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Research article

k-fractional integral inequalities of Hadamard type for exponentially (s,m)-convex functions

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Abstract: The aim of this article is to present fractional versions of the Hadamard type inequalities for exponentially (s,m)-convex functions via k-analogue of Riemann-Liouville fractional integrals. The results provide generalizations of various known fractional integral inequalities. Some special cases are analyzed in the form of corollaries and remarks.

Keywords: convex functions; Hadamard inequality; Riemann-Liouville fractional integral operators
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1. Introduction

Fractional integral inequalities are useful generalizations of classical inequalities. The Hadamard inequality is the geometric interpretation of convex functions which has been analyzed by many researchers for fractional integral and differentiation operators. For fractional versions of the Hadamard inequality we refer the researchers to [1–9]. Convex functions proved very useful for the establishment of new inequalities which have interesting consequences in the theory of classical inequalities. The Hadamard inequality is the most classical inequality for convex functions which is stated in the undermentioned theorem:

Theorem 1. *[9] If $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In recent years the theory of mathematical inequalities is analyzed via fractional integral operators of different kinds (see, [1–15] and references in there). Inequalities have a significant role in the field of convex analysis, while the classical Hadamard inequality is equivalent to the definition of convex functions.

Definition 1. *A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is said to be convex function if*

$$f(rx + (1 - r)y) \leq rf(x) + (1 - r)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $r \in [0, 1]$.

In [16], Qiang et al. introduced the notion of exponentially (s, m)-convex function as follows:

Definition 2. *Let $s \in [0, 1]$ and $I \subseteq [0, \infty)$ be an interval. A function $f : I \to \mathbb{R}$ is said to be exponentially (s, m)-convex function if*

$$f(rx + m(1 - r)y) \leq r^s \frac{f(x)}{e^{\eta x}} + m(1 - r)^s \frac{f(y)}{e^{\eta y}}$$

holds for all $m \in [0, 1]$ and $\eta \in \mathbb{R}$.

Remark 1. *By selecting suitable values of parameters s, m and η, the above definition reproduces the well-known functions as follows:*

(i) By setting $\eta = 0$, (s, m)-convex function [17] can be obtained.
(ii) By setting $\eta = 0$ and $s = 1$, m-convex function [18] can be obtained.
(iii) By setting $\eta = 0$ and $m = 1$, s-convex function [19] can be obtained.
(iv) By setting $\eta = 0$, $s = 1$ and $m = 1$, convex function [20] can be obtained.
(v) By setting $s = 1$, exponentially m-convex function [21] can be obtained.
(vi) By setting $m = 1$, exponentially s-convex function [19] can be obtained.
(vii) By setting $s = 1$ and $m = 1$, exponentially convex function [22] can be obtained.

The well known beta function is frequently used in the presented results, defined as follows:

Definition 3. *[23] The beta function of two variables x and y are define as:*

$$\beta(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt$$

for $Re(x) > 0$, $Re(y) > 0$.

The objective of this article is to obtain k-fractional integral inequalities for a generalized class of convex functions namely exponentially (s, m)-convex functions. The classical fractional integral operators namely Riemann-Liouville (RL) fractional integrals are defined as follows:

Definition 4. *Let $f \in L_1[a, b]$. Then RL fractional integrals $I^{\alpha}_{a^+} f$ and $I^{\alpha}_{b^-} f$ of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ of f are defined by*

$$I^{\alpha}_{a^+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x - r)^{1-\alpha} f(r)dr, \quad x > a$$

and

$$I^{\alpha}_{b^-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (r - x)^{1-\alpha} f(r)dr, \quad x < b$$

respectively. Here $\Gamma(\alpha)$ is the gamma function and $I^0_{a^+} f(x) = I^0_{b^-} f(x) = f(x)$.

In [24], Mubeen and Habibullah gave the Riemann-Liouville k-fractional integrals as follows:

Definition 5. *Let $f \in L_1[a,b]$. Then RL k-fractional integrals $I_{a^+}^{\alpha,k} f$ and $I_{b^-}^{\alpha,k} f$ of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ of f are defined by*

$$I_{a^+}^{\alpha,k} f(x) := \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-r)^{\frac{\alpha}{k}-1} f(r)dr, \quad x > a$$

and

$$I_{b^-}^{\alpha,k} f(x) := \frac{1}{k\Gamma_k(\alpha)} \int_x^b (r-x)^{\frac{\alpha}{k}-1} f(r)dr, \quad x < b$$

respectively. Here $\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{\frac{-t^k}{k}} dt$ and $I_{a^+}^{0,1} f(x) = I_{b^-}^{0,1} f(x) = f(x)$.

In Section 2, we prove k-fractional integral inequality of Hadamard type for exponentially (s,m)-convex functions and deduce some related results. In Section 3, we prove a version of k-fractional integral inequality of Hadamard type for differentiable functions f so that $|f'|$ is exponentially (s,m)-convex. In Section 4, we give some particular cases of results given in Sections 2 & 3.

2. Main results

In the undermentioned theorem, we give k-fractional integral inequality of Hadamard type for exponentially (s,m)-convex functions.

Theorem 2. *Let $f : [0,\infty) \to \mathbb{R}$ be an exponentially (s,m)-convex function with $m \in (0,1]$, $\eta \in \mathbb{R}$ with $f \in L_1[a,b]$, $0 \le a < b$. If $\frac{a}{m}, \frac{a}{m^2}, mb \in [a,b]$, then we will have*

$$\begin{aligned} \frac{1}{h(\eta)} f\left(\frac{bm+a}{2}\right) &\le \frac{\Gamma_k(\alpha+k)}{2^s(mb-a)^{\frac{\alpha}{k}}} \left[m^{\frac{\alpha}{k}+1} I_{b^-}^{\alpha,k} f\left(\frac{a}{m}\right) + I_{a^+}^{\alpha,k} f(mb)\right] \\ &\le \frac{\alpha}{k2^s} \left\{ \left[m^2 \frac{f(\frac{a}{m^2})}{e^{\frac{\eta a}{m^2}}} + m\frac{f(b)}{e^{\eta b}}\right] \beta\left(\frac{\alpha}{k}, s+1\right) + \left[m\frac{f(b)}{e^{\eta b}} + \frac{f(a)}{e^{\eta a}}\right] \frac{k}{\alpha+ks}\right\} \end{aligned} \tag{2.1}$$

where $h(\eta) = \frac{1}{e^{\eta b}}$ for $\eta < 0$ and $h(\eta) = \frac{1}{e^{\frac{\eta a}{m}}}$ for $\eta \ge 0$.

Proof. Since f is an exponentially (s,m)-convex function, we have

$$f\left(\frac{um+v}{2}\right) \le \frac{1}{2^s}\left(\frac{mf(u)}{e^{\eta u}} + \frac{f(v)}{e^{\eta v}}\right), \quad u,v \in [a,b].$$

Since $\frac{a}{m}, mb \in [a,b]$, for $r \in [0,1]$, $(1-r)\frac{a}{m}+rb \le b$ and $(1-r)mb+ra \ge a$. By setting $u = (1-r)\frac{a}{m}+rb \le b$ and $v = m(1-r)b+ra \ge a$ in the above inequality, then by integrating over $[0,1]$ after multiplying with $r^{\frac{\alpha}{k}-1}$, we have

$$\begin{aligned} &f\left(\frac{bm+a}{2}\right)\int_0^1 r^{\frac{\alpha}{k}-1}dr \\ &\le \frac{1}{2^s}\left[\int_0^1 r^{\frac{\alpha}{k}-1} \frac{mf\left((1-r)\frac{a}{m}+rb\right)}{e^{\eta\left((1-r)\frac{a}{m}+rb\right)}}dr + \int_0^1 r^{\frac{\alpha}{k}-1}\frac{f(m(1-r)b+ra)}{e^{\eta(m(1-r)b+ra)}}dr\right]. \end{aligned}$$

Now, if we let $w = (1-r)\frac{a}{m} + rb$ and $z = m(1-r)b + ra$ in right hand side of above inequality, we get

$$\begin{aligned}&f\left(\frac{bm+a}{2}\right)\frac{k}{\alpha}\\&\leq \frac{1}{2^s}\left[\int_{\frac{a}{m}}^{b}\left(\frac{w-\frac{a}{m}}{b-\frac{a}{m}}\right)^{\frac{\alpha}{k}-1}\frac{mf(w)dw}{e^{\eta w}\left(b-\frac{a}{m}\right)} + \int_{a}^{mb}\left(\frac{mb-z}{mb-a}\right)^{\frac{\alpha}{k}-1}\frac{f(z)dz}{e^{\eta z}(mb-a)}\right].\end{aligned}$$

Further, it gives the following inequality which provide the first inequality of (2.1):

$$f\left(\frac{bm+a}{2}\right) \leq \frac{h(\eta)\Gamma_k(\alpha+k)}{2^s(mb-a)^{\frac{\alpha}{k}}}\left[m^{\frac{\alpha}{k}+1}I_{b^-}^{\alpha,k}f\left(\frac{a}{m}\right) + I_{a^+}^{\alpha,k}f(mb)\right].$$

On the other hand by using exponentially (s,m)-convexity of f, we have

$$\begin{aligned}&mf\left((1-r)\frac{a}{m}+rb\right) + f(m(1-r)b+ra)\\&\leq m^2(1-r)^s\frac{f(\frac{a}{m^2})}{e^{\frac{\eta a}{m^2}}} + mr^s\frac{f(b)}{e^{\eta b}} + m(1-r)^s\frac{f(b)}{e^{\eta b}} + r^s\frac{f(a)}{e^{\eta a}}.\end{aligned}$$

By multiplying both sides of above inequality with $\alpha\left(\frac{1}{2}\right)^s r^{\frac{\alpha}{k}-1}$ and integrating over $[0,1]$, after some calculations we get

$$\begin{aligned}&\frac{\Gamma_k(\alpha+k)}{2^s(mb-a)^{\frac{\alpha}{k}}}\left[m^{\frac{\alpha}{k}+1}I_{b^-}^{\alpha,k}f\left(\frac{a}{m}\right) + I_{a^+}^{\alpha,k}f(mb)\right]\\&\leq \frac{\alpha}{k2^s}\left\{\left[m^2\frac{f(\frac{a}{m^2})}{e^{\frac{\eta a}{m^2}}} + m\frac{f(b)}{e^{\eta b}}\right]\int_0^1 r^{\frac{\alpha}{k}-1}(1-r)^s dr + \left[m\frac{f(b)}{e^{\eta b}} + \frac{f(a)}{e^{\eta a}}\right]\int_0^1 r^{\frac{\alpha}{k}-1}r^s dr\right\}.\end{aligned}$$

By using definition of the beta function, from aforementioned inequality the second inequality of (2.1) is obtained. □

In the following we give consequences of above theorem:

Corollary 1. *The undermentioned inequality holds for exponentially (s,m)-convex functions via RL fractional integrals*

$$\begin{aligned}&\frac{1}{h(\eta)}f\left(\frac{bm+a}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2^s(mb-a)^{\alpha}}\left[m^{\alpha+1}I_{b^-}^{\alpha}f\left(\frac{a}{m}\right) + I_{a^+}^{\alpha}f(mb)\right]\\&\leq \frac{\alpha}{2^s}\left\{\left[m^2\frac{f(\frac{a}{m^2})}{e^{\frac{\eta a}{m^2}}} + m\frac{f(b)}{e^{\eta b}}\right]\beta\left(\frac{\alpha}{k}, s+1\right) + \left[m\frac{f(b)}{e^{\eta b}} + \frac{f(a)}{e^{\eta a}}\right]\frac{1}{\alpha+s}\right\}.\end{aligned} \tag{2.2}$$

Proof. By setting $k = 1$ in inequality (2.1) of Theorem 2, we get the above inequality (2.2). □

Corollary 2. *The undermentioned result holds for convex functions via RL k-fractional integrals*

$$f\left(\frac{b+a}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{b^-}^{\alpha,k}f(a) + I_{a^+}^{\alpha,k}f(b)\right] \leq \frac{f(a)+f(b)}{2}. \tag{2.3}$$

Proof. By setting $\eta = 0$, $s = 1$ and $m = 1$ in (2.1) of Theorem 2, we get the above inequality (2.3) which is given in [3]. □

Corollary 3. *The undermentioned result holds for convex functions via RL fractional integrals*

$$f\left(\frac{b+a}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I_{b^-}^{\alpha} f(a) + I_{a^+}^{\alpha} f(b)\right] \leq \frac{f(a)+f(b)}{2}. \tag{2.4}$$

Proof. By setting $\eta = 0$, $s = 1$, $m = 1$ and $k = 1$ in (2.1) of Theorem 2, we get the above inequality (2.4) which is given in [9]. □

3. Bounds of Hadamard inequality

In this section k-fractional integral inequalities of Hadamard type for exponentially (s, m)-convex function in terms of the first derivatives has been obtained. For the proof of next result we will use the undermentioned lemma.

Lemma 1. *[3] Let function $f : [a, b] \to \mathbb{R}$ be differentiable on interval (a, b). If $f' \in L[a, b]$, then one has*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)\right] \\
&= \frac{b-a}{2}\int_0^1 \left[(1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}}\right] f'(ra+(1-r)b)dr.
\end{aligned}$$

Theorem 3. *Let $f : [0, \infty) \to \mathbb{R}$ be a differential function such that $[a, b] \subset [0, \infty)$, and $f' \in L_1[a, b]$. If $|f'|$ is an exponentially (s, m)-convex function with $m \in (0, 1]$, $\eta \in \mathbb{R}$, $q > 1$. Then for RL k-fractional integrals we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)\right]\right| \\
&\leq \frac{(b-a)\left(\frac{m|f'(\frac{b}{m})|}{e^{\frac{\eta b}{m}}} + \frac{|f'(a)|}{e^{\eta a}}\right)}{2} \\
&\left\{\left[\frac{2^{\frac{\alpha}{k}+s+1}-2}{2^{\frac{\alpha}{k}+s+1}(\frac{\alpha}{k}+s+1)} - \left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right.\right. \\
&\left.\left. + \left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right]\right\}
\end{aligned} \tag{3.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemma 1, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)\right]\right|$$

$$\leq \frac{b-a}{2}\int_0^1 |(1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}}||f'(ra+(1-r)b)|dr.$$

By using exponentially (s,m)-convexity of $|f'|$ we will get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)]\right| \\
&\leq \frac{b-a}{2}\int_0^{\frac{1}{2}}[(1-r)^{\frac{\alpha}{k}} - r^{\frac{\alpha}{k}}]\left[r^s\frac{|f'(a)|}{e^{\eta a}} + m(1-r)^s\frac{\left|f'\left(\frac{b}{m}\right)\right|}{e^{\frac{\eta b}{m}}}\right]dr \\
&+\int_{\frac{1}{2}}^1[r^{\frac{\alpha}{k}} - (1-r)^{\frac{\alpha}{k}}]\left[r^s\frac{|f'(a)|}{e^{\eta a}} + m(1-r)^s\frac{\left|f'\left(\frac{b}{m}\right)\right|}{e^{\frac{\eta b}{m}}}\right]dr \\
&= \frac{b-a}{2}\left\{\frac{|f'(a)|}{e^{\eta a}}\int_0^{\frac{1}{2}}(1-r)^{\frac{\alpha}{k}}r^s dr - \frac{|f'(a)|}{e^{\eta a}}\int_0^{\frac{1}{2}} r^{\frac{\alpha}{k}}r^s dr\right. \\
&+\frac{m\left|f'\left(\frac{b}{m}\right)\right|}{e^{\frac{\eta b}{m}}}\int_0^{\frac{1}{2}}(1-r)^{\frac{\alpha}{k}}(1-r)^s dr - \frac{m\left|f'\left(\frac{b}{m}\right)\right|}{e^{\frac{\eta b}{m}}}\int_0^{\frac{1}{2}} r^{\frac{\alpha}{k}}(1-r)^s dr \\
&-\frac{|f'(a)|}{e^{\eta a}}\int_{\frac{1}{2}}^1(1-r)^{\frac{\alpha}{k}}r^s dr + \frac{|f'(a)|}{e^{\eta a}}\int_{\frac{1}{2}}^1 r^{\frac{\alpha}{k}}r^s dr \\
&\left.-\frac{m\left|f'\left(\frac{b}{m}\right)\right|}{e^{\frac{\eta b}{m}}}\int_{\frac{1}{2}}^1(1-r)^{\frac{\alpha}{k}}(1-r)^s dr + \frac{m\left|f'\left(\frac{b}{m}\right)\right|}{e^{\frac{\eta b}{m}}}\int_{\frac{1}{2}}^1 r^{\frac{\alpha}{k}}(1-r)^s dr\right\}.
\end{aligned} \tag{3.2}$$

Now, by using Holder inequality, one has

$$\int_0^{\frac{1}{2}}(1-r)^{\frac{\alpha}{k}}r^s dr \leq \left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}},$$

$$\int_{\frac{1}{2}}^1(1-r)^{\frac{\alpha}{k}}r^s dr \leq \left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}},$$

$$\int_0^{\frac{1}{2}} r^{\frac{\alpha}{k}}(1-r)^s dr \leq \left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}},$$

and

$$\int_{\frac{1}{2}}^1 r^{\frac{\alpha}{k}}(1-r)^s dr \leq \left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}.$$

By using the above inequalities in the right hand side of (3.2), we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)]\right| \\
&\leq \frac{b-a}{2}\left\{\frac{|f'(a)|}{e^{\eta a}}\left[\frac{2^{\frac{\alpha}{k}+s+1}-2}{2^{\frac{\alpha}{k}+s+1}(\frac{\alpha}{k}+s+1)} - \left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right.\right.
\end{aligned}$$

$$+\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\Bigg]$$

$$+\frac{m\left|f'\left(\frac{b}{m}\right)\right|}{e^{\frac{\eta b}{m}}}\left[\frac{2^{\frac{\alpha}{k}+s+1}-2}{2^{\frac{\alpha}{k}+s+1}(\frac{\alpha}{k}+s+1)}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right]\Bigg\}.$$

□

Corollary 4. *The undermentioned inequality holds for exponentially (s, m)-convex functions of Riemann-Liouville fractional integrals*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[I^{\alpha}_{a^+}f(b)+I^{\alpha}_{b^-}f(a)]\right|\\&\leq\frac{(b-a)\left(\frac{m\left|f'\left(\frac{b}{m}\right)\right|}{e^{\frac{\eta b}{m}}}+\frac{|f'(a)|}{e^{\eta a}}\right)}{2}\\&\left\{\left[\frac{2^{\alpha+s+1}-2}{2^{\alpha+s+1}(\alpha+s+1)}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\alpha p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right.\right.\\&\left.\left.+\left[\frac{2^{\alpha p+1}-1}{2^{\alpha p+1}(\alpha p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right]\right\}.\end{aligned}\tag{3.3}$$

Proof. By setting $k = 1$ in inequality (3.1) of Theorem 3 we get the above inequality (3.3). □

4. Results for (s, m)-convex functions

In this section we discuss some particular cases of the results established in Sections 2 and 3.

Theorem 4. *Let $f : [0, \infty) \to \mathbb{R}$ be an (s, m)-convex function with $m \in (0, 1]$, $f \in L_1[a, b]$, $a, b \in [0, \infty)$ where $\frac{a}{m}, \frac{a}{m^2}, mb \in [a, b]$. Then we will have the undermentioned inequality:*

$$\begin{aligned}&f\left(\frac{bm+a}{2}\right)\leq\frac{\Gamma_k(\alpha+k)}{2^s(mb-a)^{\frac{\alpha}{k}}}\left[m^{\frac{\alpha}{k}+1}I^{\alpha,k}_{b^-}f\left(\frac{a}{m}\right)+I^{\alpha,k}_{a^+}f(mb)\right]\\&\leq\frac{\alpha}{k2^s}\left\{\left[m^2f\left(\frac{a}{m^2}\right)+mf(b)\right]\beta\left(\frac{\alpha}{k},s+1\right)\right.\\&\left.+[mf(b)+f(a)]\frac{k}{\alpha+ks}\right\}.\end{aligned}\tag{4.1}$$

Proof. Its proof is alike to the proof of Theorem 2 or directly (4.1) can be obtained from (2.1) by taking $\eta = 0$. □

Corollary 5. *The undermentioned inequality holds for m-convex functions via RL k-fractional integrals*

$$
\begin{aligned}
& f\left(\frac{bm+a}{2}\right) \le \frac{\Gamma_k(\alpha+k)}{2(mb-a)^{\frac{\alpha}{k}}}\left[m^{\frac{\alpha}{k}+1} I_{b^-}^{\alpha,k} f\left(\frac{a}{m}\right) + I_{a^+}^{\alpha,k} f(mb)\right] \\
& \le \frac{\alpha}{2k}\left\{\frac{k\left[mf(b)+f(a)\right]}{\alpha+k} + \left[m^2 f\left(\frac{a}{m^2}\right) + mf(b)\right]\beta\left(\frac{\alpha}{k},\ 2\right)\right\}.
\end{aligned} \tag{4.2}
$$

Proof. By setting $s = 1$ in inequality (4.1) of Theorem 4 we get the above inequality (4.2). □

Corollary 6. *The undermentioned inequality holds for s-convex functions via RL k-fractional integrals*

$$
\begin{aligned}
& f\left(\frac{b+a}{2}\right) \le \frac{\Gamma_k(\alpha+k)}{2^s(b-a)^{\frac{\alpha}{k}}}\left[I_{b^-}^{\alpha,k} f(a) + I_{a^+}^{\alpha,k} f(b)\right] \\
& \le \frac{\alpha}{k2^s}\left[f(a)+f(b)\right]\left\{\frac{k}{\alpha+ks} + \beta\left(\frac{\alpha}{k}, s+1\right)\right\}.
\end{aligned} \tag{4.3}
$$

Proof. By setting $m = 1$, in inequality (4.1) of Theorem 4 we get the above inequality (4.3). □

Theorem 5. *Let $f : [0,\infty) \to \mathbb{R}$ be a function and $[a,b] \subset [0,\infty)$ with $f \in L_1[a,b]$. If $|f'|$ is an (s,m)-convex function with $m \in (0,1]$ and $q > 1$. Then for RL k-fractional integrals we have*

$$
\begin{aligned}
& \left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)]\right| \\
& \le \frac{(b-a)\left(m\left|f'\left(\frac{b}{m}\right)\right| - |f'(a)|\right)}{2} \\
& \left\{\left[\frac{2^{\frac{\alpha}{k}+s+1}-2}{2^{\frac{\alpha}{k}+s+1}(\frac{\alpha}{k}+s+1)} - \left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right.\right. \\
& \left.\left. + \left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right]\right\}
\end{aligned} \tag{4.4}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Its proof is alike to the proof of Theorem 3, or directly (4.4) can be obtained from (3.1) by taking $\eta = 0$. □

Corollary 7. *The undermentioned inequality holds for m-convex functions via RL k-fractional integrals*

$$
\begin{aligned}
& \left|\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k} f(b) + I_{b^-}^{\alpha,k} f(a)]\right| \\
& \le \frac{(b-a)\left(m\left|f'\left(\frac{b}{m}\right)\right| - |f'(a)|\right)}{2} \\
& \left\{\left[\frac{2^{\frac{\alpha}{k}+2}-2}{2^{\frac{\alpha}{k}+2}(\frac{\alpha}{k}+2)} - \left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{q+1}-1}{2^{q+1}(q+1)}\right]^{\frac{1}{q}}\right.\right. \\
& \left.\left. + \left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{q+1}(q+1)}\right]^{\frac{1}{q}}\right]\right\}.
\end{aligned} \tag{4.5}
$$

Proof. By setting $s = 1$ in inequality (4.4) of Theorem 5, we get the above inequality (4.5). □

Corollary 8. *The undermentioned inequality holds for s-convex functions via RL k-fractional integrals*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)]\right|\\&\leq\frac{(b-a)\left(|f'(b)|-|f'(a)|\right)}{2}\\&\left\{\left[\frac{2^{\frac{\alpha}{k}+s+1}-2}{2^{\frac{\alpha}{k}+s+1}(\frac{\alpha}{k}+s+1)}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{qs+1}-1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right.\right.\\&\left.\left.+\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{qs+1}(qs+1)}\right]^{\frac{1}{q}}\right]\right\}.\end{aligned}\tag{4.6}$$

Proof. By setting $m = 1$ in inequality (4.4) of Theorem 5, we get the above inequality (4.6). □

Corollary 9. *The undermentioned inequality holds for convex functions via RL k-fractional integrals*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}[I_{a^+}^{\alpha,k}f(b)+I_{b^-}^{\alpha,k}f(a)]\right|\\&\leq\frac{(b-a)\left(|f'(b)|-|f'(a)|\right)}{2}\\&\left\{\left[\frac{2^{\frac{\alpha}{k}+2}-2}{2^{\frac{\alpha}{k}+2}(\frac{\alpha}{k}+2)}-\left[\frac{1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{2^{q+1}-1}{2^{q+1}(q+1)}\right]^{\frac{1}{q}}\right.\right.\\&\left.\left.+\left[\frac{2^{\frac{\alpha}{k}p+1}-1}{2^{\frac{\alpha}{k}p+1}(\frac{\alpha}{k}p+1)}\right]^{\frac{1}{p}}\left[\frac{1}{2^{q+1}(q+1)}\right]^{\frac{1}{q}}\right]\right\}.\end{aligned}\tag{4.7}$$

Proof. By setting $m = 1$ and $s = 1$ in inequality (4.4) of Theorem 5 we get the above inequality (4.7). □

5. Conclusions

In this article we have presented fractional versions of the Hadamard inequality for exponentially (s, m)-convex functions. By applying definitions of exponentially (s, m)-convex function and Riemann-Liouville fractional integrals we have obtained Hadamard type inequalities in different forms. An identity is used to get error estimations of these Hadamard inequalities. Connections of the results of this paper with already known results are also established. In our future work we are finding the refinements of fractional integral inequalities.

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Conflict of interest

It is declared that the author have no competing interests.

Authors contribution

Ghulam Farid and Atiq Ur Rehman proposed the work with the consultation of Chahn Yong Jung, Sidra Bibi make calculations and verifications of results along with Shin Min Kang. Ultimately, all authors have equal contributions.

Agreement on corresponding authors

All authors have agreement on Shin Min Kang and Chahn Yong Jung as the corresponding authors.

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