\frac{2n}{2-2\alpha+n}}}\|\Lambda^s v\|_{L^2} + C\|\Lambda^s v\|_{L^{\frac{2n}{n-2\beta}}}\|\nabla u\|_{L^{\frac{n}{\beta}}}\|\Lambda^s v\|_{L^2}\\
&\le C\|v\|_{L^{\frac{n}{\alpha-1}}}\|\Lambda^{s+\alpha} u\|_{L^2}\|\Lambda^s v\|_{L^2} + C\|\Lambda^{s+\beta} v\|_{L^2}\|\Lambda^{\frac{n}{2}-\beta+1} u\|_{L^2}\|\Lambda^s v\|_{L^2}\\
&\le \varepsilon\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \varepsilon\|\Lambda^{s+\beta} v\|_{L^2} + C\|\Lambda^{\frac{n}{2}-\beta+1} u\|_{L^2}^2\|\Lambda^s v\|_{L^2}^2 + C\|v\|_{L^{\frac{n}{\alpha-1}}}^2\|\Lambda^s v\|_{L^2}^2\\
&\le \varepsilon\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \varepsilon\|\Lambda^{s+\beta} v\|_{L^2} + C(\|u\|_{L^2}^2 + \|\Lambda^{\alpha+1} u\|_{L^2}^2)\|\Lambda^s v\|_{L^2}^2\\
&\quad + C(\|v\|_{L^2}^2 + \|\Lambda^{\beta+1} v\|_{L^2}^2)\|\Lambda^s v\|_{L^2}^2\\
&\le \varepsilon\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \varepsilon\|\Lambda^{s+\beta} v\|_{L^2} + C(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\Lambda^{\alpha+1} u\|_{L^2}^2\\
&\quad + \|\Lambda^{\beta+1} v\|_{L^2}^2)\|\Lambda^s v\|_{L^2}^2.
\end{aligned}
$$

While if $\alpha > 1+\frac{n}{2}$ and $\beta > \frac{n}{2}$, one has

$$
\begin{aligned}
K_2 + K_5 &\le C\|v\|_{L^2}\|\Lambda^s\nabla u\|_{L^\infty}\|\Lambda^s v\|_{L^2} + C\|\Lambda^s v\|_{L^\infty}\|\nabla u\|_{L^2}\|\Lambda^s v\|_{L^2}\\
&\le C\|v\|_{L^2}\|\Lambda^s u\|_{L^2}^{\frac{2\alpha-n-2}{2\alpha}}\|\Lambda^{s+\alpha} u\|_{L^2}^{\frac{n+2}{2\alpha}}\|\Lambda^s v\|_{L^2}\\
&\quad + C(\|\nabla v\|_{L^2} + \|\Lambda^{s+\beta} v\|_{L^2})\|\nabla u\|_{L^2}\|\Lambda^s v\|_{L^2}\\
&\le \varepsilon\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \varepsilon\|\Lambda^{s+\beta} v\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2)\|\Lambda^s v\|_{L^2}^2\\
&\quad + C\|\Lambda^s u\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2.
\end{aligned}
$$

Summarizing the above two cases, we can get

$$
K_2 + K_5 \le \varepsilon\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \varepsilon\|\Lambda^{s+\beta} v\|_{L^2}^2 + C\|\Lambda^s u\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2 + C(\|u\|_{L^2}^2
$$

$$+ \|v\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\Lambda^{\alpha+1}u\|_{L^2}^2 + \|\Lambda^{\beta+1}v\|_{L^2}^2)\|\Lambda^s v\|_{L^2}^2. \tag{2.25}$$

Substituting (2.19), (2.21)–(2.25) into (2.18), we derive that

$$\begin{aligned}
&\frac{d}{dt}(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2) + \|\Lambda^{s+\alpha}u\|_{L^2}^2 + \|\Lambda^{s+\beta}v\|_{L^2}^2 + \|\Lambda^{s+\gamma}\theta\|_{L^2}^2 \\
&\leq C(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^{\frac{4\alpha}{4\alpha-n}} \\
&\quad + \|\nabla v\|_{L^2}^{\frac{4\beta}{2\alpha+2\beta-n}} + \|\nabla\theta\|_{L^2}^{\frac{4\gamma}{2\alpha+2\gamma-n}} + \|\Lambda^{\alpha+1}u\|_{L^2}^2 + \|\Lambda^{\beta+1}v\|_{L^2}^2 + 1) \\
&\quad \times (\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s\theta\|_{L^2}^2) + C(\|\nabla v\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2),
\end{aligned}$$

this together with Gronwall's inequality, (2.5) and (2.17) yields (2.1).

Now we turn to show the uniqueness part of Theorem 1.1. For clarity, we describe this part as the following proposition.

Proposition 2.2. *Let $T > 0$. Assume that $(u^{(1)}, v^{(1)}, \theta^{(1)})$ and $(u^{(2)}, v^{(2)}, \theta^{(2)})$ are two solutions of* (1.1) *satisfying*

$$\begin{aligned}
&(u^{(i)}, v^{(i)}, \theta^{(i)}) \in L^\infty(0, T; H^1(R^n)),\ for\ i = 1, 2, \\
&(\Lambda^{\alpha+1}u^{(2)}, \Lambda^{\beta+1}v^{(2)}, \Lambda^{\gamma+1}\theta^{(2)}) \in L^2(0, T; L^2(R^n)).
\end{aligned}$$

Then $(u^{(1)}, v^{(1)}, \theta^{(1)}) = (u^{(2)}, v^{(2)}, \theta^{(2)})$ on $R^n \times (0, T)$.

Proof. Let $p^{(1)}$ and $p^{(2)}$ be the pressures associated with $(u^{(1)}, v^{(1)}, \theta^{(1)})$ and $(u^{(2)}, v^{(2)}, \theta^{(2)})$, respectively. Then the differences $(\tilde{u}, \tilde{v}, \tilde{\theta})$ between these two solutions

$$\tilde{u} = u^{(1)} - u^{(2)},\ \tilde{v} = v^{(1)} - v^{(2)},\ \tilde{\theta} = \theta^{(1)} - \theta^{(2)},\ \tilde{p} = p^{(1)} - p^{(2)},$$

satisfy

$$\begin{cases}
\tilde{u}_t + u^{(1)}\cdot\nabla\tilde{u} + \tilde{u}\cdot\nabla u^{(2)} + \Lambda^{2\alpha}\tilde{u} + \nabla\tilde{p} + \mathrm{div}(v^{(1)}\otimes v^{(1)} - v^{(2)}\otimes v^{(2)}) = 0, \\
\tilde{v}_t + u^{(1)}\cdot\nabla\tilde{v} + \tilde{u}\cdot\nabla v^{(2)} + \nabla\tilde{\theta} + \Lambda^{2\beta}\tilde{v} + \tilde{v}\cdot\nabla u^{(1)} + v^{(2)}\cdot\nabla\tilde{u} = 0, \\
\tilde{\theta}_t + u^{(1)}\cdot\nabla\tilde{\theta} + \tilde{u}\cdot\nabla\theta^{(2)} + \Lambda^{2\gamma}\tilde{\theta} + \mathrm{div}\tilde{v} = 0, \\
\mathrm{div}\tilde{u} = 0, \\
(\tilde{u}, \tilde{v}, \tilde{\theta})(x, 0) = 0.
\end{cases}$$

Taking the L^2-inner product to the first three equations with $\tilde{u}$, $\tilde{v}$ and $\tilde{\theta}$ respectively, and adding the results together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\tilde{u}(t)\|_{L^2}^2 + \|\tilde{v}(t)\|_{L^2}^2 + \|\tilde{\theta}(t)\|_{L^2}^2) + \|\Lambda^\alpha\tilde{u}\|_{L^2}^2 + \|\Lambda^\beta\tilde{v}\|_{L^2}^2 + \|\Lambda^\gamma\tilde{\theta}\|_{L^2}^2 \\
&= -\int_{R^n}\tilde{u}\cdot\nabla u^{(2)}\cdot\tilde{u}\,dx - \int_{R^n}\mathrm{div}(v^{(1)}\otimes v^{(1)} - v^{(2)}\otimes v^{(2)})\cdot\tilde{u}\,dx \\
&\quad - \int_{R^n}\tilde{u}\cdot\nabla v^{(2)}\cdot\tilde{v}\,dx - \int_{R^n}\nabla\tilde{\theta}\cdot\tilde{v}\,dx - \int_{R^n}\tilde{v}\cdot\nabla u^{(1)}\cdot\tilde{v}\,dx \\
&\quad - \int_{R^n}v^{(2)}\cdot\nabla\tilde{u}\cdot\tilde{v}\,dx - \int_{R^n}\tilde{u}\cdot\nabla\theta^{(2)}\tilde{\theta}\,dx - \int_{R^n}\tilde{\theta}\mathrm{div}\tilde{v}\,dx
\end{aligned}$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8,$$

where we have used the following facts

$$\int_{R^n} u^{(1)} \cdot \nabla\tilde{u} \cdot \tilde{u}\, dx = 0, \ \int_{R^n} u^{(1)} \cdot \nabla\tilde{v} \cdot \tilde{v}\, dx = 0,$$
$$\int_{R^n} u^{(1)} \cdot \nabla\tilde{\theta} \cdot \tilde{\theta}\, dx = 0, \ \int_{R^n} \nabla\tilde{p} \cdot \tilde{u}\, dx = 0.$$

Note that

$$\begin{aligned} I_4 + I_8 &= -\int_{R^n} \nabla\tilde{\theta} \cdot \tilde{v}\, dx - \int_{R^n} \tilde{\theta}\mathrm{div}\tilde{v}\, dx \\ &= \int_{R^n} \tilde{\theta}\mathrm{div}\tilde{v}\, dx - \int_{R^n} \tilde{\theta}\mathrm{div}\tilde{v}\, dx = 0. \end{aligned}$$

By Hölder's, Young's and Gagliardo-Nirenberg inequalities, together with Sobolev embedding theorem, we decuce that if $\alpha < \frac{n}{2}$,

$$\begin{aligned} I_1 &\leq \|\tilde{u}\|_{L^{\frac{n}{\alpha}}} \|\nabla u^{(2)}\|_{L^{\frac{2n}{n-2\alpha}}} \|\tilde{u}\|_{L^2} \\ &\leq C\|\tilde{u}\|_{L^2}^{\frac{4\alpha-n}{2\alpha}} \|\Lambda^\alpha \tilde{u}\|_{L^2}^{\frac{n-2\alpha}{2\alpha}} \|\Lambda^{\alpha+1} u^{(2)}\|_{L^2} \|\tilde{u}\|_{L^2} \\ &\leq \varepsilon\|\Lambda^\alpha \tilde{u}\|_{L^2}^2 + C\|\Lambda^{\alpha+1} u^{(2)}\|_{L^2}^{\frac{4\alpha}{6\alpha-n}} \|\tilde{u}\|_{L^2}^2, \end{aligned}$$

and if $\alpha > \frac{n}{2}$,

$$\begin{aligned} I_1 &\leq \|\tilde{u}\|_{L^2} \|\nabla u^{(2)}\|_{L^2} \|\tilde{u}\|_{L^\infty} \\ &\leq C\|\tilde{u}\|_{L^2} \|\nabla u^{(2)}\|_{L^2} \|\tilde{u}\|_{L^2}^{\frac{2\alpha-n}{2\alpha}} \|\Lambda^\alpha \tilde{u}\|_{L^2}^{\frac{n}{2\alpha}} \\ &\leq \varepsilon\|\Lambda^\alpha \tilde{u}\|_{L^2}^2 + C\|\nabla u^{(2)}\|_{L^2}^{\frac{4\alpha}{4\alpha-n}} \|\tilde{u}\|_{L^2}^2. \end{aligned}$$

Next, we estimate the term I_3. If $\beta < \frac{n}{2}$, noting that $\alpha + \beta \geq 1 + \frac{n}{2}$, we know that

$$\begin{aligned} I_3 &\leq C\|\tilde{v}\|_{L^2} \|\nabla v^{(2)}\|_{L^{\frac{2n}{n-2\beta}}} \|\tilde{u}\|_{L^{\frac{n}{\beta}}} \\ &\leq C\|\tilde{v}\|_{L^2} \|\Lambda^{\beta+1} v^{(2)}\|_{L^2} \|\tilde{u}\|_{L^2}^{\frac{2\alpha+2\beta-n}{2\alpha}} \|\Lambda^\alpha \tilde{u}\|_{L^2}^{\frac{n-2\beta}{2\alpha}} \\ &\leq \varepsilon\|\Lambda^\alpha \tilde{u}\|_{L^2}^2 + C\|\Lambda^{\beta+1} v^{(2)}\|_{L^2}^{\frac{4\alpha}{4\alpha+2\beta-n}} (\|\tilde{u}\|_{L^2}^2 + \|\tilde{v}\|_{L^2}^2). \end{aligned}$$

If $\beta > \frac{n}{2}$, one obtains

$$\begin{aligned} I_3 &\leq \|\tilde{v}\|_{L^\infty} \|\tilde{u}\|_{L^2} \|\nabla v^{(2)}\|_{L^2} \\ &\leq C\|\tilde{v}\|_{L^2}^{\frac{2\beta-n}{2\beta}} \|\Lambda^\beta \tilde{v}\|_{L^2}^{\frac{n}{2\beta}} \|\tilde{u}\|_{L^2} \|\nabla v^{(2)}\|_{L^2} \\ &\leq \varepsilon\|\Lambda^\beta \tilde{v}\|_{L^2}^2 + C\|\tilde{v}\|_{L^2}^{\frac{2(2\beta-n)}{4\beta-n}} \|\nabla v^{(2)}\|_{L^2}^{\frac{4\beta}{4\beta-n}} \|\tilde{u}\|_{L^2}^{\frac{4\beta}{4\beta-n}} \\ &\leq \varepsilon\|\Lambda^\beta \tilde{v}\|_{L^2}^2 + C\|\nabla v^{(2)}\|_{L^2}^{\frac{4\beta}{4\beta-n}} (\|\tilde{u}\|_{L^2}^2 + \|\tilde{v}\|_{L^2}^2). \end{aligned}$$

Similarly, for I_7, we infer that if $\gamma < \frac{n}{2}$,

$$I_7 \le \varepsilon\|\Lambda^\alpha \tilde{u}\|_{L^2}^2 + C\|\Lambda^{\gamma+1}\theta^{(2)}\|_{L^2}^{\frac{4\alpha}{4\alpha+2\gamma-n}}(\|\tilde{u}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2),$$

and if $\gamma > \frac{n}{2}$,

$$I_7 \le \varepsilon\|\Lambda^\gamma \tilde{\theta}\|_{L^2}^2 + C\|\nabla\theta^{(2)}\|_{L^2}^{\frac{4\gamma}{4\gamma-n}}(\|\tilde{u}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2).$$

Now we are in a position to deal with I_2, I_5 and I_6 as follows.

$$\begin{aligned}
I_2 &= -\int_{R^n} \operatorname{div}(v^{(1)} \otimes v^{(1)} - v^{(2)} \otimes v^{(2)}) \cdot \tilde{u}\, dx \\
&= \int_{R^n} (v^{(1)} \cdot v^{(1)} \cdot \nabla\tilde{u} - v^{(2)} \cdot v^{(2)} \cdot \nabla\tilde{u})\, dx \\
&= \int_{R^n} (v^{(1)} \cdot v^{(1)} \cdot \nabla\tilde{u} - v^{(2)} \cdot v^{(1)} \cdot \nabla\tilde{u} + v^{(2)} \cdot v^{(1)} \cdot \nabla\tilde{u} - v^{(2)} \cdot v^{(2)} \cdot \nabla\tilde{u})\, dx \\
&= \int_{R^n} (\tilde{v} \cdot v^{(1)} \cdot \nabla\tilde{u} + v^{(2)} \cdot \tilde{v} \cdot \nabla\tilde{u})\, dx.
\end{aligned}$$

Then

$$\begin{aligned}
I_2 + I_5 + I_6 &= \int_{R^n} (\tilde{v} \cdot v^{(1)} \cdot \nabla\tilde{u} - \tilde{v} \cdot \nabla u^{(1)} \cdot \tilde{v})\, dx \\
&= \int_{R^n} (\tilde{v} \cdot v^{(1)} \cdot \nabla\tilde{u} - \tilde{v} \cdot v^{(2)} \cdot \nabla\tilde{u} + \tilde{v} \cdot v^{(2)} \cdot \nabla\tilde{u} - \tilde{v} \cdot \nabla u^{(1)} \cdot \tilde{v})\, dx \\
&= \int_{R^n} (\tilde{v} \cdot \tilde{v} \cdot \nabla\tilde{u} + \tilde{v} \cdot v^{(2)} \cdot \nabla\tilde{u} - \tilde{v} \cdot \nabla u^{(1)} \cdot \tilde{v})\, dx \\
&= \int_{R^n} \tilde{v} \cdot v^{(2)} \cdot \nabla\tilde{u}\, dx - \int_{R^n} \tilde{v} \cdot \nabla u^{(2)} \cdot \tilde{v}\, dx \\
&:= Q_1 + Q_2.
\end{aligned}$$

For Q_1, if $\beta < \frac{n}{2} - 1$, applying Hölder's, Young's and Gagliardo-Nirenberg inequalities yields

$$\begin{aligned}
Q_1 &\le C\|\tilde{v}\|_{L^2}\|v^{(2)}\|_{L^{\frac{2n}{n-2\beta-2}}}\|\nabla\tilde{u}\|_{L^{\frac{n}{\beta+1}}} \\
&\le C\|\tilde{v}\|_{L^2}\|\Lambda^{\beta+1}v^{(2)}\|_{L^2}\|\tilde{u}\|_{L^2}^{\frac{2\alpha+2\beta-n}{2\alpha}}\|\Lambda^\alpha\tilde{u}\|_{L^2}^{\frac{n-2\beta}{2\alpha}} \\
&\le \varepsilon\|\Lambda^\alpha\tilde{u}\|_{L^2}^2 + C\|\tilde{v}\|_{L^2}^{\frac{4\alpha}{4\alpha+2\beta-n}}\|\Lambda^{\beta+1}v^{(2)}\|_{L^2}^{\frac{4\alpha}{4\alpha+2\beta-n}}\|\tilde{u}\|_{L^2}^{\frac{2(2\alpha+2\beta-n)}{4\alpha+2\beta-n}} \\
&\le \varepsilon\|\Lambda^\alpha\tilde{u}\|_{L^2}^2 + C\|\Lambda^{\beta+1}v^{(2)}\|_{L^2}^{\frac{4\alpha}{4\alpha+2\beta-n}}(\|\tilde{v}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2),
\end{aligned}$$

where we have used $\alpha + \beta \ge 1 + \frac{n}{2}$. And if $\beta > \frac{n}{2} - 1$,

$$\begin{aligned}
Q_1 &\le C\|\tilde{v}\|_{L^2}\|v^{(2)}\|_{L^\infty}\|\nabla\tilde{u}\|_{L^2} \\
&\le C\|\tilde{v}\|_{L^2}(\|\nabla v^{(2)}\|_{L^2} + \|\Lambda^{\beta+1}v^{(2)}\|_{L^2})(\|\tilde{u}\|_{L^2} + \|\Lambda^\alpha\tilde{u}\|_{L^2}) \\
&\le \varepsilon\|\Lambda^\alpha\tilde{u}\|_{L^2}^2 + C\|\tilde{u}\|_{L^2}^2 + C(\|\Lambda^{\beta+1}v^{(2)}\|_{L^2}^2 + \|\nabla v^{(2)}\|_{L^2}^2)\|\tilde{v}\|_{L^2}^2.
\end{aligned}$$

For Q_2, if $\beta < \frac{n}{2}$, noting that $\alpha + \beta \ge 1 + \frac{n}{2}$, one infers

$$\begin{aligned} Q_2 &\le \|\tilde{v}\|_{L^2}\|\nabla u^{(2)}\|_{L^{\frac{n}{\beta}}}\|\tilde{v}\|_{L^{\frac{2n}{n-2\beta}}} \\ &\le C\|\tilde{v}\|_{L^2}\|\Lambda^{\frac{n}{2}-\beta+1}u^{(2)}\|_{L^2}\|\Lambda^{\beta}\tilde{v}\|_{L^2} \\ &\le \varepsilon\|\Lambda^{\beta}\tilde{v}\|_{L^2}^2 + C\|\tilde{v}\|_{L^2}^2\|\Lambda^{\frac{n}{2}-\beta+1}u^{(2)}\|_{L^2}^2 \\ &\le \varepsilon\|\Lambda^{\beta}\tilde{v}\|_{L^2}^2 + C(\|\nabla u^{(2)}\|_{L^2}^2 + \|\Lambda^{\alpha+1}u^{(2)}\|_{L^2}^2)\|\tilde{v}\|_{L^2}^2. \end{aligned}$$

If $\beta > \frac{n}{2}$, one deduces

$$\begin{aligned} Q_2 &\le \|\tilde{v}\|_{L^2}\|\nabla u^{(2)}\|_{L^2}\|\tilde{v}\|_{L^\infty} \\ &\le C\|\tilde{v}\|_{L^2}\|\nabla u^{(2)}\|_{L^2}\|\tilde{v}\|_{L^2}^{\frac{2\beta-n}{2\beta}}\|\Lambda^{\beta}\tilde{v}\|_{L^2}^{\frac{n}{2\beta}} \\ &\le \varepsilon\|\Lambda^{\beta}\tilde{v}\|_{L^2}^2 + C\|\nabla u^{(2)}\|_{L^2}^{\frac{4\beta}{4\beta-n}}\|\tilde{v}\|_{L^2}^2. \end{aligned}$$

Putting all the above estimates together gives

$$\begin{aligned} &\frac{d}{dt}(\|\tilde{u}\|_{L^2}^2 + \|\tilde{v}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2) + \|\Lambda^{\alpha}\tilde{u}\|_{L^2}^2 + \|\Lambda^{\beta}\tilde{v}\|_{L^2}^2 + \|\Lambda^{\gamma}\tilde{\theta}\|_{L^2}^2 \\ &\le C(\|\nabla u^{(2)}\|_{L^2}^2 + \|\nabla u^{(2)}\|_{L^2}^{\frac{4\alpha}{4\alpha-n}} + \|\nabla u^{(2)}\|_{L^2}^{\frac{4\beta}{4\beta-n}} + \|\nabla v^{(2)}\|_{L^2}^2 + \|\nabla v^{(2)}\|_{L^2}^{\frac{4\beta}{4\beta-n}} \\ &\quad + \|\nabla \theta^{(2)}\|^{\frac{4\gamma}{4\gamma-n}} + \|\Lambda^{\alpha+1}u^{(2)}\|_{L^2}^2 + \|\Lambda^{\alpha+1}u^{(2)}\|_{L^2}^{\frac{4\alpha}{6\alpha-n}} + \|\Lambda^{\beta+1}v^{(2)}\|_{L^2}^2 \\ &\quad + \|\Lambda^{\beta+1}v^{(2)}\|_{L^2}^{\frac{4\alpha}{4\alpha+2\beta-n}} + \|\Lambda^{\gamma+1}\theta^{(2)}\|_{L^2}^{\frac{4\alpha}{4\alpha+2\gamma-n}} + 1)(\|\tilde{u}\|_{L^2}^2 + \|\tilde{v}\|_{L^2}^2 + \|\tilde{\theta}\|_{L^2}^2). \end{aligned}$$

Then Gronwall's inequality immediately yields the uniqueness and the proof of Proposition 2.2 is completed.

3. Conclusions

In this paper, we investigate the global regularity for the n-dimensional tropical climate model with fractional diffusion. When the nonnegative constants α, β and γ satisfy $\alpha \ge \frac{1}{2}+\frac{n}{4}$, $\alpha+\beta \ge 1+\frac{n}{2}$, $\alpha+\gamma \ge 1 + \frac{n}{2}$, by using the energy methods, we obtain the global existence and uniqueness of solution for the system. In the special case $\theta = 0$, we could obtain that system (1.4) has a unique global solution provide that $\alpha \ge \frac{1}{2}+\frac{n}{4}, \alpha+\beta \ge 1+\frac{n}{2}$ and $(u_0, v_0) \in H^s(s \ge 1)$, which generalizes the result in [25]. Let us mention that when $n = 2$, by examining the proof in section 2 and making a slight adjustment to Theorem 1.1, we can establish the global existance and uniqueness of the solutions with $\alpha > 1, \alpha + \beta \ge 2, \alpha + \gamma \ge 2$ or $\alpha = 1, \beta > 1, \gamma > 1$, where $(u_0, v_0, \theta_0) \in H^s(s \ge 1)$.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. D. M. W. Frierson, A. J. Majda, O. M. Pauluis, Large scale dynamics of precipitation fronts in the tropical atmosphere: A novel relaxation limit, *Commun. Math. Sci.*, **2** (2004), 591–626.
2. C. C. Ma, Z. H. Jiang, R. H. Wan, Local well-posedness for the tropical climate model with fractional velocity diffusion, *Kinet. Relat. Mod.*, **9** (2016), 551–570.
3. J. K. Li, E. Titi, Global well-posedness of strong solutions to a tropical climate model, *DCDS*, **36** (2016), 4495–4516.
4. B. Q. Dong, W. J. Wang, J. H. Wu, H. Zhang, Global regularity results for the climate model with fractional dissipation, *DCDS-B*, **24** (2019), 211–229.
5. Z. Ye, Global regularity for a class of 2D tropical climate model, *J. Math. Anal. Appl.*, **446** (2017), 307–321.
6. R. H. Wan, Global small solutions to a tropical climate model without thermal diffusion, *J. Math. Phys.*, **57** (2016), 021507.
7. C. C. Ma, R. H. Wan, Spectral analysis and global well-posedness for a viscous tropical climate model with only a damp term, *Nonlinear Anal.-Real.*, **39** (2018), 554–567.
8. B. Q. Dong, J. H. Wu, Z. Ye, 2D tropical climate model with fractional dissipation and without thermal diffusion, *Commun. Math. Sci.*, **18** (2020), 259–292.
9. X. Ye, M. X. Zhu, Global strong solutions of the tropical climate model with temperature-dependent diffusion on the barotropic mode, *Appl. Math. Lett.*, **89** (2019), 8–14.
10. X. Ye, M. X. Zhu, Global strong solutions of the 2D tropical climate system with temperature-dependent viscosity, *Z. Angew. Math. Phys.*, **71** (2020), 97.
11. F. Wu, Regularity criteria for the 3D tropical climate model in Morrey-Campanato space, *Electron. J. Qual. Theo.*, **48** (2019), 1–11.
12. Y. Wang, S. Zhang, N. Pan, Regularity and global existence on the 3D tropical climate model, *Bull. Malays. Math. Sci. Soc.*, **43** (2020), 641–650.
13. M. X. Zhu, Global regularity for the tropical climate model with fractional diffusion on barotropic mode, *Appl. Math. Lett.*, **81** (2018), 99–104.
14. Z. Ye, Global regularity of 2D tropical climate model with zero thermal diffusion, *Z. Angew. Math. Mech.* **100** (2020), e201900132.
15. Z. X. Li, L. H. Deng, H. F. Shang, Global well-posedness and large time decay for the d-dimensional tropical climate model, *AIMS Mathematics*, **6** (2021), 5581–5595.
16. C. V. Tran, X. W. Yu, Z. C. Zhai, On global regularity of 2D generalized magnetohydrodynamic equation, *J. Differ. Equ.*, **254** (2013), 4194–4216.
17. Q. S. Jiu, D. J. Niu, Mathematical results related to a two-dimensional magneto-hydrodynamic equations, *Acta Math. Sci.*, **26** (2006), 744–756.

18. Y. Zhou, J. S. Fan, A regularity criterion for the 2D MHD system with zero magnetic diffusivity, *J. Math. Anal. Appl.*, **378** (2011), 169–172.

19. Z. H. Jiang, Y. N. Wang, Y. Zhou, On regularity criteria for the 2D generalized MHD system, *J. Math. Fluid Mech.*, **18** (2016), 331–341.

20. W. H. Wang, T. G. Qin, Q. Y. Bie, Global well-posedness and analyticity results to 3-D generalized magnetohydrodynamic equations, *Appl. Math. Lett.*, **59** (2016), 65–70.

21. Y. Z. Wang, K. Y. Wang, Global well-posedness of the three dimensional magnetohydrodynamics equations, *Nonlinear Anal.-Real.*, **17** (2014), 245–251.

22. Q. Y. Bie, Q. R. Wang, Z. A. Yao, Regularity criteria for the 3D MHD equations in term of velocity, *Math. Methods Appl. Sci.*, **38** (2015), 2506–2516.

23. K. Yamazaki, Global regularity of logarithmically supercritical MHD system with zero diffusivity, *Appl. Math. Lett.*, **29** (2014), 46–51.

24. J. H. Wu, Global regularity for a class of generalized Magnetohydrodynamic equations, *J. Math. Fluid Mech.*, **13** (2011), 295–305.

25. J. H. Wu, Generalized MHD equations, *J. Differ. Equ.*, **195** (2003), 284–312.

26. J. Singh, D. Kumar, D. Baleanu, A new analysis of fractional fish farm model associated with Mittag-Leffler-type kernel, *Int. J. Biomath.*, **13** (2020), 2050010.

27. J. Singh, Analysis of fractional blood alcohol model with composite fractional derivative, *Chaos Soliton. Fract.*, **140** (2020), 110127.

28. H. Singh, D. Baleanu, J. Singh, H. Dutta, Computational study of fractional order smoking model, *Chaos Soliton. Fract.*, **142** (2021), 110440.

29. F. Souna, S. Djilali, A. Lakmeche, Spatiotemporal behavior in a predator-prey model with herd behavior and cross-diffusion and fear effect, *Eur. Phys. J. Plus*, **136** (2021), 474.

30. S. Djilali, S. Bentout, B. Ghanbari, S. Kumar, Spatial patterns in a vegetation model with internal competition and feedback regulation, *Eur. Phys. J. Plus*, **136** (2021), 256.

31. L. Guin, S. Pal, S. Chakravarty, S. Djilali, Pattern dynamics of a reaction-diffusion predator-prey system with both refuge and harvesting, *Int. J. Biomath.*, **14** (2021), 2050084.

32. C. E. Kenig, G. Ponce, L. Vega, Well-posedness of the initial value problem for the Korteweg-de Vries equation, *J. Amer. Math. Soc.*, **4** (1991), 323–347.

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