Mathematics

Research article

Input-to-state stability in the meaning of switching for delayed feedback switched stochastic financial system

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Abstract: Financial system is essentially chaotic and unstable if there is not any external inputs. By means of Lyapunov function method, design of switching law, novel fuzzy assumption, L^p estimation technique and Laplace semigroup theory, the author presents the boundedness and LMI-based (globally) asymptotical input-to-state stability criteria of financial systems. Particularly, the globally asymptotical stability in the meaning of switching implies that when the time t is big enough, the dynamic of any subsystem must approach its unique equilibrium point. Besides, the global stability in the classical sense is not applicable to eruption of the periodical financial crisis. So the stability in the meaning of switching proposed in this paper is suitable and appropriate. Numerical examples illuminate the effectiveness of the obtained results.

1. Introduction

In recent decades, a kind of complex financial system model consisting of production sub-blocks, currency sub-blocks, securities sub-blocks and labor sub-blocks has become a hot topic for many researchers because it well simulates the dynamic behavior of important economic indicators in the real financial market ([1–9] and the references therein), and a series of beneficial economic macro-control and stability analysis are obtained, which are conducive to macro-control and market self-regulation with theoretical guidance ([1–10]). But the previous research only involves the stability of the equilibrium point with a fixed interest rate, while the financial market is complex and

changeable, and the market will adjust itself, and the financial management department will carry out macro-control according to the market situation. Its purpose is not necessarily to pursue the stability of the equilibrium point with a certain fixed interest rate, but to constantly consciously or unconsciously switch to equilibrium points of different interest rates (positive or negative or zero) to achieve the prosperity and stability of the financial market. Besides, the important economic indicators, such as savings, the unit investment cost, and the elasticity of commodity demand, has certain uncertainty. It is necessary to deal with the important economic index parameters obtained by previous statistics or different ways with fuzzy method. Takagi-Sugeno (T-S) fuzzy model will be considered in this paper, for Takagi-Sugeno fuzzy models have excellent ability in analyzing, synthesizing, and approximating complex dynamical behaviors by a set of IF-THEN rules ([14–18]). Thereby, switched models and fuzzy models are usually investigated in various control systems ([11–21]). Due to the random inflows and outputs of foreign capital will produce real-time or delayed disturbance to economic indicators, Ito stochastic models are often considered in the modeling of various control systems ([22–27]). In real financial markets, the government purchase will promote economic prosperity and stability, which will randomly improve some important economic indicators, such as savings, the unit investment cost, and represents the elasticity of commodity demand. Because of the lag of macro-control effect in the real market, the effect of macro-control is not obvious when the market is switched to one subsystem and only takes effect when the market is switched to another subsystem. Therefore, one of the main tasks of this paper is to design such a switching rule that the whole switching system is stable although some subsystems are unstable.

On the other hand, the financial system is the nonlinear system with non-Lipschitz functions, which brings about the true difficulties. That is, the globally asymptotical stability in the meaning of switching introduced by [32, Definition 3] may truly be not available for the financial system. In fact, according to [33,34], the unique existence of the solution of a system can be guaranteed if the active functions are Lipschitz continuous and growing linearly. However, the financial systems studied by this paper are complex and chaos due to the non-Lipschitz functions with superlinear growth. To overcome this difficulty, a suitable Lyapunov-Krasovskii function is given to create conditions making the switched financial system stable globally.

It is found from the study of financial mathematical model that the essence of financial system is chaotic and unstable. This is also in line with the unpredictability of the real financial market. In fact, the cyclical outbreak of the world economic crisis has confirmed this view. If the financial market is to be stable, the government must be active in taking remedial measures at a certain time. Thereby, input-to-state stability of switched financial system is studied in this paper. Due to the difficulties brought about by non-Lipschitz functions of delayed (fuzzy) stochastic switched financial systems, free weight matrix criteria can not be derived. However, by means of Lyapunov function method, design of switching law, novel fuzzy assumption, L^p estimation technique and Laplace semigroup theory, the author presents the boundedness and LMI-based (globally) asymptotical input-to-state stability criteria of financial systems. Particularly, the globally asymptotical stability in the meaning of switching implies that when the time t is big enough, the dynamic of any subsystem must approach its unique equilibrium point. Motivated by some methods of the related literature ([19–32, 35–43]), the author presents the following highlights in this paper:

♦ It is the first paper to design a novel fuzzy condition and a state-dependent switching law for the

globally asymptotical input-to-state stability in the meaning of switching for fuzzy switched stochastic financial system with delayed feedback;

♦ It is the first time to utilize the Laplace semigroup to derive the boundedness for fuzzy switched stochastic financial system with delayed feedback, which reveals the relation between ordinary differential equations model and partial differential equations model.

Throughout of this paper, for any symmetric matrices A and B, the matrix-form inequality $A < B$ represents that the symmetric matrix $(B - A)$ is a positive definite matrix.

2. System description

The following financial system has been investigated in many existing literature([1–9]),

$$\begin{cases} \dot{x} = z + (y - a)x \\ \dot{y} = 1 - by - x^2 \\ \dot{z} = -x - cz, \end{cases} \tag{2.1}$$

where x represents the interest rate, y represents the investment demand, z represents the price index, a represents savings, b represents the unit investment cost, and c represents the elasticity of commodity demand. It is well known that if $c - b - abc \leqslant 0$, the financial system (2.1) has the unique equilibrium point $P_2(0, \frac{1}{b}, 0)$; if $c - b - abc > 0$, the financial system (2.1) owns three equilibrium point $P_1(\theta, \frac{1+ac}{c}, \frac{-\theta}{c})$, $P_2(0, \frac{1}{b}, 0)$, $P_3(-\theta, \frac{1+ac}{c}, \frac{\theta}{c})$, where $\theta = \sqrt{\frac{c-b-abc}{c}}$. Chaos appears in the financial system (2.1) if $c - b - abc > 0$ and $c + a - \frac{1}{b} < 0$. For example, let $a = 0.9, b = 0.2, c = 1.2$, then there is a chaos phenomenon in the financial system (2.1) (see, e.g. [2,3]). Throughout this paper, we assume that $c - b - abc > 0$.

For the equilibrium point $P_1(\theta, \frac{1+ac}{c}, \frac{-\theta}{c})$, we may set

$$\begin{cases} X_1 = x - \theta \\ X_2 = y - \dfrac{1 + ac}{c} \\ X_3 = z + \dfrac{\theta}{c}, \end{cases} \tag{2.2}$$

then the financial system (2.1) is translated into the following system:

$$\begin{cases} \dot{X}_1 = \dfrac{1}{c}X_1 + \theta X_2 + X_3 + X_1 X_2 \\ \dot{X}_2 = -2\theta X_1 - bX_2 - X_1^2 \\ \dot{X}_3 = -X_1 - cX_3, \end{cases} \tag{2.3}$$

or

$$\dot{X}(t) = H_1 X(t) + f(X(t)), \tag{2.4}$$

where the equilibrium point $P_1(\theta, \frac{1+ac}{c}, \frac{-\theta}{c})$ of the financial system (2.2) is corresponding to the null

solution of the system (2.3) or (2.4), and A and f are defined as follows,

$$H_1 = \begin{pmatrix} \frac{1}{c} & \theta & 1 \\ -2\theta & -b & 0 \\ -1 & 0 & -c \end{pmatrix}, \quad f(X) = \begin{pmatrix} X_1 X_2 \\ -X_1^2 \\ 0 \end{pmatrix}, \tag{2.5}$$

Similarly, for the equilibrium point $P_2(0, \frac{1}{b}, 0)$, let

$$\begin{cases} X_1 = x \\ X_2 = y - \dfrac{1}{b} \\ X_3 = z, \end{cases} \tag{2.6}$$

$$\begin{cases} \dot{X}_1 = (\dfrac{1}{b} - a)X_1 + X_3 + X_1 X_2 \\ \dot{X}_2 = -bX_2 - X_1^2 \\ \dot{X}_3 = -X_1 - cX_3, \end{cases} \tag{2.7}$$

or

$$\dot{X} = H_2 X + f(X), \tag{2.8}$$

where $X = (X_1, X_2, X_3)^T$,

$$H_2 = \begin{pmatrix} \frac{1}{b} - a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix} \tag{2.9}$$

For the equilibrium point $P_3(-\theta, \frac{1+ac}{c}, \frac{\theta}{c})$, we may set

$$\begin{cases} X_1 = x + \theta \\ X_2 = y - \dfrac{1 + ac}{c} \\ X_3 = z - \dfrac{\theta}{c}, \end{cases} \tag{2.10}$$

$$\begin{cases} \dot{X}_1 = \dfrac{1}{c}X_1 - \theta X_2 + X_3 + X_1 X_2 \\ \dot{X}_2 = 2\theta X_1 - bX_2 - X_1^2 \\ \dot{X}_3 = -X_1 - cX_3, \end{cases} \tag{2.11}$$

or

$$\dot{X}(t) = H_3 X(t) + f(X(t)), \tag{2.12}$$

where

$$H_3 = \begin{pmatrix} \frac{1}{c} & -\theta & 1 \\ 2\theta & -b & 0 \\ -1 & 0 & -c \end{pmatrix}, \tag{2.13}$$

In the real financial market, the economic system is a dynamic market with randomness. Economic activities do not necessarily pursue a fixed interest rate. The null solution of the following switched system

$$\dot{X}(t) = H_\sigma X(t) + f(X(t)), \quad \sigma \in \mathfrak{T}, \text{ where } \mathfrak{T} = \{1\}, \text{ or } \{1, 2\}, \text{ or } \{1, 2, 3\} \tag{2.14}$$

corresponds to a series of equilibrium points P_σ with respective interest rates. The switched system (2.14) means that the financial system is adjusted macroscopically, or the market adjusts itself when the economic situation changes to a critical value. For example, when the economic crisis breaks out, in order to promote economic development, the interest rate is adjusted to zero or even negative. When the economy is booming, banks raise interest rates to finance, and the interest rate converges to a positive equilibrium point. Thereby, the global asymptotic stability of the switched system (2.14) reflects the activity of real financial market and the stability. That is, when the time t is big enough, the dynamical behavior of the switched system converges to an equilibrium point with a fixed positive interest at the stage of booming, and converges to another equilibrium point with a fixed non-positive interest at the stage of depression. In financial management, the Uncertainty of parameters sampling conforms to the fuzzy model to a great extent. So we consider the following T-S fuzzy rule for the financial system (2.14).

Fuzzy Rule r: **IF** $\hat{\omega}_1(t)$ is v_{r1} and $\cdots$ and $\hat{\omega}_\kappa(t)$ is $v_{r\kappa}$ **THEN**

$$\dot{X}(t) = H_{\sigma r} X(t) + f(X(t)), \tag{2.15}$$

where $\hat{\omega}_k(t)(k = 1, 2, \cdots, \kappa)$ is the premise variable, $v_{rk}(r = 1, 2, \cdots, n; k = 1, 2, \cdots, \kappa)$ is the fuzzy set that is characterized by membership function. And n is the number of the **IF-THEN** rules, κ is the number of the premise variables.

In view of a standard fuzzy inference method, (2.15) can be inferred as follows,

$$\dot{X}(t) = \sum_{r=1}^{n} \varrho_r(\hat{\omega}(t)) H_{\sigma r} X(t) + f(X(t)), \tag{2.16}$$

where $\hat{\omega}(t) = [\hat{\omega}_1(t), \hat{\omega}_2(t), \cdots, \hat{\omega}_\kappa(t)]$, $\varrho_r(\hat{\omega}(t)) = \frac{\bar{w}_r(\hat{\omega}(t))}{\sum\limits_{k=1}^{n} \bar{w}_k(\hat{\omega}(t))}$, $\bar{w}_r(\hat{\omega}(t)) = \prod_{j=1}^{\kappa} v_{rj}(\hat{\omega}(t)) : R^\kappa \to [0, 1](r = 1, 2, \cdots, n)$ denotes the grade of membership of $\hat{\omega}(t)$ in v_{rj} with respect to the fuzzy rule r. ϱ_r can be regarded as the normalized weight of each **IF-THEN** rule, satisfying

$$\varrho_r(\hat{\omega}(t)) \geqslant 0, \quad \sum_{r=1}^{n} \varrho_r(\hat{\omega}(t)) = 1.$$

Remark 1. Since the important economic parameters, such as savings, the unit investment cost, and the elasticity of commodity demand, usually come from some small sample statistics, we hope that the financial system can be stabilized in a certain range of parameter ambiguity.

Since some stochastic disturbance factors in the real financial market, such as the input and outflow of foreign capital, these additional funds will produce immediate and delayed stochastic disturbance to some important financial parameters, such as savings, the unit investment cost, and the elasticity of commodity demand. So, in this paper, we have to consider the following delayed feedback stochastic financial system:

$$\begin{cases} dX(t) = \Big[\sum_{r=1}^{n} \varrho_r(\hat{\omega}(t)) H_{\sigma r} X(t) + f(X(t)) + v(t) \Big] dt + \varphi_\sigma(X(t), X(t - \tau(t))) dw(t), & t \geqslant 0 \\ X(s) = \xi(s), & s \in [-\tau, 0] \end{cases} \tag{2.17}$$

where v is the external input, time delay $\tau(t)$ satisfies $\tau(t) \in [0, \tau]$, and $X(t - \tau(t))$ is the delayed feedback state variable. The noise perturbation $\varphi : R^3 \times R^3 \to R^{3\times3}$ is a Borel measurable function with $\varphi_\sigma(X(t), X(t - \tau(t))) = (\varphi_1, \varphi_2, \varphi_3)^T$ and $\varphi_i = (\varphi_{\sigma i1}(X_i(t), X_i(t - \tau(t))), \varphi_{\sigma i2}(X_i(t), X_i(t - \tau(t))), \varphi_{\sigma i3}(X_i(t), X_i(t - \tau(t))))$, and $w(t) = (w_1(t), w_2(t), w_3(t))^T \in R^3$ is a 3-dimensional Brownian motion defined on a complete probability space $(\Upsilon, \mathcal{F}, \mathbb{P})$ with a natural filtration $\{F_t\}_{t\geqslant0}$ generated by $\{w(s) : 0 \leqslant s \leqslant t\}$, where we associate Υ with the canonical space generated by $w(t)$, and denoted by $\mathcal{F}$ the associated σ-algebra generated by $w(t)$ with the probability measure $\mathbb{P}$, and $\mathbb{E}\{dw(t)\} = 0$. Besides, $\mathbb{E}\{dw_i(t)dw_j(t)\} = dt$ if $i = j$, and $\mathbb{E}\{dw_i(t)dw_j(t)\} = 0$ if $i \neq j$, where $i, j = 1, 2, 3$.

If the fuzzy factors are ignored, the system (2.17) is degenerated into the following system:

$$\begin{cases} dX(t) = \Big[H_\sigma X(t) + f(X(t)) + v(t) \Big] dt + \varphi_\sigma(X(t), X(t - \tau(t))) dw(t), & t \geqslant 0 \\ X(s) = \xi(s), & s \in [-\tau, 0] \end{cases} \tag{2.18}$$

Definition 1 ([32, Definition 3]). The switched system (2.16) is said to be globally asymptotically stable in the meaning of switching if the zero solution of the switched system (2.18) is be globally asymptotically stable. Particularly, in the case of $\mathfrak{T} = \{1\}$, the switched system (2.16) is globally asymptotically stable (in the classical meaning) if the zero solution of the switched system (2.16) is be globally asymptotically stable.

Remark 2. The globally asymptotical stability in the meaning of switching means that when the time $t > 0$ is big enough, and the system is switched into any ith subsystem, the dynamic of the subsystem must approach its equilibrium point P_i. For example, the financial crisis breaks out, the financial system is switched into a subsystem, say, the 2th subsystem whose equilibrium point is P_2, and the dynamic of the subsystem must approach P_2 if the time $t > 0$ is big enough. Moreover, if the stability is local, then the dynamic of the subsystem may not approach P_2 due to various different initial values. Besides, the global financial crisis often erupts periodically, which illuminates that the global stability in the classical sense is actually meaningless. So the stability in the meaning of switching proposed in this paper is suitable and appropriate. Note that Peng Li and Xiaodi Li presented the definition of global asymptotical input-to-state stability under the assumption that v is a measurable locally bounded disturbance input ([44]). Similarly, in this paper, v is also a measurable bounded disturbance input, satisfying the bounded assumption (3.1).

Definition 2. System (2.18) is said to be global asymptotical input-to-state stability if there exist functions $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that for each $t_0 \geqslant 0$, $\xi(s) \in R^3$ and each input v, the solution satisfies

$$|X(t)| \leqslant \beta(\|\xi(s)\|_\tau, t - t_0) + \gamma(\|v\|), \quad \forall t \geqslant t_0. \tag{2.19}$$

In this paper, we assume

(A1) There are constant symmetric matrices $\Lambda_{\sigma 1}$ and $\Lambda_{\sigma 2}$ with $\Lambda_{\sigma 2} < \alpha_\sigma I$ such that

$$\text{trace}[\varphi_\sigma^T(u, v)\varphi_\sigma(u, v)] \leqslant u^T \Lambda_{\sigma 1} u + v^T \Lambda_{\sigma 2} v,$$

where I represents the identity matrix.

The following estimate condition for fuzzy parameters is considered for the fuzzy system (2.17):

(A2) For any given mode σ, there exists correspondingly a constant matrix E_σ such that

$$\sum_{r=1}^n \varrho_r(\hat{\omega}(t))\left(H_{\sigma r} + H_{\sigma r}^T\right) < \frac{1}{n} \sum_{r=1}^n \left(H_{\sigma r} + H_{\sigma r}^T\right) + E_\sigma + E_\sigma^T.$$

Remark 3. The fuzzy estimate condition is feasible, which can be verified in (5.3)–(5.5) of Example 5.2 below.

Lemma 1.1([28,29]). Let $\Omega \subset \mathbb{R}^N (N \in \mathbb{N})$ be a bounded domain with smooth boundary and let Δ denote the Laplacian in $L^s(\Omega)$ with domain

$$\{z \in W^{2,s}(\Omega) | \nabla \cdot v = 0 \text{ on } \partial\Omega\}$$

for $s \in (1, \infty)$. Then the operator $-\Delta + 1$ is sectorial and possesses closed fractional powers $(-\Delta + 1)^\delta, \delta \in (0, 1)$, with dense domain $D((-\Delta + 1)^\delta)$. Moreover, the following three properties hold.

(i) If $m \in \{0, 1\}, p \in [1, \infty]$ and $q \in (1, \infty)$, then there exists a constant $C_1 > 0$ such that for all $z \in D((-\Delta + 1)^\delta)$,

$$\|z\|_{W^{m,p}(\Omega)} \leqslant C_1 \|(-\Delta + 1)^\delta z\|_{L^q(\Omega)},$$

(ii) Suppose $p \in [1, \infty)$. Then the associated heat semigroup $(e^{t\Delta})_{t \geqslant 0}$ maps $L^p(\Omega)$ into $D((-\Delta + 1)^\delta)$ in $L^p(\Omega)$, and there exist constants $C_2 > 0$ and $\lambda_2 > 0$ such that

$$\|(-\Delta + 1)^\delta e^{t(\Delta - 1)} z\|_{L^p(\Omega)} \leqslant C_2 t^{-\delta} e^{-\lambda_2 t} \|z\|_{L^p(\Omega)}$$

for all $z \in L^p(\Omega)$ and all $t > 0$.

3. Main result in the case of non-fuzzy factors

Define the switching law for the switched system (2.18) as follows,

switching law $\mathfrak{F}_a$: At each switching we determine the next mode according to the following minimum law :

$$\sigma(t) = \arg\min \sum_{r=1}^n \xi^T \begin{pmatrix} Q + PH_\sigma + H_\sigma^T P + (\varepsilon P + \varepsilon^{-1}\alpha_1^2 P) + \rho\Lambda_{\sigma 1} + P\overline{D}_\sigma + \overline{D}_\sigma^T P + \Psi & 0 \\ 0 & \rho\Lambda_{\sigma 2} - (1 - \alpha)Q \end{pmatrix} \xi.$$

($\mathfrak{F}_{a1}$) Choose the initial mode $\sigma(t) = i_0$, if $\xi(t_0) \in \Upsilon_{i_0}$.

($\mathfrak{F}_{a2}$) For each $t > t_0$, if $\sigma(t^-) = i$ and $\xi \in \Upsilon_i$, keep $\sigma(t) = i$. On the other hand, if $\sigma(t^-) = i$ but $\xi \notin \Upsilon_i$. i.e., hitting a switching surface, choose the next mode by applying the above-mentioned minimum law and begin to switch.

Here, Υ_σ is defined as follows,

$$\Upsilon_\sigma = \left\{ \xi \in R^6 \Big| \xi^T \begin{pmatrix} Q + PH_\sigma + H_\sigma^T P + (\varepsilon P + \varepsilon^{-1}\alpha_1^2 P) + \rho\Lambda_{\sigma 1} + P\overline{D}_\sigma + \overline{D}_\sigma^T P + \Psi & 0 \\ 0 & \rho\Lambda_{\sigma 2} - (1-\alpha)Q \end{pmatrix} \xi < 0 \right\},$$

where $\varepsilon > 0, \rho > 0$ are two given positive numbers, $P = diag(p_1, p_2, p_3)$ is a given positive definite diagonal matrix with $p_1 = p_2$, and $Q > 0, \Psi > 0$ are given symmetric matrices, respectively.

(A3) Since X_1 represents the interest rate which is actually bounded in real world, there exists the positive number α_1 such that $|X_1| \leqslant \alpha_1$.

(A4) There are constant matrices $\underline{D}_\sigma$ and $\overline{D}_\sigma$ such that the external input satisfies $v(t) = D_\sigma(t)X(t)$ with

$$\underline{D}_\sigma \leqslant D_\sigma(t) \leqslant \overline{D}_\sigma. \tag{3.1}$$

Theorem 3.1. Suppose the conditions (A1),(A3)and (A4) hold. Besides, $\dot{\tau}(t) \leqslant \alpha$, and there is a positive definite diagonal matrix $P \leqslant \rho I$ with $P = diag(p_1, p_2, p_3)$ with $p_1 = p_2$, a positive definite matrix Q, and a sequence of non-negative numbers $\beta_\sigma(\sigma \in \mathfrak{I})$ with $\sum_{\sigma \in \mathfrak{I}} \beta_\sigma = 1$ such that

$$\begin{pmatrix} \sum_{\sigma \in \mathfrak{I}} \beta_\sigma(PH_\sigma + H_\sigma^T P + \rho\Lambda_{\sigma 1} + P\overline{D}_\sigma + \overline{D}_\sigma^T P) + (\varepsilon P + \varepsilon^{-1}\alpha_1^2 P) + Q + \Psi & 0 \\ 0 & \rho \sum_{\sigma \in \mathfrak{I}} \Lambda_{\sigma 2} - (1-\alpha)Q \end{pmatrix} < 0, \tag{3.2}$$

then the zero solution of the system (2.18) is the global asymptotical input-to-state stability. That is, the switched system (2.18) is said to be the global asymptotical stability in the meaning of switching.

Proof. Consider the following Lyapunov-Krasovskii function

$$V(t, X) = X^T(t)PX(t) + \int_{t-\tau(t)}^{t} X^T(s)QX(s)ds.$$

The condition (A3) and $P = diag(p_1, p_2, p_3)$ with $p_1 = p_2$ yield

$$f(X) = \begin{pmatrix} f_1(X) \\ f_2(X) \\ f_3(X) \end{pmatrix} = \begin{pmatrix} X_1X_2 \\ -X_1^2 \\ 0 \end{pmatrix}, \quad X^T Pf(X) + [\sqrt{P}f(X)]^T(\sqrt{P}X) \leqslant \varepsilon X^T PX + \varepsilon^{-1}f^T(X)Pf(X), \ P = diag(p_1, p_2, p_3), \ p_1 = p_2$$

$$\Rightarrow X^T Pf(X) + f^T(X)PX \leqslant \varepsilon X^T PX + \varepsilon^{-1}[X_1^2(p_2X_1^2 + p_1X_2^2)] \leqslant \varepsilon X^T PX + \varepsilon^{-1}[\alpha_1^2(p_1X_1^2 + p_2X_2^2)] \leqslant X^T(\varepsilon P + \varepsilon^{-1}\alpha_1^2 P)X. \tag{3.3}$$

Let R_+ be the set of positive numbers. Denote by $C^{1,2}(R_+ \times R^n \to R_+)$ the family of nonnegative functions $V(t, X)$ on $R_+ \times R^n$ which are continuous once differentiable in t and twice differentiable in X. For each such V, one can define an operator $\mathcal{L}V$ associated with equations (2.18) as

$$\begin{aligned}
\mathcal{L}V =& V_t(t, X(t)) + V_X(t, X(t))\Big[H_\sigma X(t) + f(X(t)) + v(t)\Big] \\
& + \frac{1}{2}\mathrm{trace}[\varphi_\sigma^T(X(t), X(t - \tau(t)))V_{XX}(t, X(t))\varphi_\sigma(X(t), X(t - \tau(t)))] \\
\leqslant& X^T(t)QX(t) - (1 - \dot{\tau}(t))X^T(t - \tau(t))QX(t - \tau(t)) + 2X^T(t)P\Big[H_\sigma X(t) + f(X(t)) + D_\sigma(t)X(t)\Big] \\
& + \rho\,\mathrm{trace}[\varphi_\sigma^T(X(t), X(t - \tau(t)))\varphi_\sigma(X(t), X(t - \tau(t)))] \\
\leqslant& X^T(t)\Big[Q + PH_\sigma + H_\sigma^T P + (\varepsilon P + \varepsilon^{-1}\alpha_1^2 P) + \rho\Lambda_{\sigma 1} + P\overline{D}_\sigma + \overline{D}_\sigma^T P\Big]X(t) + X^T(t - \tau(t))\Big(\rho\Lambda_{\sigma 2} - (1 - \alpha)Q\Big)X(t - \tau(t)).
\end{aligned}$$
(3.4)

where

$$V_X(t, X(t)) = \left(\frac{\partial V(t, X(t))}{\partial X_1}, \frac{\partial V(t, X(t))}{\partial X_2}, \frac{\partial V(t, X(t))}{\partial X_3}\right), \quad V_{XX}(t, X(t)) = \left(\frac{\partial^2 V(t, X(t))}{\partial X_i \partial X_j}\right)_{3\times 3}.$$

Set $\xi = (X^T(t), X^T(t - \tau(t)))^T$, one can claim

$$\begin{aligned}
\mathcal{L}V \leqslant& \xi^T \begin{pmatrix} Q + PH_\sigma + H_\sigma^T P + (\varepsilon P + \varepsilon^{-1}\alpha_1^2 P) + \rho\Lambda_{\sigma 1} + P\overline{D}_\sigma + \overline{D}_\sigma^T P & 0 \\ 0 & \rho\Lambda_{\sigma 2} - (1 - \alpha)Q \end{pmatrix} \xi \\
<& \xi^T \begin{pmatrix} -\Psi & 0 \\ 0 & 0 \end{pmatrix} \xi < 0.
\end{aligned}$$
(3.5)

In fact, it follows by (3.2) and the proof by contradiction that $\bigcup_{\sigma \in \mathfrak{I}} \Upsilon_\sigma = R^6 \setminus \{0\}$. On one hand, when $\sigma(t^-) = i$ and $\xi(t) \in \Upsilon_i$, then keep $\sigma(t) = i$, and (3.5) holds. On the other hand, when $\sigma(t^-) = i$ and $\xi(t) \notin \Upsilon_i$, which means that the trajectory hits a switching surface, and is chosen by the minimum law (3.2). At the same time, $\xi \in \Upsilon_j (j \neq i)$, and then (3.5) holds.

Besides,

$$\begin{aligned}
dV(t) =& \mathcal{L}V(t)dt + V_X(t)\varphi_\sigma(X(t), X(t - \tau(t)))dw(t) \\
\leqslant& \xi^T(t)\Upsilon_\sigma \xi(t) + 2X^T(t)P\varphi_\sigma(X(t), X(t - \tau(t)))dw(t).
\end{aligned}$$
(3.6)

So it follows by (3.6) that

$$\begin{aligned}
\mathbb{E}V(t + \varepsilon) - \mathbb{E}V(t) =& \int_t^{t+\varepsilon} \mathbb{E}\mathcal{L}V(s)ds + \mathbb{E}\int_t^{t+\varepsilon} 2X^T(s)P\varphi_\sigma(X(s), X(s - \tau(s)))dw(s) \\
=& \int_t^{t+\varepsilon} \mathbb{E}\mathcal{L}V(s)ds,
\end{aligned}$$
(3.7)

which implies that

$$D^+\mathbb{E}V(t) < 0.$$
(3.8)

Moreover, it is obvious that

$$(\lambda_{\min}P)|X(t)|^2 \leqslant X^T(t)PX(t) \leqslant V(t) \leqslant (\lambda_{\max}P + \tau\lambda_{\max}Q)\|X_t\|_\tau,$$
(3.9)

where

$$\|X_t\|_\tau = \sup_{s \in [-\tau, 0]} |X(t + s)|.$$

Combining (3.8) and (3.9) results in that the zero solution of the system (2.18) is the global asymptotical input-to-state stability. That is, the switched system (2.18) is said to be the global asymptotical stability in the meaning of switching.

$\square$

Remark 4. The global stability in the meaning of switching has practical significance. In fact, the global financial crisis often erupts periodically, which illuminates that the global stability in the classical sense is actually meaningless. For example, in [45], there were only some numerical simulations and stability analysis of dynamics of the delayed feedback financial system. Besides, in [46] and [47], only the stability and instability of some equilibrium points of financial system were analyzed. And in [7], only global stabilization of an equilibrium point of the financial system was considered. All the results obtained in the mentioned literature are not suitable for real finance, for economic crisis always appears periodically, which is the essence of financial market in the real world. However, the switching system model of this paper accords with this essential characteristic. Therefore, Theorem 3.1 has this novelty and originality.

4. Main result in the case of fuzzy system

Define the switching law as follows,
switching law $\mathfrak{F}$: At each switching we determine the next mode according to the following minimum law :

$$\sigma(t) = \arg\min \sum_{r=1}^{n} X^T \left[\frac{1}{n} \sum_{r=1}^{n} \left(H_{\sigma r} + H_{\sigma r}^T \right) + E_\sigma + E_\sigma^T + \Lambda_{\sigma 1} + e^{\varsigma \tau} q \alpha_\sigma I + \overline{D}_\sigma + \overline{D}_\sigma^T + \Psi \right] X. \qquad (4.1)$$

$(\mathfrak{F}_1)$ Choose the initial mode $\sigma(t) = i_0$, if $X(t_0) \in \Upsilon_{i_0}$.

$(\mathfrak{F}_2)$ For each $t > t_0$, if $\sigma(t^-) = i$ and $X \in \Upsilon_i$, keep $\sigma(t) = i$. On the other hand, if $\sigma(t^-) = i$ but $X \notin \Upsilon_i$. i.e., hitting a switching surface, choose the next mode by applying (4.1) and begin to switch.

Here, Ψ is a positive definite symmetric matrix with $\lambda_{\min} \Psi = \lambda > 0$, and Υ_σ is defined as follows,

$$\Upsilon_\sigma = \left\{ X \in R^3 \Big| X^T \left(\frac{1}{n} \sum_{r=1}^{n} \left(H_{\sigma r} + H_{\sigma r}^T \right) + E_\sigma + E_\sigma^T + \Lambda_{\sigma 1} + e^{\varsigma \tau} q \alpha_\sigma I + \overline{D}_\sigma + \overline{D}_\sigma^T + \Psi \right) X < 0 \right\},$$

where $\lambda_{\min} \Psi$ represents the minimum of all the eigenvalues of the symmetric matrix $\Psi > 0$.

Theorem 4.1. Suppose that the conditions (A1),(A2) and (A4) hold, and there is a sequence of nonnegative constants β_σ ($\sigma = 1, 2, 3$) with $\sum_{\sigma=1}^{3} \beta_\sigma = 1$ and $0 \leqslant \beta_\sigma \leqslant 1$ and positive constants $\varsigma > 0$ and $q > 1$ such that

$$\sum_{\sigma=1}^{3} \beta_\sigma \left[\frac{1}{n} \sum_{r=1}^{n} \left(H_{\sigma r} + H_{\sigma r}^T \right) + E_\sigma + E_\sigma^T + \Lambda_{\sigma 1} + e^{\varsigma \tau} q \alpha_\sigma I + \overline{D}_\sigma + \overline{D}_\sigma^T \right] + \Psi < 0, \qquad (4.2)$$

then the fuzzy switched system (2.17) is globally asymptotical input-to-state stability.

Proof. Consider the following Lyapunov function:

$$V(t) = V(t, X(t)) = X^T(t)X(t).$$

Let R_+ be the set of positive numbers. Denote by $C^{1,2}(R_+ \times R^n \to R_+)$ the family of nonnegative functions $V(t, X)$ on $R_+ \times R^n$ which are continuous once differentiable in t and twice differentiable in X. For each such V, we define an operator $\mathcal{L}V$ associated with equations (2.17) as

$$
\begin{aligned}
\mathcal{L}V =& V_t(t, X(t)) + V_X(t, X(t))[(\sum_{r=1}^{n} \varrho_r(\hat{\omega}(t))H_{\sigma r})X(t) + f(X(t)) + v(t)] \\
& + \frac{1}{2}\text{trace}[\varphi_\sigma^T(X(t), X(t - \tau(t)))V_{XX}(t, X(t))\varphi_\sigma(X(t), X(t - \tau(t)))] \\
\leqslant& X^T(t)\Big(\frac{1}{n}\sum_{r=1}^{n}\Big(H_{\sigma r} + H_{\sigma r}^T\Big) + E_\sigma + E_\sigma^T + \overline{D}_\sigma + \overline{D}_\sigma^T\Big)X(t) \\
& + \Big(X^T(t)\Lambda_{\sigma 1}X(t) + X^T(t - \tau(t))\Lambda_{\sigma 2}X(t - \tau(t))\Big),
\end{aligned}
$$

where

$$V_X(t, X(t)) = \left(\frac{\partial V(t, X(t))}{\partial X_1}, \frac{\partial V(t, X(t))}{\partial X_2}, \frac{\partial V(t, X(t))}{\partial X_3}\right), \quad V_{XX}(t, X(t)) = \left(\frac{\partial^2 V(t, X(t))}{\partial X_i \partial X_j}\right)_{3\times 3}.$$

Let γ be a positive number with $0 < \gamma < \min\{\lambda, \varsigma, \frac{1}{2}\lambda_2\}$ and

$$U(t, X(t)) = \begin{cases} e^{\gamma t}X^T(t)X(t), & t \geqslant 0, \\ X^T(t)X(t), & t \in [-\tau, 0], \end{cases}$$

where $\lambda_2 > 0$ is the first positive eigenvalue of the following Neumann boundary problem

$$
\begin{cases}
(-\Delta + 1)z = \lambda z, & \text{in } \Omega, \\
\partial_\nu z = 0, & \text{on } \partial\Omega,
\end{cases}
\tag{4.3}
$$

and Ω is a bounded domain in R^3 with smooth boundary $\partial\Omega$.

It is obvious that U is continuous for $t \geqslant -\tau$. For $t \geqslant 0$ and $\gamma > 0$,

$$dU(t, X(t)) = e^{\gamma t}\Big(\gamma X^T(t)X(t)dt + d[X^T(t)X(t)]\Big).$$

Ito formula yields

$$dV(t) = \mathcal{L}V(t)dt + V_X(t)\varphi_\sigma(X(t), X(t - \tau(t)))dw(t),$$

and hence

$$dU(t, X(t)) = e^{\gamma t}\Big(\gamma X^T(t)X(t)dt + \mathcal{L}V(t)dt + V_X(t)\varphi_\sigma(X(t), X(t - \tau(t)))dw(t)\Big).$$

Integrating the above inequality from t to $t + \varepsilon$, and taking the mathematical expectation, one may derive that, for all $t \geqslant 0$ and any $\varepsilon > 0$,

$$\mathbb{E}U(t + \varepsilon, X(t + \varepsilon)) - \mathbb{E}U(t, X(t)) = \mathbb{E}\int_{\varepsilon}^{t+\varepsilon} e^{\gamma s}\Big(\gamma X^T(s)X(s) + \mathcal{L}V(s)\Big)ds$$

and then

$$D^+\mathbb{E}U(t, X(t)) = e^{\gamma t}\Big(\gamma\mathbb{E}[X^T(t)X(t)] + \mathbb{E}\mathcal{L}V(t)\Big) \tag{4.4}$$

Now we claim that there is a positive constants $C_0 > 1$ and $K \in R$ with $K > 1$ such that

$$\mathbb{E}U(t, X(t)) \leqslant KC_0\mathbb{E}\|\xi\|_\tau^2, \quad \forall\, t \geqslant 0, \tag{4.5}$$

where $\|\xi\|_\tau = \sup\limits_{s\in[-\tau,0]} \sqrt{\xi^T(s)\xi(s)}$.

Indeed, suppose this assertion is not true, then it is not difficult to prove (see, e.g. [7]) that there exists positive constant q with $K \geqslant q > 1$ such that

$$\mathbb{E}U(t, X(t)) \leqslant KC_0\mathbb{E}\|\xi\|_\tau^2, \quad t \in [-\tau, t^*] \tag{4.6}$$

and

$$\mathbb{E}U(t^{**}, X(t^{**})) = \frac{1}{q}KC_0\mathbb{E}\|\xi\|_\tau^2\,;\ \mathbb{E}U(t^{**}, X(t^{**})) \leqslant \mathbb{E}U(t, X(t)) \leqslant \mathbb{E}U(t^*, X(t^*)) = KC_0\mathbb{E}\|\xi\|_\tau^2, \quad \forall\, t \in [t^{**}, t^*]. \tag{4.7}$$

It follows from (4.6), (4.7) and the definition of $U(t, X(t))$ that for $s \in [-\tau, 0]$ and $t \in [t^{**}, t^*]$

$$\begin{aligned}
e^{s^s}\mathbb{E}[X^T(t+s)X(t+s)] &\leqslant e^{\gamma s}\mathbb{E}[X^T(t+s)X(t+s)]\\
&= \begin{cases} e^{-\gamma t}\mathbb{E}U(t+s, X(t+s)), & t+s \geqslant 0 \\ e^{\gamma s}\mathbb{E}U(t+s, X(t+s)), & t+s \leqslant 0 \end{cases}\\
&\leqslant e^{-\gamma t}\mathbb{E}U(t+s, X(t+s))\\
&\leqslant e^{-\gamma t}qe^{\gamma t}\mathbb{E}[X^T(t)X(t)]\\
&= q\mathbb{E}[X^T(t)X(t)],
\end{aligned} \tag{4.8}$$

which yields that for any $s \in [-\tau, 0]$,

$$\mathbb{E}[X^T(t-\tau(t))X(t-\tau(t))] \leqslant e^{s\tau}q\mathbb{E}X^T(t)X(t). \tag{4.9}$$

Combining (4.9) and (A1) results in

$$\mathbb{E}[X^T(t-\tau(t))\Lambda_{\sigma 2}X(t-\tau(t))] \leqslant e^{s\tau}q\alpha_\sigma\mathbb{E}[X^T(t)X(t)], \tag{4.10}$$

which derives

$$\begin{aligned}
\mathbb{E}\mathcal{L}V &\leqslant \mathbb{E}[X^T(t)\Big(\frac{1}{n}\sum_{r=1}^{n}\Big(H_{\sigma r} + H_{\sigma r}^T\Big) + E_\sigma + E_\sigma^T + \overline{D}_\sigma + \overline{D}_\sigma^T\Big)X(t)]\\
&\quad + \mathbb{E}\Big(X^T(t)\Lambda_{\sigma 1}X(t) + X^T(t-\tau(t))\Lambda_{\sigma 2}X(t-\tau(t))\Big)\\
&\leqslant \mathbb{E}\Big[X^T(t)\Big(\frac{1}{n}\sum_{r=1}^{n}\Big(H_{\sigma r} + H_{\sigma r}^T\Big) + E_\sigma + E_\sigma^T + \Lambda_{\sigma 1} + e^{s\tau}q\alpha_\sigma I + \overline{D}_\sigma + \overline{D}_\sigma^T\Big)X(t)\Big]
\end{aligned} \tag{4.11}$$

For any given $t \geqslant t_0$, according to the switching law $\mathfrak{F}$, when $\sigma(t^-) = i$ and $X(t) \in \Upsilon_i$, then keep $\sigma(t) = i$, and we can conclude that

$$\begin{aligned}
\mathbb{E}\mathcal{L}V(t, X(t)) &\leqslant \mathbb{E}\Big[X^T(t)\Big(\frac{1}{n}\sum_{r=1}^{n}\Big(H_{\sigma r} + H_{\sigma r}^T\Big) + E_\sigma + E_\sigma^T + \Lambda_{\sigma 1} + e^{s\tau}q\alpha_\sigma I + \overline{D}_\sigma + \overline{D}_\sigma^T\Big)X(t)\Big]\\
&\leqslant \mathbb{E}\Big[X^T(t)(-\Psi)X(t)\Big] \leqslant -\lambda\mathbb{E}[X^T X] = -\lambda\mathbb{E}V(t, X(t)).
\end{aligned} \tag{4.12}$$

When $\sigma(t^-) = i$ and $X(t) \notin \Upsilon_i$, which means that the trajectory hits a switching surface. On the other hand, it is not difficult to deduce from (4.2) that $\bigcup_{i=1}^{3} \Upsilon_i = R^3 \setminus \{0\}$, which together with the minimum law (4.1) yields (4.12), too.

(4.12) together with (4.4) implies

$$
\begin{aligned}
D^+\mathbb{E}U(t, X(t)) &= e^{\gamma t}\Big(\gamma\mathbb{E}[X^T(t)X(t)] + \mathbb{E}\mathcal{L}V(t)\Big) \\
&\leqslant e^{\gamma t}\Big((\gamma - \lambda)\mathbb{E}[X^T(t)X(t)]\Big) = (\gamma - \lambda)\mathbb{E}U(t, X(t)) \leqslant 0,
\end{aligned}
\tag{4.13}
$$

which derives that $\mathbb{E}U(t^*, X(t^*)) \leqslant \mathbb{E}U(t^{**}, X(t^{**}))$. However, (4.7) yields

$$
\mathbb{E}U(t^{**}, X(t^{**})) = \frac{1}{q}KC_0\mathbb{E}\|\xi\|_\tau^2 < KC_0\mathbb{E}\|\xi\|_\tau^2 = \mathbb{E}U(t^*, X(t^*)).
$$

So we have prove (4.5), which means

$$
e^{\gamma t}X^T(t)X(t) \leqslant KC_0\mathbb{E}\|\xi\|_\tau^2, \quad \forall\, t \geqslant 0,
$$

or

$$
X^T(t)X(t) \leqslant KC_0\mathbb{E}\|\xi\|_\tau^2 e^{-\gamma t}, \quad \forall\, t \geqslant 0,
$$

which together with Definition 2 implies that the fuzzy switched system (2.17) is globally asymptotical input-to-state stability. $\qquad\square$

Remark 5. It is the first paper to design and investigate the stability of switched fuzzy delayed feedback financial system with stochastic perturbation, which is in line with the fact that a prosperous and stable financial market may not necessarily pursue a equilibrium point with a fixed interest rate. In most cases, financial market is more stable through switching law under macro control and market self-regulation.

Remark 6. Although our ordinary differential equations model successfully explains the financial market stability criterion under switching law, state variables are still related to regions, such as certain countries and regions. So we consider the partial differential equations model for the financial system (2.17).

$$
\begin{cases}
du(t, X) = \Delta u(t, X)dt + [\sum_{r=1}^{n} \varrho_r(\hat{\omega}(t))H_{\sigma r}u(t, X) + f(u(t, X)) - v(t, X)]dt \\
\qquad\qquad + \varphi_\sigma(u(t, X), u(t - \tau(t), X))dw(t), \quad t \geqslant 0, X \in \Omega \\
u(s, X) = \xi(s, X), \quad s \in [-\tau, 0], X \in \Omega \\
\dfrac{\partial u_i}{\partial X_j} = 0, \quad X_j \in \partial\Omega, i, j = 1, 2, 3,
\end{cases}
\tag{4.14}
$$

where Δ is the Laplace operator(see, e.g. [28, 30,31]).

Remark 7. Considering that the interest rate, the investment demand and the price index are usually invariable in a country, we see, the state variable $u(t, X) = (u_1(t, X), u_2(t, X), u_3(t, X))^T$ is actually

uniform with respect to the region, which implies that the system (4.14) is exactly the system (2.17). That is, $u_i(t, X)$ of the system (4.14) is exactly equivalent to $X_i(t)$ in the system (2.17).

Remark 8. From [40, Lemma 2.3] in the case of $p = 2$, we know that the fuzzy system (4.14) is exponentially stable under the same assumptions of Theorem 4.1.

To obtain the boundedness criterion of the system (4.14) or (2.17), we need the following assumption on the Laplace operator.

(A3) There exist $M > 0$ and $\zeta > 0$ such that $\|e^{t\Delta}\| \leq Me^{-\zeta t}$.

Below, we assume that the condition (A3) holds. Besides, due to the meaning of Remark 7, we propose the boundedness definition on the system (2.17) or (4.14).

Definition 3. The fuzzy system (2.17) is said to be bounded under the meaning of L^∞ if for any given $T > \tau_1 > 0$ such that for all $t \in [\tau_1, T]$, the each state variable $u_i(t, X)$ of the system (4.14) satisfies

$$\|u_i(t, \cdot)\|_{L^\infty(\Omega)} \leq C, \quad \forall i = 1, 2, 3.$$

Remark 9. The initial function $\xi(s)$ is bounded for $s \in [-\tau, 0]$,

Theorem 4.2. Under the assumptions of Theorem 4.1, the fuzzy system (2.17) is bounded under the meaning of L^∞.

In order to verify the correctness of Theorem 4.2, we may firstly present the following technical lemma with regard to the system (4.14).

Lemma 4.3. Under the same assumptions of Theorem 4.1, the fuzzy system (4.14) is bounded under the meaning of L^∞.

Proof. For the convenience of the proof, we may rewrite the system (4.14) as follows,

$$\begin{cases} du_i(t, X) = \Delta u_i(t, X)dt + [\sum_{r=1}^{n} \varrho_r(\hat{\omega}(t)) \sum_{j=1}^{3} h_{\sigma r i j} u_j(t, X) + f_i(u(t, X)) + v_i(t, X)]dt \\ + \sum_{j=1}^{3} \varphi_{\sigma i j}(u_i(t, X), u_i(t - \tau(t), X))dw_j(t), \quad t \geq 0, X \in \Omega \\ u(s, X) = \xi(s, x), \quad s \in [-\tau, 0], x \in \Omega \\ \dfrac{\partial u_i}{\partial X_j} = 0, \quad X_j \in \partial\Omega, i, j = 1, 2, 3, \end{cases} \quad (4.15)$$

where $H_{\sigma r} = (h_{\sigma r i j})_{3 \times 3}$.

It follows from the system (4.15) that for any given $\tau_1 > 0$,

$$\begin{cases} du_i(t, X) = [\Delta u_i(t, X) - u_i(t, X)]dt + \Big[\sum_{r=1}^{n} \varrho_r(\hat{\omega}(t)) \sum_{j=1}^{3} h_{\sigma r i j} u_j(t, X) + f_i(u(t, X)) \\ \quad + v_i(t, X) + u_i(t, X) \Big]dt + \sum_{j=1}^{3} \varphi_{\sigma i j}(u_i(t, X), u_i(t - \tau(t), X))dw_j(t), \quad t \geqslant 0, X \in \Omega \\ u(s, X) = \xi(s, X), \quad s \in [-\tau, 0], x \in \Omega \\ \dfrac{\partial u_i}{\partial X_j} = 0, \quad X_j \in \partial\Omega, i, j = 1, 2, 3, \end{cases}$$

which implies that for any given $\tau_1 > 0$,

$$\begin{aligned}
u_i(t, X) =& e^{t(\Delta-1)}u_i(\tau_1, X) + \int_{\tau_1}^{t} e^{(t-s)(\Delta-1)} \sum_{r=1}^{n} \varrho_r(\hat{\omega}(s)) \sum_{j=1}^{3} h_{\sigma r i j} u_j(s, X)ds \\
&+ \int_{\tau_1}^{t} e^{(t-s)(\Delta-1)}[f_i(u(s, X)) + v_i(t, X) + u_i(s, X)]ds \\
&+ \int_{\tau_1}^{t} e^{(t-s)(\Delta-1)} \sum_{j=1}^{3} \varphi_{\sigma i j}(u_i(s, X), u_i(s - \tau(s), X))dw_j(s).
\end{aligned} \tag{4.16}$$

We can see it from the proof of Theorem 4.1 that

$$e^{\gamma t} X^T(t) X(t) \leqslant K C_0 \mathbb{E}\|\xi\|_\tau^2, \quad \forall\, t \geqslant 0,$$

or

$$e^{\gamma t} \sum_{i=1}^{3} X_i^2 \leqslant K C_0 \mathbb{E}\|\xi\|_\tau^2, \quad \forall\, t \geqslant 0,$$

which implies that

$$e^{2\gamma t} f_i^2(X_i) \leqslant e^{2\gamma t} f^T(X) f(X) = e^{2\gamma t}(X_1^2 X_2^2 + X_1^4) \leqslant e^{2\gamma t} \sum_{i=1}^{3} X_i^2 \sum_{i=1}^{3} X_i^2 \leqslant (K C_0 \mathbb{E}\|\xi\|_\tau^2)^2$$

then we know from Remark 8 that

$$e^{\gamma t} \sum_{i=1}^{3} \|u_i(t, X)\|_{L^2(\Omega)}^2 \leqslant C, \text{ and } e^{2\gamma t}\|f_i^2(u_i(t, X))\|_{L^2(\Omega)}^2 \leqslant C^2, \quad t \geqslant 0, \tag{4.17}$$

where $2\gamma < \lambda_2$, and $C > 0$ is a constant which is big enough.

Since C is big enough, we can make a series of estimates with Lemma 1.1,

$$\begin{aligned}
&\mathbb{E}\|e^{t\Delta}u_i(\tau_1, X)\|_{L^\infty(\Omega)} \\
&\leqslant C_1 C_2 M t^{-\delta} e^{-\lambda_2 t} \mathbb{E}\|u_i(\tau_1, X)\|_{L^2(\Omega)} \leqslant C,
\end{aligned} \tag{4.18}$$

where $\delta \in (\frac{1}{3}, \frac{2}{3})$, and λ_2 is defined in (4.3).

$$\mathbb{E}\| \int_{\tau_1}^t e^{(t-s)(\Delta-1)} \sum_{r=1}^n \varrho_r(\hat{\omega}(s)) \sum_{j=1}^3 h_{\sigma rij} u_j(s, X) ds\|_{L^\infty(\Omega)}$$

$$\leqslant \sum_{r=1}^n \sum_{j=1}^3 |h_{\sigma rij}| C_1 C_2 M \mathbb{E} \int_{\tau_1}^t (t-s)^{-\delta} e^{-\lambda_2(t-s)} \|u_j(s, X)\|_{L^2(\Omega)} ds \qquad (4.19)$$

$$\leqslant \sum_{r=1}^n \sum_{j=1}^3 |h_{\sigma rij}| C_1 C_2 C.$$

Similarly, we can get by (4.17)

$$\mathbb{E}\| \int_{\tau_1}^t e^{(t-s)(\Delta-1)} [f_i(u(s, X)) + v_i(t, X) + u_i(s, X)] ds\|_{L^\infty(\Omega)}$$

$$\leqslant C_1 C_2 M \mathbb{E} \int_{\tau_1}^t (t-s)^{-\delta} e^{-\lambda_2(t-s)} \|f_i(u(s, X))\|_{L^2(\Omega)} ds \qquad (4.20)$$

$$+ C_1 C_2 M \mathbb{E} \int_{\tau_1}^t (t-s)^{-\delta} e^{-\lambda_2(t-s)} (\|v_i(s, X)\|_{L^2(\Omega)} + \|u_i(s, X)\|_{L^2(\Omega)}) ds$$

$$\leqslant C_1 C_2 C^2 + C_1 C_2 C^2$$

Since the constant C is big enough, the condition (A1) yields

$$(\sum_{j=1}^3 \varphi_{\sigma ij}(u_i(s, X), u_i(s - \tau(s), X)))^2 \leqslant C \sum_{i=1}^3 [u_i^2(s, X) + u_i^2(s - \tau(s), X)].$$

Moreover, since the constant C is big enough, we can estimate

$$\mathbb{E}\| \int_{\tau_1}^t e^{(t-s)(\Delta-1)} \sum_{j=1}^3 \varphi_{\sigma ij}(u_i(s, X), u_i(s - \tau(s), X)) dw_j(s)\|_{L^\infty(\Omega)}$$

$$\leqslant C\mathbb{E} \sqrt{C_1 \int_{\tau_1}^t \left\| (-\Delta + 1)^\delta e^{2(t-s)(\Delta-1)} \sum_{i=1}^3 [u_i^2(s, X) + u_i^2(s - \tau(s), X)] \right\|_{L^2(\Omega)} ds} \qquad (4.21)$$

$$\leqslant C \sum_{i=1}^3 \mathbb{E} \sqrt{C_1 C_2 M \int_{\tau_1}^t (2t - 2s)^{-\delta} e^{-2\lambda_2(t-s)} [\|u_i^2(s, X)\|_{L^2(\Omega)} + \|u_i^2(s - \tau(s), X)\|_{L^2(\Omega)}] ds}$$

$$\leqslant C^2.$$

Finally, combining (4.18)–(4.21) results in

$$\|u_i(t, X)\|_{L^\infty(\Omega)} \leqslant C^3, \ i = 1, 2, 3.$$

And then the proof is completed.

$\square$

Now we can know from Remark 7 that Theorem 4.2 has been proved.

5. Numerical example

Example 5.1. Let $\mathfrak{T} = \{1, 2\}$ in the system (2.18).

In the case $\sigma = 1$,

$$\begin{cases} dX(t) = \left[H_1 X(t) + f(X(t)) + v(t)\right]dt + \varphi_\sigma(X(t), X(t - \tau(t)))dw(t), & t \geqslant 0 \\ X(s) = \xi(s), & s \in [-\tau, 0] \end{cases} \quad (5.1a)$$

In the case $\sigma = 2$,

$$\begin{cases} dX(t) = \left[H_2 X(t) + f(X(t)) + v(t)\right]dt + \varphi_\sigma(X(t), X(t - \tau(t)))dw(t), & t \geqslant 0 \\ X(s) = \xi(s), & s \in [-\tau, 0] \end{cases} \quad (5.1b)$$

Let $a = 2, b = 0.4, c = 2$, then

$$H_1 = \begin{pmatrix} \frac{1}{c} & \theta & 1 \\ -2\theta & -b & 0 \\ -1 & 0 & -c \end{pmatrix} = \begin{pmatrix} 0.5000 & 0 & 1.0000 \\ 0 & -0.4000 & 0 \\ -1.0000 & 0 & -2.0000 \end{pmatrix} = \begin{pmatrix} \frac{1}{b} - a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix} = H_2.$$

Besides, set $\alpha_1 = 0.85, \varepsilon = 0.5, \psi = 0.001I, \Lambda_{\sigma 1} \equiv 0.0001I \equiv \Lambda_{\sigma 2}$, and
and

$$\overline{D}_1 = 0, \ \overline{D}_2 = \begin{pmatrix} -2.5 & -1.5 & -1.2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using the computer Matlab LMI toolbox to solve the LMI condition (3.2) results in the following feasible data:

$$\beta_1 = 0.0107, \beta_2 = 0.9893, P = \begin{pmatrix} 0.9993 & 0 & 0 \\ 0 & 0.9993 & 0 \\ 0 & 0 & 0.9976 \end{pmatrix}; Q = \begin{pmatrix} 0.6873 & 0 & 0.0011 \\ 0 & 0.5987 & 0.0001 \\ 0.0011 & 0.0001 & 0.6676 \end{pmatrix}$$

Now, according to Theorem 3.1, the switched system (5.1) is said to be the global asymptotical stability in the meaning of switching (see Figure 1 and 2).
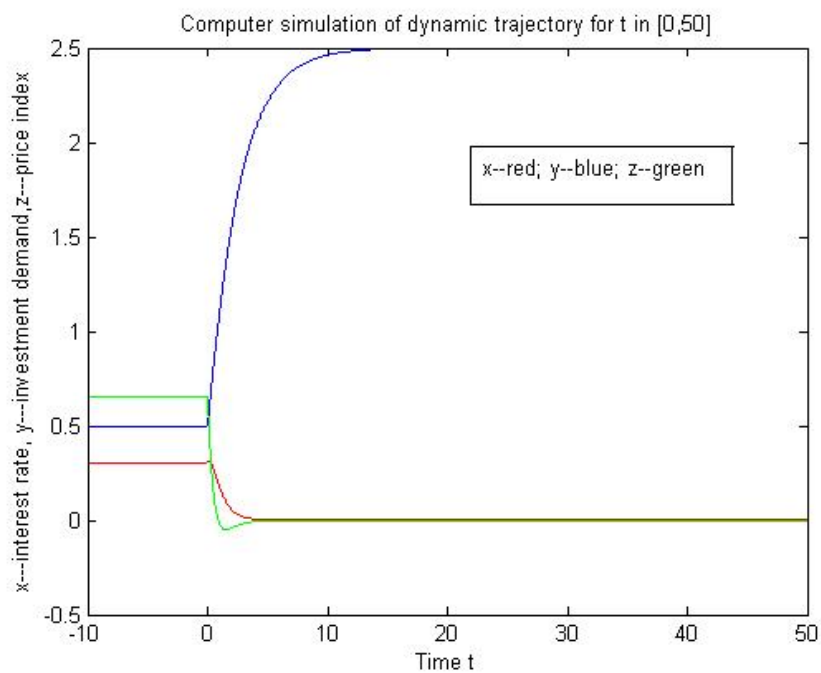
Figure 1. State trajectories of the interest rate (red), investment demand (blue), and price index (green).
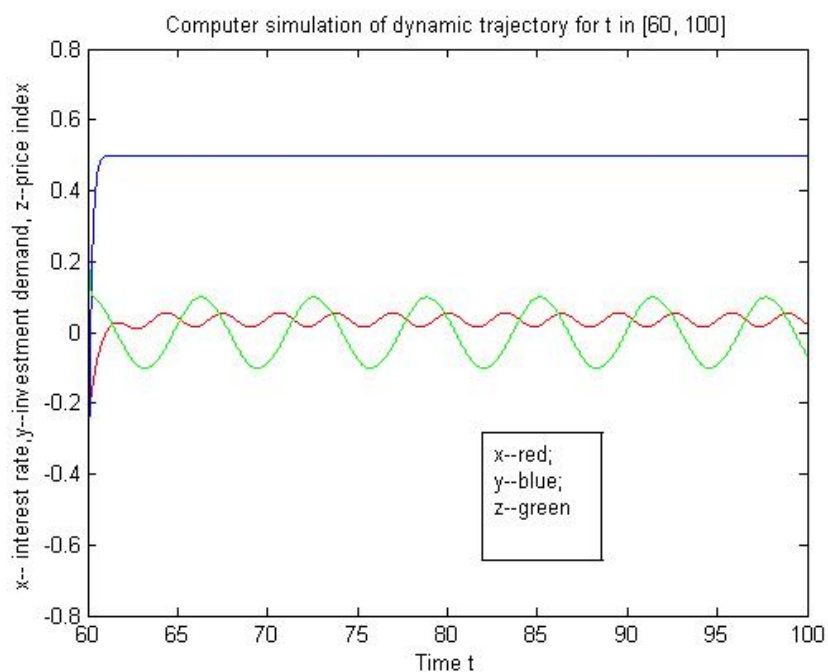

Figure 2. State trajectories of the interest rate (red), investment demand (blue), and price index (green).

Example 5.2. Consider the following fuzzy switched financial system with $n = 2$ (the number of the **IF-THEN** rules),

Fuzzy Rule 1: **IF** $\hat{\omega}_1(t)$ is v_{11} and $\hat{\omega}_2(t)$ is v_{12} **THEN**

$$\begin{cases} dX(t) = [H_{\sigma 1}X(t) + f(X(t)) + v(t)]dt + \varphi_\sigma(X(t), X(t - \tau(t)))dw(t), & t \geq 0 \\ X(s) = \xi(s), & s \in [-\tau, 0] \end{cases}$$ (5.2a)

In the case $r = 1$ equipped with $a = 0.9, b = 0.63, c = 1.5$, then

$$H_{11} = \begin{pmatrix} \frac{1}{c} & \theta & 1 \\ -2\theta & -b & 0 \\ -1 & 0 & -c \end{pmatrix} = \begin{pmatrix} 0.6667 & 0.1140 & 1.0000 \\ -0.2280 & -0.6300 & 0 \\ -1.0000 & 0 & -1.5000 \end{pmatrix}$$

$$H_{21} = \begin{pmatrix} \frac{1}{b} - a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix} = \begin{pmatrix} 0.6873 & 0 & 1.0000 \\ 0 & -0.6300 & 0 \\ -1.0000 & 0 & -1.5000 \end{pmatrix}$$

$$H_{31} = \begin{pmatrix} \frac{1}{c} & -\theta & 1 \\ 2\theta & -b & 0 \\ -1 & 0 & -c \end{pmatrix} = \begin{pmatrix} 0.6667 & -0.1140 & 1.0000 \\ 0.2280 & -0.6300 & 0 \\ -1.0000 & 0 & -1.5000 \end{pmatrix}$$

Fuzzy Rule 2: **IF** $\hat{\omega}_1(t)$ is v_{21} and $\hat{\omega}_2(t)$ is v_{22} **THEN**

$$\begin{cases} dX(t) = [H_{\sigma 2}X(t) + f(X(t)) + v(t)]dt + \varphi_\sigma(X(t), X(t - \tau(t)))dw(t), & t \geq 0 \\ X(s) = \xi(s), & s \in [-\tau, 0] \end{cases}$$ (5.2b)

In the case $r = 2$ equipped with $a = 0.8, b = 0.6, c = 1.2$, then

$$H_{12} = \begin{pmatrix} 0.8333 & 0.1414 & 1.0000 \\ -0.2828 & -0.6000 & 0 \\ -1.0000 & 0 & -1.2000 \end{pmatrix}$$

$$H_{22} = \begin{pmatrix} 0.8667 & 0 & 1.0000 \\ 0 & -0.6000 & 0 \\ -1.0000 & 0 & -1.2000 \end{pmatrix}$$

$$H_{32} = \begin{pmatrix} 0.8333 & -0.1414 & 1.0000 \\ 0.2828 & -0.6000 & 0 \\ -1.0000 & 0 & -1.2000 \end{pmatrix}$$

Set

$$E_1 = \begin{pmatrix} 0.1000 & 0.06385 & 0 \\ 0.06385 & 0.0500 & 0 \\ 0 & 0 & 0.3000 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0.0900 & 0 & 0 \\ 0 & 0.0160 & 0 \\ 0 & 0 & 0.1600 \end{pmatrix} \; E_3 = \begin{pmatrix} 0.1000 & -0.06385 & 0 \\ -0.06385 & 0.1150 & 0 \\ 0 & 0 & 0.3000 \end{pmatrix}$$

$$\overline{D}_1 = 0 = \overline{D}_3, \; \overline{D}_2 = \begin{pmatrix} -1.0544 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Due to $0 \leqslant \varrho_r(\hat{\omega}(t)) \leqslant 1$ and $\sum\limits_{r=1}^{2} \varrho_r(\hat{\omega}(t)) = 1$, we can get by computing

$$
\begin{aligned}
\sum_{r=1}^{2} \varrho_r(\hat{\omega}(t))\Big(H_{1r} + H_{1r}^T\Big) &= \varrho_1(\hat{\omega}(t))(H_{11} + H_{11}^T) + \varrho_2(\hat{\omega}(t))(H_{12} + H_{12}^T)] \\
&= \varrho_1(\hat{\omega}(t))\begin{pmatrix} 1.3333 & -0.1140 & 0 \\ -0.1140 & -1.2600 & 0 \\ 0 & 0 & -3.0000 \end{pmatrix} + \varrho_2(\hat{\omega}(t))\begin{pmatrix} 1.6667 & -0.1414 & 0 \\ -0.1414 & -1.2000 & 0 \\ 0 & 0 & -2.4000 \end{pmatrix} \\
&\leqslant \max_{s \in [0,1]}\left[s\begin{pmatrix} 1.3333 & -0.1140 & 0 \\ -0.1140 & -1.2600 & 0 \\ 0 & 0 & -3.0000 \end{pmatrix} + (1-s)\begin{pmatrix} 1.6667 & -0.1414 & 0 \\ -0.1414 & -1.2000 & 0 \\ 0 & 0 & -2.4000 \end{pmatrix}\right] \\
&< \begin{pmatrix} 1.7000 & 0 & 0 \\ 0 & -1.1300 & 0 \\ 0 & 0 & -2.1000 \end{pmatrix} \\
&= \begin{pmatrix} 1.5000 & -0.1277 & 0 \\ -0.1277 & -1.2300 & 0 \\ 0 & 0 & -2.7000 \end{pmatrix} + 2\begin{pmatrix} 0.1 & 0.06385 & 0 \\ 0.06385 & 0.0500 & 0 \\ 0 & 0 & 0.3 \end{pmatrix} \\
&= \frac{1}{2}\sum_{r=1}^{2}\Big(H_{1r} + H_{1r}^T\Big) + E_1 + E_1^T
\end{aligned}
\tag{5.3}
$$

Similarly, we get

$$
\begin{aligned}
\sum_{r=1}^{2} \varrho_r(\hat{\omega}(t))\Big(H_{2r} + H_{2r}^T\Big) &= \varrho_1(\hat{\omega}(t))(H_{21} + H_{21}^T) + \varrho_2(\hat{\omega}(t))(H_{22} + H_{22}^T) \\
&= \varrho_1(\hat{\omega}(t))\begin{pmatrix} 1.3746 & 0 & 0 \\ 0 & -1.2600 & 0 \\ 0 & 0 & -3.0000 \end{pmatrix} + \varrho_2(\hat{\omega}(t))\begin{pmatrix} 1.7333 & 0 & 0 \\ 0 & -1.2000 & 0 \\ 0 & 0 & -2.4000 \end{pmatrix} \\
&\leqslant \begin{pmatrix} 1.7333 & 0 & 0 \\ 0 & -1.2000 & 0 \\ 0 & 0 & -2.4000 \end{pmatrix} \\
&< \begin{pmatrix} 1.7339 & 0 & 0 \\ 0 & -1.1980 & 0 \\ 0 & 0 & -2.3800 \end{pmatrix} \\
&= \begin{pmatrix} 1.5539 & 0 & 0 \\ 0 & -1.2300 & 0 \\ 0 & 0 & -2.7000 \end{pmatrix} + 2\begin{pmatrix} 0.0900 & 0 & 0 \\ 0 & 0.0160 & 0 \\ 0 & 0 & 0.1600 \end{pmatrix} \\
&= \frac{1}{2}\sum_{r=1}^{2}\Big(H_{2r} + H_{2r}^T\Big) + E_2 + E_2^T
\end{aligned}
\tag{5.4}
$$

and

$$\sum_{r=1}^{2} \varrho_r(\hat{\omega}(t))\left(H_{3r} + H_{3r}^T\right) = \varrho_1(\hat{\omega}(t))(H_{31} + H_{31}^T) + \varrho_2(\hat{\omega}(t))(H_{32} + H_{32}^T)]$$

$$= \varrho_1(\hat{\omega}(t))\begin{pmatrix} 1.3333 & 0.1140 & 0 \\ 0.1140 & -1.2600 & 0 \\ 0 & 0 & -3.0000 \end{pmatrix} + \varrho_2(\hat{\omega}(t))\begin{pmatrix} 1.6667 & 0.1414 & 0 \\ 0.1414 & -1.2000 & 0 \\ 0 & 0 & -2.4000 \end{pmatrix}$$

$$\leqslant \max_{s \in [0,1]}\left[s\begin{pmatrix} 1.3333 & 0.1140 & 0 \\ 0.1140 & -1.2600 & 0 \\ 0 & 0 & -3.0000 \end{pmatrix} + (1-s)\begin{pmatrix} 1.6667 & 0.1414 & 0 \\ 0.1414 & -1.2000 & 0 \\ 0 & 0 & -2.4000 \end{pmatrix}\right]$$

$$< \begin{pmatrix} 1.7000 & 0 & 0 \\ 0 & -1.0000 & 0 \\ 0 & 0 & -2.1000 \end{pmatrix}$$

$$= \begin{pmatrix} 1.5000 & 0.1277 & 0 \\ 0.1277 & -1.2300 & 0 \\ 0 & 0 & -2.7000 \end{pmatrix} + 2\begin{pmatrix} 0.1000 & -0.06385 & 0 \\ -0.06385 & 0.115 & 0 \\ 0 & 0 & 0.3000 \end{pmatrix}$$

$$= \frac{1}{2}\sum_{r=1}^{2}\left(H_{3r} + H_{3r}^T\right) + E_3 + E_3^T.$$

(5.5)

Combining (5.3)–(5.5) means that the fuzzy estimate condition (A2) is satisfied.
Set

$$\varphi_\sigma(X(t), X(t-\tau(t))) \equiv \begin{pmatrix} 0.15X_1(t-\tau(t)) & 0 & 0 \\ 0 & 0.15X_2(t-\tau(t)) & 0 \\ 0 & 0 & 0.13X_3(t-\tau(t)) \end{pmatrix},$$

then we let

$$\Lambda_{\sigma 1} \equiv 0.0001I \equiv \Lambda_{\sigma 2}, \quad \alpha_\sigma \equiv 0.03,$$

which implies that the condition (A2) holds.

Let $\varsigma = 0.05, q = 1.2, \tau = 5$, and then employing computer Matlab LMI toolbox to solve (4.2) results in

$$\beta_1 = 0.0150, \beta_2 = 0.9750, \beta_3 = 0.0100, \quad \Psi = \begin{pmatrix} 0.01856 & 0 & 0 \\ 0 & 0.01396 & 0 \\ 0 & 0 & 0.01766 \end{pmatrix} > 0.$$

According to Theorem 4.1, the fuzzy switched system (5.2) is globally exponential input-to-state stability in the meaning of switching.

6. Conclusions and further considerations

Although the non-Lipschitz functions make the free weight matrix technique unsuitable for financial systems, in this paper, LMI-based criterion of globally asymptotical stability in the meaning of switching for chaos delayed feedback switched stochastic financial system was obtained, and the globally asymptotical stability in the meaning of switching is more suitable for the financial system,

which implies that when the time t is big enough, the dynamic of any subsystem must approach its unique equilibrium point. Besides, the global financial crisis often erupts periodically, which illuminates that the global stability in the classical sense is actually meaningless. In addition, boundedness result and the exponential stability criterion was also derived for fuzzy financial system in the meaning of switching. Numerical examples shows the effectiveness of the proposed methods.

Very recently, Xiaodi Li and his coauthors consider the finite-time stability and event-triggered impulsive control on dynamical systems ([37, 40]) . Since finite-time stability of the financial system, or event-triggered impulsive control of the financial system is seldom investigated, how to propose a suitable finite-time stability criterion ? It is an interesting problem.

Acknowledgments

Conflict of interest

It is declared that both of the authors have no competing interests.

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