AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(1): 1024–1039.
DOI:10.3934/math.2021061
Received: 11 May 2020
Accepted: 14 October 2020
Published: 06 November 2020

Research article

Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions

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Abstract: In the present investigation, our aim is to define a generalized subclass of analytic and bi-univalent functions associated with a certain q-integral operator in the open unit disk $\mathbb{U}$. We estimate bounds on the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for normalized analytic functions f in the open unit disk by considering the function f and its inverse $g = f^{-1}$. Furthermore, we derive special consequences of the results presented here, which would apply to several (known or new) subclasses of analytic and bi-univalent functions.

Keywords: analytic functions; univalent functions; Taylor-Maclaurin series expansions; Taylor-Maclaurin initial coefficients; bi-univalent functions; q-derivative (or q-difference) operator; q-integral operator
Mathematics Subject Classification: Primary 05A30, 30C45; Secondary 11B65, 47B38

1. Introduction and definitions

By $\mathcal{H}(\mathbb{U})$ we denote the analytic function class in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\},$$

where $\mathbb{C}$ represents the set of complex numbers.

The class $\mathcal{A}$ of normalized analytic functions consists of functions $f \in \mathcal{H}(\mathbb{U})$, which have the following Taylor-Maclaurin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\forall\, z \in \mathbb{U}) \tag{1.1}$$

and satisfy the normalization condition given by

$$f(0) = f'(0) - 1 = 0.$$

Further, a noteworthy subclass of $\mathcal{A}$, which contains all univalent functions in the open unit disk $\mathbb{U}$, is denoted by $\mathcal{S}$.

All functions $f \in \mathcal{S}$ that satisfy the following condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{U}) \tag{1.2}$$

are placed in the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$.

For regular functions f and g in the unit disk $\mathbb{U}$, we say that the function f is subordinate to the function g, and write

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z),$$

if there exists a Schwarz function w of the class $\mathcal{B}$, where

$$\mathcal{B} = \{w : w \in \mathcal{A}, \quad w(0) = 0 \text{ and } |w(z)| < 1 \quad (\forall\, z \in \mathbb{U})\}, \tag{1.3}$$

such that

$$f(z) = g(w(z)).$$

Specifically, when the given function g is regular in $\mathbb{U}$, then the following equivalence holds true:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

We next introduce the class $\mathcal{P}$ which consists of functions p, which are analytic in $\mathbb{U}$ and normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{1.4}$$

such that

$$\Re(p(z)) > 0.$$

In the theory of analytic functions, the vital role of the function class $\mathcal{P}$ is obvious from the fact that there are many subclasses of analytic functions which are related to this class of functions denoted by $\mathcal{P}$.

In connection with functions in the class $\mathcal{S}$, on the account of the Koebe one-quarter theorem (see [9]), it is clear that, under every function $f \in \mathcal{S}$, the image of $\mathbb{U}$ contains a disk of radius $\frac{1}{4}$. Consequently, every univalent function $f \in \mathcal{S}$ has an inverse f^{-1} given by

$$f^{-1}(f(z)) = z = f(f^{-1}(z)) \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.5}$$

A function $f \in \mathcal{S}$ such that both f and its inverse function $g = f^{-1}$ are univalent in $\mathbb{U}$ is known as bi-univalent in $\mathbb{U}$. The class of bi-univalent functions in $\mathbb{U}$ is symbolized by Σ. In their pioneering work, Srivastava et al. [46] basically resuscitated the study of the analytic and bi-univalent function class Σ in recent years. In fact, as sequels to their investigation in [46], a number of different subclasses of Σ have since then been presented and studied by many authors (see, for example, [2, 5–8, 11, 25, 26, 35, 38, 40–42, 47, 51–53, 55–57]). However, except for a few of the cited works using the Faber polynomial expansion method for finding upper bounds for the *general* Taylor-Maclaurin coefficients, most of these investigations are devoted to the study of non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ of the Taylor-Maclaurin series expansion.

Some important elementary concept details and definitions of the q-calculus which play vital role in our presentation will be recalled next.

Definition 1. Let $q \in (0, 1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \dfrac{1 - q^\lambda}{1 - q} & (\lambda \in \mathbb{C}) \\[2ex] \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 2. Let $q \in (0, 1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1 & (n = 0) \\[2ex] \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 3. The generalized q-Pochhammer symbol is defined, for $t \in \mathbb{R}$ and $n \in \mathbb{N}$, by

$$[t]_{n,q} = [t]_q [t+1]_q [t+2]_q \cdots [t + (n-1)]_q.$$

Also, for $t > 0$, let the q-gamma function be defined as follows:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \qquad \text{and} \qquad \Gamma_q(1) = 1,$$

where

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=0}^{\infty} \left(\frac{1-q^{n+1}}{1-q^{n+t}} \right).$$

Definition 4. (see [13] and [14]) For a function f in the class $\mathcal{A}$, the q-derivative (or q-difference) operator D_q is defined, in a given subset of $\mathbb{C}$, by

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\[2ex] f'(0) & (z = 0). \end{cases} \tag{1.6}$$

We note from Definition 4 that the q-derivative operator D_q converges to the ordinary derivative operator as follows:

$$\lim_{q \longrightarrow 1-} \left(D_q f \right)(z) = \lim_{q \longrightarrow 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

for a differentiable function f in a given subset of $\mathbb{C}$. Further, taking (1.1) and (1.6) into account, it is easy to observe that

$$\left(D_q f \right)(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}. \tag{1.7}$$

Recently, the study of the q-calculus has fascinated the intensive devotion of researchers. The great concentration is because of its advantages in many areas of mathematics and physics. The significance of the q-derivative operator D_q is quite obvious by its applications in the study of several subclasses of analytic functions. Initially, in the year 1990, Ismail et al. [12] gave the idea of q-starlike functions. Nevertheless, a firm foothold of the usage of the q-calculus in the context of Geometric Function Theory was effectively established, and the use of the generalized basic (or q-) hypergeometric functions in Geometric Function Theory was made by Srivastava (see, for details, [29, pp. 347 *et seq.*]). After that, notable studies have been made by numerous mathematicians which offer a momentous part in the advancement of Geometric Function Theory. For instance, Srivastava et al. [44] examined the family of q-starlike functions associated with conic region, and in [22] the estimate on the third Hankel determinant was settled. Recently, a set of articles were published by Srivastava et al. (see, for example, [20, 43, 49, 50]) in which they studied various families of q-starlike functions related with the Janowski functions from different aspects. For some more recent investigations about q-calculus, we may refer the reader to [1, 3, 4, 15, 16, 27, 33, 36, 48].

We remark in passing that, in the aforementioned recently-published survey-cum-expository review article [33], the so-called (p, q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [33, p. 340]). Srivastava [33] also pointed out how the Hurwitz-Lerch Zeta function as well as its *multi-parameter* extension, which is popularly known as the λ-generalized Hurwitz-Lerch Zeta function (see, for details, [30]), have motivated the studies of several other families of extensively- and widely-investigated linear convolution operators which emerge essentially from the Srivastava-Attiya operator [37] (see also [31] and [32]).

Definition 5. (see [12]) A function f in the function class $\mathcal{A}$ is said to belong to the function class $\mathcal{S}_q^*$ if

$$f(0) = f'(0) - 1 = 0 \tag{1.8}$$

and

$$\left|\frac{z}{f(z)}\left(D_q f\right)z - \frac{1}{1-q}\right| \leqq \frac{1}{1-q}. \tag{1.9}$$

Then on the account of last inequality, it is obvious that, in limit case when $q \to 1-$

$$\left|w - \frac{1}{1-q}\right| \leqq \frac{1}{1-q}$$

the above closed disk is merely the right-half plane and the class $\mathcal{S}_q^*$ of q-starlike functions turns into the familiar class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$. Analogously, in view of the principle of subordination, one may express the relations in (1.8) and (1.9) as follows: (see [54]):

$$\frac{z\left(D_q f\right)(z)}{f(z)} \prec \widehat{m}(z) \qquad \left(\widehat{m}(z) = \frac{1+z}{1-qz}\right).$$

In recent years, many integral and derivative operators were defined and studied from different viewpoints and different perspectives (see, for example, [10, 19, 23]). Motivated by the ongoing researches, Srivastava et al. [45] introduced the q-version of the Noor integral operator as follows.

Definition 6. (see [45]) Let a function $f \in \mathcal{A}$. Then the q-integral operator is given by

$$\mathcal{F}_{q,\lambda+1}^{-1}(z) * \mathcal{F}_{q,\lambda+1}(z) = zD_q f(z)$$

and

$$\begin{aligned} \mathcal{I}_q^\lambda f(z) &= f(z) * \mathcal{F}_{q,\lambda+1}^{-1}(z) \\ &= z + \sum_{n=2}^{\infty} \Psi_{n-1} a_n z^n \qquad (z \in \mathbb{U};\ \lambda > -1), \end{aligned} \tag{1.10}$$

where

$$\mathcal{F}_{q,\lambda+1}^{-1}(z) = z + \sum_{n=2}^{\infty} \Psi_{n-1} z^n$$

and

$$\Psi_{n-1} = \frac{[n]_q!\Gamma_q(1+\lambda)}{\Gamma_q(n+\lambda)} = \frac{[n]_q!}{[\lambda+1]_{q,n-1}}.$$

Specifically, we notice that

$$\mathcal{I}_q^0 f(z) = zD_q f(z) \quad \text{and} \quad \mathcal{I}_q^1 f(z) = f(z).$$

Clearly, in limit case when $q \to 1-$, th above q-integral operator simply becomes to the Noor integral operator (see [24]). It is straightforward to verify the following identity:

$$zD_q\left(\mathcal{I}_q^{\lambda+1} f(z)\right) = \left(1 + \frac{[\lambda]_q}{q^\lambda}\right)\mathcal{I}_q^\lambda f(z) - \frac{[\lambda]_q}{q^\lambda}\mathcal{I}_q^{\lambda+1} f(z). \tag{1.11}$$

If $q \to 1-$, the equality (1.11) implies that

$$z\left(\mathcal{I}^{\lambda+1} f(z)\right)' = (1+\lambda)\,\mathcal{I}^\lambda f(z) - \lambda \mathcal{I}^{\lambda+1} f(z),$$

which is the known recurrence formula for the Noor integral operator (see [24]).

Motivated by the works, which we have mentioned above, we now define subfamilies of the normalized univalent function class $\mathcal{S}$ by means of the operator $\mathcal{I}_q^\lambda$ and the principle of subordination between analytic functions as follows.

Definition 7. Let a function $f \in \mathcal{S}$. Then f belongs to the function class $\mathcal{H}_q(\lambda, \widehat{p})$ if it satisfies the following conditions:

$$\frac{zD_q\left(\mathcal{I}_q^\lambda f\right)(z)}{\left(\mathcal{I}_q^\lambda f\right)(z)} \prec \widehat{p}(z) \qquad (\lambda > -1;\ z \in \mathbb{U}) \tag{1.12}$$

and

$$\frac{wD_q\left(\mathcal{I}_q^\lambda g\right)(w)}{\left(\mathcal{I}_q^\lambda g\right)(w)} \prec \widehat{p}(w) \quad (\lambda > -1;\ w \in \mathbb{U}), \tag{1.13}$$

where the function $\widehat{p}(z)$ is analytic and has positive real part in $\mathbb{U}$. Moreover, $\widehat{p}(0) = 1$, $\widehat{p}'(0) > 0$, and $\widehat{p}(\mathbb{U})$ is symmetric with respect to the real axis. Consequently, it has a series expansion of the form given by

$$\widehat{p}(z) = 1 + \widehat{p}_1 z + \widehat{p}_2 z^2 + \widehat{p}_3 z^3 + \cdots \qquad (\widehat{p}_1 > 0), \tag{1.14}$$

noticing that $g(w) = f^{-1}(w)$.

In order to drive the main results in this paper, the following known lemma is needed.

Lemma 1. (see [9]) Let the function $p \in \mathcal{P}$ and let it have the form (1.4). Then

$$|p_n| \leqq 2 \qquad (n \in \mathbb{N})$$

and the bound is sharp.

2. A set of main results

We begin by estimating the upper bound for the Taylor-Maclaurin coefficients of functions in the function class $\mathcal{H}_q(\lambda, \widehat{p})$.

Theorem 1. *If the function $f \in \mathcal{H}_q(\lambda, \widehat{p})$ has the power series given by* (1.1), *then*

$$|a_2| \leqq \sqrt{\frac{\widehat{p}_1^3[\lambda+1]_q}{\left|\widehat{p}_1^2\left(q(q+1)^2 - [\lambda+1]_q\right) + (\widehat{p}_1 - \widehat{p}_2)[\lambda+1]_q\right|}} \tag{2.1}$$

and

$$|a_3| \leqq \widehat{p}_1\left(\widehat{p}_1 + \frac{[\lambda+1]_q}{q(q+1)^2}\right). \tag{2.2}$$

Proof. Since $f \in \mathcal{H}_q(\lambda, \widehat{p})$ and $f^{-1} = g$, by means of Definition 7 and by using the principle of subordination, there exit functions $s(z)$, $r(z) \in \mathcal{B}$ such that

$$\frac{zD_q\left(\mathcal{I}_q^\lambda f\right)(z)}{\left(\mathcal{I}_q^\lambda f\right)(z)} = \widehat{p}(r(z)) \qquad \text{and} \qquad \frac{wD_q\left(\mathcal{I}_q^\lambda g\right)(w)}{\left(\mathcal{I}_q^\lambda g\right)(w)} = \widehat{p}(s(w)). \tag{2.3}$$

We define the following two functions:

$$p_1(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + \sum_{n=1}^{\infty} r_n z^n$$

and

$$p_2(z) = \frac{1 + s(z)}{1 - s(z)} = 1 + \sum_{n=1}^{\infty} s_n z^n.$$

Then it is clear that $p_j \in \mathcal{P}$ for $j = 1, 2$. Equivalently, the last relations in terms of $r(z)$ and $s(z)$ can be restated as follows:

$$r(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left[r_1 z + \left(r_2 - \frac{r_1^2}{2}\right) z\right] + \cdots \tag{2.4}$$

and

$$s(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2}\left[s_1 z + \left(s_2 - \frac{s_1^2}{2}\right) z\right] + \cdots . \tag{2.5}$$

Therefore, by means of (2.4), (2.5) and (2.3), if we take (1.14) into account, we have

$$\widehat{p}\,(r(z)) = \widehat{p}\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}\widehat{p}_1 r_1 z + \left[\frac{1}{2}\widehat{p}_1\left(r_2 - \frac{r_1^2}{2}\right) + \frac{1}{4}\widehat{p}_2 r_1^2\right] z^2 + \cdots \tag{2.6}$$

and

$$\widehat{p}(s(z)) = \widehat{p}\left(\frac{p_2(z) - 1}{p_2(z) + 1}\right) = 1 + \frac{1}{2}\widehat{p}_1 s_1 w + \left[\frac{1}{2}\widehat{p}_1\left(s_2 - \frac{s_1^2}{2}\right) + \frac{1}{4}\widehat{p}_2 s_1^2\right] w^2 + \cdots . \tag{2.7}$$

Now, upon expanding the right-hand sides of both equations in (2.3), we find that

$$\frac{zD_q\left(\mathcal{I}_q^{\lambda} f\right)(z)}{\left(\mathcal{I}_q^{\lambda} f\right)(z)} = 1 + a_2 z + \left(\frac{q(q+1)^2}{[\lambda + 1]_q} a_3 - a_2^2\right) z^2 + \cdots \tag{2.8}$$

and

$$\frac{wD_q\left(\mathcal{I}_q^{\lambda} g\right)(w)}{\left(\mathcal{I}_q^{\lambda} g\right)(w)} = 1 - a_2 w + \left(\frac{q(q+1)^2}{[\lambda + 1]_q}(2a_2^2 - a_3) - a_2^2\right) w^2 + \cdots . \tag{2.9}$$

Substituting from (2.6), (2.7), (2.8) and (2.9) into (2.3) and then by equating the corresponding coefficients of z, z^2, w and w^2, we get

$$a_2 = \frac{1}{2}\widehat{p}_1 r_1, \tag{2.10}$$

$$\frac{q(q+1)^2}{[\lambda + 1]_q} a_3 - a_2^2 = \frac{1}{2}\widehat{p}_1\left(r_2 - \frac{r_1^2}{2}\right) + \frac{1}{4}\widehat{p}_2 r_1^2, \tag{2.11}$$

$$a_2 = -\frac{1}{2}\widehat{p}_1 s_1 \tag{2.12}$$

and

$$\frac{q(q+1)^2}{[\lambda+1]_q}(2a_2^2-a_3)-a_2^2=\frac{1}{2}\widehat{p}_1\left(s_2-\frac{s_1^2}{2}\right)+\frac{1}{4}\widehat{p}_2 s_1^2.$$

From (2.10) and (2.12), it immediately follows that

$$r_1=-s_1 \tag{2.13}$$

and

$$a_2^2=\frac{1}{8}\widehat{p}_1^2(r_1^2+s_1^2). \tag{2.14}$$

Addition of (2.11) and (2.14) yields

$$2\left[\frac{q(q+1)^2}{[\lambda+1]_q}-1\right]a_2^2=\frac{1}{2}\widehat{p}_1\left[r_2+s_2-\frac{1}{2}\left(r_1^2+s_1^2\right)\right]+\frac{1}{4}\widehat{p}_2\left(r_1^2+s_1^2\right).$$

Also, by using (2.14) in the last equation, we get

$$a_2^2=\frac{\widehat{p}_1^3\,[\lambda+1]_q\,(r_2+s_2)}{4\widehat{p}_1^2\left[q(q+1)^2-[\lambda+1]_q\right]+(\widehat{p}_1-\widehat{p}_2)[\lambda+1]_q}, \tag{2.15}$$

which, in view of Lemma 1, yields the required bound on $|a_2|$ as asserted in (2.1).

Further, in order to find the estimate on $|a_3|$, we subtract (2.14) from (2.11). Further computations by using (2.13) lead us to

$$a_3=a_2^2+\frac{1}{4}\frac{[\lambda+1]_q}{q(q+1)^2}\widehat{p}_1(r_2-s_2). \tag{2.16}$$

Finally, by using (2.14) in conjunction with Lemma 1 on the coefficients of r_2 and s_2, we are led to the assertion given in (2.2). This completes our proof of Theorem 1. □

In the next result, we solve the Fekete-Szegö problem for the function class $\mathcal{H}_q(\lambda,\widehat{p})$ by making use of the coefficients a_2, a_3 and a complex parameter ν.

Theorem 2. *Let the function f belong to the class $\mathcal{H}_q(\lambda,\widehat{p})$ and let $\nu\in\mathbb{C}$. Then*

$$\left|a_3-\nu a_2^2\right|\leqq\begin{cases}\dfrac{\widehat{p}_1[\lambda+1]_q}{q(1+q)^2} & \left(0\leqq\Theta(\nu)<\frac{1}{4q(1+q)^2}\right)\\[2ex] 4\widehat{p}_1[\lambda+1]_q\Theta(\nu) & \left(\Theta(\nu)\geqq\frac{1}{4q(1+q)^2}\right),\end{cases} \tag{2.17}$$

where

$$\Theta(\nu)=\frac{\widehat{p}_1^2(1-\nu)}{4\widehat{p}_1^2\left[q(q+1)^2-[\lambda+1]_q\right]+(\widehat{p}_1-\widehat{p}_2)[\lambda+1]_q}. \tag{2.18}$$

Proof. On the account of (2.15) and (2.16), we have

$$a_3-\nu a_2^2=\frac{[\lambda+1]_q}{4q(1+q)^2}\widehat{p}_1(r_2-s_2)+(1-\nu)a_2^2,$$

which can be written in the following equivalent form:

$$a_3 - \nu a_2^2 = \frac{[\lambda+1]_q}{4q(1+q)^2}\,\widehat{p}_1(r_2 - s_2) + \frac{\widehat{p}_1^3\,[\lambda+1]_q\,(r_2+s_2)}{4\widehat{p}_1^2\left[q(q+1)^2 - [\lambda+1]_q\right] + (\widehat{p}_1 - \widehat{p}_2)[\lambda+1]_q}.$$

Some suitable computations would yield

$$a_3 - \nu a_2^2 = \widehat{p}_1[\lambda+1]_q\left[\left(\Theta(\nu) + \frac{1}{4q(1+q)^2}\right)r_2 + \left(\Theta(\nu) - \frac{1}{4q(1+q)^2}\right)s_2\right],$$

where $\Theta(\nu)$ is defined in (2.18). Since all $\widehat{p}_j$ $(j = 1, 2)$ are real and $\widehat{p}_1 > 0$, we obtain

$$\left|a_3 - \nu a_2^2\right| = 2\widehat{p}_1[\lambda+1]_q\left|\left(\Theta(\nu) + \frac{1}{4q(1+q)^2}\right) + \left(\Theta(\nu) - \frac{1}{4q(1+q)^2}\right)\right|$$

The proof of Theorem 2 is thus completed. □

Remark 1. It follows from Theorem 2 when $\nu = 1$ that, if $f \in \mathcal{H}_q(\lambda, \widehat{p})$, then

$$\left|a_3 - a_2^2\right| \leqq \frac{\widehat{p}_1[\lambda+1]_q}{q(1+q)^2}.$$

If we first set $\lambda = 1$ and then apply limit as $q \to 1-$, then we have following consequence of Theorem 2.

Corollary 1. (see [57]) *Let a function f belong to the class given by*

$$\lim_{q\to 1-} \mathcal{H}_q(1, \widehat{p}) =: \mathcal{ST}_\sigma(\phi)$$

and $\nu \in \mathbb{C}$. Then

$$\left|a_3 - \nu a_2^2\right| \leqq \begin{cases} \dfrac{\widehat{p}_1}{2} & \left(0 \leqq \Theta_1(\nu) < \frac{1}{8}\right) \\ \\ 4\widehat{p}_1\Theta_1(\nu) & \left(\Theta_1(\nu) \geqq \frac{1}{8}\right), \end{cases}$$

where

$$\Theta_1(\nu) = \frac{\widehat{p}_1^2(1-\nu)}{4[\widehat{p}_1^2 + (\widehat{p}_1 - \widehat{p}_2)]}.$$

3. Applications of the main results

For the class of q-starlike functions of order α with $0 < \alpha \leqq 1$, the function $\widehat{p}$ is given by

$$\widehat{p}(z) = \frac{1 + (1 - (1+q)\alpha)z}{1 - qz} = 1 + (1+q)(1-\alpha)z + q(1+q)(1-\alpha)z^2 + \cdots.$$

Then Definition 7 of the bi-univalent function class $\mathcal{H}_q(\lambda, \widehat{p})$ yields a presumably new class $\mathcal{H}_q^1(\lambda, \alpha)$, which is given below.

Definition 8. A function $f \in A$ is said to be in the class $\mathcal{H}_q^1(\lambda, \alpha)$ if it satisfies the following conditions:

$$\left| \frac{zD_q\left(\mathcal{I}_q^\lambda f\right)(z)}{\left(\mathcal{I}_q^\lambda f\right)(z)} - \frac{1-\alpha q}{1-q} \right| \leqq \frac{1-\alpha}{1-q} \qquad (z \in \mathbb{U})$$

and

$$\left| \frac{wD_q\left(\mathcal{I}_q^\lambda g\right)(w)}{\left(\mathcal{I}_q^\lambda g\right)(w)} - \frac{1-\alpha q}{1-q} \right| \leqq \frac{1-\alpha}{1-q},$$

where $g(w) - f^{-1}(w)$.

Hence, upon setting

$$\widehat{p_1} = (1+q)(1-\alpha) \quad \text{and} \quad \widehat{p_2} = q(1+q)(1-\alpha)$$

in Definition 8, we are led to the following corollaries of Theorem 1 stated below.

Corollary 2. *Let the function $f \in \mathcal{H}_q^1(\lambda, \alpha)$ have the form* (1.1)*. Then*

$$|a_2| \leqq \frac{(1+q)(1-\alpha)\sqrt{[\lambda+1]_q}}{\sqrt{\left|(1+q)(1-\alpha)\left(q(q+1)^2 - [\lambda+1]_q\right) + (1-q)[\lambda+1]_q\right|}}$$

and

$$|a_3| \leqq \frac{(1-\alpha)\left[q(q+1)^2(1-\alpha) + [\lambda+1]_q\right]}{q(q+1)}.$$

Corollary 3. *Let the function $f \in \mathcal{H}_q^1(\lambda, \alpha)$ have the form* (1.1)*. Then*

$$\left|a_3 - a_2^2\right| \leqq \frac{(1-\alpha)[\lambda+1]_q}{q(1+q)}.$$

In Corollary 2, we set $\lambda = 1$. Then we arrive at the following result.

Corollary 4. *Let the function f be in the class given by*

$$\mathcal{H}_q^2(\lambda, \alpha) := \mathcal{H}_q^1(1, \alpha)$$

and have the form (1.1)*. Then*

$$|a_2| \leqq \frac{(1-\alpha)\sqrt{(q+1)}}{\sqrt{\left|(1-\alpha)\left[q(q+1)-1\right] + q - 1\right|}}$$

and

$$|a_3| \leqq \frac{1}{q}(1-\alpha)\left[q(q+1)(1-\alpha) + 1\right].$$

On the other hand, for $0 < \alpha \leqq 1$, if we set

$$\widehat{p}(z) = \left(\frac{1+z}{1-qz}\right)^{\alpha} = 1 + (1+q)\alpha z + \frac{(1+q)((1+q)\alpha + q - 1)\alpha}{2} z^2 + \cdots,$$

then Definition 7 of the bi-univalent function class $\mathcal{H}_q(\lambda, \widehat{p})$ gives a new class $\mathcal{H}_q^3(\lambda, \alpha)$, which is given below.

Definition 9. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{H}_q^3(\lambda, \alpha)$ if the following inequalities hold true:

$$\left| \arg\left(\frac{zD_q\left(\mathcal{I}_q^\lambda f\right)(z)}{\left(\mathcal{I}_q^\lambda f\right)(z)} \right) \right| \leqq \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U})$$

and

$$\left| \arg\left(\frac{wD_q\left(\mathcal{I}_q^\lambda g\right)(w)}{\left(\mathcal{I}_q^\lambda g\right)(w)} \right) \right| \leqq \frac{\alpha\pi}{2},$$

where the inverse function is given by $f^{-1}(w) = g(w)$.

Using the parameter setting given by

$$\widehat{p}_1 = (1+q)\alpha \qquad \text{and} \qquad \widehat{p}_2 = \frac{[(1+q)\alpha + q - 1](1+q)\alpha}{2}$$

in Definition 9, it leads to the following consequences of Theorem 1.

Corollary 5. *Let the function $f \in \mathcal{H}_q^3(\lambda, \alpha)$ be given by* (1.1). *Then*

$$|a_2| \leqq \frac{(q+1)\alpha\sqrt{2[\lambda+1]_q}}{\sqrt{\left|2(q+1)\alpha\left[q(q+1)^2 - [\lambda+1]_q\right] + [3 - q - (1+\alpha)][\lambda+1]_q\right|}}$$

and

$$|a_3| \leqq \frac{\left[q(q+1)^3\alpha + [\lambda+1]_q\right]\alpha}{q(q+1)}.$$

Corollary 6. *Let the function $f \in \mathcal{H}_q^3(\lambda, \alpha)$ be given by* (1.1). *Then*

$$\left|a_3 - a_2^2\right| \leqq \frac{[\lambda+1]_q}{q(1+q)}\alpha.$$

Next, if we take

$$\widehat{p}(z) = \frac{1 + \mathcal{T}^2 z}{1 - \mathcal{T} z - \mathcal{T}^2 z^2}, \tag{3.1}$$

where

$$\mathcal{T} = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

The function given in (3.1) is not univalent in $\mathbb{U}$. However, it is univalent in

$$|z| \leqq \frac{3-\sqrt{5}}{2} \approx 0.38.$$

It is noteworthy that

$$\frac{1}{|\mathcal{T}|} = \frac{|\mathcal{T}|}{1 - |\mathcal{T}|},$$

which ensures the division of the interval [0, 1] by the above-mentioned number $|\mathcal{T}|$ such that it fulfills the golden section. Since the equation:

$$\mathcal{T}^2 = 1 + \mathcal{T}$$

holds true for $\mathcal{T}$, in order to attain higher powers $\mathcal{T}^n$ as a linear function of the lower powers, this relation can be used. In fact, it can be decomposed all the way down to a linear combination of $\mathcal{T}$ and 1. The resulting recurrence relations yield the Fibonacci numbers $\mathfrak{u}_n$:

$$\mathcal{T}^n = \mathfrak{u}_n \mathcal{T} + \mathfrak{u}_{n-1}.$$

For the function $\widehat{p}$ represented in (3.1), Definition 7 of the bi-univalent function class $\mathcal{H}_q(\lambda, \widehat{p})$ gives a new class $\mathcal{H}_q^4(\lambda, \widehat{p})$, which (by using the principle of subordination) gives the following definition.

Definition 10. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}_q^4(\lambda, \alpha)$ if it satisfies the following subordination conditions:

$$\frac{zD_q\left(\mathcal{I}_q^\lambda f\right)(z)}{\left(\mathcal{I}_q^\lambda f\right)(z)} \prec \frac{1 + \mathcal{T}^2 z}{1 - \mathcal{T} z - \mathcal{T}^2 z^2} \qquad (z \in \mathbb{U})$$

and

$$\frac{wD_q\left(\mathcal{I}_q^\lambda g\right)(w)}{\left(\mathcal{I}_q^\lambda g\right)(w)} \prec \frac{1 + \mathcal{T}^2 w}{1 - \mathcal{T} w - \mathcal{T}^2 w^2}$$

where $g(w) = f^{-1}(w)$.

Using similar arguments as in proof of Theorem 1, we can obtain the the upper bounds on the Taylor-Maclaurin coefficients a_2 and a_3 given in Corollary 7.

Corollary 7. *Let the function $f \in \mathcal{H}_q^4(\lambda, \alpha)$ have the form* (1.1). *Then*

$$|a_2| \leqq \frac{\mathcal{T}\sqrt{[\lambda + 1]_q}}{\sqrt{\left|\mathcal{T}\left[q(q+1)^2 - [\lambda + 1]_q\right] + [1 + 3\mathcal{T}][\lambda + 1]_q\right|}}$$

and

$$|a_3| \leqq \frac{\mathcal{T}\left[\mathcal{T} q(q+1)^2 + [\lambda + 1]_q\right]}{q(q+1)^2}.$$

4. Concluding remarks and observations

Here, in our present investigation, we have successfully examined the applications of a certain q-integral operator to define several new subclasses of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For each of these newly-defined analytic and bi-univalent function classes, we have settled the problem of finding the upper bounds on the coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion subject to a gap series condition. By means of corollaries of our main theorems, we have also highlighted some known consequences and some applications of our main results.

Studies of the coefficient problems (including the Fekete-Szegö problems and the Hankel determinant problems) continue to motivate researchers in Geometric Function Theory of Complex Analysis. With a view to providing incentive and motivation to the interested readers, we have chosen to include several recent works (see, for example, [17, 18, 21, 28, 33, 34, 39]), on various bi-univalent function classes as well as the ongoing usages of the q-calculus in the study of other analytic or meromorphic univalent and multivalent function classes.

Acknowledgements

The fourth author is supported by UKM Grant: GUP-2019-032.

Conflicts of interest

The authors declare that they have no competing interests.

References

1. Q. Z. Ahmad, N. Khan, M. Raza, M. Tahir, B. Khan, Certain q-difference operators and their applications to the subclass of meromorphic q-starlike functions, *Filomat*, **33** (2019), 3385–3397.
2. H. Aldweby, M. Darus, On a subclass of bi-univalent functions associated with the q-derivative operator, *J. Math. Comput. Sci.*, **19** (2019), 58–64.
3. M. Arif, O. Barkub, H. M. Srivastava, S. Abdullah, S. A. Khan, Some Janowski type harmonic q-starlike functions associated with symmetrical points, *Mathematics*, **8** (2020), 1–16.
4. M. Arif, H. M. Srivastava, S. Umar, Some applications of a q-analogue of the Ruscheweyh type operator for multivalent functions, *RACSAM*, **113** (2019), 1211–1221.
5. M. Çaglar, E. Deniz, Initial coefficients for a subclass of bi-univalent functions defined by Salagean differential operator, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, **66** (2017), 85–91.
6. E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, *J. Class. Anal.*, **2** (2013), 49–60.
7. E. Deniz, J. M. Jahangiri, S. G. Hamidi, S. K. Kina, Faber polynomial coefficients for generalized bi-subordinate functions of complex order, *J. Math. Inequal.*, **12** (2018), 645–653.
8. E. Deniz, H. T. Yolcu, Faber polynomial coefficients for meromorphic bi-subordinate functions of complex order, *AIMS Mathematics*, **5** (2020), 640–649.
9. P. L. Duren, *Univalent functions*, New York, Berlin, Heidelberg and Tokyo: Springer-Verlag, 1983.
10. D. E. Edmunds, V. Kokilashvili, A. Meskhi, *Bounded and compact integral operators*, Dordrecht, Boston and London: Kluwer Academic Publishers, 2002.
11. H. Ö. Güney, G. Murugusundaramoorthy, H. M. Srivastava, The second Hankel determinant for a certain class of bi-close-to-convex functions, *Results Math.*, **74** (2019), 1–13.

12. M. E. H. Ismail, E. Merkes, D. Styer, A generalization of starlike functions, *Complex Variables Theory Appl.*, **14** (1990), 77–84.

13. F. H. Jackson, On q-definite integrals, *Quart. J. Pure Appl. Math.*, **41** (1910), 193–203.

14. F. H. Jackson, q-difference equations, *Am. J. Math.*, **32** (1910), 305–314.

15. B. Khan, Z. G. Liu, H. M. Srivastava, N. Khan, M. Darus, M. Tahir, A study of some families of multivalent q-starlike functions involving higher-order q-Derivatives, *Mathematics*, **8** (2020), 1–12.

16. B. Khan, H. M. Srivastava, N. Khan, M. Darus, M. Tahir, Q. Z. Ahmad, Coefficient estimates for a subclass of analytic functions associated with a certain leaf-like domain, *Mathematics*, **8** (2020), 1–15.

17. N. Khan, M. Shafiq, M. Darus, B. Khan, Q. Z. Ahmad, Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with Lemniscate of Bernoulli, *J. Math. Inequal.*, **14** (2020), 51–63.

18. Q. Khan, M. Arif, M. Raza, G. Srivastava, H. Tang, S. U. Rehman, et al. Some applications of a new integral operator in q-analog for multivalent functions, *Mathematics*, **7** (2019), 1–13.

19. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral operators in non-standard function spaces*, Basel and Boston: Birkhäuser, 2016.

20. S. Mahmood, Q. Z. Ahmad, H. M. Srivastava, N. Khan, B. Khan, M. Tahir, A certain subclass of meromorphically q-starlike functions associated with the Janowski functions, *J. Inequal. Appl.*, **2019** (2019), 1–11.

21. S. Mahmood, N. Raza, E. S. A. Abujarad, G. Srivastava, H. M. Srivastava, S. N. Malik, Geometric properties of certain classes of analytic functions associated with a q-integral operator, *Symmetry*, **11** (2019), 1–14.

22. S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan, I. Ali, Upper bound of the third Hankel determinant for a subclass of q-starlike functions, *Symmetry*, **11** (2019), 1–13.

23. G. V. Milovanović, M. T. Rassias, *Analytic number theory*, *approximation theory*, *and special functions*: *In honor of Hari M. Srivastava*, Berlin, Heidelberg and New York: Springer, 2014.

24. K. I. Noor, On new classes of integral operators, *J. Natur. Geom.*, **16** (1999), 71–80.

25. S. Porwal, M. Darus, On a new subclass of bi-univalent functions, *J. Egyptian Math. Soc.*, **21** (2013), 190–193.

26. M. S. Rehman, Q. Z. Ahmad, B. Khan, M. Tahir, N. Khan, Generalisation of certain subclasses of analytic and univalent functions, *Maejo Int. J. Sci. Technol.*, **13** (2019), 1–9.

27. M. S. Rehman, Q. Z. Ahmad, H. M. Srivastava, B. Khan, N. Khan, Partial sums of generalized q-Mittag-Leffler functions, *AIMS Mathematics*, **5** (2019), 408–420.

28. L. Shi, Q. Khan, G. Srivastava, J. L. Liu, M. Arif, A study of multivalent q-starlike functions connected with circular domain, *Mathematics*, **7** (2019), 1–12.

29. H. M. Srivastava, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, In: *Univalent functions*, *fractional calculus*, *and their applications*, Chichester: Halsted Press (Ellis Horwood Limited), 329–354, 1989.

30. H. M. Srivastava, A new family of the λ-generalized Hurwitz-Lerch zeta functions with applications, *Appl. Math. Inform. Sci.*, **8** (2014), 1485–1500.

31. H. M. Srivastava, Some general families of the Hurwitz-Lerch Zeta functions and their applications: Recent developments and directions for further researches, *Proc. Inst. Math. Mech. Nat. Acad. Sci. Azerbaijan*, **45** (2019), 234–269.

32. H. M. Srivastava, The Zeta and related functions: Recent developments, *J. Adv. Engrg. Comput.*, **3** (2019), 329–354.

33. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. A Sci.*, **44** (2020), 327–344.

34. H. M. Srivastava, Q. Z. Ahmad, N. Khan, S. Kiran, B. Khan, Some applications of higher-order derivatives involving certain subclasses of analytic and multivalent functions, *J. Nonlinear Var. Anal.*, **2** (2018), 343–353.

35. H. M. Srivastava, Ş. Altınkaya, S. Yalçin, Certain subclasses of bi-univalent functions associated with the Horadam polynomials, *Iran. J. Sci. Technol. Trans. A Sci.*, **43** (2019), 1873–1879.

36. H. M. Srivastava, M. K. Aouf, A. O. Mostafa, Some properties of analytic functions associated with fractional q-calculus operators, *Miskolc Math. Notes*, **20** (2019), 1245–1260.

37. H. M. Srivastava, A. A. Attiya, An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination, *Integr. Transf. Spec. Funct.*, **18** (2007), 207–216.

38. H. M. Srivastava, D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, *J. Egyptian Math. Soc.*, **23** (2015), 242–246.

39. H. M. Srivastava, D. Bansal, Close-to-convexity of a certain family of q-Mittag-Leffler functions, *J. Nonlinear Var. Anal.*, **1** (2017), 61–69.

40. H. M. Srivastava, S. Bulut, M. Çağlar, N. Yağmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, *Filomat*, **27** (2013), 831–842.

41. H. M. Srivastavaa, S. S. Eker, R. M. Ali, Coefficient bounds for a certain class of analytic and bi-univalent functions, *Filomat*, **29** (2015), 1839–1845.

42. H. M. Srivastava, S. M. El-Deeb, The Faber polynomial expansion method and the Taylor-Maclaurin coefficient estimates of bi-close-to-convex functions connected with the q-convolution, *AIMS Mathematics*, **5** (2020), 7087–7106.

43. H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad, Coefficient inequalities for q-starlike functions associated with the Janowski functions, *Hokkaido Math. J.*, **48** (2019), 407–425.

44. H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad, M. Tahir, A generalized conic domain and its applications to certain subclasses of analytic functions, *Rocky Mountain J. Math.*, **49** (2019), 2325–2346.

45. H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan, S. Hussain, The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator, *Stud. Univ. Babeş-Bolyai Math.*, **63** (2018), 419–436.

46. H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.*, **23** (2010), 1188–1192.

47. H. M. Srivastava, A. Motamednezhad, E. A. Adegan, Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator, *Mathematics*, **8** (2020), 1–12.

48. H. M. Srivastava, N. Raza, E. S. A. AbuJarad, G. Srivastava, M. H. AbuJarad, Fekete-Szegö inequality for classes of (p, q)-starlike and (p, q)-convex functions, *RACSAM*, **113** (2019), 3563–3584.

49. H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general classes of q-starlike functions associated with the Janowski functions, *Symmetry*, **11** (2019), 1–14.

50. H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general families of q-starlike functions associated with the Janowski functions, *Filomat*, **33** (2019), 2613–2626.

51. H. M. Srivastava, A. K. Wanas, Initial Maclaurin coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions defined by a linear combination, *Kyungpook Math. J.*, **59** (2019), 493–503.

52. T. S. Taha, *Topics in univalent function theory*, Ph. D. Thesis, University of London, London, 1981.

53. M. Tahir, N. Khan, Q. Z. Ahmad, B. Khan, G. Mehtab, Coefficient estimates for some subclasses of analytic and bi-univalent functions associated with conic domain, *SCMA*, **16** (2019), 69–81.

54. H. E. Ö. Uçar, Coefficient inequality for q-starlike functions, *Appl. Math. Comput.*, **276** (2016), 122–126.

55. Q. H. Xu, Y. C. Gui, H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.*, **25** (2012), 990–994.

56. Q. H. Xu, H. G. Xiao, H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.*, **218** (2012), 11461–11465.

57. P. Zaprawa, On the Fekete-Szegö problem for classes of bi-univalent functions, *Bull. Belg. Math. Soc. Simon Stevin*, **21** (2014), 169–178.