AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(1): 868–881.
DOI:10.3934/math.2021051
Received: 20 September 2020
Accepted: 27 October 2020
Published: 03 November 2020

Research article

Sign-changing solutions for a class of fractional Kirchhoff-type problem with logarithmic nonlinearity

Qing Yang and Chuanzhi Bai*

School of Mathematical Science, Huaiyin Normal University, Huai'an, Jiangsu Province, 223300, China

* **Correspondence:** Email: czbai@hytc.edu.cn; Tel:13952342299; Fax: +1051783525012.

Abstract: In this paper, we are interested the following fractional Kirchhoff-type problem with logarithmic nonlinearity

$$\begin{cases} \left(a + b \iint_{\Omega^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)(-\Delta)^s u + V(x)u = Q(x)|u|^{p-2}u \ln u^2, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N > 2s$ $(0 < s < 1)$, $(-\Delta)^s$ is the fractional Laplacian, V, Q are continuous, $V, Q \geq 0$. $a, b > 0$ are constants, $4 < p < 2_s^* := \frac{2N}{N-2s}$. By using constraint variational method, a quantitative deformation lemma and some analysis techniques, we obtain the existence of ground state sign-changing solutions for above problem.

Keywords: fractional Kirchhoff-Schrodinger-type equation; sign-changing solutions; logarithmic nonlinearity; variation methods
Mathematics Subject Classification: 35J20, 35J65, 35R11

1. Introduction

In this paper, we consider the following fractional Kirchhoff-Schrödinger-type problem with logarithmic nonlinearity

$$\begin{cases} \left(a + b \iint_{\Omega^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)(-\Delta)^s u + V(x)u = Q(x)|u|^{p-2}u \ln u^2, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N > 2s$ $(0 < s < 1)$, $(-\Delta)^s$ is the fractional Laplacian, defined for any $u \in C_c^\infty(\mathbb{R}^N)$ by

$$(-\Delta)^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{B_\varepsilon(x)^c} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

$a, b > 0$ are constants, $4 < p < 2_s^* := \frac{2N}{N-2s}$, and $V, Q : \Omega \to \mathbb{R}$ satisfy

(H) $V, Q \in C(\Omega, [0, \infty))$, and $V, Q \neq 0$.

We know that logarithmic nonlinearities have many applications in quantum optics, quantum mechanics, transport, nuclear physics and diffusion phenomena etc (see [1] and the reference therein). Recently, many authors have investigated the following logarithmic Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = Q(x)|u|^{p-2}u \ln u^2, & \text{in } \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Many results about logarithmic Schrödinger equation like (1.2) have been obtained, see [2–7] and reference therein. In [8], Chen and Tang studied the ground state sign-changing solutions to elliptic equations with logarithmic nonlinearity of (1.2). The fractional Kirchhoff equation was first introduced in [9]. Recently, Li, Wang and Zhang [10] considered the existence of ground state sign-changing solutions for following p-Laplacian Kirchhoff-type problem with logarithmic nonlinearity

$$\begin{cases} \left(a + b \int_\Omega |\nabla u|^p dx\right) \Delta_p u = |u|^{q-2}u \ln u^2, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

We refer to [11, 12] for a study of existence of sign-changing solutions to (1.2), or more general problems like (1.2) with a logarithmic nonlinearity. Variational methods for non-local operators of elliptic type was first introduced by Fiscela and Valdinoci in [13]. In these years, nonlinear problems involving nonlocal operator have been extent studied, see for instance [14–22] and the references therein. However, to the best of our knowledge, there seem no results on sign-changing solutions for logarithmic fractional Kirchhoff-type problem.

Motivated and inspired by [8, 10] and the aforementioned works, in this paper, we investigate the existence of sign-changing solutions to logarithmic fractional Kirchhoff-type problem (1.1). The main results we get are based on constraint variational method, some analysis techniques and a quantitative deformation lemma. Our result extends the theorem of Chen and Tang [8] from elliptic equations with logarithmic nonlinearity to fractional Kirchhoff-type problem with logarithmic nonlinearity. This article is organized as follows. In Section 2, we give some notations and preliminaries. Section 3 is devoted to the proof of our main result.

2. Preliminaries

For any $s \in (0, 1)$, we define $W^{s,2}(\Omega)$ as a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in $W^{s,2}(\Omega)$ belongs to $L^p(\Omega)$ and

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy < \infty.$$

Equip $W^{s,2}(\Omega)$ with the norm

$$\|u\|_{W^{s,2}(\Omega)} = \|u\|_p + \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/p}.$$

Then $W^{s,2}(\Omega)$ is a Banach space. The space $W_0^{s,2}(\Omega) = \{u \in W^{s,2}(\Omega) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$ endowed with the norm

$$[u] = \left(\iint_{\Omega^2} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{1/2}.$$

Let

$$E := \left\{u \in W_0^{s,2}(\Omega) : \int_\Omega V(x)|u|^2dx < +\infty\right\}$$

endowed with the norm

$$\|u\|_a := \left(a[u]^2 + \int_\Omega V(x)|u|^2dx\right)^{1/2}.$$

Now, we define the energy functional $\mathcal{J} : E \to \mathbb{R}$ associated with problem (1.1) by

$$\mathcal{J}(u) = \frac{1}{2}\|u\|_a^2 + \frac{b}{4}[u]^4 + \frac{2}{p^2}\int_\Omega Q(x)|u|^pdx - \frac{1}{p}\int_\Omega Q(x)|u|^p \ln u^2dx. \tag{2.1}$$

For each $q \in (p, 2_s^*)$, one has that

$$\lim_{t\to 0}\frac{Q(x)|t|^{p-1}\ln t^2}{|t|} = 0, \quad \lim_{t\to\infty}\frac{Q(x)|t|^{p-1}\ln t^2}{|t|^{q-1}} = 0.$$

Then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$Q(x)|t|^{p-1}|\ln t^2| \le \varepsilon|t| + C_\varepsilon|t|^{q-1}, \quad \forall x \in \Omega,\ t \in \mathbb{R}. \tag{2.2}$$

By (2.2), we know that $\mathcal{J}$ is well defined and $\mathcal{J} \in C^1(E, \mathbb{R})$ with

$$\begin{aligned}\langle \mathcal{J}'(u), v\rangle = (a + b[u]^2)&\iint_{\Omega^2}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy\\ &+ \int_\Omega V(x)uvdx - \int_\Omega Q(x)|u|^{p-2}uv\ln u^2dx, \quad \forall u, v \in E.\end{aligned} \tag{2.3}$$

Obviously, if $u \in E$ is a critical point of $\mathcal{J}$, then u is a weak solution of (1.1).

If $u \in E$ is a solution of (1.1) and $u^\pm \neq 0$, then u is a sign-changing solution of (1.1), where

$$u^+(x) := \max\{u(x), 0\}, \quad u^-(x) := \min\{u(x), 0\}.$$

The Nehari manifold for $\mathcal{J}$ is defined as

$$\mathcal{N} = \{u \in E \setminus \{0\} : \langle \mathcal{J}'(u), u\rangle = 0\}.$$

Moreover, we define the nodal set

$$\mathcal{M} := \{w \in \mathcal{N} : w^\pm \neq 0,\ \langle \mathcal{J}'(w), w^+\rangle = \langle \mathcal{J}'(w), w^-\rangle = 0\}.$$

Lemma 2.1. *The following inequalities hold :*

(1). $2(1-x^p)+px^p\ln x^2\geq 0,\quad \forall x\in[0,1)\cup(1,+\infty),\quad p>2;$

(2). $\dfrac{1-x^2}{2}-\dfrac{1-x^p}{p}>0,\quad \forall x\in[0,1)\cup(1,+\infty),\quad p>2;$

(3). $1-xy-\dfrac{2-x^p-y^p}{p}\geq 0,\quad \forall x,y\geq 0,\quad p>2;$

(4). $\dfrac{1-x^4}{4}-\dfrac{1-x^p}{p}\geq 0,\quad \forall x\geq 0,\quad p>4;$

(5). $\dfrac{1-x^2y^2}{2}-\dfrac{2-x^p-y^p}{p}\geq 0,\quad \forall x,y\geq 0,\quad p>4;$

(6). $1-x^3y-\dfrac{4-3x^p-y^p}{p}\geq 0,\quad \forall x,y\geq 0,\quad p>4.$

Proof. Here we only prove (6) holds, the proof of other cases are similar, we can omit it. Let

$$f(x,y)=1-x^3y-\frac{4-3x^p-y^p}{p},\quad x,y\geq 0.$$

The critical points of f must satisfy the system of equations :

$$0=f_1(x,y)=-3x^2y+3x^{p-1},$$

$$0=f_2(x,y)=-x^3+y^{p-1}.$$

Hence, the critical points of f are $(0,0)$ and $(1,1)$. Since $A=f_{11}(1,1)=3(p-3)>0$, $B=f_{12}(1,1)=-3$, $C=f_{22}(1,1)=p-1$, and $B^2-AC=9-3(p-3)(p-1)<0$, which implies that f has a local minimum value at $(1,1)$, and $f(1,1)=0$. Obviously, $f(0,0)=1-\frac{4}{p}>0$. So, for any $x,y\geq 0$, we have that $f(x,y)\geq \min f(x,y)=f(1,1)=0$.

Lemma 2.2. *For each $u\in E$ and $\alpha,\beta\geq 0$, we have*

$$\begin{aligned}\mathcal{J}(u)\geq{}&\mathcal{J}(\alpha u^++\beta u^-)+\frac{1-\alpha^p}{p}\langle\mathcal{J}'(u),u^+\rangle+\frac{1-\beta^p}{p}\langle\mathcal{J}'(u),u^-\rangle\\&+\left(\frac{1-\alpha^2}{2}-\frac{1-\alpha^p}{p}\right)\|u^+\|_a^2+\left(\frac{1-\beta^2}{2}-\frac{1-\beta^p}{p}\right)\|u^-\|_a^2\\&+b\left(\frac{1-\alpha^4}{4}-\frac{1-\alpha^p}{p}\right)[u^+]^4+b\left(\frac{1-\beta^4}{4}-\frac{1-\beta^p}{p}\right)[u^-]^4\\&+b\left(\frac{1-\alpha^2\beta^2}{2}-\frac{1-\alpha^p}{p}-\frac{1-\beta^p}{p}\right)[u^+]^2[u^-]^2.\end{aligned}\tag{2.4}$$

Proof. From (2.3) in [8], one has

$$\begin{aligned}&\int_\Omega Q(x)|\alpha u^++\beta u^-|^p\ln(\alpha u^++\beta u^-)^2dx\\&\quad=\int_\Omega Q(x)[|\alpha u^+|^p\ln(\alpha u^+)^2+|\beta u^-|^p\ln(\beta u^-)^2]dx.\end{aligned}\tag{2.5}$$

By a direct calculation, we easily obtain that

$$\begin{aligned}\|\alpha u^+ + \beta u^-\|_a^2 = \alpha^2\left(a[u^+]^2 + \int_\Omega V(x)|u^+|^2dx\right) + \beta^2\left(a[u^-]^2 + \int_\Omega V(x)|u^-|^2dx\right)\\ - 2\alpha\beta\int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy,\end{aligned} \tag{2.6}$$

$$\begin{aligned}[\alpha u^+ + \beta u^-]^4 = {} & \alpha^4[u^+]^4 + \beta^4[u^-]^4 + 2\alpha^2\beta^2[u^+]^2[u^-]^2\\ & - 4\alpha\beta(\alpha^2[u^+]^2 + \beta^2[u^-]^2)\int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy\\ & + 4\alpha^2\beta^2\left(\int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy\right)^2,\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\langle \mathcal{J}'(u), u^\pm\rangle = {} & (a + b[u]^2)\left([u^\pm]^2 - \int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy\right)\\ & + \int_\Omega V(x)(u^\pm)^2dx - \int_\Omega Q(x)|u^\pm|^p\ln(u^\pm)^2dx\\ = {} & a[u^\pm]^2 + b[u^\pm]^2([u^+]^2 + [u^-]^2)\\ & - (a + b([u^+]^2 + [u^-]^2 + 2[u^\pm]^2))\int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy\\ & + 2b\left(\int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy\right)^2\\ & + \int_\Omega V(x)(u^\pm)^2dx - \int_\Omega Q(x)|u^\pm|^p\ln(u^\pm)^2dx.\end{aligned} \tag{2.8}$$

Thus, it follows from (2.5)–(2.8), Lemma 2.1 and $u^+(x)u^-(y) + u^+(y)u^-(x) \le 0$ that

$$\begin{aligned}\mathcal{J}(u) - \mathcal{J}(\alpha u^+ + \beta u^-) = {} & \frac{1}{2}(\|u^+ + u^-\|_a^2 - \|\alpha u^+ + \beta u^-\|_a^2)\\ & + \frac{b}{4}([u^+ + u^-]^4 - [\alpha u^+ + \beta u^-]^4) + \frac{2}{p^2}\int_\Omega Q(x)[|u^+ + u^-|^p - |\alpha u^+ + \beta u^-|^p]dx\\ & - \frac{1}{p}\int_\Omega Q(x)[|u^+ + u^-|^p\ln(u^+ + u^-)^2 - |\alpha u^+ + \beta u^-|^p\ln(\alpha u^+ + \beta u^-)^2]dx\\ = {} & \frac{1-\alpha^p}{p}\langle \mathcal{J}'(u), u^+\rangle + \frac{1-\beta^p}{p}\langle \mathcal{J}'(u), u^-\rangle\\ & + \left(\frac{1-\alpha^2}{2} - \frac{1-\alpha^p}{p}\right)\|u^+\|_a^2 + \left(\frac{1-\beta^2}{2} - \frac{1-\beta^p}{p}\right)\|u^-\|_a^2\\ & - a\left(1 - \alpha\beta - \frac{1-\alpha^p}{p} - \frac{1-\beta^p}{p}\right)\int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy\end{aligned}$$

$$
\begin{aligned}
&+ b\left(\frac{1-\alpha^4}{4} - \frac{1-\alpha^p}{p}\right)[u^+]^4 + b\left(\frac{1-\beta^4}{4} - \frac{1-\beta^p}{p}\right)[u^-]^4 \\
&+ b\left(\frac{1-\alpha^2\beta^2}{2} - \frac{1-\alpha^p}{p} - \frac{1-\beta^p}{p}\right)[u^+]^2[u^-]^2 \\
&- b\left(1-\alpha^3\beta - \frac{3(1-\alpha^p)}{p} - \frac{1-\beta^p}{p}\right)[u^+]^2 \int_\Omega\int_\Omega \frac{u^+(x)u^-(y)+u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy \\
&- b\left(1-\alpha\beta^3 - \frac{1-\alpha^p}{p} - \frac{3(1-\beta^p)}{p}\right)[u^-]^2 \int_\Omega\int_\Omega \frac{u^+(x)u^-(y)+u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy \\
&+ b\left(1-\alpha^2\beta^2 - \frac{2(1-\alpha^p)}{p} - \frac{2(1-\beta^p)}{p}\right)[u^-]^2 \left(\int_\Omega\int_\Omega \frac{u^+(x)u^-(y)+u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy\right)^2 \\
&+ \left(\frac{2(1-\alpha^p)}{p^2} + \frac{\alpha^p\ln\alpha^2}{p}\right)\int_\Omega Q(x)|u^+|^p dx + \left(\frac{2(1-\beta^p)}{p^2} + \frac{\beta^p\ln\beta^2}{p}\right)\int_\Omega Q(x)|u^-|^p dx \\
\geq\ & \frac{1-\alpha^p}{p}\langle \mathcal{J}'(u), u^+\rangle + \frac{1-\beta^p}{p}\langle \mathcal{J}'(u), u^-\rangle \\
&+ \left(\frac{1-\alpha^2}{2} - \frac{1-\alpha^p}{p}\right)\|u^+\|_a^2 + \left(\frac{1-\beta^2}{2} - \frac{1-\beta^p}{p}\right)\|u^-\|_a^2 \\
&+ b\left(\frac{1-\alpha^4}{4} - \frac{1-\alpha^p}{p}\right)[u^+]^4 + b\left(\frac{1-\beta^4}{4} - \frac{1-\beta^p}{p}\right)[u^-]^4 \\
&+ b\left(\frac{1-\alpha^2\beta^2}{2} - \frac{1-\alpha^p}{p} - \frac{1-\beta^p}{p}\right)[u^+]^2[u^-]^2,
\end{aligned}
$$

which implies that (2.4) holds for all $u \in E$ and $\alpha, \beta \geq 0$.

According to Lemma 2.2, we have the following corollaries.

Corollary 2.3. *For each $u \in E$ and $t \geq 0$, we get that*

$$\mathcal{J}(u) \geq \mathcal{J}(tu) + \frac{1-t^p}{p}\langle \mathcal{J}'(u), u\rangle + \left(\frac{1-t^2}{2} - \frac{1-t^p}{p}\right)\|u\|_a^2.$$

Corollary 2.4. *For each $u \in \mathcal{M}$, there holds*

$$\mathcal{J}(u^+ + u^-) = \max_{\alpha,\beta\geq 0} \mathcal{J}(\alpha u^+ + \beta u^-).$$

Corollary 2.5. *For each $u \in \mathcal{N}$, we have that*

$$\mathcal{J}(u) = \max_{t\geq 0} \mathcal{J}(tu).$$

Lemma 2.6. *Let $4 < p < 2_s^*$. For each $u \in E$, we have*

(i) If $u \neq 0$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$;

(ii) If $u^\pm \neq 0$, there exists a unique pair (α_u, β_u) of positive numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}$.

Proof. (i) For any $u \in E \setminus \{0\}$, set

$$
\begin{aligned}
f_u(t) &= \langle \mathcal{J}_\lambda'(tu), tu\rangle \\
&= t^2\|u\|_a^2 + bt^4[u]^4 - t^p\int_\Omega Q(x)|u|^p\ln(tu)^2 dx, \quad t > 0.
\end{aligned}
\tag{2.9}
$$

From (2.2), $p > 4$ and (2.9), it is easy to see that $\lim_{t\to 0^+} f_u(t) = 0$, $f_u(t) > 0$ for $t > 0$ small and $f_u(t) < 0$ for t large. Thanks to the continuity of $f_u(t)$, there is $t_u > 0$ such that $f_u(t) = 0$. In the following, we prove that t_u is unique. Arguing by contradiction, we assume that there exist two positive constants $t_1 \neq t_2$ such that $f_u(t_1) = f_u(t_2) = 0$, that is $t_1 u, t_2 u \in \mathcal{N}$. By Corollary 2.3 and Lemma 2.1 (2), we get

$$\begin{aligned}\mathcal{J}(t_1 u) &\geq \mathcal{J}(t_2 u) + \frac{1-\left(\frac{t_2}{t_1}\right)^p}{p}\langle \mathcal{J}'(t_1 u), t_1 u\rangle \\ &\quad + t_1^2\left(\frac{1-\left(\frac{t_2}{t_1}\right)^2}{2} - \frac{1-\left(\frac{t_2}{t_1}\right)^p}{p}\right)\|u\|_a^2 > \mathcal{J}(t_2 u)\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}(t_2 u) &\geq \mathcal{J}(t_1 u) + \frac{1-\left(\frac{t_1}{t_2}\right)^p}{p}\langle \mathcal{J}'(t_2 u), t_2 u\rangle \\ &\quad + t_2^2\left(\frac{1-\left(\frac{t_1}{t_2}\right)^2}{2} - \frac{1-\left(\frac{t_1}{t_2}\right)^p}{p}\right)\|u\|_a^2 > \mathcal{J}(t_1 u),\end{aligned}$$

which is absurd. Thus, $t_u > 0$ is unique.

(ii) For each $u \in E$ with $u^{\pm} \neq 0$, in view of Lemma 2.6 (i), there exists a pair (α_u, β_u) of positive numbers such that $\alpha_u u^+, \beta_u u^- \in \mathcal{N}$. Let

$$\begin{aligned}H(\alpha,\beta) &= \langle \mathcal{J}(\alpha u^+ + \beta u^-), \alpha u^+\rangle \\ &= \alpha^2\|u^+\|_a^2 + b\alpha^4[u^+]^4 + b\alpha^2\beta^2[u^+]^2[u^-]^2 \\ &\quad - b\alpha\beta(3\alpha^2[u^+]^2 + \beta^2[u^-]^2)\int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy \\ &\quad + 2b\alpha^2\beta^2\left(\int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy\right)^2 \\ &\quad - \int_\Omega Q(x)|\alpha u^+|^p \ln(\alpha u^+)^2 dx,\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}K(\alpha,\beta) &= \langle \mathcal{J}(\alpha u^+ + \beta u^-), \beta u^-\rangle \\ &= \beta^2\|u^-\|_a^2 + b\beta^4[u^-]^4 + b\alpha^2\beta^2[u^+]^2[u^-]^2 \\ &\quad - b\alpha\beta(\alpha^2[u^+]^2 + 3\beta^2[u^-]^2)\int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy \\ &\quad + 2b\alpha^2\beta^2\left(\int_\Omega\int_\Omega \frac{u^+(x)u^-(y) + u^+(y)u^-(x)}{|x-y|^{N+2s}}dxdy\right)^2 \\ &\quad - \int_\Omega Q(x)|\beta u^-|^p \ln(\beta u^-)^2 dx.\end{aligned} \tag{2.11}$$

Since $4 < p < 2_s^*$, it follows from (2.2) that

$$H(\alpha,\alpha) > 0, \quad K(\alpha,\alpha) > 0, \quad \text{for } \alpha > 0 \text{ small enough,}$$

$$H(\beta,\beta) < 0, \quad K(\beta,\beta) < 0, \quad \text{for } \beta > 0 \text{ large enough.}$$

So, there exist $0 < t_1 < t_2$ such that

$$H(t_1, t_1) > 0, \quad K(t_1, t_1) > 0, \quad H(t_2, t_2) < 0, \quad K(t_2, t_2) < 0. \tag{2.12}$$

Combining (2.10), (2.11) with (2.12), we obtain that

$$H(t_1,\beta) > 0, \quad H(t_2,\beta) < 0, \quad \forall \beta \in [t_1, t_2] \tag{2.13}$$

and

$$K(\alpha, t_1) > 0, \quad K(\alpha, t_2) < 0, \quad \forall \alpha \in [t_1, t_2]. \tag{2.14}$$

Hence, thanks to (2.13), (2.14) and Miranda's Theorem [23], there exists some pair (α_u,β_u) with $t_1 < \alpha_u,\beta_u < t_2$ such that

$$H(\alpha_u,\beta_u) = K(\alpha_u,\beta_u) = 0.$$

These show that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}$. The proof of unique of (α_u,β_u) is similar to that of (i), we omit detail here.

From Corollaries 2.4, 2.5, and Lemma 2.6, we can deduce the following lemma.

Lemma 2.7. *The following minimax characterization hold*

$$\inf_{u\in\mathcal{N}} \mathcal{J}(u) =: c = \inf_{u\in E, u\neq 0} \max_{t\geq 0} \mathcal{J}(tu)$$

and

$$\inf_{u\in\mathcal{M}} \mathcal{J}(u) =: m = \inf_{u\in E, u\neq 0} \max_{\alpha,\beta\geq 0} \mathcal{J}(\alpha u^+ + \beta u^-).$$

Lemma 2.8. $c > 0$ *and* $m > 0$ *are achieved.*

Proof. We only prove that $m > 0$ and is achieved since the other case is similar. For each $u \in \mathcal{M}$, one has $\langle \mathcal{J}'(u), u\rangle = 0$ and then by (2.2) and fractional Sobolev embedding theorem, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} a[u]^2 \leq a\|u\|_a^2 \leq a\|u\|_a^2 + b[u]^4 &= \int_\Omega Q(x)|u|^p \ln u^2 dx \\ &\leq \frac{a}{2}[u]^2 + C_1[u]^q, \quad u \in \mathcal{M}. \end{aligned} \tag{2.15}$$

Since $q > p > 4$, by (2.15), there exists a constant $\rho > 0$ such that $[u] \geq \rho$ for each $u \in \mathcal{M}$.

Let $\{u_n\} \subset \mathcal{M}$ be such that $\mathcal{J}(u_n) \to m$. From (2.1) and (2.3), we have

$$\begin{aligned} m + o(1) &= \mathcal{J}(u_n) - \frac{1}{p}\langle \mathcal{J}'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_a^2 + \left(\frac{b}{4} - \frac{b}{p}\right)[u_n]^4 + \frac{2}{p^2}\int_\Omega Q(x)|u|^p dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_a^2, \end{aligned} \tag{2.16}$$

which implies that $\{u_n\}$ is bounded. Thus, there exists u_*, in subsequence sense, such that $u_n^\pm \rightharpoonup u_*^\pm$ in E and $u_n^\pm \to u_*^\pm$ in $L^r(\Omega)$ for $2 \le r < 2_s^*$. Since $\{u_n\} \subset \mathcal{M}$, we have $\langle \mathcal{J}'(u_n), u_n^\pm \rangle = 0$, which yields that

$$\begin{aligned}
a\rho^2 &\le a\|u_n^\pm\|_a^2 \le a\|u_n^\pm\|_a^2 + b[u_n^\pm]^4 + b[u_n^+]^2[u_n^-]^2 \\
&\quad - (a + b(u_n^+]^2 + [u_n^-]^2 + 2[u_n^\pm]^2)) \int_\Omega \int_\Omega \frac{u_n^+(x)u_n^-(y) + u_n^+(y)u_n^-(x)}{|x-y|^{N+2s}} dxdy \\
&\quad + 2\left(\int_\Omega \int_\Omega \frac{u_n^+(x)u_n^-(y) + u_n^+(y)u_n^-(x)}{|x-y|^{N+2s}} dxdy \right)^2 \\
&= \int_\Omega Q(x)|u_n^\pm|^p \ln(u_n^\pm)^2 dx \\
&\le \varepsilon \int_\Omega |u_n^\pm| dx + C_\varepsilon \int_\Omega |u_n^\pm|^q dx \\
&\le C_2 \int_\Omega |u_n^\pm|^q dx.
\end{aligned} \tag{2.17}$$

By the compactness of the embedding $W_0^{s,2}(\Omega) \hookrightarrow L^r(\Omega)$, we obtain

$$\int_\Omega |u_*^\pm|^q dx \ge C_3 \rho^2,$$

which implies $u_*^\pm \ne 0$. By the Lebesgue dominated convergence theorem and the weak semicontinuity of norm, one has

$$\begin{aligned}
&a\|u_*^\pm\|_a^2 + b[u_*^\pm]^4 + b[u_*^+]^2[u_*^-]^2 \\
&\quad - (a + b([u_*^+]^2 + [u_*^-]^2 + 2[u_*^\pm]^2)) \int_\Omega \int_\Omega \frac{u_*^+(x)u_*^-(y) + u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}} dxdy \\
&\quad + 2\left(\int_\Omega \int_\Omega \frac{u_*^+(x)u_*^-(y) + u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}} dxdy \right)^2 \\
&\le \liminf_{n\to\infty} \Big[a\|u_n^\pm\|_a^2 + b[u_n^\pm]^4 + b[u_n^+]^2[u_n^-]^2 \\
&\quad -(a + b(u_n^+]^2 + [u_n^-]^2 + 2[u_n^\pm]^2)) \int_\Omega \int_\Omega \frac{u_n^+(x)u_n^-(y) + u_n^+(y)u_n^-(x)}{|x-y|^{N+2s}} dxdy \\
&\quad +2\left(\int_\Omega \int_\Omega \frac{u_n^+(x)u_n^-(y) + u_n^+(y)u_n^-(x)}{|x-y|^{N+2s}} dxdy \right)^2 \Big] \\
&= \liminf_{n\to\infty} \int_\Omega Q(x)|u_n^\pm|^p \ln(u_n^\pm)^2 dx \\
&= \int_\Omega Q(x)|u_*^\pm|^p \ln(u_*^\pm)^2 dx,
\end{aligned}$$

which yields that

$$\langle \mathcal{J}'(u_*), u_*^+ \rangle \le 0 \quad \langle \mathcal{J}'(u_*), u_*^- \rangle \le 0.$$

In view of Lemma 2.6 (ii), there exist constants $\alpha, \beta > 0$ such that $\alpha u_*^+ + \beta u_*^- \in \mathcal{M}$. Thus, from (2.1), (2.3), (2.4), Lemma 2.1 and the weak semicontinuity of norm, we obtain that

$$\begin{aligned}
m &= \lim_{n\to\infty}\left[\mathcal{J}(u_n) - \frac{1}{p}\langle \mathcal{J}'(u_n), u_n\rangle\right] \\
&= \lim_{n\to\infty}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|_a^2 + \left(\frac{b}{4}-\frac{b}{p}\right)[u_n]^4 + \frac{2}{p^2}\int_\Omega Q(x)|u_n|^p dx\right] \\
&\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u_*\|_a^2 + \left(\frac{b}{4}-\frac{b}{p}\right)[u_*]^4 + \frac{2}{p^2}\int_\Omega Q(x)|u_*|^p dx \\
&= \mathcal{J}(u_*) - \frac{1}{p}\langle \mathcal{J}'(u_*), u_*\rangle \\
&\geq \mathcal{J}(\alpha u_*^+ + \beta u_*^-) + \frac{1-\alpha^p}{p}\langle \mathcal{J}'(u_*), u_*^+\rangle + \frac{1-\beta^p}{p}\langle \mathcal{J}'(u_*), u_*^-\rangle - \frac{1}{p}\langle \mathcal{J}'(u_*), u_*\rangle \\
&\geq m - \frac{\alpha^p}{p}\langle \mathcal{J}'(u_*), u_*^+\rangle - \frac{\beta^p}{p}\langle \mathcal{J}'(u_*), u_*^-\rangle \geq m,
\end{aligned}$$

which shows

$$\langle \mathcal{J}'(u_*), u_*^\pm\rangle = 0, \quad \mathcal{J}(u_*) = m.$$

Moreover, it follows from $u_*^\pm \neq 0$, $\langle \mathcal{J}'(u_*), u_*\rangle = 0$ and (2.6) that

$$\begin{aligned}
m = \mathcal{J}(u_*) &= \mathcal{J}(u_*) - \frac{1}{p}\langle \mathcal{J}'(u_*), u_*\rangle \\
&= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_*\|_a^2 + \left(\frac{b}{4}-\frac{b}{p}\right)[u_*]^4 + \frac{2}{p^2}\int_\Omega Q(x)|u_*|^p dx \\
&\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u_*\|_a^2 \geq \left(\frac{1}{2}-\frac{1}{p}\right)(\|u_*^+\|_a^2 + \|u_*^-\|_a^2) > 0.
\end{aligned}$$

3. Main result

In this section, we will give the main result and proof.

Lemma 3.1. *The minimizers of* $\inf_{\mathcal{N}}\mathcal{J}$ *and* $\inf_{\mathcal{M}}\mathcal{J}$ *are critical points of* $\mathcal{J}$.

Proof. Thanks to Lemma 2.8, we prove the minimizer u_* of $\inf_{\mathcal{M}}\mathcal{J}$ is critical point of $\mathcal{J}$. Arguing by contradiction, we assume that $u_* = u_*^+ + u_*^- \in \mathcal{M}$, $\mathcal{J}(u_*) = m$ and $\mathcal{J}'(u_*) \neq 0$. Then there exist $\delta > 0$ and $\gamma > 0$ such that

$$\|\mathcal{J}'(u)\| \geq \gamma, \quad \text{for all } \|u - u_*\| \leq 3\delta \text{ and } u \in E.$$

Set $D = \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right)$. By Lemma 2.2, one has

$$\varrho := \max_{(\alpha,\beta)\in\partial D} \mathcal{J}(\alpha u_*^+ + \beta u_*^-) < m.$$

Let $\varepsilon := \min\{(m-\varrho)/3, \delta\gamma/8\}$ and $S_\delta := B(u_*, \delta)$. By applying the Lemma 2.3 in Ref. [24], there exists a deformation $\eta \in C([0,1]\times E, E)$ such that

(i) $\eta(1, v) = v$ if $v \notin \mathcal{J}^{-1}([m-2\varepsilon, m+2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, \mathcal{J}^{m+\varepsilon} \cap S_\delta) \subset \mathcal{J}^{m-\varepsilon}$;

(iii) $\mathcal{J}(\eta(1,v)) \leq \mathcal{J}(v)$, $\forall v \in E$.

From (iii) and Lemma 2.2, for each $\alpha,\beta > 0$ with $|\alpha-1|^2 + |\beta-1|^2 \geq \delta^2/\|u_*\|^2$, one has

$$\mathcal{J}(\eta(1,\alpha u_*^+ + \beta u_*^-)) \leq \mathcal{J}(\alpha u_*^+ + \beta u_*^-) < \mathcal{J}(u_*) = m. \tag{3.1}$$

By Corollary 2.4, we have $\mathcal{J}(\alpha u_*^+ + \beta u_*^-) \leq \mathcal{J}(u_*) = m$ for $\alpha,\beta > 0$. According to (ii), one has

$$\mathcal{J}(\eta(1,\alpha u_*^+ + \beta u_*^-)) \leq m - \varepsilon, \quad \forall \alpha,\beta > 0,\ |\alpha-1|^2 + |\beta-1|^2 < \delta^2/\|u_*\|^2. \tag{3.2}$$

Thus, from (3.1) and (3.2), we obtain

$$\max_{(\alpha,\beta)\in\bar{D}} \mathcal{J}(\eta(1,\alpha u_*^+ + \beta u_*^-)) < m. \tag{3.3}$$

Let $h(\alpha,\beta) = \alpha u_*^+ + \beta u_*^-$, we will prove that $\eta(1,h(D)) \cap \mathcal{J} \neq \emptyset$.
Define

$$k(\alpha,\beta) := \eta(1,h(\alpha,\beta)),$$

$$\Phi(\alpha,\beta) := (\langle \mathcal{J}'(h(\alpha,\beta)), u_*^+\rangle, \langle \mathcal{J}'(h(\alpha,\beta)), u_*^-\rangle) := (\Phi_1(\alpha,\beta), \Phi_2(\alpha,\beta))$$

$$\Psi(\alpha,\beta) := \left(\frac{1}{\alpha}\langle \mathcal{J}'(k(\alpha,\beta)), (k(\alpha,\beta))^+\rangle, \frac{1}{\beta}\langle \mathcal{J}'(k(\alpha,\beta)), (k(\alpha,\beta))^-\rangle\right).$$

Obviously, Φ is a C^1 functions. Moreover, we have by a direct calculation that

$$\begin{aligned}\frac{\partial \Phi_1(\alpha,\beta)}{\partial \alpha}|_{(1,1)} = {} & \|u_*^+\|_a^2 + 3b[u_*^+]^4 + b[u_*^+]^2[u_*^-]^2 \\ & - 6b[u_*^+]^2 \int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y) + u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy \\ & + 2b\left(\int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y) + u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy\right)^2 \\ & - (p-1)\int_\Omega Q(x)|u_*^+|^p \ln(u_*^+)^2 dx - 2\int_\Omega Q(x)|u_*^+|^p dx,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \Phi_1(\alpha,\beta)}{\partial \beta}|_{(1,1)} = {} & 2b[u_*^+]^2[u_*^-]^2 - [a + 3b([u_*^+]^2 + [u_*^-]^2)]\int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y) + u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy \\ & + 4b\left(\int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y) + u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy\right)^2.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\frac{\partial \Phi_2(\alpha,\beta)}{\partial \beta}|_{(1,1)} = {} & \|u_*^-\|_a^2 + 3b[u_*^-]^4 + b[u_*^+]^2[u_*^-]^2 \\ & - 6b[u_*^-]^2 \int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y) + u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy \\ & + 2b\left(\int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y) + u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy\right)^2 \\ & - (p-1)\int_\Omega Q(x)|u_*^-|^p \ln(u_*^-)^2 dx - 2\int_\Omega Q(x)|u_*^-|^p dx,\end{aligned}$$

and

$$\frac{\partial\Phi_2(\alpha,\beta)}{\partial\alpha}|_{(1,1)} = 2b[u_*^+]^2[u_*^-]^2 - [a+3b([u_*^+]^2+[u_*^-]^2)]\int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y)+u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy$$
$$+4b\left(\int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y)+u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy\right)^2.$$

It is easy to check that

$$\begin{vmatrix} \frac{\partial\Phi_1(\alpha,\beta)}{\partial\alpha}|_{(1,1)} & \frac{\partial\Phi_2(\alpha,\beta)}{\partial\alpha}|_{(1,1)} \\ \frac{\partial\Phi_1(\alpha,\beta)}{\partial\beta}|_{(1,1)} & \frac{\partial\Phi_2(\alpha,\beta)}{\partial\beta}|_{(1,1)} \end{vmatrix} \neq 0.$$

Thus, by degree theory [25, 26], we can derive that $\Psi(\alpha_0,\beta_0) = 0$ for some $(\alpha_0,\beta_0) \in D$, so that $\eta(1,h(\alpha_0,\beta_0)) = k(\alpha_0,\beta_0) \in \mathcal{M}$. This contradicts (3.3) and shows that $\mathcal{J}'(u_*) = 0$. Similarly, we can prove that any minimizer of $\inf_{\mathcal{N}}\mathcal{J}$ is a critical point of $\mathcal{J}$.

Now, we are in a position to prove our main result.

Theorem 3.2. *Suppose that condition (H) holds. If* $4 < p < 2_s^*$, *then problem (1.1) has a solution* $u_0 \in \mathcal{N}$ *and a sign-changing solution* $u_* \in \mathcal{M}$ *such that*

$$\inf_{\mathcal{M}}\mathcal{J} = \mathcal{J}(u_*) \geq 2\mathcal{J}(u_0) = 2\inf_{\mathcal{N}}\mathcal{J} > 0.$$

Proof. By Lemmas 2.8 and 3.1, there exist $u_0 \in \mathcal{N}$ and $u_* \in \mathcal{M}$ such that $\mathcal{J}(u_0) = c$ with $\mathcal{J}'(u_0) = 0$, and $\mathcal{J}(u_*) = m$ with $\mathcal{J}'(u_*) = 0$. That is, problem (1.1) has a solution $u_0 \in \mathcal{N}$ and a sign-changing solution $u_* \in \mathcal{M}$. Moreover, by (2.5)–(2.7), Corollary 2.4 and Lemma 2.7, we get

$$\begin{aligned} m = \mathcal{J}(u_*) &= \sup_{\alpha,\beta\geq 0}\mathcal{J}(\alpha u_*^+ + \beta u_*^-) \\ &= \sup_{\alpha,\beta\geq 0}\Big[\frac{1}{2}\|\alpha u_*^+ + \beta u_*^-\|_a^2 + \frac{b}{4}[\alpha u_*^+ + \beta u_*^-]^4 + \frac{2}{p^2}\int_\Omega |\alpha u_*^+ + \beta u_*^-|^p dx \\ &\quad -\frac{1}{p}\int_\Omega Q(x)|\alpha u_*^+ + \beta u_*^-|^p \ln(\alpha u_*^+ + \beta u_*^-)^2 dx\Big] \\ &= \sup_{\alpha,\beta\geq 0}\Big[\mathcal{J}(\alpha u_*^+) + \mathcal{J}(\beta u_*^-) - \alpha\beta\int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y)+u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy \\ &\quad +\frac{b}{2}\alpha^2\beta^2[u_*^+]^2[u_*^-]^2 + b\alpha^2\beta^2\left(\int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y)+u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy\right)^2 \\ &\quad -b\alpha\beta(\alpha^2[u_*^+]^2 + \beta^2[u_*^-]^2)\int_\Omega\int_\Omega \frac{u_*^+(x)u_*^-(y)+u_*^+(y)u_*^-(x)}{|x-y|^{N+2s}}dxdy\Big] \\ &\geq \sup_{\alpha\geq 0}\mathcal{J}(\alpha u_*^+) + \sup_{\beta\geq 0}\mathcal{J}(\beta u_*^-) \geq 2c > 0. \end{aligned}$$

Remark 3.3. In [8, 10], (1.2) and (1.3) has a sign-changing solution with precisely two nodal domains has been proved respectively. By Theorem 3.2, we know that (1.1) has a sign-changing solution. But according to the method is used in [8, 10], we cannot prove that the sign-changing solution of (1.1) has precisely two nodal domains.

Acknowledgments

The authors thanks editor and anonymous referees for their remarkable comments, suggestion that help to improve this paper. This work is supported by Natural Science Foundation of China (11571136).

Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

References

1. K. G. Zloshchastiev, Logarithmic nonlinearity in theories of quantum gravity: Origin of time and observational consequences, *Grav. Cosmol.*, **16** (2010), 288–297.
2. P. D'Avenia, E. Montefusco, M. Squassina, On the logarithmic Schrodinger equation, *Commun. Contemp. Math.*, **16** (2014), 313–402.
3. M. Squassina, A. Szulkin, Multiple solutions to logarithmic Schrodinger equations with periodic potential, *Calc. Var.*, **54** (2015), 585–597.
4. W. C. Troy, Uniqueness of positive ground state solutions of the logarithmic Schrodinger equation, *Arch. Ration. Mech. Anal.*, **222** (2016), 1581–1600.
5. S. Tian, Multiple solutions for the semilinear elliptic equations with the sign-changing logarithmic nonlinearity, *J. Math. Anal. Appl.*, **454** (2017), 816–228.
6. C. Ji, A. Szulkin, A logarithmic Schrodinger equation with asymptotic conditions on the potential, *J. Math. Anal. Appl.*, **437** (2016), 241–254.
7. C. O. Alves, D. C. de Morais Filho, Existence and concentration of positive solutions for a Schrodinger logarithmic equation, *Z. Angew. Math. Phys.*, **69** (2018), 144.
8. S. Chen, X. H. Tang, Ground state sign-changing solutions for elliptic equations with logarithmic nonlinearity, *Acta Math. Hungar.*, **157** (2019), 27–38.
9. A. Fiscela, E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, *Nonlinear Anal.*, **94** (2014), 156–170.
10. Y. Li, D. Wang, J. Zhang, Sign-changing solutions for a class of p-Laplacian Kirchhoff-type problem with logarithmic nonlinearity, *AIMS Mathematics*, **5** (2020), 2100–2112.
11. L. Wen, X. H. Tang, S. Chen, Ground state sign-changing solutions for Kirchhoff equations with logarithmic nonlinearity, *Electron. J. Qual. Theor.*, **47** (2019), 1–13.
12. W. Shuai, Multiple solutions for logarithmic Schrodinger equations, *Nonlinearity*, **32** (2019), 2201–2225.
13. R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discret. Contin. Dyn. Syst.*, **33** (2013), 2105–2137.
14. Q. Zhou, K. Wang, Existence and multiplicity of solutions for nonlinear elliptic problems with the fractional Laplacian, *Fract. Calc. Appl. Anal.*, **18** (2015), 133–145.

15. D. Yang, C. Bai, Multiplicity of weak positive solutions for fractional $p\&q$ Laplacian problem with singular nonlinearity, *J. Funct. Space.*, **2020** (2020), 1–8.

16. G. M. Bisci, V. D. Radulescu, R. Servadei, *Variational methods for nonlocal fractional problems*, Cambridge: Cambridge University Press, 2016.

17. E. Cinti, F. Colasuonno, A nonlocal supercritical Neumann problem, *J. Differ. Equations*, **268** (2020), 2246–2279.

18. V. Ambrosio, J. Mawhin, G. M. Bisci, (Super)Critical nonlocal equations with periodic boundary conditions, *Sel. Math. New Ser.*, **24** (2018), 3723–3751.

19. C. O. Alves, C. E. T. Ledesma, Fractional elliptic problem in exterior domains with nonlocal Neumann condition, *Nonlinear Anal.*, **195** (2020), 111732.

20. T. Isernia, Sign-changing solutions for a fractional Kirchhoff equation, *Nonlinear Anal.*, **190** (2020), 111623.

21. L. M. Del Pezzo, A. Quaas, A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian, *J. Differ. Equations*, **263** (2017), 765–778.

22. X. Shang, J. Zhang, Y. Yang, Positive solutions of nonhomogeneous fractional Laplacian problem with critical exponent, *Commun. Pure Appl. Anal.*, **13** (2014), 567–584.

23. C. Miranda, Un'osservazione su un teorema di Brouwer, *Bol. Un. Mat. Ital.*, **3** (1940), 5–7.

24. M. Willem, *Minimax theorems*, Bosten: Birkhauser, 1996.

25. K. Deimling, *Nonlinear functional analysis*, Berlin: Springer-Verlag, 1985.

26. D. Guo, *Nonlinear functional analysis*, Beijing: Higher Education Press, 2015.

AIMS Press