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Research article

Cardiac memory phenomenon, time-fractional order nonlinear system and bidomain-torso type model in electrocardiology

Aziz Belmiloudi*

Institut de Recherche Mathématique de Rennes (IRMAR), Université Européenne de Bretagne, 20 avenue Buttes de Coësmes, CS 70839, 35708 Rennes Cédex 7, France

* **Correspondence:** Email: aziz.belmiloudi@math.cnrs.fr.

Abstract: Cardiac memory, also known as the Chatterjee phenomenon, refers to the persistent but reversible T-wave changes on ECG caused by an abnormal electrical activation pattern. After a period of abnormal ventricular activation in which the myocardial repolarization is altered and delayed (such as with artificial pacemakers, tachyarrhythmias with wide QRS complexes or ventricular pre-excitation), the heart remembers and mirrors its repolarization in the direction of the vector of "abnormally" activated QRS complexes. This phenomenon alters patterns of gap junction distribution and generates changes in repolarization seen at the level of ionic-channel, ionic concentrations, ionic-current gating and action potential. In this work, we propose a mathematical model of cardiac electrophysiology which takes into account cardiac memory phenomena. The electrical activity in heart through torso, which is dependent on the prior history of accrued heartbeats, is mathematically modeled by a modified bidomain system with time fractional-order dynamics (which are used to describe processes that exhibit memory). This new bidomain system, that I name " *memory bidomain system*", is a degenerate nonlinear coupled system of reaction-diffusion equations in shape of a fractional-order differential equation coupled with a set of time fractional-order partial differential equations. Cardiac memory is represented via fractional-order capacitor (associate to capacitive current) and fractional-order cellular membrane dynamics. First, mathematical model is introduced. Afterward, results on generalized Gronwall inequality within the framework of coupled fractional differential equations are developed. Next, the existence and uniqueness of solution of state system are proved as well as stability result. Further, some preliminary numerical applications are performed to show that memory reproduced by fractional-order derivatives can play a significant role on key dependent electrical properties including APD, action potential morphology and spontaneous activity.

Keywords: fractional-order dynamics; heart-torso coupling; cellular heterogeneity; integral inequality; weak solution; cardiac memory; memory induced T-wave; ionic models
Mathematics Subject Classification: 35R11, 34A08, 35K57, 35B30, 92C50, 35Q92

1. Introduction and mathematical setting of the problem

1.1. Modeling motivation

The heart is an electrically controlled mechanical pump which drives blood flow through the circulatory system vessels (through deformation of its walls), where electrical impulses trigger mechanical contraction (of various chambers of heart) and whose dysfunction is incompatible with life. The coordinated contraction of heart and the maintenance of heartbeat are controlled by a complex network of interconnected cardiac cells electrically coupled by gap junctions and voltage-gated ion channels. The evaluation of bioelectrical activity in heart is then a very complex process which uses different phenomenological mechanism and subject to various perturbations, and physiological and pathophysiological variations.

The electrical system of a normal heart is highly organized in a steady rhythmic pattern. This normal heartbeat is called sinus rhythm. Irregular or abnormal heartbeats, called arrhythmias, are caused by a change in propagation and/or formation of electrical impulses, that regulate a steady heartbeat, causing a heartbeat that is too fast or too slow, that can remain stable or become chaotic (irregular and disorganized). Many times, arrhythmias are harmless and can occur in healthy people without heart disease; however, some of these rhythms can be serious and require special and efficiency treatments. Fibrillation is one type of arrhythmia and is considered the most serious cardiac rhythm disturbance. It occurs when the heart beats with rapid, erratic electrical impulses (highly disorganized almost chaotic activation). This leads to quivering (or fibrillation) of heart chambers rather than normal contraction, which then leads the heart to lose its ability to pump enough blood through circulatory system. The treatment therapy of these diseases, when it becomes troublesome or when it can present a danger, often uses electrical impulses to stabilize cardiac function and restore the sinus rhythm, by implanting the patients with active cardiac devices (electrotherapy). For example, in case of cardiac rhythms that are too slow, the devices transmit electronic impulses and ensure that periodic contractions of heart are maintained at a hemodynamically sufficient rate; and in the case of a fast or irregular heart rate, the devices monitor heart rate and, if needed, treat episodes of tachyarrhythmia (including tachycardia and/or fibrillation) by transmitting automatically impulses to either give defibrillation shocks or cause overstimulation (via an ICDs*) or synchronize contraction of left and right ventricles.

After cessation of a transient period of abnormal ventricular activation (arrhythmia or pacing) in which the myocardial repolarization is altered and delayed (such as with artificial pacemakers [82], tachyarrhythmias with wide QRS complexes, intermittent left bundle branch block or ventricular pre-excitation observed in Wolff-Parkinson-White syndrome [45]), the heart remembers and mirrors its repolarization in the direction of vector of "abnormally" activated QRS complexes [66]. This remodeling of electrical properties of myocardium is characterized by persistent but reversible T-wave changes on the surface electrocardiogram (ECG). The scope, significance and direction of T-wave deviation depend on duration and direction of abnormal electrical activation. Moreover, these changes are often confused with pathological conditions manifesting with T-wave deviations, such as (acute) myocardial ischemia or infarction. This cumulative and complex phenomenon is named cardiac memory and can persist up for several weeks after normal ventricular conduction is restored. Heart is considered as network of cardiac oscillators communicating via gap junctions between neighboring cells and through voltage gated ion channels (that are activated by changes in electrical membrane

*The so-called implantable cardioverter defibrillators potential near channel): this phenomenon alters patterns of gap junction distribution and generates changes in repolarization seen at the level of ionic-channel, ionic concentrations, ionic-current gating and action potential.

The existence of cardiac memory has been known for many years and resulted in a large number of publications, see for example [23, 37, 44, 58–60, 67, 72, 73, 75] and the references therein. Yet despite all this, this phenomemon is under-recognized and there is still limited information regarding its physiological significance and practical clinical implications because this phenomenon is considered to be a relatively benign pathophysiologic finding. Unfortunately, the work of past few years has shown that despite the mostly benign nature of cardiac memory in healthy individuals under some conditions, it can be the trigger of more complex arrhythmias and then requires emergency care of the patient. Other works estimate that clinically administered antiarrhythmic drugs alter expression of cardiac memory and that generate changes in repolarization could in turn alter the effects of these drugs (see e.g. [4, 63]). Moreover, cardiac memory may also lead to unnecessary and invasive diagnostic investigation that put patients under unnecessary risks (see e.g., [71]).

Then, recognition of cardiac memory as a serious potential cause of T-wave changes and the analysis of T-wave morphology (throughout the ECG during narrow- and wide-QRS rhythms) are critical to help differentiate T-wave changes due to myocardial ischemia from those induced by cardiac memory, and consequently sustainably establish the clinical relevance of this phenomenon, facilitates diagnosis and increases efficiency of cardiac disorders treatment.

Consequently, this has greatly emphasized the need for model and methodologies capable of predicting and understanding the dynamic mechanisms of different sources of electrical instability in heart like cardiac action potential (AP) repolarization alternans which is influenced by memory. At cellular level, alternans is generally manifests as cyclic, beat-to-beat alternations between long and short action potential and/or intracellular Ca transient, and is frequently associated with the development of ventricular tachycardia and fibrillation. It is generally agreed that disturbances of bi-directional (mutual) relationship between transmembrane potential (or membrane voltage) ϕ and Ca-sensitive ionic currents play a key role in generation of alternans. Because membrane voltage ϕ is strongly affected by Ca-sensitive ionic currents and intracellular Ca loading in turn is strongly influenced by ϕ-dependent ionic currents. This complex bidirectional coupling influences and controls the amplitude and duration of action potentials (APD) through various time- and voltage-dependent ionic currents.

The classical bidomain system is commonly used for modeling propagation of electrophysiological waves in cardiac tissue. Motivated by above discussions, to take into account the critical effect of memory in propagation of electrophysiological waves, together with other critical cardiac material parameters, we propose and analyze a new bidomain model, that I name "*memory bidomain system*" by incorporating memory effects. In next section, we shall present derivation of this memory bidomain model.

1.2. Modeling and formulation of the problem

Mathematical and computational cardiac electrophysiological modeling is now an important field in applied mathematics. Indeed, nowadays, heart and cardiovascular diseases are still the leading cause of death and disability all over the world. That is why we need to improve our knowledge about heart behavior, and more particularly about its electrical behavior. Consequently we want strong methods to compute electrical fluctuations in the myocardium to prevent cardiac disorders (as arrhythmia). Tissue-level cardiac electrophysiology, which can provide a bridge between electrophysiological cell models at smaller scales, and tissue mechanics, metabolism and blood flow at larger scales, is usually modeled using the coupled bidomain equations, originally derived in [78], which represent a homogenization of the intracellular and extracellular medium, where electrical currents are governed by Ohm's law (see also e.g. [46] for a review and an introduction to this field). The model was modified and extended to include heart tissue surrounded by a conductive bath or a conductive body (see e.g. [64] and [74]). From mathematical viewpoint, the classical bidomain system is commonly formulated in terms of intracellular and extracellular electrical potentials of anisotropic cardiac tissue (macroscale), ϕ_i and ϕ_e, (or, equivalently, extracellular potential ϕ_e and the transmembrane voltage $\phi = \varphi_i - \varphi_e$) coupled with cellular state variables u describing cellular membrane dynamics and torso potential state variable ϕ_s. This is a system of non-linear partial differential equations (PDEs) coupled with ordinary differential equations (ODEs), in the physical region Ω (occupied by excitable cardiac tissue, which is an open, bounded, and connected subset of d-dimensional Euclidean space $\mathbf{R}^d$, $d \leq 3$). The PDEs describe the propagation of electrical potentials and ODEs describe the electrochemical processes.

Cardiac memory can affect considerably the resulting electrical activity in heart and thus the cardiac disorders therapeutic treatment. It is then necessary to introduce the impact of memory on dynamical behaviors of such a system. Memory terms can cause dynamical instabilities (as Alternans) and give rise to highly complex behavior including oscillations and chaos.

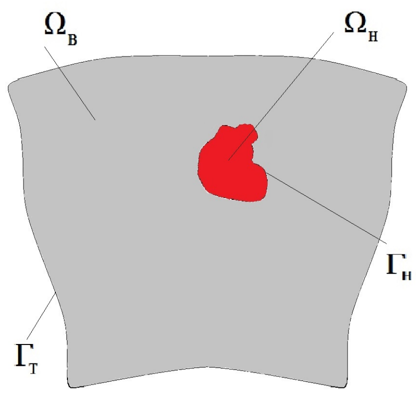

Figure 1. The derived "*Memory bidomain model*" is defined on heart domain Ω_H, while Ω_B is the rest of body.

In order to take into account the influence of cardiac memory and inward movement of u into the cell which prolongs depolarization phase of action potential, we propose a new bidomain model. In this new model, classical bidomain model has been modified via time fractional-order dynamical system, arising due to cellular heterogeneity, which are used to describe processes that exhibit memory. More precisely, the derived system with memory (or history), is a nonlinear coupled reaction-diffusion model in shape of a set of time fractional-order differential equations (FDE) coupled with a set of time fractional-order partial differential equations, in the torso-heart's spatial domain Ω (Figure 1) which is a bounded open subset with a sufficiently regular boundary $\partial\Omega$, and during the final fixed time horizon $T > 0$, as follows

$$\begin{aligned}
&\mathfrak{c}_\alpha \partial^\alpha_{0^+}\phi + \mathcal{I}(.;\phi,u) - div(\mathcal{K}_i\nabla\varphi_i) = I_i, \text{ in } Q_{\mathrm{H}} = \Omega_{\mathrm{H}} \times (0,T)\\
&\mathfrak{c}_\alpha \partial^\alpha_{0^+}\phi + \mathcal{I}(.;\phi,u) + div(\mathcal{K}_e\nabla\varphi_e) = -I_e, \text{ in } Q_{\mathrm{H}}\\
&-div(\mathcal{K}_s\nabla\varphi_s) = 0, \text{ in } Q_{\mathrm{B}} = \Omega_{\mathrm{B}} \times (0,T)\\
&\partial^\beta_{0^+}u + \mathcal{G}(.;\phi,u) = 0, \text{ in } Q_{\mathrm{H}}\\
&\text{subject to initial and boundary conditions (1.4),}
\end{aligned} \tag{1.1}$$

or equivalently

$$\begin{aligned}
&\mathfrak{c}_\alpha \partial^\alpha_{0^+}\phi + \mathcal{I}(.;\phi,u) - div(\mathcal{K}_i\nabla\phi) = div(\mathcal{K}_i\nabla\varphi_e) + I_i, \text{ in } Q_{\mathrm{H}}\\
&-div((\mathcal{K}_e + \mathcal{K}_i)\nabla\varphi_e) = div(\mathcal{K}_i\nabla\phi) + I, \text{ in } Q_{\mathrm{H}}\\
&-div(\mathcal{K}_s\nabla\varphi_s) = 0, \text{ in } Q_{\mathrm{B}}\\
&\partial^\beta_{0^+}u + \mathcal{G}(.;\phi,u) = 0, \text{ in } Q_{\mathrm{H}}\\
&\text{subject to initial and boundary conditions (1.4).}
\end{aligned} \tag{1.2}$$

Here, $\partial^\alpha_{0^+}$ and $\partial^\beta_{0^+}$ denote the forward Caputo fractional derivatives with α and β be real values in $]0,1]$ and the unknowns are the potentials φ_i, φ_e, φ_s and a single ionic variable u (e.g. gating variable, concentration, etc.).

The heart's spatial domain is represented by Ω_{H} which is a bounded open subset, and by $\Gamma_{\mathrm{H}} = \partial\Omega_{\mathrm{H}}$ we denote its piecewise smooth boundary. A distinction is made between the intracellular and extracellular tissues which are separated by the cardiac cellular membrane. The surrounding tissue within the thorax is modeled by a volume conduction Ω_{B} with a piecewise smooth boundary $\Gamma_{\mathrm{B}} = \Gamma_{\mathrm{H}} \cup \Gamma_T$ where Γ_T is the thorax surface. The whole domain is denoted by the $\Omega = \overline{\Omega}_{\mathrm{H}} \cup \Omega_{\mathrm{B}}$. In Ω_{H}, the transmembrane potential is $\phi = \varphi_i - \varphi_e$ where φ_e and φ_i are the transmembrane, extracellular and intracellular potentials, respectively, and in Ω_{B}, φ_s is thoracic medium electric potential. The parameter $\mathfrak{c}_\alpha$ is $\mathfrak{c}_\alpha = \kappa C_\alpha > 0$, where C_α is the membrane capacitance per unit area and κ is the surface area-to-volume ratio (homogenization parameter). The membrane is assumed to be passive, so the capacitance C_α can be assumed to be not a function of state variables. Classically this membrane is assimilate to a simple parallel resistor-capacitor circuit. However various studies showed clearly that a passive membrane may be more appropriately modeled with a non-ideal capacitor, in which the current-voltage relationship is given by a fractional-order derivative (see e.g [84]). So, according to the passivity of tissue we can assimilate this membrane to electrical circuit with a resistor associate to the ionic current (I_{ion}) and a capacitor associate to the capacitive current, $I^\alpha_c = \mathfrak{c}_\alpha \partial^\alpha_{0^+}\phi$, in parallel, with α usually ranging from 0.5 to 1 (Figure 2). Moreover, since the electrical restitution curve (ERC)[†] is affected by the action potential history through ionic memory, we have represented the memory via ionic variable u by a time fractional-order dynamic term $\partial^\beta_{0^+}u$.

[†]which traditionally describes the recovery of APD as a function of the interbeat interval

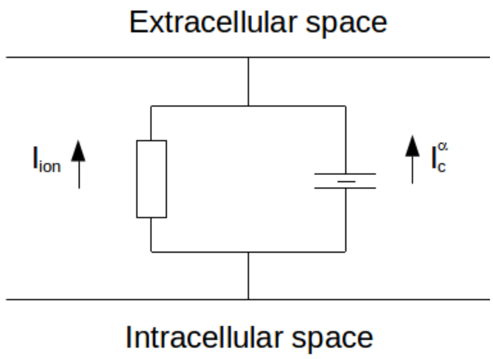

Figure 2. Modeling of the membrane as resistor and non-ideal capacitor coupled in parallel (where $0 < \alpha \leq 1$ is the fractional order of capacitor).

The coupling between equations in Ω_{H} and equation in Ω_{B} (in systems (1.1) and (1.2)) is operate at the heart-torso interface Γ_{H}. The continuity conditions at the interface can be given by

$$\begin{aligned} &\varphi_e = \varphi_s, \text{ in } \Sigma_{\mathrm{H}} = \Gamma_{\mathrm{H}} \times (0, T) \\ &(\mathcal{K}_e \nabla \varphi_e) \cdot \mathbf{n} = (\mathcal{K}_s \nabla \varphi_s) \cdot \mathbf{n}, \quad \text{on } \Sigma_{\mathrm{H}}. \end{aligned} \tag{1.3}$$

where $\mathbf{n}$ is the outward normal to Γ_{H}.

The tensors $\mathcal{K}_i$ and $\mathcal{K}_e$ are the conductivity tensors describing anisotropic intracellular and extracellular conductive media, and tensor $\mathcal{K}_s$ represents the conductivity tensor of thoracic medium. The electrophysiological ionic state u describes a cumulative way of effects of ion transport through cell membranes (which describes e.g., the dynamics of ion-channel and ion concentrations in different cellular compartments). The operator $\mathcal{I}$ is equal to κI_{ion}, where the nonlinear operator I_{ion} describes the sum of transmembrane ionic currents across cell membrane with u. The nonlinear operator $\mathcal{G}$ is representing the ionic activity in myocardium. Functional forms for $\mathcal{I}$ and $\mathcal{G}$ are determined by an electrophysiological cell model. The source terms are $I_i = \kappa f_i$, $I_e = \kappa f_e$ and $I = I_i + I_e$, where f_i and f_e describe intracellular and extracellular stimulation currents, respectively.

To close the system, we impose the following initial and boundary conditions

$$\begin{aligned} &\text{initial conditions} \\ &\phi(., t = 0) = \phi_0, \quad u(., t = 0) = u_0, \text{ in } \Omega_{\mathrm{H}} \\ &\text{and boundary conditions} \\ &\varphi_e = \varphi_s, \text{ in } \Sigma_{\mathrm{H}} \\ &(\mathcal{K}_e \nabla \varphi_e) \cdot \mathbf{n} = (\mathcal{K}_s \nabla \varphi_s) \cdot \mathbf{n}, \quad \text{on } \Sigma_{\mathrm{H}} \\ &(\mathcal{K}_i \nabla \varphi_i) \cdot \mathbf{n} = 0, \quad \text{on } \Sigma_{\mathrm{H}} \\ &(\mathcal{K}_s \nabla \varphi_s) \cdot \mathbf{n}_T = 0, \quad \text{on } \Sigma_T = \Gamma_T \times (0, T) \end{aligned} \tag{1.4}$$

where $\mathbf{n}_T$ is the outward normal to Γ_T.

In absence of a grounded electrode, the bidomain equations are a naturally singular problem since φ_e and φ_s, in system (1.2), only appear in the equations and boundary conditions through their gradients. Moreover, the states φ_e and φ_s are only defined up to the same constant. Such problems have compatibility conditions determining whether there are any solution to the PDEs. This is easily found by integrating the second and third equations of (1.2) over the domain and using the divergence theorem with boundary conditions. Then the following conservation of the total current is derived (*a.e. in* $(0, T)$)

$$\int_{\Omega_H} I d\mathbf{x} = 0. \tag{1.5}$$

Consequently, we must choose current I such that the compatibility condition (1.5) is satisfied. For the choose of intracellular stimulus, it is natural and usual (as is common) to take $I_i = -I_e$ (denoted in the sequel by f_H) i.e., setting the total current I to zero. It is clear that this condition does not correspond to zero extracellular stimulus, but that the extracellular stimulus current and the intracellular stimulus current are equal in magnitude and opposite in direction. Moreover, the functions φ_e and φ_s are defined within a class of equivalence, regardless of the same time-dependent function. This function can be fixed, for example, by setting the condition, (a.e. in (0, T))

$$\int_{\Gamma_H} \varphi_e d\mathbf{x} = \int_{\Gamma_H} \varphi_s d\mathbf{x} = p_H \text{ where } p_H \text{ is a fixed time-dependent function.} \tag{1.6}$$

Remark 1.1. *1. Condition (1.6) is used for pressure in oceanography (see e.g. [15]).*

2. The functions $\mathcal{K}_i$, $\mathcal{K}_e$, $\mathcal{H}$ and $\mathcal{G}$ depend on the fiber extension ratio.

3. We can suppose, for example, that f_H is only a time dependent source and is of the form

$$f_H(\mathbf{x}, t) = \theta(t)(\chi_{\Omega_H^{(1)}}(\mathbf{x}) - \chi_{\Omega_H^{(2)}}(\mathbf{x})), \tag{1.7}$$

where $\chi_{\Omega_H^{(i)}}$ is the characteristic function of set $\Omega_H^{(i)}$, $i = 1, 2$. The support regions $\Omega_H^{(1)}$ and $\Omega_H^{(2)}$ can be considered to represent an anode (positive electrode) and a cathode (negative electrode) respectively. □

In recent years, various problems concerning biological rhythmic phenomena and memory processes which can be included in a cardiac model in many ways (via delay model or time fractional dynamical model), have been studied. Via delayed system we can cited e.g., [11, 17, 20, 40, 41, 43, 69] and the references therein. For problems associated with bidomain models with time-delay, the literature is limited, to our knowledge, to [8, 9, 27]. In these references, in order to take into account the influence of disturbance in data and the time-varying delays on propagation of electrophysiological waves in heart, the authors have developed a new mathematical model and have considered the theoretical analysis as well as numerical simulations (with real data) and validation of developed model. Via time fractional dynamical model, the literature is also limited, we can cite e.g. [26, 31]. In [26], the authors study, using a minimal cardiac cell model, the effects of a fractional-order time diffusion for the voltage and for the ionic current gating. They have shown numerically the interest of modeling memory through fractional-order and that with the model it is able to analyze the influence of memory on some electrical properties as spontaneous activity and alternans. Concerning problems associated with classical bidomain models various methods and technique, as evolution variational inequalities approach, semi-group theory, Faedo-Galerkin method and others, the studies of well-posedness of solutions have been derived in the literature (see e.g., [12, 16, 18, 19, 24, 80] and the references therein); for development of multiscale mathematical and computational modeling of bioelectrical activity in myocardial tissue and the formation of cardiac disorders (as arrhythmias), and their numerical simulations, which are based on methods as finite difference method, finite element method or lattice Boltzmann method, have been receiving a significant amount of attention (see e.g., [5, 6, 21, 28–30, 35, 38, 46, 51, 65, 79, 81] and the references therein).

The rest of paper is organized as follows. In next section, we give some preliminaries results useful in the sequel. In Section 3, some preliminaries results concerning fractional calculus are given and results on generalized Gronwall inequality to a coupled fractional differential equations are developed. In Section 4 we shall prove the existence, stability and uniqueness of weak solutions of derived model, under some hypotheses for data and some regularity of nonlinear operators. Numerical experiments, using a modified Lattice Boltzmann Method for numerical simulations of the derived memory bidomain systems, are described in Section 5. In Section 6, conclusions are discussed.

2. Assumptions, notations and some fundamental inequalities

Let $\mho \subset \mathbb{R}^m$, $m \geq 1$, be an open and bounded set with a smooth boundary and $\mho_T = \mho \times (0, T)$. We use the standard notation for Sobolev spaces (see [1]), denoting the norm of $W^{q,p}(\mho)$ ($q \in \mathbb{N}$, $p \in [1, \infty]$) by $\| \, . \, \|_{W^{q,p}}$. In the special case $p = 2$, we use $H^q(\mho)$ instead of $W^{q,2}(\mho)$. The duality pairing of a Banach space X with its dual space X' is given by $\langle ., . \rangle_{X',X}$. For a Hilbert space Y the inner product is denoted by $(., .)_Y$ and the inner product in $L^2(\Omega)$ is denoted by $(., .)$. For any pair of real numbers $r, s \geq 0$, we introduce the Sobolev space $H^{r,s}(\mho_T)$ defined by $H^{r,s}(\mho_T) = L^2(0, T; H^r(\mho)) \cap H^s(0, T; L^2(\mho))$, which is a Hilbert space normed by $\left(\| v \|^2_{L^2(0,T;H^r(\mho))} + \| v \|^2_{H^s(0,T;L^2(\mho))}\right)^{1/2}$, where $H^s(0, T; L^2(\mho))$ denotes the Sobolev space of order s of functions defined on $(0, T)$ and taking values in $L^2(\mho)$, and defined (see [52]) by $H^s(0, T; L^2(\mho)) = [H^q(0, T, L^2(\mho)), L^2(\mho_T)]_\theta$, for $s = (1 - \theta)q$ with $\theta \in (0, 1)$ and $q \in \mathbb{N}$, and $H^q(0, T; L^2(\mho)) = \left\{v \in L^2(\mho_T) | \ \frac{\partial^j v}{\partial t^j} \in L^2(\mho_T), \text{for } 1 \leq j \leq q\right\}$. For a given Banach space $\mathbb{X}$, with norm $\|.\|_{\mathbb{X}}$, of functions integrable on $\mho$, we define its subspace $\mathbb{X}|_{\mathbb{R}} = \left\{u \in \mathbb{X}, \int_\mho u = 0\right\}$ that is a Banach space with norm $\|.\|_{\mathbb{X}}$, and we denote by $[u]$ the projection of $u \in \mathbb{X}$ on $\mathbb{X}|_{\mathbb{R}}$ such that $[u] = u - \frac{1}{mes(\mho)} \int_\mho u d\mathbf{x}$ (with $mes(\mho)$ standing for Lebesgue measure of $\mho$).

Lemma 2.1. *(Poincaré-Wirtinger inequality) Assume that* $1 \leq p \leq \infty$ *and that* $\mho$ *is a bounded connected open subset of* $\mathbb{R}^d$ *with a sufficiently regular boundary* $\partial\Omega$ *(e.g., a Lipschitz boundary). Then there exists a Poincaré constant* C *, depending only on* Ω *and* p*, such that for every function* u *in Sobolev space* $W^{1,p}(\mho)$*, we have*

$$\|[u]\|_{L^p(\mho)} \leq C\|\nabla u\|_{L^p(\mho)}.$$

Remark 2.1. *From the Poincaré-Wirtinger inequality, we can deduce that the* H^1 *semi-norm and the* H^1 *norm are equivalent in the space* $H^1(\mho)|_{\mathbb{R}}$. □

Remark 2.2. *Let q be a nonnegative integer. We have the following results (see e.g. [1])*

(i) $H^q(\mho) \subset L^p(\mho)$, $\forall p \in [1, \frac{2m}{m-2q}]$*, with continuous embedding (with the exception that if* $2q = m$*, then* $p \in [1, +\infty[$ *and if* $2q > m$*, then* $p \in [1, +\infty]$ *).*

(ii) (Gagliardo-Nirenberg inequalities) There exists $C > 0$ *such that*

$$\| v \|_{L^p(\mho)} \leq C \| v \|^{\theta}_{H^q(\mho)} \| v \|^{1-\theta}_{L^2(\mho)}, \forall v \in H^q(\mho),$$

where $0 \leq \theta < 1$ *and* $p = \frac{2m}{m-2\theta q}$ *(with the exception that if* $q - m/2$ *is a nonnegative integer, then* θ *is restricted to 0).* □

Finally, we introduce the spaces:

- $\mathbb{H}_{\mathrm{H}} = L^2(\Omega_{\mathrm{H}})$, $\mathbb{V}_{\mathrm{H}} = H^1(\Omega_{\mathrm{H}})$, $\mathbb{V} = H^1(\Omega)$ (endowed with their usual norms) and $\mathbb{U}_{\mathrm{H}} = \mathbb{V}_{\mathrm{H}}|_{\mathbb{R}}$,
- $\mathbb{U}_{\mathrm{HB}} = \left\{\psi \in H^1(\Omega) \mid \int_{\Omega_{\mathrm{H}}} \psi d\mathbf{x} = 0\right\}$.

We will denote by $\mathbb{V}'_{\mathrm{H}}$ (resp. $\mathbb{U}'_{\mathrm{H}}$) the dual of $\mathbb{V}_{\mathrm{H}}$ (resp. of $\mathbb{U}_{\mathrm{H}}$). We have the following continuous embeddings, where $p \geq 2$ if $d = 2$ and $2 \leq p \leq 6$ if $d = 3$, p' is such that $\dfrac{1}{p'} + \dfrac{1}{p} = 1$

$$\begin{aligned} &\mathbb{V}_{\mathrm{H}} \subset \mathbb{H}_{\mathrm{H}} \subset \mathbb{V}'_{\mathrm{H}}\,,\ \mathbb{U}_{\mathrm{H}} \subset \mathbb{H}_{\mathrm{H}}|_{\mathbb{R}} \subset \mathbb{U}'_{\mathrm{H}}, \\ &\mathbb{V}_{\mathrm{H}} \subset L^p(\Omega_{\mathrm{H}}) \subset \mathbb{H}_{\mathrm{H}} \equiv (\mathbb{H}_{\mathrm{H}})' \subset L^{p'}(\Omega_{\mathrm{H}}) \subset \mathbb{V}'_{\mathrm{H}} \end{aligned} \tag{2.1}$$

and the injections $\mathbb{V}_{\mathrm{H}} \subset \mathbb{H}_{\mathrm{H}}$ and $\mathbb{U}_{\mathrm{H}} \subset \mathbb{H}_{\mathrm{H}}|_{\mathbb{R}}$ are compact. We can now introduce the following spaces (where $q > 1$, $p \geq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$)

$$\mathbb{D}_p(0,T) = L^p(Q_{\mathrm{H}}) \cap L^2(0,T;\mathbb{V}_{\mathrm{H}}), \text{ and its dual } \mathbb{D}'_p(0,T) = L^{p'}(Q_{\mathrm{H}}) + L^2(0,T;\mathbb{V}'_{\mathrm{H}}) \subset L^{p'}(0,T;\mathbb{V}'_{\mathrm{H}}).$$

Remark 2.3. *Space* $\mathbb{D}_p(0,T)$ *is equipped with norm:* $\| u \|_{\mathbb{D}_p} = \max(\| u \|_{L^p(Q_{\mathrm{H}})}, \| u \|_{L^2(0,T;\mathbb{V}_{\mathrm{H}})})$ *and its dual with norm:* $\| v \|_{\mathbb{D}'_p} = \inf_{v=v_1+v_2} (\| v_1 \|_{L^{p'}(Q_{\mathrm{H}})} + \| v_2 \|_{L^2(0,T;\mathbb{V}'_{\mathrm{H}})})$. □

Definition 2.1. *A real valued function* $\mathcal{H}$ *defined on* $D \times \mathbb{R}^q$, $q \geq 1$, *is a Carathéodory function iff* $\mathcal{H}(.;\mathbf{v})$ *is measurable for all* $\mathbf{v} \in \mathbb{R}^q$ *and* $\mathcal{H}(y;.)$ *is continuous for almost all* $y \in D$.

Our study involves the following fundamental inequalities, which are repeated here for review:

(i) *Hölder's inequality*: $\int_D \Pi_{i=1,k} f_i dx \leq \Pi_{i=1,k} \| f_i \|_{L^{q_i}(D)}$,

where $\| f_i \|_{L^{q_i}(D)} = \left(\int_D | f_i |^{q_i} dx\right)^{1/q_i}$ and $\sum_{1 \leq i \leq k} \dfrac{1}{q_i} = 1$.

(ii) *Young's inequality* ($\forall a, b > 0$ and $\epsilon > 0$): $ab \leq \frac{\epsilon}{p} a^p + \frac{\epsilon^{-q/p}}{q} b^q,\ for\ p, q \in]1, +\infty[\ and\ \frac{1}{p} + \frac{1}{q} = 1.$

(iii) *Minkowski's integral inequality*:

$$\left[\int_\Omega \left(\int_0^t | f(x,s) | ds\right)^p dx\right]^{1/p} \leq \int_0^t \left(\int_\Omega | f(x,s) |^p dx\right)^{1/p} ds, \text{ for } p \in]1, +\infty[\text{ and } t > 0.$$

Finally, we denote by $\mathfrak{L}(A;B)$ the set of linear and continuous operators from a vectorial space A into a vectorial space B, and by $\mathcal{R}^*$ the adjoint operator to a linear operator $\mathcal{R}$ between Banach spaces.

From now on, we assume that the following assumptions hold for the nonlinear operators and tensor functions appearing in our model.

(H1) We assume that the conductivity tensor functions $\mathcal{K}_\theta \in W^{1,\infty}(\overline{\Omega}_{\rm H})$, $\theta \in \{i, e\}$ and $\mathcal{K}_s \in W^{1,\infty}(\overline{\Omega}_{\rm B})$ are symmetric, positive definite matrix functions and that they are uniformly elliptic, i.e., there exist constants $0 < K_1 < K_2$ such that $(\forall \psi \in \mathbb{R}^d)$

$$\begin{aligned} K_1\|\psi\|^2 &\leq \psi^T \mathcal{K}_\theta \psi \leq K_2\|\psi\|^2 \ \text{ in } \overline{\Omega}_{\rm H}, \\ K_1\|\psi\|^2 &\leq \psi^T \mathcal{K}_s \psi \leq K_2\|\psi\|^2 \ \text{ in } \overline{\Omega}_{\rm B}. \end{aligned} \tag{2.2}$$

Remark 2.4. *We can emphasize a specificity of tensors $\mathcal{K}_e$ and $\mathcal{K}_i$ (see e.g., [25]).*

1. *The tensors $\mathcal{K}_e(\mathbf{x})$ and $\mathcal{K}_i(\mathbf{x})$ have the same basis of eigenvectors $Q(\mathbf{x}) = (q_k(\mathbf{x}))_{1\leq k\leq d}$ in $\mathbb{R}^d$, which reflect the organization of muscle in fibers, and consequently $\mathcal{K}_i(\mathbf{x}) = Q(\mathbf{x})\Lambda_i(\mathbf{x})Q(\mathbf{x})^T$ and $\mathcal{K}_e(\mathbf{x}) = Q(\mathbf{x})\Lambda_e(\mathbf{x})Q(\mathbf{x})^T$, where $\Lambda_i(\mathbf{x}) = diag((\lambda_{i,k})_{1\leq k\leq d})$ and $\Lambda_e(\mathbf{x}) = diag((\lambda_{e,k})_{1\leq k\leq d})$.*
2. *The muscle fibers are tangent to Γ so that (for $\theta \in \{i, e\}$) : $\mathcal{K}_\theta \mathbf{n} = \lambda_{\theta,d}\mathbf{n}$, a.e., in Γ, with $\lambda_{\theta,d}(\mathbf{x}) \geq \lambda > 0$, λ a constant.* □

The operators $\mathcal{I}$ and $\mathcal{G}$ which describe electrophysiological behavior of the system can be taken as follows (affine functions with respect to u)

$$\begin{aligned} \mathcal{I}(\mathbf{x}, t; \phi, u) &= \mathcal{I}_0(\mathbf{x}, t; \phi) + \mathcal{I}_1(\mathbf{x}, t; \phi)u, \\ \mathcal{G}(\mathbf{x}, t; \phi, u) &= \mathcal{I}_2(\mathbf{x}, t; \phi) + \hbar(\mathbf{x}, t)u, \end{aligned} \tag{2.3}$$

where $\hbar$ is a sufficiently regular function and $\mathcal{I}_0(\mathbf{x}, t; \phi) = g_1(\mathbf{x}; \phi) + g_2(\mathbf{x}, t; \phi)$ with g_1 is an increasing function on ϕ. Moreover, the operators $\mathcal{I}_0$, $\mathcal{I}_1$ and $\mathcal{I}_2$ appearing in $\mathcal{I}$ and $\mathcal{G}$, are supposed to satisfy the following assumptions.

$\textbf{(H2)}_{\rm p}$ The operators $\mathcal{I}_0$, $\mathcal{I}_1$ and $\mathcal{I}_2$ are Carathéodory functions from $(\Omega \times \mathbb{R}) \times \mathbb{R}$ into $\mathbb{R}$ and continuous on ϕ (as in [12]). Furthermore, for some $p \geq 2$ if $d = 2$ and $p \in [2, 6]$ if $d = 3$, the following requirements hold

(i) there exist constants $\beta_i \geq 0$ $(i = 1, \ldots, 10)$ such that for any $v \in \mathbb{R}$

$$\begin{aligned} &|\mathcal{I}_0(.; v)| \leq \beta_1 + \beta_2|v|^{p-1}, \\ &|\mathcal{I}_1(.; v)| \leq \beta_3 + \beta_4|v|^{p/2-1}, \\ &|\mathcal{I}_2(.; v)| \leq \beta_5 + \beta_6|v|^{p/2}, \\ &|E_g(.; v)| \leq \beta_7 + \beta_8|v|^p, |g_2(.; v)| \leq \beta_9 + \beta_{10}|v|^{p-2}, \end{aligned} \tag{2.4}$$

where E_g is the primitive of g_1.

(ii) there exist constants $\mu_1 > 0, \mu_2 > 0, \mu_i \geq 0$ (for $i = 3, 7$) such that for any $(v, w) \in \mathbb{R}^2$:

$$\begin{aligned} &\mu_1 v\mathcal{I}(.; v, w) + w\mathcal{G}(.; v, w) \geq \mu_2|v|^p - \mu_3\left(\mu_1|v|^2 + |w|^2\right) - \mu_4, \\ &E_g(., v) \geq \mu_5|v|^p - \mu_6|v|^2 - \mu_7. \end{aligned} \tag{2.5}$$

In order to assure the uniqueness of solution we assume that

(H3) The Nemytskii operators $\mathcal{I}$ and $\mathcal{G}$ satisfy Carathéodory conditions and there exists some $\mu > 0$ such the operator $F_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F_\mu(.; \mathbf{v}) = \begin{pmatrix} \mu(\mathcal{I}(.; \mathbf{v})) \\ \mathcal{G}(.; \mathbf{v}) \end{pmatrix}, \ \forall \mathbf{v} = (v, w) \in \mathbb{R}^2, \tag{2.6}$$

satisfies a one-sided Lipschitz condition (see e.g., [14, 70]): there exists a constant $C_L > 0$ such that $(\forall \mathbf{v}_i = (v_i, w_i) \in \mathbb{R}^2,\ i = 1, 2)$

$$\Big(F_\mu(.; \mathbf{v}_1) - F_\mu(.; \mathbf{v}_2)\Big) \cdot (\mathbf{v}_1 - \mathbf{v}_2) \geq -C_L \|\mathbf{v}_1 - \mathbf{v}_2\|^2. \tag{2.7}$$

Lemma 2.2. *([9]) Assume that F_μ is differentiable with respect to (ϕ, u) and denote by $\lambda_1(\phi, u) \leq \lambda_2(\phi, u)$ the eigenvalues of symmetrical part of Jacobian matrix $\nabla F_\mu(\phi, u)$:*

$$Q_\mu(\phi, u) = \frac{1}{2}\left(\nabla F_\mu(\phi, u)^T + \nabla F_\mu(\phi, u)\right).$$

If there exists a constant C_F independent of ϕ and u such as:

$$C_F \leq \lambda_1(\phi, u) \leq \lambda_2(\phi, u), \tag{2.8}$$

then F_μ satisfies the hypothesis **(H3)**.

Lemma 2.3. *([9]) Let assumptions* (2.3), **(H1)** *and* $(\mathbf{H2})_{\mathrm{p}}$ *be fulfilled. For $(\phi, u) \in L^p(\Omega) \times \mathbb{H}$ and a.e., t, there exist constants $C_i > 0$ ($i = 1, 6$) such that*

$$\begin{aligned} \|\mathcal{I}(., t; \phi, u)\|_{L^{p'}(\Omega_{\mathrm{H}})} &\leq C_1 + C_2 \|\phi\|^{p/p'}_{L^p(\Omega_{\mathrm{H}})} + C_3 \|u\|^{2/p'}_{\mathbb{H}_{\mathrm{H}}}, \\ \|\mathcal{G}(., t; \phi, u)\|_{L^2(\Omega_{\mathrm{H}})} &\leq C_4 + C_5 \|\phi\|^{p/2}_{L^p(\Omega_{\mathrm{H}})} + C_6 \|u\|_{\mathbb{H}_{\mathrm{H}}}, \end{aligned} \tag{2.9}$$

where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

In this work we assume that $p = 4$. The considered functions $\mathcal{I}_i$, in this paper, include the three classical type models in which assumptions **(H1)**, $(\mathbf{H2})_4$ and **(H3)** are satisfied namely the Rogers-McCulloch [57] (RM), Fitz-Hugh-Nagumo [39] (FHN) and Aliev-Panfilov [61](LAP) models as follows. The function $\mathcal{I}_0$ is defined by a cubic reaction term of the form $\mathcal{I}_0(.; v) = b_1(.)v(v - r)(v - 1)$, and the functions $\mathcal{I}_1$ and $\mathcal{I}_2$ are given by

(a) for RM type model : $\mathcal{I}_1(.; v) = b_2(.)v,\ \ \mathcal{I}_2(; , v) = -b_3(.)v$,

(b) for FHN type model : $\mathcal{I}_1(.; v) = b_2(.),\ \ \mathcal{I}_2(; , v) = -b_3(.)v$,

(c) for LAP type model : $\mathcal{I}_1(.; v) = b_2(.)v,\ \ \mathcal{I}_2(; , v) = b_3(.)v(r + 1 - v)$,

where $b_i \in W^{1,\infty}(Q)$, $i = 1, 3$, are sufficiently regular functions from Q into $\mathbb{R}^{+,*}$ and $r \in [0, 1]$. We obtain easily the following Lemma.

Lemma 2.4. *The following properties hold. For all v_1, v_2 in $\mathbb{R}$ we have*

$\mathcal{I}_0(.; v_1) - \mathcal{I}_0(.; v_2) = b_1(v_1 - v_2)\left(v_1^2 + v_2^2 + v_1 v_2 - (r + 1)(v_1 + v_2) + r\right)$ *and*

(a) for RM type model : $\mathcal{I}_1(.; v_1) - \mathcal{I}_1(.; v_2) = b_2(v_1 - v_2),\ \ \mathcal{I}_2(.; v_1) - \mathcal{I}_2(.; v_2) = -b_3(v_1 - v_2)$,

(b) for FHN type model : $\mathcal{I}_1(.; v_1) - \mathcal{I}_1(.; v_2) = 0,\ \ \mathcal{I}_2(.; v_1) - \mathcal{I}_2(.; v_2) = -b_3(v_1 - v_2)$,

(c) for LAP type model : $\mathcal{I}_1(.; v_1) - \mathcal{I}_1(.; v_2) = b_2(v_1 - v_2)$,

$$\mathcal{I}_2(.; v_1) - \mathcal{I}_2(.; v_2) = b_3(v_1 - v_2)((r + 1) - v_1 - v_2).$$

For the sake of simplicity, we shall write $\mathcal{I}_i(\psi)$, $\mathcal{I}(\psi, v)$ and $\mathcal{G}(\psi, v)$ in place of $\mathcal{I}_i(\mathbf{x}, t; \psi)$, $\mathcal{I}(\mathbf{x}, t; \psi, v)$ and $\mathcal{G}(\mathbf{x}, t; \psi, v)$, respectively (for $i = 0, 2$).

3. Fractional calculus and a generalized Gronwall's inequality

Fractional differential equations have been studied by many investigations in recent years and the idea of defining a derivative of fractional order (non-integer order) dates back to Leibnitz [49]. Since 19th century, different authors have considered this problem e.g., Riemann, Liouville, Hadmard, Hardy, Littlewood and Caputo among others. Fractional integrals and derivatives have proved to be useful in real applications, since they arise naturally in many biological phenomena such as viscoelasticity, neurobiology and chaotic systems, see for instance [3, 34, 36, 55, 62, 76, 83].

The classical and the most used form of fractional calculus is given by the Riemann-Liouville and Caputo derivatives. In contrast to this nonlocal Riemann-Liouville derivative operator, when solving differential equations, is the use of Caputo fractional derivative [22] in which it is not necessary to define the fractional order initial conditions.

3.1. Definitions and basic results

The object of this section is to give a brief introduction to some definitions and basic results in fractional calculus in the Riemann-Liouville sense and Caputo sense. Let $\gamma \in]0,1]$ and X be a Banach space, we start from a formal level and assume the given functions $f : t \in (a,T) \to f(t) \in X$ and $g : t \in (a,T) \to f(t) \in X$ are sufficiently smooth (with $-\infty < a < T < \infty$).

Definition 3.1. *The forward and backward Riemann-Liouville fractional integrals of fractional order γ on (a,T) are defined, respectively, by ($t \in (a,T)$)*

$$\begin{aligned} I^{\gamma}_{a^+}[f](t) &= \frac{1}{\mathbf{\Gamma}(\gamma)}\int_a^t (t-\tau)^{\gamma-1} f(\tau)d\tau, \\ I^{\gamma}_{T^-}[f](t) &= \frac{1}{\mathbf{\Gamma}(\gamma)}\int_t^T (\tau-t)^{\gamma-1} f(\tau)d\tau, \end{aligned} \tag{3.1}$$

where $\mathbf{\Gamma}(z) = \displaystyle\int_0^{\infty} e^{\tau}\tau^{z-1}d\tau$ is the Euler $\mathbf{\Gamma}$-function.

The basic equality for the fractional integral is (from Fubini's Theorem and the relationship between $\mathbf{\Gamma}$-function and β-function)

$$I^{\gamma_1}_{a^+}\left[I^{\gamma_2}_{a^+}[f]\right] = I^{\gamma_1+\gamma_2}_{a^+}[f] \tag{3.2}$$

and holds for a L^p-function f $(1 \le p \le \infty)$.

Definition 3.2. *The forward Riemann-Liouville and Caputo derivatives of fractional order γ on (a,T) are defined, respectively, by ($t \in (a,T)$)*

$$\begin{aligned} D^{\gamma}_{a^+}[f](t) &= \frac{d}{dt}(I^{1-\gamma}_{a^+}[f](t)) = \frac{1}{\mathbf{\Gamma}(1-\gamma)}\frac{d}{dt}(\int_a^t (t-\tau)^{-\gamma} f(\tau)d\tau), \\ \partial^{\gamma}_{a^+}[f](t) &= I^{1-\gamma}_{a^+}\left[\frac{df(t)}{dt}\right] = \frac{1}{\mathbf{\Gamma}(1-\gamma)}\int_a^t (t-\tau)^{-\gamma}\frac{df}{dt}(\tau)d\tau. \end{aligned} \tag{3.3}$$

From (3.2) we can deduce the following relation between fractional integral and Caputo derivative

$$f(t) = f(a) + I^{\gamma}_{a^+}\left[\partial^{\gamma}_{a^+} f\right](t) = f(a) + \frac{1}{\mathbf{\Gamma}(\gamma)}\int_a^t (t-\tau)^{\gamma-1}\partial^{\gamma}_{a^+} f(\tau)d\tau. \tag{3.4}$$

Definition 3.3. *The backward Riemann-Liouville and Caputo derivatives of fractional order γ, on (a, T) are defined, respectively, by ($t \in (a, T)$)*

$$\begin{aligned} D^{\gamma}_{T^-} f(t) &= -\frac{d}{dt}(I^{1-\gamma}_{T^-}[f](t)) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\left(\int_t^T (\tau - t)^{-\gamma} f(\tau) d\tau\right), \\ \partial^{\gamma}_{T^-}[f](t) &= -I^{1-\gamma}_{T^-}\left[\frac{df(t)}{dt}\right] = -\frac{1}{\Gamma(1-\gamma)}\int_t^T (\tau - t)^{-\gamma}\frac{df}{dt}(\tau) d\tau. \end{aligned} \tag{3.5}$$

By substitution $a \to -\infty$, the following definition is obtained.

Definition 3.4. *The Liouville-Weyl fractional integral and the Caputo fractional derivative on the real axis are defined, respectively, by ($t \in (-\infty, T)$)*

$$\begin{aligned} I^{\gamma}_{-\infty}[f](t) &= \frac{1}{\Gamma(\gamma)}\int_{-\infty}^t (t-\tau)^{\gamma-1} f(\tau) d\tau, \\ \partial^{\gamma}_{-\infty} f(t) &= I^{1-\gamma}_{-\infty}\left[\frac{df}{dt}(t)\right] = \frac{1}{\Gamma(1-\gamma)}\int_{-\infty}^t (t-\tau)^{-\gamma}\frac{df}{dt}(\tau) d\tau. \end{aligned} \tag{3.6}$$

Remark 3.1. *1. For $\gamma \longrightarrow 1$ the forward (resp. backward) Riemann-Liouville and Caputo derivatives of fractional order γ of f converge to the classical derivative $\frac{df}{dt}$ (resp. to $-\frac{df}{dt}$). Moreover, Riemann-Liouville fractional derivative of fractional order γ of constant function $t \longrightarrow f(t) = k$ is not 0 since $D^{\gamma}_{a^+} f(t) = \frac{k}{\Gamma(1-\gamma)}\frac{d}{dt}(\int_a^t (t-\tau)^{-\gamma} d\tau) = \frac{k(t-a)^{-\gamma}}{\Gamma(1-\gamma)}$.*

2. It is possible to show that the difference between Riemann-Liouville and Caputo fractional derivatives depends only on the values of f on endpoints a and T. More precisely, for $f \in C^1([a, T], X)$ we have

$$\begin{aligned} D^{\gamma}_{a^+} f(t) &= \partial^{\gamma}_{a^+} f(t) + \frac{f(a)(t-a)^{-\gamma}}{\Gamma(1-\gamma)}, \\ D^{\gamma}_{T^-} f(t) &= \partial^{\gamma}_{T^-} f(t) + \frac{f(T)(T-t)^{-\gamma}}{\Gamma(1-\gamma)}. \end{aligned} \tag{3.7}$$

From [42], we have the following results

Lemma 3.1. *(Continuity properties of fractional integral in L^p spaces on (a, T))*
The fractional integral $I^{\gamma}_{a^+}$ is a continuous operator from

(i) *$L^p(a, T)$ into $L^p(a, T)$, for any $p \geq 1$,*

(ii) *$L^p(a, T)$ into $L^r(0, T)$, for any $p \in (1, 1/\gamma)$ and $r \in [1, p/(1-\gamma p)]$,*

(iii) *$L^p(a, T)$ into $C^{0,\gamma-1/p}([a, T])$, for any $p \in]1/\gamma, +\infty[$,*

(iii) *$L^{1/\gamma}(a, T)$ into $L^p(a, T)$, for any $p \in [1, +\infty)$*

(iv) *$L^{\infty}(a, T)$ into $C^{0,\gamma}([a, T])$.*

From Lemma 3.1 and (3.4) we can deduce the following corollary.

Lemma 3.2. *Let X be a Banach space and $\gamma \in]0, 1]$. Suppose the Caputo derivative $\partial^{\gamma}_{a^+} f \in L^p(a, T; X)$ and $p > \frac{1}{\gamma}$, then $f \in C^{0,\gamma-1/p}([a, T]; X)$.*

We also need for our purposes the fractional integration by parts in the formulas (see for instance [2, 47, 86])

Lemma 3.3. *Let $0 < \gamma \le 1$ and $p, q \ge 1$ with $1/p + 1/q \le 1 + \gamma$. Then*

(i) *if g is a L^p-function on (a, T) and g is a L^q-function on (a, T), then*

$$\int_a^T (f(\tau), I^{\gamma}_{a^+}[g](\tau))_X d\tau = \int_a^T (g(\tau), I^{\gamma}_{T^-}[f](\tau))_X d\tau,$$

(ii) *if $f \in I^{\gamma}_{T^-}(L^p)$ and $g \in I^{\gamma}_{a^+}(L^q)$, then*

$$\int_a^T (f(\tau), D^{\gamma}_{a^+} g(\tau))_X d\tau = \int_a^T (g(\tau), D^{\gamma}_{T^-} f(\tau))_X d\tau.$$

Lemma 3.4. *Let $0 < \gamma \le 1$, and g be L^p-function on (a, T) (for $p \ge 1$) and f be absolutely continuous function on $[a, T]$. Then*

(i) $\displaystyle\int_a^T (\partial^{\gamma}_{a^+} f(\tau), g(\tau))_X d\tau = -\int_a^T (D^{\gamma}_{T^-} g(\tau), f(\tau))_X d\tau + |(I^{1-\gamma}_{T^-}[g](\tau), f(\tau))_X|_a^T,$

(ii) $\displaystyle\int_a^T (D^{\gamma}_{a^+} f(\tau), g(\tau))_X d\tau = -\int_a^T (D^{\gamma}_{T^-} g(\tau), f(\tau))_X d\tau + (I^{1-\gamma}_{T^-}[g](T^-), f(T))_X,$

(iii) $\displaystyle\int_a^T (D^{\gamma}_{T^-} f(\tau), g(\tau))_X d\tau = -\int_a^T (D^{\gamma}_{a^+} g(\tau), f(\tau))_X d\tau - (I^{1-\gamma}_{a^+}[g](a^+), f(a))_X.$

From [85], we can deduce the following Lemma.

Lemma 3.5. *(A generalized Gronwall's inequality)*
Assume $\gamma > 0$, h is a nonnegative function locally integrable on $(0, T)$ and b is a nonnegative, bounded, nondecreasing continuous function defined on $[0, T)$. Let f be a nonnegative and locally integrable function on $(0, T)$ with

$$f(t) \le h(t) + b(t) I^{\gamma}_{0^+}[f](t).$$

Then (for $t \in (0, T)$)

$$f(t) \le h(t) + \int_0^t \sum_{k=1}^{\infty} h(\tau)(t-\tau)^{k\gamma-1} \frac{(b(t))^k}{\Gamma(k\gamma)} d\tau.$$

If in addition h is a nondecreasing function on $(0, T)$, then

$$f(t) \le h(t) E_{\gamma,1}(b(t) t^{\gamma}).$$

The used function E_{θ_1,θ_2} is the classical two-parametric Mittag-Leffler function (usually denoted by E_{θ_1} if $\theta_2 = 1$) which is defined by

$$E_{\theta_1,\theta_2}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\theta_1 + \theta_2)} z^k. \tag{3.8}$$

The function E_{θ_1,θ_2} is an entire function of the variable z for any $\theta_1, \theta_2 \in \mathbb{C}$, $Re(\theta_1) > 0$.

Next we declare a compactness theorem in Hilbert spaces. Assume that $\mathcal{X}_0$, $\mathcal{X}_1$ and $\mathcal{X}$ are Hilbert spaces with

$$\mathcal{X}_0 \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{X}_1 \text{ being continuous and } \mathcal{X}_0 \hookrightarrow \mathcal{X} \text{ is compact.} \tag{3.9}$$

The Fourier transform of $f : \mathbb{R} \to \mathcal{X}_1$ is defined by $\widehat{f}(\tau) = \displaystyle\int_{-\infty}^{\infty} \exp(-2i\pi s\tau) f(s) ds$ and we have Lemma (see e.g., [68])

Lemma 3.6. *Let $\gamma \in (0,1)$ and f be a L^1-function in $\mathbb{R}$ with compact support. Then*

(i) $\widehat{I^{\gamma}_{-\infty}f}(\tau) = (2i\pi\tau)^{-\gamma}\widehat{f}(\tau)$,

(ii) $\widehat{\partial^{\gamma}_{-\infty}f}(\tau) = (2i\pi\tau)^{\gamma}\widehat{f}(\tau)$.

We define the Hilbert space $\mathcal{W}^{\gamma}(\mathbb{R};\mathcal{X}_0,\mathcal{X}_1)$, for a given $\gamma > 0$, by

$$\mathbb{W}^{\gamma}(\mathbb{R};\mathbb{X}_0,\mathcal{X}_1) = \{v \in L^2(\mathbb{R},\mathcal{X}_0) \mid \partial^{\gamma}_{-\infty}f \in L^2(\mathbb{R},\mathcal{X}_1)\},$$

endowed with the norm

$$\| v \|_{\mathbb{W}^{\gamma}} = \left(\| v \|_{L^2\left(\mathbb{R},\mathcal{X}_0\right)} + \|| \tau |^{\gamma}\, \widehat{v} \|_{L^2\left(\mathbb{R},\mathcal{X}_1\right)}\right)^{1/2}.$$

For any subset K of $\mathbb{R}$, we define the subspace $\mathbb{W}^{\gamma}_K$ of $\mathbb{W}^{\gamma}$ by

$$\mathbb{W}^{\gamma}_K(\mathbb{R};\mathcal{X}_0,\mathcal{X}_1) = \{v \in \mathbb{W}^{\gamma}(\mathbb{R};\mathcal{X}_0,\mathcal{X}_1) \mid \text{support of } v \subset K\}.$$

From Lemma 3.6 and similar arguments to drive Theorem 2.2 in [77], we have the following compactness result.

Theorem 3.1. *Let $\mathcal{X}_0$, $\mathcal{X}_1$ and $\mathcal{X}$ be Hilbert spaces with the injection* (3.9). *Then for any bounded set K and any $\gamma > 0$, the injection of $\mathbb{W}^{\gamma}_K(\mathbb{R};\mathcal{X}_0,\mathcal{X}_1)$ into $L^2(\mathbb{R};\mathcal{X})$ is compact.*

3.2. A generalized Gronwall inequality to a coupled fractional differential equation

In this section we present a new generalized Gronwall inequality with singularity in the context of a coupled fractional differential equations.

Theorem 3.2. *Assume that γ_1 and γ_2 be in]0,1] with $\gamma_2 \le \gamma_1$ and Ψ, h_i, for $i = 1,2$, are nonnegative functions locally integrable on $(0,T)$. If f and g are nonnegative functions locally integrable, with $f(0) = f_0$, $g(0) = g_0$, and satisfy the inequalities (for $t \in (0,T)$)*

$$\begin{aligned} &\partial^{\gamma_1}_{0^+}f(t) \le h_1(t) + c_1(f(t)+g(t)) + \eta\Psi(t), \\ &\partial^{\gamma_2}_{0^+}g(t) + \xi\Psi(t) \le h_2(t) + c_2(f(t)+g(t)), \end{aligned} \tag{3.10}$$

where $c_i > 0$, $i = 1,2$, $\eta \ge 0$ and $\xi \ge 0$ are four constants with $\eta \le \dfrac{\xi\boldsymbol{\Gamma}(\gamma_1)}{\boldsymbol{\Gamma}(\gamma_2)T^{(\gamma_1-\gamma_2)}}$. Then we have the following estimate

$$F(t) \le F_0 E_{\gamma_2,1}(d_1 t^{\gamma_2}) + \sum_{k=0}^{\infty} d_1^k I_{0^+}^{k\gamma_2+\gamma_1}\left[h_1\right](t) + \sum_{k=0}^{\infty} d_1^k I_{0^+}^{(k+1)\gamma_2}\left[h_2\right](t), \tag{3.11}$$

where $F = f + g$, $F_0 = f_0 + g_0$, $d_0 = \dfrac{T^{(\gamma_1-\gamma_2)}\boldsymbol{\Gamma}(\gamma_2)}{\boldsymbol{\Gamma}(\gamma_1)}$ and $d_1 = c_1 d_0 + c_2$.

The function $E_{.,.}$ is the Mittag-Leffler function which is defined by (3.8).

Proof. From (3.4) we can deduce that

$$\begin{aligned}\mathbf{\Gamma}(\gamma_1)(f(t)-f_0) &\le \int_0^t (t-\tau)^{\gamma_1-1}h_1(\tau)d\tau + c_1\int_0^t (t-\tau)^{\gamma_1-1}(f(\tau)+g(\tau))d\tau \\ &\quad +\eta\int_0^t (t-\tau)^{\gamma_1-1}\Psi(\tau)d\tau, \\ \mathbf{\Gamma}(\gamma_2)(g(t)-g_0) &\le \int_0^t (t-\tau)^{\gamma_2-1}h_2(\tau)d\tau + c_2\int_0^t (t-\tau)^{\gamma_2-1}(f(\tau)+g(\tau))d\tau \\ &\quad -\xi\int_0^t (t-\tau)^{\gamma_2-1}\Psi(\tau)d\tau.\end{aligned} \tag{3.12}$$

Put $F = f + g$, then

$$\begin{aligned}(\mathbf{\Gamma}(\gamma_1)\mathbf{\Gamma}(\gamma_2))(F(t)-F_0) &\le \int_0^t (\mathbf{\Gamma}(\gamma_2)(t-\tau)^{\gamma_1-1}h_1(\tau) + \mathbf{\Gamma}(\gamma_1)(t-\tau)^{\gamma_2-1}h_2(\tau))d\tau \\ &\quad + \int_0^t \left(\eta\mathbf{\Gamma}(\gamma_2)(t-\tau)^{\gamma_1-1} - \xi\mathbf{\Gamma}(\gamma_1)(t-\tau)^{\gamma_2-1}\right)\Psi(\tau)d\tau \\ &\quad + \int_0^t \left(c_1\mathbf{\Gamma}(\gamma_2)(t-\tau)^{\gamma_1-1} + c_2\mathbf{\Gamma}(\gamma_1)(t-\tau)^{\gamma_2-1}\right)F(\tau)d\tau.\end{aligned} \tag{3.13}$$

We can deduce that (since $t \in (0, T)$ and $\gamma_1 - \gamma_2 \ge 0$)

$$\begin{aligned}(\mathbf{\Gamma}(\gamma_1)\mathbf{\Gamma}(\gamma_2))(F(t)-F_0) &\le \mathbf{\Gamma}(\gamma_2)\int_0^t (t-\tau)^{\gamma_1-1}h_1(\tau)d\tau + \mathbf{\Gamma}(\gamma_1)\int_0^t (t-\tau)^{\gamma_2-1}h_2(\tau))d\tau \\ &\quad +(\eta\mathbf{\Gamma}(\gamma_2)T^{(\gamma_1-\gamma_2)} - \xi\mathbf{\Gamma}(\gamma_1))\int_0^t (t-\tau)^{\gamma_2-1}\Psi(\tau)d\tau \\ &\quad +(c_1\mathbf{\Gamma}(\gamma_2)T^{(\gamma_1-\gamma_2)} + c_2\mathbf{\Gamma}(\gamma_1))\int_0^t (t-\tau)^{\gamma_2-1}F(\tau)d\tau.\end{aligned} \tag{3.14}$$

Since $\eta\mathbf{\Gamma}(\gamma_2)T^{(\gamma_1-\gamma_2)} - \xi\mathbf{\Gamma}(\gamma_1) \le 0$, then

$$\begin{aligned}(\mathbf{\Gamma}(\gamma_1)\mathbf{\Gamma}(\gamma_2))(F(t)-F_0) &\le \mathbf{\Gamma}(\gamma_2)\int_0^t (t-\tau)^{\gamma_1-1}h_1(\tau)d\tau + \mathbf{\Gamma}(\gamma_1)\int_0^t (t-\tau)^{\gamma_2-1}h_2(\tau))d\tau \\ &\quad +(c_1\mathbf{\Gamma}(\gamma_2)T^{(\gamma_1-\gamma_2)} + c_2\mathbf{\Gamma}(\gamma_1))\int_0^t (t-\tau)^{\gamma_2-1}F(\tau)d\tau.\end{aligned} \tag{3.15}$$

Thus

$$F(t) \le F_0 + I_{0^+}^{\gamma_1}[h_1](t) + I_{0^+}^{\gamma_2}[h_2](t) + d_1 I_{0^+}^{\gamma_2}[F](t), \tag{3.16}$$

where $d_0 = \dfrac{T^{(\gamma_1-\gamma_2)}\mathbf{\Gamma}(\gamma_2)}{\mathbf{\Gamma}(\gamma_1)}$ and $d_1 = \dfrac{c_1\mathbf{\Gamma}(\gamma_2)T^{(\gamma_1-\gamma_2)} + c_2\mathbf{\Gamma}(\gamma_1)}{\mathbf{\Gamma}(\gamma_1)} = c_1 d_0 + c_2$.

Finally, from Lemma 3.5 and relation (3.2), we get (since $I_{0^+}^{k\gamma_2}[F](0) = F_0\, I_{0^+}^{k\gamma_2}[.1] = \frac{F_0 t^{k\gamma_2}}{\mathbf{\Gamma}(1+k\gamma_2)}$)

$$F(t) \le F_0\sum_{k=0}^{\infty}\frac{1}{\mathbf{\Gamma}(1+k\gamma_2)}(d_1 t^{\gamma_2})^k + \sum_{k=0}^{\infty} d_1^k I_{0^+}^{k\gamma_2+\gamma_1}[h_1](t) + \sum_{k=0}^{\infty} d_1^k I_{0^+}^{(k+1)\gamma_2}[h_2](t).$$

This completes the proof. □

Corollary 3.1. *Assume that assumptions of Theorem 3.2 hold, and that h_i, for $i = 1, 2$, are nondecreasing functions on $(0, T)$. Then, if f and g are nonnegative functions locally integrable, with $f(0) = f_0$, $g(0) = g_0$, and satisfy the inequalities* (3.10)*, we have the following estimate*

$$F(t) \le F_0\, E_{\gamma_2,1}(d_1 t^{\gamma_2}) + t^{\gamma_1} h_1(t) E_{\gamma_2,\gamma_1+1}(d_1 t^{\gamma_2}) + t^{\gamma_2} h_2(t) E_{\gamma_2,\gamma_2+1}(d_1 t^{\gamma_2}), \tag{3.17}$$

where $F = f + g$, $F_0 = f_0 + g_0$, $d_0 = \dfrac{T^{(\gamma_1-\gamma_2)}\mathbf{\Gamma}(\gamma_2)}{\mathbf{\Gamma}(\gamma_1)}$ and $d_1 = \dfrac{c_1\mathbf{\Gamma}(\gamma_2)T^{(\gamma_1-\gamma_2)} + c_2\mathbf{\Gamma}(\gamma_1)}{\mathbf{\Gamma}(\gamma_1)} = c_1 d_0 + c_2$.

Proof. Since h_i, for $i = 1, 2$, are nondecreasing functions then

$$I_{0^+}^{k\gamma_2+\gamma_i}[h_i](t) \le h_i(t)\frac{1}{\mathbf{\Gamma}(k\gamma_2+\gamma_i)}\int_0^t (t-\tau)^{k\gamma_2+\gamma_i-1} d\tau = h_i(t)\frac{1}{\mathbf{\Gamma}(k\gamma_2+\gamma_i+1)} t^{k\gamma_2+\gamma_i}.$$

Consequently, from (3.11) we can deduce the result. This completes the proof. □

Corollary 3.2. *Assume that assumptions of Theorem 3.2 hold, and that h_i are $L^{\frac{1}{\alpha_i}}$ nondecreasing functions on $(0, T)$, with $\alpha_i \in [0, \gamma_i[$, for $i = 1, 2$. Then, if f and g are nonnegative functions locally integrable, and satisfy the inequalities* (3.10)*, we have the following estimate*

$$\begin{aligned} F(t) \le F(0)E_{\gamma_2,1}(d_1 t^{\gamma_2}) &+ \frac{t^{\gamma_1-\alpha_1}}{r_1} E_{\gamma_2,\gamma_1}(d_1 t^{\gamma_2}) \parallel h_1 \parallel_{L^{1/\alpha_1}(0,t)} \\ &+ \frac{t^{\gamma_2-\alpha_2}}{r_2} E_{\gamma_2,\gamma_2}(d_1 t^{\gamma_2}) \parallel h_2 \parallel_{L^{1/\alpha_2}(0,t)}, \end{aligned} \tag{3.18}$$

where $r_i = (\frac{\gamma_i-\alpha_i}{1-\alpha_i})^{1-\alpha_i}$, for $i = 1, 2$, $F = f + g$, $F_0 = f_0 + g_0$, $d_0 = \dfrac{T^{(\gamma_1-\gamma_2)}\mathbf{\Gamma}(\gamma_2)}{\mathbf{\Gamma}(\gamma_1)}$ and $d_1 = \dfrac{c_1\mathbf{\Gamma}(\gamma_2)T^{(\gamma_1-\gamma_2)} + c_2\mathbf{\Gamma}(\gamma_1)}{\mathbf{\Gamma}(\gamma_1)} = c_1 d_0 + c_2$.

Proof. From Hölder's inequality, we obtain (for $i = 1, 2$)

$$\begin{aligned} \mathbf{\Gamma}(k\gamma_2+\gamma_i) I_{0^+}^{k\gamma_2+\gamma_i}[h_i](t) &\le (\int_0^t (t-\tau)^{\frac{k\gamma_2+\gamma_i-1}{1-\alpha_i}} d\tau)^{1-\alpha_i} (\int_0^t h_i(\tau)^{\frac{1}{\alpha_i}} d\tau)^{\alpha_i} \\ &\le t^{k\gamma_2} (\int_0^t (t-\tau)^{\frac{\gamma_i-1}{1-\alpha_i}} d\tau)^{1-\alpha_i} (\int_0^t h_i(\tau)^{\frac{1}{\alpha_i}} d\tau)^{\alpha_i} = \frac{t^{k\gamma_2+\gamma_i-\alpha_i}}{r_i} (\int_0^t h_i(\tau)^{\frac{1}{\alpha_i}} d\tau)^{\alpha_i}, \end{aligned} \tag{3.19}$$

where $r_i = (\frac{\gamma_i-\alpha_i}{1-\alpha_i})^{1-\alpha_i}$.

According to (3.11), we have

$$\begin{aligned} F(t) \le F_0\, E_{\gamma_2,1}(d_1 t^{\gamma_2}) &+ \sum_{k=0}^{\infty} d_1^k \frac{t^{k\gamma_2+\gamma_1-\alpha_1}}{\mathbf{\Gamma}(k\gamma_2+\gamma_1) r_1} (\int_0^t h_1(\tau)^{\frac{1}{\alpha_i}} d\tau)^{\alpha_1} \\ &+ \sum_{k=0}^{\infty} d_1^k \frac{t^{k\gamma_2+\gamma_2-\alpha_2}}{\mathbf{\Gamma}(k\gamma_2+\gamma_2) r_2} (\int_0^t h_2(\tau)^{\frac{1}{\alpha_i}} d\tau)^{\alpha_2}. \end{aligned} \tag{3.20}$$

Thus

$$\begin{aligned} F(t) \le F_0\, E_{\gamma_2,1}(d_1 t^{\gamma_2}) &+ \frac{t^{\gamma_1-\alpha_1}}{r_1} E_{\gamma_2,\gamma_1}(d_1 t^{\gamma_2}) \parallel h_1 \parallel_{L^{1/\alpha_1}(0,t)} \\ &+ \frac{t^{\gamma_2-\alpha_2}}{r_2} E_{\gamma_2,\gamma_2}(d_1 t^{\gamma_2}) \parallel h_2 \parallel_{L^{1/\alpha_2}(0,t)}\ . \end{aligned} \tag{3.21}$$

This completes the proof. □

In the sequel we will always denote C some positive constant which may be different at each occurrence.

4. Well-Posedness of the memory bidomain-torso system

In this section, we prove the existence and uniqueness of weak solution to problem (1.1), under Lipschitz and boundedness assumptions on the non-linear operators.

4.1. Variational formulation and preliminary results

First we define the state ψ, as the extracellular cardiac potential in Ω_H, and the thoracic potential in Ω_B, by

$$\psi = \varphi_e \text{ in } \Omega_H \text{ and } \psi = \varphi_s \text{ in } \Omega_B. \tag{4.1}$$

Then the potential ϕ can be written as $\phi = \varphi_i - \psi|_{\Omega_H}$. From the interface coupling condition (1.3), it follows that if $\varphi_e \in H^1(\Omega_H)$ and $\varphi_s \in H^1(\Omega_B)$ then $\psi \in H^1(\Omega)$. Similarly, we introduce the corresponding conductivity tensor $\mathcal{K}_g \in W^{1,\infty}(\overline{\Omega})$ as follows :

$$\mathcal{K}_g = \mathcal{K}_e \text{ in } \overline{\Omega}_H \text{ and } \mathcal{K}_g = \mathcal{K}_s \text{ in } \overline{\Omega}_B.$$

We now define the following forms

$$\begin{aligned} A_i(w,v) &= \int_{\Omega_H} \mathcal{K}_i \nabla w \cdot \nabla v d\mathbf{x}, \quad A_e(w,v) = \int_{\Omega_H} \mathcal{K}_e \nabla w \cdot \nabla v d\mathbf{x}, \\ A_s(w,v) &= \int_{\Omega_B} \mathcal{K}_s \nabla w \cdot \nabla v d\mathbf{x}, \quad \mathbf{A_g}(w,v) = \int_{\Omega} \mathcal{K}_g \nabla w \cdot \nabla v d\mathbf{x}. \end{aligned} \tag{4.2}$$

Proposition 4.1. *(i) The forms A_k (for $k = i, e, s$) and $\mathbf{A_g}$, are symmetric bilinear continuous forms on H^1.*
(ii) The forms A_k, (for $k = i, e, s$) and $\mathbf{A_g}$, are coercive on H^1 (we denote by ν_k, for $k = i, e, s, g$, their coercivity coefficients).

Proof. (i) and (ii) are easily obtained providing that properties of tensors $\mathcal{K}_k$, for $k = i, e, s, g$, and (2.2) are satisfied. □

We can now define the weak solution to bidomain-thorax coupled model problem (1.1).

Definition 4.1. *A quadruplet of state functions $(\varphi_i, \phi, \psi, u)$ such that : $\varphi_i \in L^2(0,T;\mathbb{V}_H)$, $\phi \in \mathbb{D}_4(0,T) \cap L^\infty(0,T;L^2(\Omega_H))$, $\partial_t^\alpha \phi \in \mathbb{D}_4'(0,T)$, $\psi \in L^2(0,T;\mathbb{V})$, $u \in L^\infty(0,T;L^3(\Omega_H))$ and $\partial_t^\beta u \in L^2(0,T;L^2(\Omega_H))$ is said to be a weak solution to system* (1.1) *if it satisfies the following weak formulation (since $I_i = -I_e = f_H$)*

$$\begin{aligned} &\langle \mathfrak{c}_\alpha \partial_{0^+}^\alpha \phi, v_i \rangle_{\mathbb{V}_H', \mathbb{V}_H} + \int_{\Omega_H} \mathcal{I}(.;\phi,u) v_i d\mathbf{x} + A_i(\varphi_i, v_i) = \int_{\Omega_H} f_H v_i d\mathbf{x}, \\ &\langle \mathfrak{c}_\alpha \partial_{0^+}^\alpha \phi, v \rangle_{\mathbb{V}', \mathbb{V}} + \int_{\Omega_H} \mathcal{I}(.;\phi,u) v d\mathbf{x} - \mathbf{A_g}(\psi, v) = \int_{\Omega_H} f_H v d\mathbf{x}, \\ &(\partial_{0^+}^\beta u, \rho)_{L^2(\Omega_H)} + \int_{\Omega_H} \mathcal{G}(.;\phi,u) \rho d\mathbf{x} = 0, \end{aligned} \tag{4.3}$$

for all $v_i \in \mathbb{V}_H$, $v \in \mathbb{V}$, $\rho \in \mathbb{V}_H$.

The results of next section concern the existence, uniqueness and regularity of solution of problem (1.1).

4.2. Existence, uniqueness and regularity results

The purpose of this section is to prove the well-posedness of the degenerate bidomain-thorax coupled model (1.1). To overcome this issue, we introduce a specific non-degenerate approximate to system (1.1).

4.2.1. Non degenerated problem study

Let $\epsilon > 0$ be a small positive number. Hence, our approximation system read as (since $I_i = -I_e = f_{\mathrm{H}}$)

$$\begin{aligned}
&\mathfrak{c}_\alpha \partial_{0^+}^\alpha \phi^\epsilon + \epsilon \partial_{0^+}^\alpha \varphi_i^\epsilon + \mathcal{I}(\phi^\epsilon, u^\epsilon) - div(\mathcal{K}_i \nabla \varphi_i^\epsilon) = f_{\mathrm{H}}, \text{ in } Q_{\mathrm{H}} \\
&\mathfrak{c}_\alpha \partial_{0^+}^\alpha \phi^\epsilon - \epsilon \partial_{0^+}^\alpha \varphi_e^\epsilon + \mathcal{I}(\phi^\epsilon, u^\epsilon) + div(\mathcal{K}_e \nabla \varphi_e^\epsilon) = f_{\mathrm{H}}, \text{ in } Q_{\mathrm{H}} \\
&\epsilon \partial_{0^+}^\alpha \varphi_s^\epsilon - div(\mathcal{K}_s \nabla \varphi_s^\epsilon) = 0, \text{ in } Q_{\mathrm{B}} \\
&\partial_{0^+}^\beta u^\epsilon + \mathcal{G}(\phi^\epsilon, u^\epsilon) = 0, \text{ in } Q_{\mathrm{H}}
\end{aligned} \tag{4.4}$$

supplemented with the same initial and boundary conditions as in (1.4).

The auxiliary initial conditions for φ_i^ϵ and φ_s^ϵ, needed by problem (4.4), are defined by introducing two arbitrary functions $\varphi_{e,0}$ and $\varphi_{s,0}$. The weak formulation of (4.4) is given by (for all $v_i \in \mathbb{V}_{\mathrm{H}}$, $v \in \mathbb{V}$, $\rho \in \mathbb{V}_{\mathrm{H}}$ and *a.e.* $t \in (0, T)$)

$$\begin{aligned}
&\langle \mathfrak{c}_\alpha \partial_{0^+}^\alpha \phi^\epsilon, v_i \rangle_{\mathbb{V}_{\mathrm{H}}', \mathbb{V}_{\mathrm{H}}} + \epsilon \langle \partial_{0^+}^\alpha \varphi_i^\epsilon, v_i \rangle_{\mathbb{V}_{\mathrm{H}}', \mathbb{V}_{\mathrm{H}}} + \int_{\Omega_{\mathrm{H}}} \mathcal{I}(\phi^\epsilon, u^\epsilon) v_i d\mathbf{x} + A_i(\varphi_i^\epsilon, v_i) = \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}} v_i d\mathbf{x}, \\
&\langle \mathfrak{c}_\alpha \partial_{0^+}^\alpha \phi^\epsilon, v \rangle_{\mathbb{V}_{\mathrm{H}}', \mathbb{V}_{\mathrm{H}}} - \epsilon \langle \partial_{0^+}^\alpha \psi^\epsilon, v \rangle_{\mathbb{V}', \mathbb{V}} + \int_{\Omega_{\mathrm{H}}} \mathcal{I}(\phi^\epsilon, u^\epsilon) v d\mathbf{x} - \mathbf{A_g}(\psi, v) = \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}} v d\mathbf{x}, \\
&(\partial_{0^+}^\beta u^\epsilon, \rho)_{L^2(\Omega_{\mathrm{H}})} + \int_{\Omega_{\mathrm{H}}} \mathcal{G}(\phi^\epsilon, u^\epsilon) \rho d\mathbf{x} = 0,
\end{aligned} \tag{4.5}$$

with the initial condition

$$(\phi^\epsilon, \varphi_i^\epsilon, \psi^\epsilon, u^\epsilon)(t = 0) = (\phi_0, \varphi_{i,0}, \psi_0, u_0),$$

where ψ^ϵ and the initial condition ψ_0 are defined as in (4.1).

The following results concern the existence and regularity of the weak solution to problem (4.4) (i.e. of (4.5)).

Theorem 4.1. *Let assumptions* **(H1)** *and* $(\mathbf{H2})_4$ *be fulfilled,* $\beta \geq \alpha$ *and* $0 < \epsilon \leq 1$ *be given. Then for* $(\varphi_{i,0}, \phi_0, \psi_0, u_0)$ *and* f_{H} *given such that* $(\varphi_{i,0}, \phi_0, \psi_0, u_0) \in \mathbb{V}_{\mathrm{H}}^2 \times \mathbb{V} \times L^3(\Omega_{\mathrm{H}})$ *and* $f_{\mathrm{H}} \in L^{\frac{2}{\alpha_1}}(0, T; L^2(\Omega_{\mathrm{H}}))$ *with* $0 < \alpha_1 < \alpha$*, there exists a solution* $(\varphi_i^\epsilon, \phi^\epsilon, \psi^\epsilon, u^\epsilon)$ *of problem* (4.5) *verifying*

$$\begin{aligned}
&\phi^\epsilon \in L^\infty(0, T; \mathbb{V}_{\mathrm{H}}), \ \ \partial_{0^+}^\alpha \phi^\epsilon \in L^2(Q_{\mathrm{H}}), \\
&\varphi_i^\epsilon \in L^\infty(0, T; \mathbb{V}_{\mathrm{H}}), \ \ \sqrt{\epsilon} \partial_{0^+}^\alpha \varphi_i^\epsilon \in L^2(Q_{\mathrm{H}}), \\
&\psi^\epsilon \in L^\infty(0, T; \mathbb{V}), \ \ \sqrt{\epsilon} \partial_{0^+}^\alpha \phi^\epsilon \in L^2(Q_{\mathrm{H}}), \\
&u^\epsilon \in L^\infty(0, T; L^2(\Omega_{\mathrm{H}})), \ \ \partial_{0^+}^\beta u^\epsilon \in L^2(Q_{\mathrm{H}}).
\end{aligned} \tag{4.6}$$

Moreover, if $\alpha > 1/2$ *and* $\beta > 1/2$ *we have the continuity of the solution at* $t = 0$.

Proof. To establish the existence result of weak solution to system (4.4) we apply the Faedo-Galerkin method, derive a priori estimates, and then pass to the limit in the approximate solutions using compactness arguments. We approximate the system equations by projecting them onto finite n dimensional subspaces, then we take the limit in n. For this, let $(w_k)_{k\geq 1}$ be a Hilbert basis and orthogonal in L^2 of $\mathbb{V}_{\mathrm{H}}$, $(f_k)_{k\geq 1}$ be a Hilbert basis of $\mathbb{U}_{\mathrm{H}}$, $(g_k)_{k\geq 1}$ be a Hilbert basis of $\mathbb{W}_{\mathrm{B}} = \{v \in H^1(\Omega_{\mathrm{B}}) \mid v|_{\Gamma_T} = 0\}$. We denote by $\tilde{f}_k$ an extension of f_k in $H^1(\Omega)$, $\tilde{g}_k$ an extension of g_k in $H^1(\Omega)$ with $\tilde{g}_k|_{\Omega_{\mathrm{H}}} = 0$ and by $(e_k)_{k\geq 1}$ a Galerkin basis of $\mathbb{U}_{\mathrm{HB}}$ which is defined as $e_{2k} = \tilde{g}_k$ and $e_{2k-1} = \tilde{f}_k$ (for all $k > 1$).

For all $n \in \mathbb{N}^*$, we denote by $W_{i,n} = span(w_1, \cdots, w_n)$, $W_{e,n} = span(f_1, \cdots, f_n)$, $W_{s,n} = span(g_1, \cdots, g_n)$ and $W_{g,n} = span(e_1, \cdots, e_{2n})$ the spaces generated, respectively, by $(w_k)_{n\geq k\geq 1}$, $(f_k)_{n\geq k\geq 1}$, $(g_k)_{n\geq k\geq 1}$ and $(e_k)_{2n\geq k\geq 1}$, and we introduce the orthogonal projector $L_{j,n}$ on the spaces $W_{j,n}$ (for $j = i, e, s, g$).

For each n, we would like to define the approximate solution $(\phi_n^\epsilon, \varphi_{i,n}^\epsilon, \psi_n^\epsilon, u_n^\epsilon)$ of the problem (4.4). Setting

$$\begin{aligned} \phi_n^\epsilon(\cdot,t) &= \sum_{l=1}^{n} \varpi_{n,l}^\epsilon(t) w_l, \quad u_n^\epsilon(\cdot,t) = \sum_{l=1}^{n} \upsilon_{n,l}^\epsilon(t) w_l, \\ \varphi_{i,n}^\epsilon(\cdot,t) &= \sum_{l=1}^{n} \varpi_{i,n,l}^\epsilon(t) w_l, \quad \psi_n^\epsilon(\cdot,t) = \sum_{l=1}^{2n} \pi_{n,l}^\epsilon(t) e_l \end{aligned} \tag{4.7}$$

where $\boldsymbol{\omega}_i^\epsilon = (\varpi_{i,n,l}^\epsilon)_{l=1,n}$, $\boldsymbol{\omega}^\epsilon = (\varpi_{n,l}^\epsilon)_{l=1,n}$, $\vartheta^\epsilon = (\upsilon_{n,l}^\epsilon)_{l=1,n}$ and $\Pi^\epsilon = (\pi_{n,m}^\epsilon)_{m=1,2n}$ are unknown functions, and replacing $(\phi^\epsilon, \varphi_i^\epsilon, \psi^\epsilon, u^\epsilon)$ by $(\phi_n^\epsilon, \varphi_{i,n}^\epsilon, \psi_n^\epsilon, u_n^\epsilon)$ in (4.4), we obtain $\forall l = 1, n$, $m = 1, 2n$ and *a.e.* $t \in (0, T)$, the system of Galerkin equations (since in Ω_{H}, $\phi_n^\epsilon = \varphi_{i,n}^\epsilon - \psi_n^\epsilon$)

$$\begin{aligned}
&(\mathfrak{c}_\alpha + \epsilon)\int_{\Omega_{\mathrm{H}}} \partial_{0^+}^\alpha \varphi_{i,n}^\epsilon w_l d\mathbf{x} - \mathfrak{c}_\alpha \int_{\Omega_{\mathrm{H}}} \partial_{0^+}^\alpha \psi_n^\epsilon w_l d\mathbf{x} \\
&\qquad = -\int_{\Omega_{\mathrm{H}}} \mathcal{I}(\varphi_{i,n}^\epsilon - \psi_n^\epsilon, u_n^\epsilon) w_l d\mathbf{x} - A_i(\varphi_{i,n}^\epsilon, w_l) + \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}} w_l d\mathbf{x} \\
&\qquad := \langle F_n^{\epsilon,i}, w_l\rangle \overset{def}{=} F_l^i(t; \boldsymbol{\omega}_i^\epsilon, \Pi^\epsilon, \vartheta^\epsilon), \\
&-\mathfrak{c}_\alpha \int_{\Omega_{\mathrm{H}}} \partial_{0^+}^\alpha \varphi_{i,n}^\epsilon e_m d\mathbf{x} + \mathfrak{c}_\alpha \int_{\Omega_{\mathrm{H}}} \partial_{0^+}^\alpha \psi_n^\epsilon e_m d\mathbf{x} + \epsilon \int_{\Omega} \partial_{0^+}^\alpha \psi_n^\epsilon e_m d\mathbf{x} \\
&\qquad = \int_{\Omega_{\mathrm{H}}} \mathcal{I}(\varphi_{i,n}^\epsilon - \psi_n^\epsilon, u_n^\epsilon) e_m d\mathbf{x} - \mathbf{A_g}(\psi_n^\epsilon, e_m) - \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}} e_m d\mathbf{x} \\
&\qquad := \langle G_n^\epsilon, e_m\rangle \overset{def}{=} G_m(t; \boldsymbol{\omega}_i^\epsilon, \Pi^\epsilon, \vartheta^\epsilon), \\
&\int_{\Omega_{\mathrm{H}}} \partial_{0^+}^\beta u_n^\epsilon w_l d\mathbf{x} = -\int_{\Omega_{\mathrm{H}}} \mathcal{G}(\varphi_{i,n}^\epsilon - \psi_n^\epsilon, u_n^\epsilon) w_l d\mathbf{x} \\
&\qquad\qquad := \langle H_n^\epsilon, w_l\rangle \overset{def}{=} H_l(t; \boldsymbol{\omega}_i^\epsilon, \Pi^\epsilon, \vartheta^\epsilon),
\end{aligned} \tag{4.8}$$

with the initial condition

$$(\phi_n^\epsilon, \varphi_{n,i}^\epsilon, \psi_n^\epsilon, u_n^\epsilon)(t=0) = (L_{i,n}\phi_0, L_{i,n}\varphi_{i,0}, L_{g,n}\psi_0, L_{i,n}u_0)$$

where $(L_{i,n}\phi_0, L_{i,n}\varphi_{i,0}, L_{g,n}\psi_0, L_{i,n}u_0)$ satisfies (by construction)

$$(L_{i,n}\phi_0, L_{i,n}\varphi_{i,0}, L_{g,n}\psi_0, L_{i,n}u_0) \overset{n\longrightarrow\infty}{\longrightarrow} (\phi_0, \varphi_{i,0}, \psi_0, u_0) \text{ strongly in } \mathbb{V}_{\mathrm{H}}^2 \times \mathbb{V} \times L^3(\Omega_{\mathrm{H}}). \tag{4.9}$$

The function $F_n^{\epsilon,i}$, G_n^{ϵ} and H_n^{ϵ} are defined by (for $(v,w,\rho)\in\mathbb{V}_{\mathrm{H}}\times\mathbb{V}\times\mathbb{V}_{\mathrm{H}}$)

$$\begin{aligned}\langle F_n^{\epsilon,i},v\rangle &= -\int_{\Omega_{\mathrm{H}}}\mathcal{I}(\phi_n^\epsilon,u_n^\epsilon)v d\mathbf{x} - A_i(\varphi_{i,n}^\epsilon,v) + \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}} v d\mathbf{x},\\ \langle G_n^\epsilon,w\rangle &= \int_{\Omega_{\mathrm{H}}}\mathcal{I}(\phi_n^\epsilon,u_n^\epsilon)w d\mathbf{x} - \mathbf{A_g}(\psi_n^\epsilon,w) - \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}} w d\mathbf{x},\\ \langle H_n^\epsilon,\rho\rangle &= -\int_{\Omega_{\mathrm{H}}}\mathcal{G}(\phi_n^\epsilon,u_n^\epsilon)\rho d\mathbf{x}.\end{aligned}\tag{4.10}$$

Step 1. We show first that, for every n, the system (4.8) admits a unique local solution. The system (4.8) is equivalent to an initial-value for a system of nonlinear fractional differential equations for functions $(\omega_i^\epsilon,\Pi^\epsilon,\vartheta^\epsilon)$

$$\begin{aligned}&\mathcal{M}_n(\partial_{0^+}^\alpha\omega_i^\epsilon,\partial_{0^+}^\alpha\Pi^\epsilon,\partial_{0^+}^\beta\vartheta^\epsilon)^T = \mathcal{F}_n(t;\omega_i^\epsilon,\Pi^\epsilon,\vartheta^\epsilon),\\ &(\omega_i^\epsilon,\Pi^\epsilon,\vartheta^\epsilon)(t=0) = (\omega_{i,0}^\epsilon,\Pi_0^\epsilon,\vartheta_0^\epsilon),\end{aligned}\tag{4.11}$$

where $\mathcal{F}_n:[0,T]\times\mathbb{R}^{4n}\to\mathbb{R}^{4n}$ is a known nonlinear vector function and is given by

$$\mathcal{F}_n(t;\omega_i^\epsilon,\Pi^\epsilon,\vartheta^\epsilon) = ((F_l^i(t;\omega_i^\epsilon,\Pi^\epsilon,\vartheta^\epsilon))_{l=1,n},(G_m(t;\omega_i^\epsilon,\Pi^\epsilon,\vartheta^\epsilon))_{m=1,2n},(H_l(t;\omega_i^\epsilon,\Pi^\epsilon,\vartheta^\epsilon))_{l=1,n})^T.$$

Thanks to assumptions $(\mathbf{H2})_4$ the function $\mathcal{F}_n$ is Carathéodory function.

Since, by construction the matrix $\mathcal{M}\in\mathbb{R}^{4n\times 4n}$ is symmetric and invertible, then by applying $\mathcal{M}_n^{-1}$ to the equation of problem (4.11), we obviously get

$$\begin{aligned}&(\partial_{0^+}^\alpha\omega_i^\epsilon,\partial_{0^+}^\alpha\Pi^\epsilon,\partial_{0^+}^\beta\vartheta^\epsilon)^T = \mathcal{S}_n(t;\omega_i^\epsilon,\Pi^\epsilon,\vartheta^\epsilon),\\ &(\omega_i^\epsilon,\Pi^\epsilon,\vartheta^\epsilon)(t=0) = (\omega_{i,0}^\epsilon,\Pi_0^\epsilon,\vartheta_0^\epsilon),\end{aligned}\tag{4.12}$$

where $\mathcal{S}_n = \mathcal{M}_n^{-1}\mathcal{F}_n$.

The existence of a local absolutely continuous function $(\omega_i^\epsilon,\Pi^\epsilon,\vartheta^\epsilon)$ on an interval $[0,\tilde{T}]$, with $\tilde{T}\in]0,T]$ is insured by the standard FODE theory see [32, 33] (see also e.g. [48]). Thus, we have a local solution $(\phi^\epsilon,\varphi_i^\epsilon,\psi^\epsilon,u^\epsilon)$ of problem (4.8) on $[0,\tilde{T}]$.

N.B. The constants which will be introduced in the sequel, are independent of n and ϵ unless otherwise specified.

Step 2. We next derive a priori estimates for functions $\phi^\epsilon,\varphi_i^\epsilon,\psi^\epsilon,u^\epsilon$, which entail that $\tilde{T}=T$, by applying iteratively the step 1. For simplicity, in next step we omit the "~" on T. Now, we set

$$\mathbf{h}_n(\cdot,t)=\sum_{k=1}^n\vartheta_{n,k}(t)w_k,\quad \mathbf{v}_n(\cdot,t)=\sum_{l=1}^n\zeta_{n,l}(t)w_l,\quad \mathbf{e}_n(\cdot,t)=\sum_{p=1}^{2n}\pi_{n,p}(t)e_p.$$

where $\vartheta_{n,k}$, $\pi_{n,p}$ and $\zeta_{n,l}$ are absolutely continuous coefficients. Then, from (4.8), the approximation solution satisfies the following weak formulation

$$
\begin{aligned}
&(\mathfrak{c}_\alpha + \epsilon) \int_{\Omega_{\rm H}} \partial^\alpha_{0^+} \varphi^\epsilon_{i,n} \mathbf{h}_n d\mathbf{x} - \mathfrak{c}_\alpha \int_{\Omega_{\rm H}} \partial^\alpha_{0^+} \psi^\epsilon_n \mathbf{h}_n d\mathbf{x} \\
&\qquad = - \int_{\Omega_{\rm H}} \mathcal{I}(\varphi^\epsilon_{i,n} - \psi^\epsilon_n, u^\epsilon_n) \mathbf{h}_n d\mathbf{x} - A_i(\varphi^\epsilon_{i,n}, \mathbf{h}_n) + \int_{\Omega_{\rm H}} f_{\rm H} \mathbf{h}_n d\mathbf{x}, \\
&-\mathfrak{c}_\alpha \int_{\Omega_{\rm H}} \partial^\alpha_{0^+} \varphi^\epsilon_{i,n} \mathbf{e}_n d\mathbf{x} + \mathfrak{c}_\alpha \int_{\Omega_{\rm H}} \partial^\alpha_{0^+} \psi^\epsilon_n \mathbf{e}_n d\mathbf{x} + \epsilon \int_{\Omega} \partial^\alpha_{0^+} \psi^\epsilon_n \mathbf{e}_n d\mathbf{x} \\
&\qquad = \int_{\Omega_{\rm H}} \mathcal{I}(\varphi^\epsilon_{i,n} - \psi^\epsilon_n, u^\epsilon_n) \mathbf{e}_n d\mathbf{x} - \mathbf{A_g}(\psi^\epsilon_n, \mathbf{e}_n) - \int_{\Omega_{\rm H}} f_{\rm H} \mathbf{e}_n d\mathbf{x}, \\
&\int_{\Omega_{\rm H}} \partial^\beta_{0^+} u^\epsilon_n \mathbf{v}_n d\mathbf{x} = - \int_{\Omega_{\rm H}} \mathcal{G}(\varphi^\epsilon_{i,n} - \psi^\epsilon_n, u^\epsilon_n) \mathbf{v}_n d\mathbf{x},
\end{aligned}
\tag{4.13}
$$

with the initial condition

$$
(\phi^\epsilon_n, \varphi^\epsilon_{n,i}, \psi^\epsilon_n, u^\epsilon_n)(t=0) = (L_{i,n}\phi_0, L_{i,n}\varphi_{i,0}, L_{g,n}\psi_0, L_{i,n}u_0).
$$

Taking $\mathbf{h}_n = \varphi^\epsilon_{i,n}$, $\mathbf{e}_n = \psi^\epsilon_n$ and $\mathbf{v}_n = u^\epsilon_n$ and using the uniform coercivity of the forms A_i and $\mathbf{A_g}$, we can deduce (since $\psi^\epsilon_n = \varphi^\epsilon_{e,n}$ and $\phi^\epsilon_n = \varphi^\epsilon_{i,n} - \varphi^\epsilon_{e,n}$ in $\Omega_{\rm H}$)

$$
\begin{aligned}
&\frac{1}{2}\left(\mathfrak{c}_\alpha \partial^\alpha_{0^+} \| \phi^\epsilon_n \|^2_{L^2(\Omega_{\rm H})} + \epsilon(\partial^\alpha_{0^+} \| \psi^\epsilon_n \|^2_{L^2(\Omega)} + \partial^\alpha_{0^+} \| \varphi^\epsilon_{i,n} \|^2_{L^2(\Omega_{\rm H})}) \right) \\
&\qquad + \nu_i \| \nabla \varphi^\epsilon_{i,n} \|^2_{L^2(\Omega_{\rm H})} + \nu_g \| \nabla \psi^\epsilon_n \|^2_{L^2(\Omega)} + \int_{\Omega_{\rm H}} \mathcal{I}(\phi^\epsilon_n, u^\epsilon_n) \phi^\epsilon_n d\mathbf{x} \\
&\qquad \leq \int_{\Omega_{\rm H}} f_{\rm H} \phi^\epsilon_n d\mathbf{x}, \\
&\frac{1}{2} \partial^\beta_{0^+} \| u^\epsilon_n \|^2_{L^2(\Omega_{\rm H})} \leq - \int_{\Omega_{\rm H}} \mathcal{G}(\phi^\epsilon_n, u^\epsilon_n) u^\epsilon_n d\mathbf{x}.
\end{aligned}
\tag{4.14}
$$

Then the following inequality

$$
\begin{aligned}
&\frac{1}{2}\left(\mathfrak{c}_\alpha \partial^\alpha_{0^+} \| \phi^\epsilon_n \|^2_{L^2(\Omega_{\rm H})} + \epsilon(\partial^\alpha_{0^+} \| \psi^\epsilon_n \|^2_{L^2(\Omega)} + \partial^\alpha_{0^+} \| \varphi^\epsilon_{i,n} \|^2_{L^2(\Omega_{\rm H})}) \right) \\
&\qquad + \nu_i \| \nabla \varphi^\epsilon_{i,n} \|^2_{L^2(\Omega_{\rm H})} + \nu_g \| \nabla \psi^\epsilon_n \|^2_{L^2(\Omega)} + \frac{1}{\mu_1} \int_{\Omega_{\rm H}} \left(\mu_1 \mathcal{I}(\phi^\epsilon_n, u^\epsilon_n) \phi^\epsilon_n + \mathcal{G}(\phi^\epsilon_n, u^\epsilon_n) u^\epsilon_n \right) d\mathbf{x} \\
&\qquad \leq \int_{\Omega_{\rm H}} f_{\rm H} \phi^\epsilon_n d\mathbf{x} - \frac{1}{\mu_1} \int_{\Omega_{\rm H}} \mathcal{G}(\phi^\epsilon_n, u^\epsilon_n) u^\epsilon_n d\mathbf{x}, \\
&\frac{1}{2} \partial^\beta_{0^+} \| u^\epsilon_n \|^2_{L^2(\Omega_{\rm H})} \leq - \int_{\Omega_{\rm H}} \mathcal{G}(\phi^\epsilon_n, u^\epsilon_n) u^\epsilon_n d\mathbf{x}.
\end{aligned}
\tag{4.15}
$$

Since, from $(\mathbf{H2})_4$, $\mu_1 \mathcal{I}(\phi^\epsilon_n, u^\epsilon_n)\phi^\epsilon_n + \mathcal{G}(\phi^\epsilon_n, u^\epsilon_n)u^\epsilon_n \geq \mu_2 \mid \phi^\epsilon_n \mid^4 - \mu_3(\mu_1 \mid \phi^\epsilon_n \mid^2 + \mid u^\epsilon_n \mid^2) - \mu_4$ and $\mid \mathcal{G}(\phi^\epsilon_n, u^\epsilon_n)u^\epsilon_n \mid \leq c_1(1 + \mid \phi^\epsilon_n \mid^2 \mid u^\epsilon_n \mid + \mid u^\epsilon_n \mid^2)$, we obtain (from Hölder's inequality)

$$
\begin{aligned}
&\mathfrak{c}_\alpha \partial^\alpha_{0^+} \| \phi^\epsilon_n \|^2_{L^2(\Omega_{\rm H})} + \epsilon(\partial^\alpha_{0^+} \| \psi^\epsilon_n \|^2_{L^2(\Omega)} + \partial^\alpha_{0^+} \| \varphi^\epsilon_{i,n} \|^2_{L^2(\Omega_{\rm H})}) \\
&\qquad + 2\nu_i \| \nabla \varphi^\epsilon_{i,n} \|^2_{L^2(\Omega_{\rm H})} + 2\nu_g \| \nabla \psi^\epsilon_n \|^2_{L^2(\Omega)} + \frac{\mu_2}{\mu_1} \int_{\Omega_{\rm H}} \mid \phi^\epsilon_n \mid^4 d\mathbf{x} \\
&\qquad \leq c_2 + c_3 \| f_{\rm H} \|^2_{L^2(\Omega_{\rm H})} + c_4(\| \phi^\epsilon_n \|^2_{L^2(\Omega_{\rm H})} + \| u^\epsilon_n \|^2_{L^2(\Omega_{\rm H})}), \\
&\partial^\beta_{0^+} \| u^\epsilon_n \|^2_{L^2(\Omega_{\rm H})} \leq c_5 + c_6(\| \phi^\epsilon_n \|^2_{L^2(\Omega_{\rm H})} + \| u^\epsilon_n \|^2_{L^2(\Omega_{\rm H})}) + \eta \int_{\Omega_{\rm H}} \mid \phi^\epsilon_n \mid^4 d\mathbf{x},
\end{aligned}
\tag{4.16}
$$

with the constants η chosen appropriately. In particular we have

$$\begin{aligned}&\partial_{0^+}^{\alpha}\Psi_n^{\epsilon}+\frac{\mu_2}{\mu_1}\int_{\Omega_H}|\,\phi_n^{\epsilon}\,|^4\,d\mathbf{x}\leq c_7(1+\|\,f_H\,\|^2_{L^2(\Omega_H)})+c_8(\Psi_n^{\epsilon}+U_n^{\epsilon}),\\&\partial_{0^+}^{\beta}U_n^{\epsilon}\leq c_9+c_{10}(\Psi_n^{\epsilon}+U_n^{\epsilon})+\eta\int_{\Omega_H}|\,\phi_n^{\epsilon}\,|^4\,d\mathbf{x},\end{aligned}\tag{4.17}$$

where $\Psi_n^{\epsilon}=\mathfrak{c}_\alpha\,\|\,\phi_n^{\epsilon}\,\|^2_{L^2(\Omega_H)}+(\|\,\sqrt{\epsilon}\psi_n^{\epsilon}\,\|^2_{L^2(\Omega)}+\|\,\sqrt{\epsilon}\varphi_{i,n}^{\epsilon}\,\|^2_{L^2(\Omega_H)})$ and $U_n^{\epsilon}=\|\,u_n^{\epsilon}\,\|^2_{L^2(\Omega_H)}$.
According to regularity $f_H\in L^{2/\alpha_1}(0,T,L^2(\Omega_H))$ and by choosing $\eta=\frac{\Gamma(\beta)\mu_2/\mu_1}{T^{\beta-\alpha}\Gamma(\alpha)}$), we can get by Corollary 3.2 and the fact that quantity $\Psi_n^{\epsilon}(0)+U_n^{\epsilon}(0)$ is uniformly bounded (from (4.9) and $0<\epsilon<1$)

$$\Psi_n^{\epsilon}(t)+U_n^{\epsilon}(t)\leq c_{11}(1+\|\,f_H\,\|^2_{L^{2/\alpha_1}(0,T,L^2(\Omega_H))}),\tag{4.18}$$

for all $t\in(0,T)$ and then $\Psi_n^{\epsilon}(t)$ and $U_n^{\epsilon}(t)$ is uniformly bounded with respect to ϵ and to n. This ensures that

$$\begin{aligned}&\text{the sequences } (\phi_n^{\epsilon},\ \sqrt{\epsilon}\varphi_{i,n}^{\epsilon},u_n^{\epsilon}) \text{ and } (\sqrt{\epsilon}\psi_n^{\epsilon}) \text{ are bounded sets}\\&\qquad\text{of } L^{\infty}(0,T;L^2(\Omega_H)) \text{ and } L^{\infty}(0,T;L^2(\Omega)), \text{ respectively.}\end{aligned}\tag{4.19}$$

Moreover, from the first inequality of (4.16), relations (3.4) and (4.9), we have that for $t\in(0,T)$ (since $\int_0^t(t-\tau)^{\alpha-1}d\tau=\frac{1}{\alpha}t^{\alpha}\leq\frac{1}{\alpha}T^{\alpha}$)

$$\begin{aligned}\boldsymbol{\Gamma}(\alpha)\Psi_n^{\epsilon}(t)+\int_0^t(t-\tau)^{\alpha-1}(2\nu_i\,\|\,\nabla\varphi_{i,n}^{\epsilon}\,\|^2_{L^2(\Omega_H)}+2\nu_g\,\|\,\nabla\psi_n^{\epsilon}\,\|^2_{L^2(\Omega)}+\frac{\mu_2}{\mu_1}\,\|\,\phi_n^{\epsilon}\,\|^4_{L^4(\Omega_H)})d\tau\\\leq c_{11}C_0\end{aligned}\tag{4.20}$$

where $C_0=(1+\|\,f_H\,\|^2_{L^{2/\alpha_1}(0,T,L^2(\Omega_H))})$.

We can deduce that (since $\phi_n^{\epsilon}=\varphi_{i,n}^{\epsilon}-\varphi_{e,n}^{\epsilon}$ and $\psi_n^{\epsilon}=\varphi_{e,n}^{\epsilon}$ in Ω_H)

$$\begin{aligned}I_{0^+}^{\alpha}\left[\|\,\nabla\phi_n^{\epsilon}\,\|^2_{L^2(\Omega_H)}\right](t)&=\frac{1}{\boldsymbol{\Gamma}(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\,\|\,\nabla\phi_n^{\epsilon}\,\|^2_{L^2(\Omega_H)}\,d\tau\leq c_{12}C_0,\\I_{0^+}^{\alpha}\left[\|\,\nabla\varphi_{i,n}^{\epsilon}\,\|^2_{L^2(\Omega_H)}\right](t)&=\frac{1}{\boldsymbol{\Gamma}(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\,\|\,\nabla\varphi_{i,n}^{\epsilon}\,\|^2_{L^2(\Omega_H)}\,d\tau\leq c_{12}C_0,\\I_{0^+}^{\alpha}\left[\|\,\nabla\psi_n^{\epsilon}\,\|^2_{L^2(\Omega)}\right](t)&=\frac{1}{\boldsymbol{\Gamma}(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\,\|\,\nabla\psi_n^{\epsilon}\,\|^2_{L^2(\Omega)}\,d\tau\leq c_{12}C_0,\\I_{0^+}^{\alpha}\left[\|\,\phi_n^{\epsilon}\,\|^4_{L^4(\Omega_H)}\right](t)&=\frac{1}{\boldsymbol{\Gamma}(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\,\|\,\phi_n^{\epsilon}\,\|^4_{L^4(\Omega_H)}\,d\tau\leq c_{12}C_0.\end{aligned}\tag{4.21}$$

Hence (since $\alpha-1\leq0$ and $(t-\tau)\in(0,T),\forall\tau\in(0,t)$ then $(t-\tau)^{\alpha-1}\geq T^{\alpha-1}$)

$$\begin{aligned}&\text{the sequences } (\varphi_{i,n}^{\epsilon}) \text{ and } (\psi_{i,n}^{\epsilon}) \text{ are bounded sets}\\&\qquad\text{of } L^2(0,T;H^1(\Omega_H)) \text{ and } L^2(0,T;H^1(\Omega)), \text{ respectively}\\&\text{and the sequence } (\phi_n^{\epsilon}) \text{ is bounded set of } L^4(0,T,L^4(\Omega_H))\cap L^2(0,T;H^1(\Omega_H)).\end{aligned}\tag{4.22}$$

Using Lemma 2.3 and results (4.19) and (4.22), we get

$$\begin{aligned}&|\int_0^T\int_{\Omega_H}\mathcal{I}(\phi_n^{\epsilon},u_n^{\epsilon})\phi_n^{\epsilon}d\mathbf{x}\,|\text{ is uniformly bounded with respect to } n,\\&|\int_0^T\int_{\Omega_H}\mathcal{G}(\phi_n^{\epsilon},u_n^{\epsilon})u_n^{\epsilon}d\mathbf{x}\,|\text{ is uniformly bounded with respect to } n.\end{aligned}\tag{4.23}$$

Now we estimate the fractional derivative $\partial_{0^+}^{\alpha}(\phi_n^{\epsilon}, \varphi_{i,n}^{\epsilon}, \psi_n^{\epsilon})$ and $\partial_{0^+}^{\beta} u_n^{\epsilon}$. Taking $\mathbf{h}_n = \partial_{0^+}^{\alpha}\varphi_{i,n}^{\epsilon}$, $\mathbf{e}_n = \partial_{0^+}^{\alpha}\psi_n^{\epsilon}$ and $\mathbf{v}_n = \partial_{0^+}^{\beta} u_n^{\epsilon}$ and using uniform coercivity of forms A_i and $\mathbf{A_g}$, we can deduce (since $\psi_n^{\epsilon} = \varphi_{e,n}^{\epsilon}$ and $\phi_n^{\epsilon} = \varphi_{i,n}^{\epsilon} - \varphi_{e,n}^{\epsilon}$ in Ω_{H})

$$\begin{aligned}
\mathfrak{c}_{\alpha} \parallel \partial_{0^+}^{\alpha}\phi_n^{\epsilon} \parallel_{L^2(\Omega_{\mathrm{H}})}^2 &+\epsilon(\parallel \partial_{0^+}^{\alpha}\psi_n^{\epsilon} \parallel_{L^2(\Omega)}^2 + \parallel \partial_{0^+}^{\alpha}\varphi_{i,n}^{\epsilon} \parallel_{L^2(\Omega_{\mathrm{H}})}^2) \\
&+\frac{\nu_i}{2}\partial_{0^+}^{\alpha} \parallel \nabla\varphi_{i,n}^{\epsilon} \parallel_{L^2(\Omega_{\mathrm{H}})}^2 +\frac{\nu_g}{2}\partial_{0^+}^{\alpha} \parallel \nabla\psi_n^{\epsilon} \parallel_{L^2(\Omega)}^2 \\
&\leq -\int_{\Omega_{\mathrm{H}}} \mathcal{I}(\phi_n^{\epsilon}, u_n^{\epsilon})\partial_{0^+}^{\alpha}\phi_n^{\epsilon} d\mathbf{x} + \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}}\partial_{0^+}^{\alpha}\phi_n^{\epsilon} d\mathbf{x}, \\
\parallel \partial_{0^+}^{\beta} u_n^{\epsilon} \parallel_{L^2(\Omega_{\mathrm{H}})}^2 &= -\int_{\Omega_{\mathrm{H}}} \mathcal{G}(\phi_n^{\epsilon}, u_n^{\epsilon})\partial_{0^+}^{\alpha} u_n^{\epsilon} d\mathbf{x}.
\end{aligned} \tag{4.24}$$

Then

$$\begin{aligned}
\mathfrak{c}_{\alpha} I_{0^+}^{\alpha}&\left[\parallel \partial_{0^+}^{\alpha}\phi_n^{\epsilon} \parallel_{L^2(\Omega_{\mathrm{H}})}^2\right](t) + \epsilon\Big(I_{0^+}^{\alpha}\left[\parallel \partial_{0^+}^{\alpha}\psi_n^{\epsilon} \parallel_{L^2(\Omega)}^2\right](t) + I_{0^+}^{\alpha}\left[\parallel \partial_{0^+}^{\alpha}\varphi_{i,n}^{\epsilon} \parallel_{L^2(\Omega_{\mathrm{H}})}^2\right](t)\Big) \\
&+\frac{\nu_i}{2} \parallel \nabla\varphi_{i,n}^{\epsilon}(t) \parallel_{L^2(\Omega_{\mathrm{H}})}^2 +\frac{\nu_g}{2} \parallel \nabla\psi_n^{\epsilon}(t) \parallel_{L^2(\Omega)}^2 \\
&\leq \frac{\nu_i}{2} \parallel \nabla\varphi_{i,n}^{\epsilon}(0) \parallel_{L^2(\Omega_{\mathrm{H}})}^2 +\frac{\nu_g}{2} \parallel \nabla\psi_n^{\epsilon}(0) \parallel_{L^2(\Omega)}^2 \\
&-\int_{\Omega_{\mathrm{H}}} I_{0^+}^{\alpha}\left[\mathcal{I}(\phi_n^{\epsilon}, u_n^{\epsilon})\partial_{0^+}^{\alpha}\phi_n^{\epsilon}\right](t) d\mathbf{x} + \int_{\Omega_{\mathrm{H}}} I_{0^+}^{\alpha}\left[f_{\mathrm{H}}\partial_{0^+}^{\alpha}\phi_n^{\epsilon}\right](t) d\mathbf{x}, \\
\int_0^t \parallel \partial_{0^+}^{\beta} u_n^{\epsilon} \parallel_{L^2(\Omega_{\mathrm{H}})}^2 d\tau &= -\int_0^t \int_{\Omega_{\mathrm{H}}} \mathcal{G}(\phi_n^{\epsilon}, u_n^{\epsilon})\partial_{0^+}^{\alpha} u_n^{\epsilon} d\mathbf{x} d\tau.
\end{aligned} \tag{4.25}$$

Let us now estimate right-hand side of equations of system (4.25).
According to $(\mathbf{H2})_4$ and regularity of $\hbar$, we can deduce that

$$\begin{aligned}
-\int_{\Omega_{\mathrm{H}}} \mathcal{G}(\phi_n^{\epsilon}, u_n^{\epsilon})\partial_{0^+}^{\beta} u_n^{\epsilon} d\mathbf{x} &\leq c_0 \int_{\Omega_{\mathrm{H}}} (1+ \mid \phi_n^{\epsilon} \mid^2 + \mid u_n^{\epsilon} \mid) \mid \partial_{0^+}^{\beta} u_n^{\epsilon} \mid d\mathbf{x} \\
&\leq \frac{1}{2} \parallel \partial_{0^+}^{\beta} u_n^{\epsilon} \parallel_{L^2(\Omega_{\mathrm{H}})}^2 +c_1(1+ \parallel \phi_n^{\epsilon} \parallel_{L^4(\Omega_{\mathrm{H}})}^4 + \parallel u_n^{\epsilon} \parallel_{L^2(\Omega_{\mathrm{H}})}^2)
\end{aligned} \tag{4.26}$$

and then from second equation of (4.25) (according to (4.19) and (4.21))

$$\parallel \partial_{0^+}^{\beta} u_n^{\epsilon} \parallel_{L^2(0,t,L^2(\Omega_{\mathrm{H}}))}^2 \leq c_2(1+ \parallel \phi_n^{\epsilon} \parallel_{L^4(Q_{\mathrm{H}})}^4 + \parallel u_n^{\epsilon} \parallel_{L^{\infty}(0,T,L^2(\Omega_{\mathrm{H}}))}^2) \leq C. \tag{4.27}$$

Before to estimate the term $\int_{\Omega_{\mathrm{H}}} I_{0^+}^{\alpha}\left[\mathcal{I}(\phi_n^{\epsilon}, u_n^{\epsilon})\partial_{0^+}^{\alpha}\phi_n^{\epsilon}\right](t) d\mathbf{x}$, we prove that u_n^{ϵ} is uniformly bounded in $L^{\infty}(0, T, L^3(\Omega_{\mathrm{H}}))$. Since u_n^{ϵ} satisfies

$$u_n^{\epsilon}(\mathbf{x}, t) = u_n^{\epsilon}(\mathbf{x}, 0) - \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \mathcal{G}(\phi_n^{\epsilon}, u_n^{\epsilon})(\mathbf{x}, \tau) d\tau,$$

then, according to regularity of $\hbar$, we have

$$\mid u_n^{\epsilon}(\mathbf{x}, t) \mid \leq c_3\Big(\mid u_n^{\epsilon}(\mathbf{x}, 0) \mid + \int_0^t (t-\tau)^{\beta-1} \mid \mathcal{I}_2(\phi_n^{\epsilon}) \mid d\tau + \int_0^t (t-\tau)^{\beta-1} \mid u_n^{\epsilon}(\mathbf{x}, \tau) \mid d\tau\Big). \tag{4.28}$$

Consequently (since from assumption (2.4) we have $|\ \mathcal{I}_2(\phi_n^\epsilon)\ |\leq \beta_5+\beta_6\ |\ \phi_n^\epsilon\ |^2$)

$$\begin{aligned} |\ u_n^\epsilon(\mathbf{x},t)\ | &\leq c_4\Big(\ |\ u_n^\epsilon(\mathbf{x},0)\ |+\int_0^t(t-\tau)^{\beta-1}d\tau\Big) \\ &+c_5\Big(\int_0^t(t-\tau)^{\beta-1}\ |\ \phi_n^\epsilon\ |^2\ d\tau+\int_0^t(t-\tau)^{\beta-1}\ |\ u_n^\epsilon(\mathbf{x},\tau)\ |\ d\tau\Big). \end{aligned}$$

This implies (since $\int_0^t(t-\tau)^{\beta-1}d\tau\leq T^\beta/\beta$)

$$\begin{aligned} \Big(\int_{\Omega_{\mathrm{H}}}|\ u_n^\epsilon(\mathbf{x},t)\ |^3\ d\mathbf{x}\Big)^{1/3} &\leq c_6\Big(1+\|\ u_n^\epsilon(\mathbf{x},0)\ \|_{L^3(\Omega)}+\Big[\int_{\Omega_{\mathrm{H}}}\Big(\int_0^t(t-\tau)^{\beta-1}\ |\ u_n^\epsilon(\mathbf{x},\tau)\ |\ d\tau\Big)^3d\mathbf{x}\Big]^{1/3} \\ &+\Big[\int_{\Omega_{\mathrm{H}}}\Big(\int_0^t(t-\tau)^{\beta-1}\ |\ \phi_n^\epsilon(\mathbf{x},s)\ |^2\ ds\Big)^3d\mathbf{x}\Big]^{1/3}\Big) \end{aligned}$$

and then (using Minkowski inequality and the continuous embedding of H^1 in L^6)

$$\begin{aligned} \|\ u_n^\epsilon(.,t)\ \|_{L^3(\Omega_{\mathrm{H}})} &\leq c_7\Big(1+\|\ u_n^\epsilon(.,0)\ \|_{L^3(\Omega)}+\int_0^t(t-\tau)^{\beta-1}\ \|\ u_n^\epsilon(.,\tau)\ \|_{L^3(\Omega_{\mathrm{H}})}\ d\tau \\ &+\int_0^t(t-\tau)^{\beta-1}\ \|\ \phi_n^\epsilon(.,\tau)\ \|^2_{H^1(\Omega_{\mathrm{H}})}\ d\tau\Big). \end{aligned}$$

Using (4.19), (4.21) and the fact, from (4.9) the quantities $u_n^\epsilon(0)$ is uniformly bounded in L^3, we obtain

$$\|\ u_n^\epsilon(.,t)\ \|_{L^3(\Omega_{\mathrm{H}})}\leq c_8+c_9I_{0^+}^\beta\left[\|\ u_n^\epsilon\ \|_{L^3(\Omega_{\mathrm{H}})}\right](t)$$

and then (from Lemma 3.5)

$$\|\ u_n^\epsilon(.,t)\ \|_{L^3(\Omega_{\mathrm{H}})}\leq c_{10}\left(1+E_{\beta,1}(c_9T^\beta)\right)\leq C. \tag{4.29}$$

Consequently

$$\text{the sequence } u_n^\epsilon \text{ is uniformly bounded in } L^\infty(0,T,L^3(\Omega_{\mathrm{H}})). \tag{4.30}$$

We can now estimate the term $\int_{\Omega_{\mathrm{H}}}I_{0^+}^\alpha\left[\mathcal{I}(\phi_n^\epsilon,u_n^\epsilon)\partial_{0^+}^\alpha\phi_n^\epsilon\right](t)d\mathbf{x}$.

Since $\mathcal{I}(\phi_n^\epsilon,u_n^\epsilon)=\mathcal{I}_0(\phi_n^\epsilon)+\mathcal{I}_1(\phi_n^\epsilon)u_n^\epsilon$ and $\mathcal{I}_0(\phi_n^\epsilon)=g_1(\phi_n^\epsilon)+g_2(\phi_n^\epsilon)$ with g_1 is an increasing function, we have first (from $(\mathbf{H2})_4$)

$$\begin{aligned} -\int_{\Omega_{\mathrm{H}}}(g_2(\phi_n^\epsilon)+\mathcal{I}_1(\phi_n^\epsilon)u_n^\epsilon)\partial_{0^+}^\alpha\phi_n^\epsilon d\mathbf{x} &\leq c_0\int_{\Omega_{\mathrm{H}}}(1+|\ \phi_n^\epsilon\ |^2+|\ u_n^\epsilon\ ||\ \phi_n^\epsilon\ |+|\ u_n^\epsilon\ |)\partial_{0^+}^\alpha\phi_n^\epsilon d\mathbf{x} \\ \leq \frac{\mathfrak{c}_\alpha}{4}\ \|\ \partial_{0^+}^\alpha\phi_n^\epsilon\ \|^2_{L^2(\Omega_{\mathrm{H}})} &+c_1(1+\|\ \phi_n^\epsilon\ \|^4_{L^4(\Omega_{\mathrm{H}})}+\|\ u_n^\epsilon\ \|^2_{L^2(\Omega_{\mathrm{H}})}+\|\ \phi_n^\epsilon\ \|^2_{L^6(\Omega_{\mathrm{H}})}\|\ u_n^\epsilon\ \|^2_{L^3(\Omega_{\mathrm{H}})}). \end{aligned} \tag{4.31}$$

Moreover, since g_1 is an increasing function then the primitive E_g of g_1 is convex. Hence, from [50], we can deduce (since g_1 is and independent on time) $\partial_{0^+}^\alpha E_g(\phi_n^\epsilon)\leq g_1(\phi_n^\epsilon)\partial_{0^+}^\alpha\phi_n^\epsilon$ and then

$$-I_{0^+}^\alpha\left[g_1(\phi_n^\epsilon)\partial_{0^+}^\alpha\phi_n^\epsilon\right](t)\leq -I_{0^+}^\alpha\left[\partial_{0^+}^\alpha E(\phi_n^\epsilon)\right](t). \tag{4.32}$$

Now, we estimate the term $-I^{\alpha}_{0^+}\left[\partial^{\alpha}_{0^+}E(\phi^{\epsilon}_n)\right]$.

$$-\int_{\Omega_{\mathrm{H}}} I^{\alpha}_{0^+}\left[\partial^{\alpha}_{0^+}E_g(\phi^{\epsilon}_n)\right](t)d\mathbf{x} = -\int_{\Omega_{\mathrm{H}}} E_g(\phi^{\epsilon}_n)(t)d\mathbf{x} + \int_{\Omega_{\mathrm{H}}} E_g(\phi^{\epsilon}_n)(0)d\mathbf{x}. \tag{4.33}$$

Since, from $(\mathbf{H2})_4$, $\mu_6 \mid \phi^{\epsilon}_n \mid^2 - \mu_7 \leq E_g(\phi^{\epsilon}_n) \leq c_2(1+ \mid \phi^{\epsilon}_n \mid^4)$ then

$$-\int_{\Omega_{\mathrm{H}}} I^{\alpha}_{0^+}\left[\partial^{\alpha}_{0^+}E(\phi^{\epsilon}_n)\right]d\mathbf{x} \leq c_3(1+ \| \phi^{\epsilon}_n(0) \|^4_{H^1(\Omega_{\mathrm{H}})} + \| \phi^{\epsilon}_n(t) \|^2_{L^2(\Omega_{\mathrm{H}})}). \tag{4.34}$$

According to (4.31), (4.32) and (4.34), we can deduce that

$$\begin{aligned}
-\int_{\Omega_{\mathrm{H}}} I^{\alpha}_{0^+}\left[\mathcal{I}(\phi^{\epsilon}_n, u^{\epsilon}_n)\partial^{\alpha}_{0^+}\phi^{\epsilon}_n\right](t)d\mathbf{x} \leq \frac{\mathfrak{c}_{\alpha}}{4} I^{\alpha}_{0^+}\left[\| \partial^{\alpha}_{0^+}\phi^{\epsilon}_n \|^2_{L^2(\Omega_{\mathrm{H}})}\right] \\
+c_4(\| u^{\epsilon}_n \|^2_{L^{\infty}(0,T,L^2(\Omega_{\mathrm{H}}))} + \| \phi^{\epsilon}_n \|^2_{L^{\infty}(0,T,L^2(\Omega_{\mathrm{H}}))}) \\
+c_5(1 + I^{\alpha}_{0^+}\left[\| \phi^{\epsilon}_n \|^4_{L^4(\Omega_{\mathrm{H}})}\right] + \| \phi^{\epsilon}_n(0) \|^4_{H^1(\Omega_{\mathrm{H}})}), \\
+c_6 I^{\alpha}_{0^+}\left[\| \phi^{\epsilon}_n \|^2_{H^1(\Omega_{\mathrm{H}})}\right] \| u^{\epsilon}_n \|^2_{L^{\infty}(0,T,L^3(\Omega_{\mathrm{H}}))}.
\end{aligned} \tag{4.35}$$

Finally for the external forcing we have (using Hölder inequality)

$$I^{\alpha}_{0^+}\left[\int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}}\partial^{\alpha}_{0^+}\psi^{\epsilon}_n d\mathbf{x}\right] \leq \frac{\mathfrak{c}_{\alpha}}{4} I^{\alpha}_{0^+}\left[\| \partial^{\alpha}_{0^+}\psi^{\epsilon}_n \|^2_{L^2(\Omega_{\mathrm{H}})}\right] + c_7 \| f_{\mathrm{H}} \|^2_{L^{2/\alpha_1}(0,T,L^2(\Omega_{\mathrm{H}}))}. \tag{4.36}$$

According to (4.35), (4.36), (4.19), (4.21), and the fact, from (4.9) and $\epsilon \in (0,1)$, the quantities $\psi^{\epsilon}_n(0)$, $\phi^{\epsilon}_n(0)$ and $\varphi^{\epsilon}_{i,n}(0)$ are uniformly bounded in H^1, we can derive from (4.25) the following estimate (for all $t \in (0,T)$)

$$\begin{aligned}
&\frac{\mathfrak{c}_{\alpha}}{2} I^{\alpha}_{0^+}\left[\| \partial^{\alpha}_{0^+}\phi^{\epsilon}_n \|^2_{L^2(\Omega_{\mathrm{H}})}\right](t) + \epsilon\left(I^{\alpha}_{0^+}\left[\| \partial^{\alpha}_{0^+}\psi^{\epsilon}_n \|^2_{L^2(\Omega)}\right](t) + I^{\alpha}_{0^+}\left[\| \partial^{\alpha}_{0^+}\varphi^{\epsilon}_{i,n} \|^2_{L^2(\Omega_{\mathrm{H}})}\right](t)\right) \\
&\quad +\frac{\nu_i}{2} \| \nabla\varphi^{\epsilon}_{i,n}(t) \|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{\nu_g}{2} \| \nabla\psi^{\epsilon}_n(t) \|^2_{L^2(\Omega)} \\
&\quad \leq c_8(1+ \| \varphi^{\epsilon}_{i,n}(0) \|^2_{H^1(\Omega_{\mathrm{H}})} + \| \psi^{\epsilon}_n(0) \|^2_{H^1(\Omega)} + \| \phi^{\epsilon}_n(0) \|^4_{H^1(\Omega_{\mathrm{H}})}) \\
&\qquad +c_9(\| f_{\mathrm{H}} \|^2_{L^{2/\alpha_1}(0,T,L^2(\Omega_{\mathrm{H}}))}) \\
&\qquad +c_{10}(\| u^{\epsilon}_n \|^2_{L^{\infty}(0,T,L^2(\Omega_{\mathrm{H}}))} + \| \phi^{\epsilon}_n \|^2_{L^{\infty}(0,T,L^2(\Omega_{\mathrm{H}}))} + I^{\alpha}_{0^+} \| \phi^{\epsilon}_n \|^4_{L^4(\Omega_{\mathrm{H}})}) \\
&\qquad +c_{11} I^{\alpha}_{0^+}\left[\| \phi^{\epsilon}_n \|^2_{H^1(\Omega_{\mathrm{H}})}\right] \| u^{\epsilon}_n \|^2_{L^{\infty}(0,T,L^3(\Omega_{\mathrm{H}}))} \leq C.
\end{aligned} \tag{4.37}$$

We can deduce that (from (4.27) and (4.37))

$$\begin{aligned}
&I^{\alpha}_{0^+}\left[\| \partial^{\alpha}_{0^+}\phi^{\epsilon}_n \|^2_{L^2(\Omega_{\mathrm{H}})}\right](t) + I^{\alpha}_{0^+}\left[\| \sqrt{\epsilon}\partial^{\alpha}_{0^+}\psi^{\epsilon}_n \|^2_{L^2(\Omega)}\right](t) + I^{\alpha}_{0^+}\left[\| \sqrt{\epsilon}\partial^{\alpha}_{0^+}\varphi^{\epsilon}_{i,n} \|^2_{L^2(\Omega_{\mathrm{H}})}\right](t) \\
&\quad + \| \nabla\varphi^{\epsilon}_{i,n}(t) \|^2_{L^2(\Omega_{\mathrm{H}})} + \| \nabla\psi^{\epsilon}_n(t) \|^2_{L^2(\Omega)} \leq C, \\
&\| \partial^{\beta}_{0^+}u^{\epsilon}_n \|^2_{L^2(0,t,L^2(\Omega_{\mathrm{H}}))} \leq C.
\end{aligned} \tag{4.38}$$

In order to show that the local solution can be extended to the whole time interval $(0;T)$, we assume that we have already defined a solution of (4.12) on $[0,T_k]$ and we shall define the local solution on $[T_k, T_{k+1}]$ (where $0 < T_{k+1} - T_k$ is small enough) by using the obtained a priori estimates and the fractional derivative $\partial^{\gamma}_{T_k^+}$ (for $\gamma = \alpha$ or β) with beginning point T_k. Consequently, by iteration process, we thus obtain that Faedo-Galerkin solutions are well-defined on $(0, T)$. So, we omit the details.

Step 3. We are now ready to prove existence of solutions to system (4.5). From results (4.19), (4.22), (4.29) and (4.38), Theorem 3.1 and compactness argumentand, it follows that there exist $(X^\epsilon; u^\epsilon) = (\phi^\epsilon, \varphi_i^\epsilon, \psi^\epsilon; u^\epsilon)$ and $(\tilde{X}^\epsilon; \tilde{u}^\epsilon) = (\tilde{\phi}^\epsilon, \tilde{\varphi}_i^\epsilon, \tilde{\psi}^\epsilon; \tilde{u}^\epsilon)$ such that there exists a subsequence of $(X_n^\epsilon; u_n^\epsilon) = (\phi_n^\epsilon, \varphi_{i,n}^\epsilon, \psi_n^\epsilon; u_n^\epsilon)$ also denoted by $(X_n^\epsilon; u_n^\epsilon)$, such that

$$\begin{aligned}
&(X_n^\epsilon; u_n^\epsilon) \longrightarrow (X^\epsilon; u^\epsilon) \text{ weakly in } L^\infty(0, T; \mathbb{V}_{\mathrm{H}} \times \mathbb{V}_{\mathrm{H}} \times \mathbb{V}) \times L^\infty(0, T; L^3(\Omega_{\mathrm{H}})),\\
&(X_n^\epsilon; u_n^\epsilon) \longrightarrow (X^\epsilon; u^\epsilon) \text{ strongly in } (L^2(Q_{\mathrm{H}}))^2 \times L^2(Q) \times L^2(Q_{\mathrm{H}}),\\
&(\partial^\alpha_{0^+} X_n^\epsilon; \partial^\beta_{0^+} u_n^\epsilon) \longrightarrow (\partial^\alpha_{0^+} \tilde{X}^\epsilon; \partial^\beta_{0^+} \tilde{u}^\epsilon) \text{ weakly in } (L^2(Q_{\mathrm{H}}))^2 \times L^2(Q) \times L^2(Q_{\mathrm{H}}).
\end{aligned} \tag{4.39}$$

First we show that $(\partial^\alpha_{0^+} X_n^\epsilon; \partial^\beta_{0^+} u_n^\epsilon)$ exists in the weak sense and that $(\partial^\alpha_{0^+} X^\epsilon; \partial^\beta_{0^+} u^\epsilon) = (\tilde{X}^\epsilon; \tilde{u}^\epsilon)$. Indeed, we take $\omega \in C_0^\infty(0, T)$ and $v \in \mathbb{V}_{\mathrm{H}}$ (then $\omega v \in \mathbb{D}_4(0, T)$), by the weak convergence and Lebesgue's dominated convergence arguments we have

$$\begin{aligned}
\langle \tilde{\phi}^\epsilon, \omega v \rangle_{\mathbb{D}_4', \mathbb{D}_4} &= \lim_{n\to\infty} \int_0^T \int_{\Omega_{\mathrm{H}}} \omega(t) \partial^\alpha_{0^+} \phi_n^\epsilon(\mathbf{x}, t) v(\mathbf{x}) d\mathbf{x} dt\\
&= \lim_{n\to\infty} \int_\Omega \left(\int_0^T \omega(t) \partial^\alpha_{0^+} \phi_n^\epsilon dt \right) v(\mathbf{x}) d\mathbf{x}\\
&= -\lim_{n\to\infty} \int_0^T D^\alpha_{T^-} \omega(t) \left(\int_\Omega \phi_n^\epsilon(\mathbf{x}, t) v(\mathbf{x}) d\mathbf{x} \right) dt - I^{1-\alpha}_{0^+} \omega(0^+) \int_\Omega \phi_n^\epsilon(\mathbf{x}, 0) v(\mathbf{x}) d\mathbf{x}\\
&= -\int_0^T D^\alpha_{T^-} \omega(t) \left(\int_\Omega \phi^\epsilon(\mathbf{x}, t) v(\mathbf{x}) d\mathbf{x} \right) dt - I^{1-\alpha}_{0^+} \omega(0^+) \int_\Omega \phi^\epsilon(\mathbf{x}, 0) v(\mathbf{x}) d\mathbf{x}\\
&= \int_0^T \int_{\Omega_{\mathrm{H}}} \omega(t) \partial^\alpha_{0^+} \phi^\epsilon(\mathbf{x}, t) v(\mathbf{x}) d\mathbf{x} dt.
\end{aligned} \tag{4.40}$$

Consequently, $\tilde{\phi}^\epsilon = \partial^\alpha_{0^+} \phi^\epsilon$ in the weak sense. In the same way we prove, in the weak sense, that $(\partial^\alpha_{0^+} \varphi_i^\epsilon, \partial^\alpha_{0^+} \psi^\epsilon, \partial^\beta_{0^+} u^\epsilon) = (\tilde{\varphi}_i^\epsilon, \tilde{\psi}^\epsilon, \tilde{u}^\epsilon)$.

Consider $\kappa \in \mathcal{D}(]0, T[)$ and $(\mathbf{h}_n, \mathbf{k}_n, \rho_n) \in W_{i,n} \times W_{g,n} \times W_{i,n}$. According to (4.8) , we can deduce that

$$\begin{aligned}
&\mathfrak{c}_\alpha \int_0^T \kappa(t) \langle \partial^\alpha_{0^+} \phi_n^\epsilon, \mathbf{h}_n \rangle_{\mathbb{V}_{\mathrm{H}}', \mathbb{V}_{\mathrm{H}}} dt + \epsilon \int_0^T \kappa(t) \langle \partial^\alpha_{0^+} \varphi_{i,n}^\epsilon, \mathbf{h}_n \rangle_{\mathbb{V}_{\mathrm{H}}', \mathbb{V}_{\mathrm{H}}} dt\\
&\qquad = -\int_0^T \kappa(t) \int_{\Omega_{\mathrm{H}}} \mathcal{I}(\phi_n^\epsilon, u_n^\epsilon) \mathbf{h}_n d\mathbf{x} dt - \int_0^T \kappa(t) A_i(\varphi_{i,n}^\epsilon, \mathbf{h}_n) dt + \int_0^T \kappa(t) \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}} \mathbf{h}_n d\mathbf{x} dt,\\
&\mathfrak{c}_\alpha \int_0^T \kappa(t) \langle \partial^\alpha_{0^+} \phi_n^\epsilon, \mathbf{k}_n \rangle_{\mathbb{V}_{\mathrm{H}}', \mathbb{V}_{\mathrm{H}}} dt - \epsilon \int_0^T \kappa(t) \langle \partial^\alpha_{0^+} \psi_n^\epsilon, \mathbf{k}_n \rangle_{\mathbb{V}', \mathbb{V}} dt\\
&\qquad\qquad = -\int_0^T \kappa(t) \int_{\Omega_{\mathrm{H}}} \mathcal{I}(\phi_n^\epsilon, u_n^\epsilon) \mathbf{k}_n d\mathbf{x} dt + \int_0^T \kappa(t) \mathbf{A}_{\mathbf{g}}(\psi_n^\epsilon, \mathbf{k}_n) dt + \int_0^T \kappa(t) \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}} \mathbf{k}_n d\mathbf{x} dt,\\
&\int_0^T \kappa(t) \langle \partial^\beta_{0^+} u_n^\epsilon, \rho_n \rangle_{\mathbb{V}_{\mathrm{H}}', \mathbb{V}_{\mathrm{H}}} dt = -\int_0^T \kappa(t) \int_{\Omega_{\mathrm{H}}} \mathcal{G}(\phi_n^\epsilon, u_n^\epsilon) \rho_n d\mathbf{x} dt.
\end{aligned}$$

According to (4.21), (4.39) and to density properties of spaces spanned by (w_l) and (e_m), and using similar arguments to derive (4.40), it is easy to pass to limit ($n \to \infty$) in linear terms. For the nonlinear terms, we take into account assumption $(\mathbf{H2})_4$ and we use the classical technique based on taking the difference between the sequence and its limit in the form of sum of two terms such that the first uses weak convergence result and the other uses strong convergence result. So we omit the details. The limit $(\phi^\epsilon, \varphi_i^\epsilon, \psi^\epsilon, u^\epsilon)$ then satisfies the following system (for all $(\mathbf{h}, \mathbf{k}, \rho) \in \mathbb{V}_{\mathrm{H}} \times \mathbb{V} \times \mathbb{V}_{\mathrm{H}}$)

$$\begin{aligned}
&\langle \mathfrak{c}_\alpha \partial_{0^+}^\alpha \phi^\epsilon, \mathbf{h}\rangle_{\mathbb{V}'_{\mathrm{H}},\mathbb{V}_{\mathrm{H}}} + \epsilon\langle \partial_{0^+}^\alpha \varphi_i^\epsilon, \mathbf{h}\rangle_{\mathbb{V}'_{\mathrm{H}},\mathbb{V}_{\mathrm{H}}} + \int_{\Omega_{\mathrm{H}}} \mathcal{I}(\phi^\epsilon, u^\epsilon)\mathbf{h} d\mathbf{x} + A_i(\varphi_i^\epsilon, \mathbf{h}) = \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}} \mathbf{h} d\mathbf{x},\\
&\langle \mathfrak{c}_\alpha \partial_{0^+}^\alpha \phi^\epsilon, \mathbf{k}\rangle_{\mathbb{V}'_{\mathrm{H}},\mathbb{V}_{\mathrm{H}}} - \epsilon\langle \partial_{0^+}^\alpha \psi^\epsilon, \mathbf{k}\rangle_{\mathbb{V}',\mathbb{V}} + \int_{\Omega_{\mathrm{H}}} \mathcal{I}(\phi^\epsilon, u^\epsilon)\mathbf{k} d\mathbf{x} - \mathbf{A_g}(\psi, v) = \int_{\Omega_{\mathrm{H}}} f_{\mathrm{H}} \mathbf{k} d\mathbf{x},\\
&(\partial_{0^+}^\beta u^\epsilon, \rho)_{L^2(\Omega_{\mathrm{H}})} + \int_{\Omega_{\mathrm{H}}} \mathcal{G}(\phi^\epsilon, u^\epsilon)\rho d\mathbf{x} = 0.
\end{aligned} \tag{4.41}$$

In the case of $\alpha > 1/2$ and $\beta > 1/2$, the continuity of solution at $t = 0$ and the equalities $\varphi_i^\epsilon(0^+) = \varphi_{i,0}$, $\phi^\epsilon(0^+) = \phi_0$, $\psi^\epsilon(0^+) = \psi_0$ and $u^\epsilon(0^+) = u_0$, is a consequence of Lemma 3.2. This completes the proof. □

We can now show the well-posedness of memory bidomain-torso (degenerate) problem (1.1).

4.3. Degenerate problem study

Similar to derive results (4.19), (4.22), (4.29) and (4.38), we have the following a priori estimates for problem (4.4).

Lemma 4.1. *Assume the conditions* **(H1)** *and* $(\mathbf{H2})_4$ *hold,* $f_{\mathrm{H}} \in L^{2/\alpha_1}(0, T; L^2(\Omega))$ *and* $u_0 \in L^3(\Omega_{\mathrm{H}})$.

(i) *If* $\phi_0, \varphi_{i,0}$ *and* ψ_0 *are* L^2*-functions, then there exists a constant* $C > 0$ *(independent on* ϵ*) such that (for all* $t \in (0, T)$*)*

$$\begin{aligned}
&\| \phi^\epsilon(t) \|^2_{L^2(\Omega_{\mathrm{H}})} + \| \sqrt{\epsilon}\psi^\epsilon(t) \|^2_{L^2(\Omega)} + \| \sqrt{\epsilon}\varphi_i^\epsilon(t) \|^2_{L^2(\Omega_{\mathrm{H}})}\\
&\quad + I_{0^+}^\alpha \left[\| \phi^\epsilon \|^4_{L^4(\Omega_{\mathrm{H}})}\right](t) + I_{0^+}^\alpha \left[\| \nabla\varphi_i^\epsilon \|^2_{L^2(\Omega_{\mathrm{H}})}\right](t) + I_{0^+}^\alpha \left[\| \nabla\psi_n^\epsilon \|^2_{L^2(\Omega)}\right](t) \le C,\\
&\| u^\epsilon \|^2_{L^\infty(0,t,L^3(\Omega_{\mathrm{H}}))} + \| \partial_{0^+}^\beta u^\epsilon \|^2_{L^2(0,t,L^2(\Omega_{\mathrm{H}}))} \le C.
\end{aligned} \tag{4.42}$$

(ii) *If* $\phi_0, \varphi_{i,0}$ *and* ψ_0 *are* H^1*-functions, then there exists a constant* $C > 0$ *(independent on* ϵ*) such that (for all* $t \in (0, T)$*)*

$$\begin{aligned}
&I_{0^+}^\alpha \left[\| \partial_{0^+}^\alpha \phi^\epsilon \|^2_{L^2(\Omega_{\mathrm{H}})}\right](t) + I_{0^+}^\alpha \left[\| \sqrt{\epsilon}\partial_{0^+}^\alpha \psi^\epsilon \|^2_{L^2(\Omega)}\right](t) + I_{0^+}^\alpha \left[\| \sqrt{\epsilon}\partial_{0^+}^\alpha \varphi_i^\epsilon \|^2_{L^2(\Omega_{\mathrm{H}})}\right](t)\\
&\qquad + \| \nabla\varphi_i^\epsilon(t) \|^2_{L^2(\Omega_{\mathrm{H}})} + \| \nabla\psi^\epsilon(t) \|^2_{L^2(\Omega)} \le C,\\
&\| u^\epsilon \|^2_{L^\infty(0,t,L^3(\Omega_{\mathrm{H}}))} + \| \partial_{0^+}^\beta u^\epsilon \|^2_{L^2(0,t,L^2(\Omega_{\mathrm{H}}))} \le C.
\end{aligned} \tag{4.43}$$

We are now ready to prove existence of weak solutions to system (1.1).

Theorem 4.2. *Let assumptions* **(H1)** *and* $(\mathbf{H2})_4$ *be fulfilled and* $\beta \ge \alpha$*. Then for* (ϕ_0, u_0) *and* f_{H} *given such that* $(\phi_0, u_0) \in \mathbb{V}_{\mathrm{H}} \times L^3(\Omega_{\mathrm{H}})$ *and* $f_{\mathrm{H}} \in L^{\frac{2}{\alpha_1}}(0, T; L^2(\Omega_{\mathrm{H}}))$ *with* $0 < \alpha_1 < \alpha$*, there exists a weak solution* $(\varphi_i, \phi, \psi, u)$ *to problem* (1.1) *verifying*

$$\begin{aligned}
&\phi \in L^\infty(0, T; \mathbb{V}_{\mathrm{H}}),\ \ \partial_{0^+}^\alpha \phi \in L^2(0, T; L^2(\Omega_{\mathrm{H}})),\\
&\psi \in L^\infty(0, T; \mathbb{V}),\ \ \psi \in L^\infty(0, T; \mathbb{V}_{\mathrm{H}}),\\
&u \in L^\infty(0, T; L^3(\Omega_{\mathrm{H}})),\ \ \partial_{0^+}^\beta u \in L^2(0, T; L^2(\Omega_{\mathrm{H}})).
\end{aligned} \tag{4.44}$$

Moreover, if $\beta > 1/2$ *then* $u \in C([0,T];L^2(\Omega_H))$ *and* $u(0^+) = u_0$ *and if* $\alpha > 1/2$ *then* $\phi \in C([0,T];L^2(\Omega_H))$ *and* $\phi(0^+) = \phi_0$.

Proof. From Lemma 4.1, similar arguments to derive (4.40), Theorem 3.1 and compactness argument, it follows that there exist $X = (\phi, \varphi_i, \psi)$ and u such that there exists a subsequence of $(X^\epsilon; u^\epsilon) = (\phi^\epsilon, \varphi_i^\epsilon, \psi^\epsilon; u^\epsilon)$ also denoted by $(X^\epsilon; u^\epsilon)$ such that

$$\begin{aligned}
&(X^\epsilon; u^\epsilon) \longrightarrow (X;u) \text{ weakly in } L^\infty(0,T;\mathbb{V}_H \times \mathbb{V}_H \times \mathbb{V}) \times L^\infty(0,T;L^3(\Omega_H)),\\
&(X^\epsilon; u^\epsilon) \longrightarrow (X;u) \text{ strongly in } (L^2(Q_H))^2 \times L^2(Q) \times L^2(Q_H),\\
&(\partial^\alpha_{0^+} X^\epsilon; \partial^\beta_{0^+} u^\epsilon) \longrightarrow (\partial^\alpha_{0^+}\phi, 0, 0; \partial^\beta_{0^+} u) \text{ weakly in } (L^2(Q_H))^2 \times L^2(Q) \times L^2(Q_H).
\end{aligned} \tag{4.45}$$

Thus, using (4.45) and similar argument to derive (4.41), we can deduce the existence of weak solution $(\phi, \varphi_i, \psi, u)$ satisfying regularity (4.44).

Finally, in case of $\alpha > 1/2$ and $\beta > 1/2$, the continuity of solution at $t = 0$ and the equalities $\phi^\epsilon(0^+) = \phi_0$, and $u^\epsilon(0^+) = u_0$, is a consequence of Lemma 3.2. This completes the proof. □.

Theorem 4.3. *Let assumptions* **(H1)**, $\mathbf{(H2)}_4$ *and* **(H3)** *be fulfilled and* $\beta \geq \alpha$. *Then the solution* $(\varphi_i, \phi, \psi, u)$ *to system* (1.1) *is unique. Moreover, let* $(\phi_0^{(i)}, u_0^{(i)}, f_H^{(i)})$ *be given such that* $(\phi_0^{(i)}, u_0^{(i)}) \in \mathbb{V}_H \times L^3(\Omega_H)$ *and* $f_H^{(i)} \in L^{\frac{2}{\alpha_1}}(0,T;L^2(\Omega_H))$, *for* $i = 1,2$, *with* $0 < \alpha_1 < \alpha$. *For* $(\varphi_i^{(j)}, \phi^{(j)}, \psi^{(j)}, u^{(j)})$ *a weak solutions to system* (1.1), *which corresponds to data* $(\phi_0^{(j)}, u_0^{(j)}, f_H^{(j)})$ *(for* $j = 1,2$*), the following Lipschitz continuity relationship is satisfied.*

(i) *If* $\alpha = \beta$, *we have the following relation*

$$\begin{aligned}
\| \phi(t) \|^2_{L^2(\Omega_H)} + \| u(t) \|^2_{L^2(\Omega_H)} &+ I^\alpha_{0^+}\left[\| \nabla\varphi_i \|^2_{L^2(\Omega_H)} + \| \nabla\psi \|^2_{L^2(\Omega_H)}\right](t)\\
&\leq C(\| \phi_0 \|^2_{L^2(\Omega_H)} + \| u_0 \|^2_{L^2(\Omega_H)} + \| f_H \|^2_{L^{2/\alpha_1}(0,T;L^2(\Omega_H))}).
\end{aligned} \tag{4.46}$$

(ii) *If* $\alpha < \beta$ *and if the following assumption holds*

$$| \mathcal{I}_2(\phi_1) - \mathcal{I}_2(\phi_2) | \leq C_I | \phi | (1 + | \phi_1 | + | \phi_2 |), \tag{4.47}$$

the relation (4.46) *is also obtained.*

Proof. As uniqueness result is a consequence of relation (4.46) (see further), we start by showing this relation.

Since $(\varphi_i^{(j)}, \phi^{(j)}, \psi^{(j)}, u^{(j)})$ is a weak solution to system (1.1), for $j = 1,2$, then $(\varphi_i, \phi, \psi, u) = (\varphi_i^{(1)} - \varphi_i^{(2)}, \phi^{(1)} - \phi^{(2)}, \psi^{(1)} - \psi^{(2)}, u^{(1)} - u^{(2)})$ satisfies

$$\begin{aligned}
&\mathfrak{c}_\alpha(\partial^\alpha_{0^+}\phi, \varphi_i)_{L^2(\Omega_H)} + \int_{\Omega_H} (\mathcal{I}(\phi_1, u_1) - \mathcal{I}(\phi_2, u_2))\varphi_i d\mathbf{x} + A_i(\varphi_i, \varphi_i) = \int_{\Omega_H} f_H \varphi_i d\mathbf{x},\\
&\mathfrak{c}_\alpha(\partial^\alpha_{0^+}\phi, \psi)_{L^2(\Omega_H)} + \int_{\Omega_H} (\mathcal{I}(\phi_1, u_1) - \mathcal{I}(\phi_2, u_2))\psi d\mathbf{x} - \mathbf{A_g}(\psi, \psi) = \int_{\Omega_H} f_H \psi d\mathbf{x},\\
&(\partial^\beta_{0^+} u, u)_{L^2(\Omega_H)} + \int_{\Omega_H} (\mathcal{G}(\phi_1, u_1) - \mathcal{G}(\phi_2, u_2)) u d\mathbf{x} = 0,\\
&u(0) = u_0, \quad \phi(0) = \phi_0,
\end{aligned} \tag{4.48}$$

where $(\phi_0, u_0, f_H) = (\phi_0^{(1)} - \phi_0^{(2)}, u_0^{(1)} - u_0^{(2)}, f_H^{(1)} - f_H^{(2)})$.

Then (since $\psi = \varphi_e$ and $\phi = \varphi_i - \varphi_e$ in $\Omega_{\rm H}$),

$$\begin{aligned}
&\mathfrak{c}_\alpha(\partial^\alpha_{0^+}\phi, \phi)_{L^2(\Omega_{\rm H})} + \int_{\Omega_{\rm H}} (\mathcal{I}(\phi_1, u_1) - \mathcal{I}(\phi_2, u_2))\phi d\mathbf{x} + A_i(\varphi_i, \varphi_i) + \mathbf{A_g}(\psi, \psi) = \int_{\Omega_{\rm H}} f_{\rm H}\phi d\mathbf{x},\\
&(\partial^\beta_{0^+}u, u)_{L^2(\Omega_{\rm H})} + \int_{\Omega_{\rm H}} (\mathcal{G}(\phi_1, u_1) - \mathcal{G}(\phi_2, u_2))u d\mathbf{x} = 0,\\
&u(0) = u_0, \;\; \phi(0) = \phi_0.
\end{aligned} \tag{4.49}$$

(i) If $\alpha = \beta$, according to **(H3)**, we have

$$\begin{aligned}
&\frac{1}{2}\min(\mu\mathfrak{c}_\alpha, 1)\partial^\alpha_{0^+}(\| \phi(t) \|^2_{L^2(\Omega_{\rm H})} + \| u(t) \|^2_{L^2(\Omega_{\rm H})})\\
&\qquad +\mu\min(\nu_i, \nu_g)(\| \nabla\varphi_i \|^2_{L^2(\Omega_{\rm H})} + \| \nabla\psi \|^2_{L^2(\Omega_{\rm H})})\\
&\qquad\qquad \leq -\int_{\Omega_{\rm H}} (F_\mu(\phi_1, u_1) - F_\mu(\phi_2, u_2)).(\phi, u)d\mathbf{x} + \int_{\Omega_{\rm H}} f_{\rm H}\phi d\mathbf{x}\\
&\qquad\qquad \leq c_0(\| \phi(t) \|^2_{L^2(\Omega_{\rm H})} + \| u(t) \|^2_{L^2(\Omega_{\rm H})} + \| f_{\rm H}(t) \|^2_{L^2(\Omega_{\rm H})}
\end{aligned} \tag{4.50}$$

and then

$$\begin{aligned}
&\partial^\alpha_{0^+}(\| \phi(t) \|^2_{L^2(\Omega_{\rm H})} + \| u(t) \|^2_{L^2(\Omega_{\rm H})}) + (\| \nabla\varphi_i \|^2_{L^2(\Omega_{\rm H})} + \| \nabla\psi \|^2_{L^2(\Omega_{\rm H})})\\
&\qquad \leq c_1(\| \phi(t) \|^2_{L^2(\Omega_{\rm H})} + \| u(t) \|^2_{L^2(\Omega_{\rm H})}) + c_2 \| f_{\rm H}(t) \|^2_{L^2(\Omega_{\rm H})} .
\end{aligned} \tag{4.51}$$

So (using Hölder inequality),

$$\begin{aligned}
&\| \phi(t) \|^2_{L^2(\Omega_{\rm H})} + \| u(t) \|^2_{L^2(\Omega_{\rm H})} + I^\alpha_{0^+}\left[\| \nabla\varphi_i \|^2_{L^2(\Omega_{\rm H})} + \| \nabla\psi \|^2_{L^2(\Omega_{\rm H})}\right](t)\\
&\qquad \leq \| \phi_0 \|^2_{L^2(\Omega_{\rm H})} + \| u_0 \|^2_{L^2(\Omega_{\rm H})} + c_3 \| f_{\rm H} \|^2_{L^{2/\alpha_1}(0,T;L^2(\Omega_{\rm H}))} t^{\alpha-\alpha_1}\\
&\qquad\quad + c_1 I^\alpha_{0^+}\left[\| \phi(t) \|^2_{L^2(\Omega_{\rm H})} + \| u(t) \|^2_{L^2(\Omega_{\rm H})})\right](t).
\end{aligned} \tag{4.52}$$

By Lemma 3.5, we can deduce that first

$$\| \phi(t) \|^2_{L^2(\Omega_{\rm H})} + \| u(t) \|^2_{L^2(\Omega_{\rm H})} \leq c_4(\| \phi_0 \|^2_{L^2(\Omega_{\rm H})} + \| u_0 \|^2_{L^2(\Omega_{\rm H})} + \| f_{\rm H} \|^2_{L^{2/\alpha_1}(0,T;L^2(\Omega_{\rm H}))})$$

and then, from (4.51)

$$\begin{aligned}
&\| \phi(t) \|^2_{L^2(\Omega_{\rm H})} + \| u(t) \|^2_{L^2(\Omega_{\rm H})} + I^\alpha_{0^+}\left[\| \nabla\varphi_i \|^2_{L^2(\Omega_{\rm H})} + \| \nabla\psi \|^2_{L^2(\Omega_{\rm H})}\right](t)\\
&\qquad \leq C(\| \phi_0 \|^2_{L^2(\Omega_{\rm H})} + \| u_0 \|^2_{L^2(\Omega_{\rm H})} + \| f_{\rm H} \|^2_{L^{2/\alpha_1}(0,T;L^2(\Omega_{\rm H}))}).
\end{aligned} \tag{4.53}$$

(ii) Assume now that $\alpha < \beta$, then, according to **(H3)**, we get

$$\begin{aligned}
&\partial^\alpha_{0^+} \| \phi(t) \|^2_{L^2(\Omega_{\rm H})} + \frac{2}{\mathfrak{c}_\alpha}\min(\nu_i, \nu_g)(\| \nabla\varphi_i \|^2_{L^2(\Omega_{\rm H})} + \| \nabla\psi \|^2_{L^2(\Omega_{\rm H})})\\
&\qquad \leq -\frac{2}{\mathfrak{c}_\alpha}\frac{1}{\mu}\int_{\Omega_{\rm H}} F_\mu(\phi_1, u_1) - F_\mu(\phi_2, u_2)).(\phi, u)d\mathbf{x}\\
&\qquad\quad +\frac{2}{\mathfrak{c}_\alpha}\frac{1}{\mu}\int_{\Omega_{\rm H}} (\mathcal{G}(\phi_1, u_1) - \mathcal{G}(\phi_2, u_2))u d\mathbf{x} + \frac{2}{\mathfrak{c}_\alpha} \| f_{\rm H}(t) \|_{L^2(\Omega_{\rm H})} \| \phi(t) \|_{L^2(\Omega_{\rm H})}\\
&\qquad \leq c_5(\| \phi(t) \|^2_{L^2(\Omega_{\rm H})} + \| u(t) \|^2_{L^2(\Omega_{\rm H})}) + c_6 \| f_{\rm H}(t) \|^2_{L^2(\Omega_{\rm H})}\\
&\qquad\quad +\frac{2}{\mathfrak{c}_\alpha}\frac{1}{\mu}\int_{\Omega_{\rm H}} (\mathcal{G}(\phi_1, u_1) - \mathcal{G}(\phi_2, u_2))\phi d\mathbf{x},\\
&\partial^\beta_{0^+} \| u(t) \|^2_{L^2(\Omega_{\rm H})} \leq -2\int_{\Omega_{\rm H}} (\mathcal{G}(\phi_1, u_1) - \mathcal{G}(\phi_2, u_2))u d\mathbf{x}.
\end{aligned} \tag{4.54}$$

On account of assumption (4.47), Hölder's inequality, continuous embedding of H^1 in L^4 and the fact that $\phi_i \in L^\infty(0,T,H^1(\Omega_H))$, for $i = 1,2$, we have

$$
\begin{aligned}
\mid \int_{\Omega_H} (\mathcal{G}(\phi_1,u_1) - \mathcal{G}(\phi_2,u_2))ud\mathbf{x} \mid \leq c_7(\| \phi(t) \|^2_{L^2(\Omega_H)} + \| u(t) \|^2_{L^2(\Omega_H)}) \\
+c_8 \int_{\Omega_H} \mid \phi \mid\mid u \mid (\mid \phi_1 \mid + \mid \phi_2 \mid)d\mathbf{x} \\
\leq c_7(\| \phi(t) \|^2_{L^2(\Omega_H)} + \| u(t) \|^2_{L^2(\Omega_H)}) \\
+c_9 \| u(t) \|_{L^2(\Omega_H)} \| \phi(t) \|_{L^4(\Omega_H)} .
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{2}{c_\alpha}\frac{1}{\mu} \int_{\Omega_H} (\mathcal{G}(\phi_1,u_1) - \mathcal{G}(\phi_2,u_2))ud\mathbf{x} \leq c_{10}(\| \phi(t) \|^2_{L^2(\Omega_H)} + \| u(t) \|^2_{L^2(\Omega_H)}) \\
+\eta_1(\| \nabla\varphi_i \|^2_{L^2(\Omega_H)} + \| \nabla\psi \|^2_{L^2(\Omega_H)}), \\
-2 \int_{\Omega_H} (\mathcal{G}(\phi_1,u_1) - \mathcal{G}(\phi_2,u_2))ud\mathbf{x} \leq c_{11}(\| \phi(t) \|^2_{L^2(\Omega_H)} + \| u(t) \|^2_{L^2(\Omega_H)}) \\
+\eta_2(\| \nabla\varphi_i \|^2_{L^2(\Omega_H)} + \| \nabla\psi \|^2_{L^2(\Omega_H)}),
\end{aligned}
\tag{4.55}
$$

with the constants η_1 and η_2 chosen appropriately. According to estimates (4.55) and by choosing $\eta_1 = \frac{1}{c_\alpha}\min(\nu_i,\nu_g)$, estimate (4.54) becomes

$$
\begin{aligned}
\partial^\alpha_{0^+} \| \phi(t) \|^2_{L^2(\Omega_H)} + \zeta\Psi(t) \leq c_{12}(\| \phi(t) \|^2_{L^2(\Omega_H)} + \| u(t) \|^2_{L^2(\Omega_H)}) + c_{13} \| f_H(t) \|^2_{L^2(\Omega_H)}, \\
\partial^\beta_{0^+} \| u(t) \|^2_{L^2(\Omega_H)} \leq c_{14}(\| \phi(t) \|^2_{L^2(\Omega_H)} + \| u(t) \|^2_{L^2(\Omega_H)}) + \eta_2\Psi(t),
\end{aligned}
\tag{4.56}
$$

with $\Psi(t) = \| \nabla\varphi_i \|^2_{L^2(\Omega_H)} + \| \nabla\psi \|^2_{L^2(\Omega)}$ and $\zeta = \frac{1}{c_\alpha}\min(\nu_i,\nu_g)$.
Consequently, by choosing $\eta_2 < \frac{\Gamma(\beta)\zeta}{T^{\beta-\alpha}\Gamma(\alpha)}$, we can deduce first from Corollary 3.2 that

$$
\| \phi(t) \|^2_{L^2(\Omega_H)} + \| u(t) \|^2_{L^2(\Omega_H)} \leq c_{15}(\| \phi_0 \|^2_{L^2(\Omega_H)} + \| u_0 \|^2_{L^2(\Omega_H)} + \| f_H \|^2_{L^{2/\alpha_1}(0,T;L^2(\Omega_H))})
$$

and then, from (4.56)

$$
\begin{aligned}
\| \phi(t) \|^2_{L^2(\Omega_H)} + \| u(t) \|^2_{L^2(\Omega_H)} + I^\alpha_{0^+}\left[\| \nabla\varphi_i \|^2_{L^2(\Omega_H)} + \| \nabla\psi \|^2_{L^2(\Omega_H)}\right](t) \\
\leq C(\| \phi_0 \|^2_{L^2(\Omega_H)} + \| u_0 \|^2_{L^2(\Omega_H)} + \| f_H \|^2_{L^{2/\alpha_1}(0,T;L^2(\Omega_H))}).
\end{aligned}
\tag{4.57}
$$

Finally, if the difference between data is null i.e., $\phi_0 = 0$, $u_0 = 0$ and $f_H = 0$, we can conclude from (4.53) (or (4.57)), (1.4) and (1.6) that $(\phi,\varphi_i,\psi,u) = (0,0,0,0)$ and then the uniqueness of solution. This completes the proof. □

Remark 4.1. *In the previous theoretical work, our main results investigate the well-posedness of system* (1.1) *with an abstract class of ionic models, including some classical models as Rogers-McCulloch, Fitz-Hugh-Nagumo and Aliev-Panfilov. We can consider other ionic model type including Mitchell-Schaeffer model,see [56] (which has a slightly different structure). This two-variable model can be defined with operators $\mathcal{I}$ and $\mathcal{G}$ as (in which the state variables are* *dimensionless and scaled)*

$$\mathcal{I}(\phi,u) = -\frac{u}{\tau_{in}}\phi^2(1-\phi) - \frac{\phi}{\tau_{out}},$$
$$\mathcal{G}(\phi,u) = \begin{cases} \dfrac{u-1}{\tau_{open}} & \text{if } \phi < \phi_{gate}, \\ \dfrac{u}{\tau_{close}} & \text{if } \phi \geq \phi_{gate}, \end{cases} \tag{4.58}$$

where $0 < \phi_{gate} < 1$, τ_{in}, τ_{out}, τ_{open} *and* τ_{close} *are given positive constants. The cubic function* $\phi^2(1-\phi)$ *describes the voltage dependence of the inward current. This model is well-known to be valid under the assumption* $0 < \tau_{in} \ll \tau_{out} \ll \min(\tau_{open}, \tau_{close})$.

In order to guarantee the well-posedness of system (1.1) *with Mitchell-Schaeffer ionical model, we can use the following regularized version of ionic operator* $\mathcal{G}$

$$\mathcal{G}_\zeta(\phi,u) = \left(\frac{1}{\tau_{close}} - (\frac{1}{\tau_{close}} - \frac{1}{\tau_{open}})h_\zeta(\phi)\right)\left(u - h_\zeta(\phi)\right), \tag{4.59}$$

where the differentiable function $0 \leq h_\zeta \leq 1$ *is given by*

$$h_\zeta(\phi) = \frac{1}{2}\left(1 - \tanh\left(\frac{\phi - \phi_{gate}}{\zeta}\right)\right),$$

with ζ *a positive parameter.*

The operator $\mathcal{G}_\zeta(\phi,u)$ *can be written as* $\mathcal{G}_\zeta(\phi,u) = \mathcal{I}_{2,\zeta}(\phi) + \hbar_\zeta(\phi)u$ *where*

$$\mathcal{I}_{2,\zeta}(\phi) = -\left(\frac{1}{\tau_{close}} - (\frac{1}{\tau_{close}} - \frac{1}{\tau_{open}})h_\zeta(\phi)\right)h_\zeta(\phi),$$
$$\hbar_\zeta(\phi) = \frac{1}{\tau_{close}} - \left(\frac{1}{\tau_{close}} - \frac{1}{\tau_{open}}\right)h_\zeta(\phi). \tag{4.60}$$

According to the definition of tanh, *we can deduce that* $\lim_{\zeta\to 0} h_\zeta(\phi) = \begin{cases} 1 \text{ if } \phi < \phi_{gate}, \\ 0 \text{ if } \phi > \phi_{gate} \end{cases}$ *and then* $\lim_{\zeta\to 0}\mathcal{G}_\zeta(\phi,u) = \mathcal{G}(\phi,u)$. *The regularized Mitchell-Schaeffer model has a slightly different structure compared to models in* (2.3) *because in this model,* $\hbar_\zeta$ *depend on* ϕ *through the function* h_ζ. *Since* h_ζ *is sufficiently regular, the arguments of this paper can be adapted with some slight necessary modifications to analyze the well-posedness of this regularized Mitchell-Schaeffer ionical model.*

In more general, the study developed in this paper remains valid (with some necessary modifications) if we consider the operator $\mathcal{G}$ *in the form of* $\mathcal{G}(.;\phi,u) = \mathcal{I}_2(.;\phi) + \hbar(.;\phi)u$ *(i.e. a general form of Hodgkin-Huxley model including Beeler-Reuter and Luo-Rudy ionic models described by continuous or regularized discontinuous functions, see [7, 53, 54]) with* $\mathcal{G}$ *Carathéodory function from* $(\Omega \times \mathbb{R}) \times \mathbb{R}^2$ *into* $\mathbb{R}$ *and locally Lipschitz continuous function on* (ϕ,u) *and,* $\mathcal{I}_2$ *and* $\hbar$ *sufficiently regulars.* □

5. Numerical applications

In this section, we shall present some preliminary numerical simulations of the problem (1.1) to illustrate our result with showing two-dimensional simulation. Two of the most used ionical models, FitzHugh-Nagumo model [39] and Mitchell-Shaeffer model [56], are considered. This numerical study investigates the impact of fractional-order ϕ dynamics and fractional-order u dynamics (i.e. the influence of the capacitor and the ionic variable memory) on key rate dependent electrical properties including APD, the amplitude of the action potential and spontaneous activity.

Although part of the theoretical analysis was limited to the case $\beta \geq \alpha$, in order to show the impact of β on the evolution of system, we will also present some results for α fixed to 1 and β variable. We will explore the behavior of the two models when first they are stimulated by a brief current and second when they are stimulated twice in sequence.

In these two situations we fix first β at 1 and we vary α, and in a second time we fix α at 1 and we vary β to analyze the impact of each time-fractional order derivative on system behavior, one after the other.

In order to solve numerically system (1.1) we have analyzed different approximation methods of Caputo fractional derivative $\partial^{\gamma}_{0^+}$, $\gamma \in]0,1]$. In this paper, we consider the following approximation (for sufficiently regular function ψ at time $t \in]t_{n-1}, t_n]$ $(n = 1, 2, \ldots)$, with $t_k = k\Delta t$ for $k = 0, 1, \ldots$ and given time step Δt)

$$\begin{aligned}
\partial^{\gamma}_{0^+}\left[\psi\right](\mathbf{x},t) &= \frac{1}{\Gamma(1-\gamma)}\int_0^t (t-\tau)^{-\gamma}\frac{\partial\psi}{\partial t}(\mathbf{x},\tau)d\tau,\\
&= \frac{1}{\Gamma(1-\gamma)}\sum_{m=0}^{n-2}\int_{t_m}^{t_{m+1}}(t-\tau)^{-\gamma}\frac{\partial\psi}{\partial t}(\mathbf{x},\tau)d\tau + \frac{1}{\Gamma(1-\gamma)}\int_{t_{n-1}}^{t}(t-\tau)^{-\gamma}\frac{\partial\psi}{\partial t}(\mathbf{x},\tau)d\tau,\\
&\approx -I^{\gamma}_{memory}(\psi_{history})(\mathbf{x},t) + \frac{\Delta t^{1-\gamma}}{\Gamma(2-\gamma)}\frac{\partial\psi}{\partial t}(\mathbf{x},t),
\end{aligned} \tag{5.1}$$

where

$$I^{\gamma}_{memory}(\psi_{history})(\mathbf{x},t) = -\frac{\Delta t^{-\gamma}}{\Gamma(2-\gamma)}\sum_{m=0}^{n-2}[(n-m)^{1-\gamma} - (n-m-1)^{1-\gamma}][\psi(\mathbf{x},t_{m+1}) - \psi(\mathbf{x},t_m)]. \tag{5.2}$$

It is clear that $I^{1}_{memory}(\psi_{history})$, for $\gamma = 1$, is a null function. The state function $\psi_{history}$ is corresponding to the prior history of ψ.

Note that the previous fractional-order dynamic is the sum of first-order dynamics and an operator I^{γ}_{memory} representing hypothetical memory effects.

The term I^{γ}_{memory} depends on the prior history of accrued heartbeats and characterize the influence of capacitive/ionic memory on the dynamical of system. This memory operator I^{γ}_{memory} can be interpreted as a new force which is added over time to the external applied forcing. So, for example, this adding force can invert at some time points the sign of the external applied stimulus [‡] and then generates changes in myocardial polarization.

Here, we perform various numerical experiments to investigate the impact of different values of ϕ fractional-order α and u fractional-order β across the time in $(0, T)$ and we present some numerical

[‡]Thus the depolarizing effect of the stimulus becomes repolarizing and vice versa

results corresponding to the middle point of heart domain Ω_H for ϕ, φ_e and u, and in the middle point of torso domain Ω_B for φ_s. We assume that the domain heart-torso is in a square region $\Omega = [-L_T/2, -L_T/2] \times [-L_T/2, -L_T/2]$ and the domain Ω_H corresponds to an approximate shape of heart, where L_T is the torso length and L_H is the characteristic length of Ω_H (Figure 3).

The boundary of domain Ω_H is defined by (for $(x, y) \in \Gamma_H$) : $|x - a|^3 + (y - b - c|x - d|^2)^2 - e = 0$, where a, b, c, d, e are given positive constants. Here we choose $a = 0.5$, $b = 3.5$, $c = 0.5$, $c = 1$, $d = 0.5$ and $e = 1/15$.

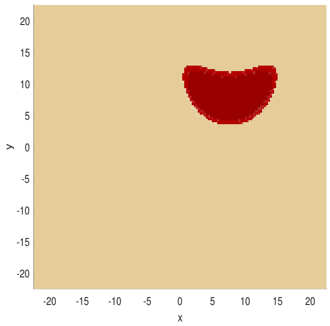

Figure 3. Heart-torso domain.

We use some biophysical parameters which are presented in Table 1 and we fix $L_H = 15cm$ and

Table 1. Cell membrane parameters.

Description	name	value (unit)
Cell surface to volume ratio	κ	200 (cm^{-1})
transmembrane capacitance	C_α	10^{-3} (F/cm^2)
Depolarization length	$\mathcal{T}_{in}$	4.5 (ms)
Repolarization length	$\mathcal{T}_{out}$	90 (ms)
Opening time constant	$\mathcal{T}_{open}$	100 (ms)
Closing time constant	$\mathcal{T}_{close}$	130 (ms)
Change-over voltage	ϕ_{gate}	−67 (mV)
Resting potential	ϕ_{min}	−80 (mV)
Maximum potential	ϕ_{max}	20 (mV)
Activation time	T_{act}	10 (ms)

$L_T = 45cm$. In those both applications, we impose the following initial conditions

$$\phi(x, y, 0) = \phi_{min}, \ U(x, y, 0) = \frac{1}{(\phi_{max} - \phi_{min})^2}. \tag{5.3}$$

Finally, we consider $T = 1$, $\mathcal{K}_i = \mathcal{K}_e = 0.003s.cm^{-1}\mathbf{I}_d$ (with $\mathbf{I}_d$ identity matrix) and we define (as in [28]) the external stimulus f_H in short time $a_i < t < a_i + T_{act}$, $i = 0, 1$ with $a_0 = 0$ (electroshock for example), as

$$f_H(\mathbf{x}, t) = I_{i,app}\chi_H(\mathbf{x})\chi_{[a_i, T_{act}+a_i]}(t)\chi_{prop}(\mathbf{x}, t)\Phi(\mathbf{x}), \text{ on } (a_i, a_i + T_{act}) \tag{5.4}$$

where I_{app} is the amplitude of external applied stimulus with $I_{i,app} = b_i 10^4$, and the functions χ_H, $\chi_{[a_i,T_{act}+a_i]}$ and χ_{prop} are defined by

$$\chi_H(\mathbf{x}) = \begin{cases} 1 \text{ if } \mathbf{x} \in \Omega, \\ 0 \text{ else} \end{cases} \tag{5.5}$$

$$\chi_{[a_i,T_{act}+a_i]}(t) = \begin{cases} 1 \text{ if } a_i < t < T_{act} + a_i, \\ 0 \text{ else} \end{cases} \tag{5.6}$$

$$\chi_{prop}(\mathbf{x}, t) = \begin{cases} 1 \text{ if } x + y < 2t/T_{act}, \\ 0 \text{ else.} \end{cases} \tag{5.7}$$

The last characteristic is associated to wave propagation as a diagonal which evolves from to left-bottom side to right-top side of Ω_H. The function Φ, which corresponds to the shape of electrical wave, is defined by

$$\Phi(\mathbf{x}) = 1 - \frac{1}{L_H}\left(x - \frac{L_H}{2}\right)^2 - \frac{1}{L_H}\left(y - \frac{L_H}{2}\right)^2. \tag{5.8}$$

To perform the numerical simulations, we have developed a numerical scheme reliable, efficient, stable and easy to implement in context of such "memory bidomain systems" by generalizing the modified Lattice Boltzmann Method introduced in previous works [9, 27, 28].

Nota Bene: In all figures first-order dynamic (corresponding to the case $\alpha = \beta = 1$) is shown in black and the last curve drawn is in dotted line.

5.1. FitzHugh-Nagumo ionical model

The first ionical model we choose is the simplest FitzHugh-Nagumo phenomenological model (see [39]). This model can be defined with the operators $\mathcal{I}$ and $\mathcal{G}$ as:

$$\mathcal{I}(\phi, u) \quad = -0.0004 \frac{\phi - \phi_{min}}{\phi_{max} - \phi_{min}} \left(U(\phi_{max} - \phi_{min})^2 + \frac{(\phi - \phi_{gate})(\phi - \phi_{max})}{(\phi_{max} - \phi_{min})^2} \right), \tag{5.9}$$

and

$$\mathcal{G}(\phi, u) = 0.63 \frac{\phi - \phi_{min}}{\phi_{max} - \phi_{min}} - 0.013 U(\phi_{max} - \phi_{min})^2. \tag{5.10}$$

For this model, we investigate only the impact of membrane capacitive memory on action potential properties in absence of ionic variable memory (i.e., we fix β to 1 and we vary α). In Figures 4-6 (response to single stimulus) and in Figures 7-9 (response to two stimuli in sequence) we plot the time evolution of ϕ, φ_e and u, at midpoint of heart, for different values of α with $\beta = 1$. We observe first that we have always the same kind of behavior. Second, we find that the amplitude varies depending on α values. For state φ_e, the effect fractional-order dynamic is negligible except for the extremum of φ_e (as the value of α becomes smaller, the value of the minimum point becomes smaller too).

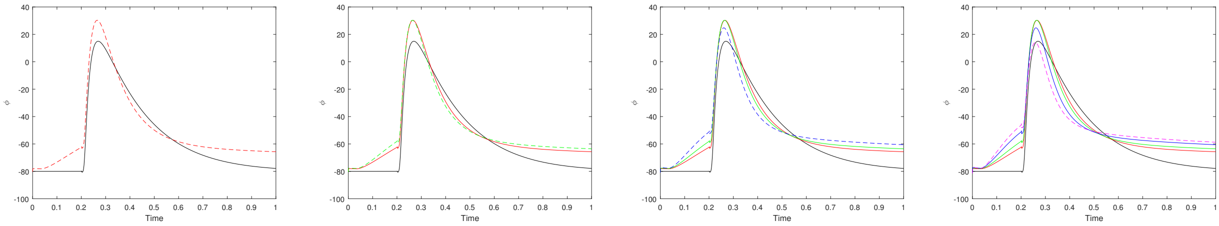

Figure 4. Response to a single stimulus. Time evolution of the transmembrane potential ϕ at midpoint of heart with $\beta = 1$ and $\alpha =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85, ▬ 0.8.

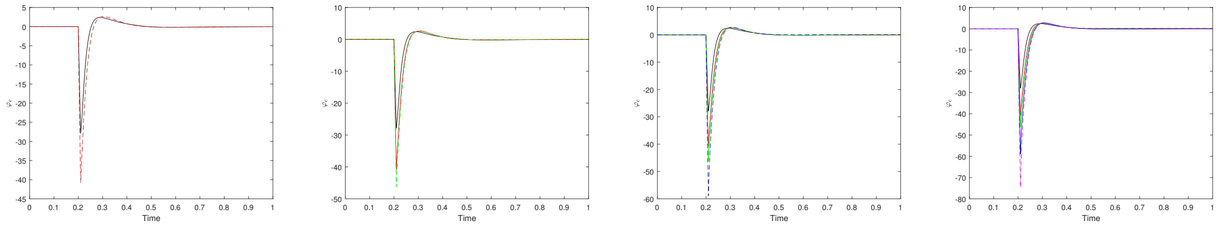

Figure 5. Response to a single stimulus. Time evolution of the extracellular potential φ_e at midpoint of heart with $\beta = 1$ and $\alpha =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85, ▬ 0.8.

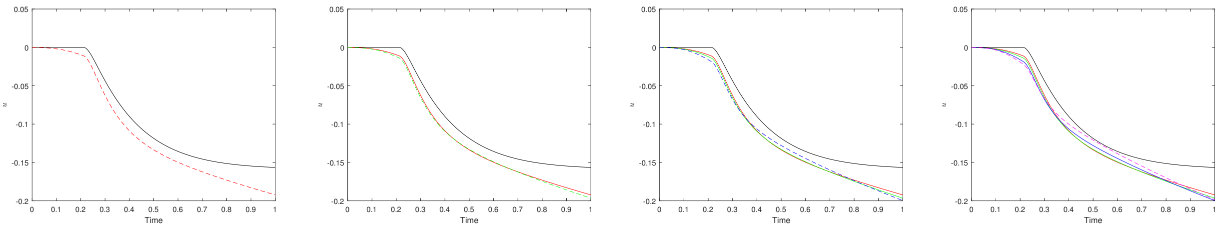

Figure 6. Response to a single stimulus. Time evolution of the ionic variable u at midpoint of heart $\beta = 1$ and $\alpha =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85, ▬ 0.8.

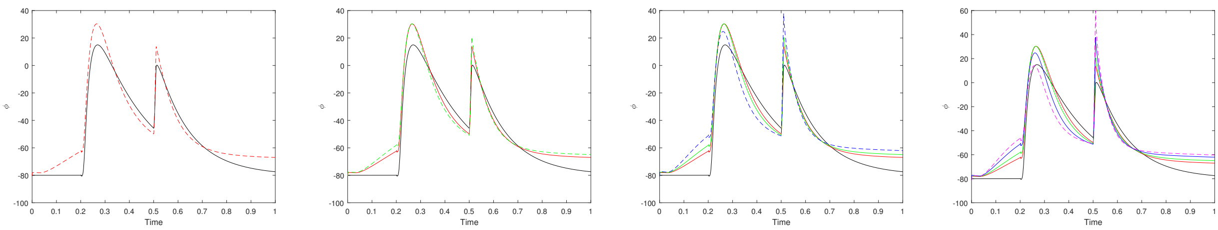

Figure 7. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the transmembrane potential ϕ at midpoint of heart $\beta = 1$ and $\alpha =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85, ▬ 0.8.

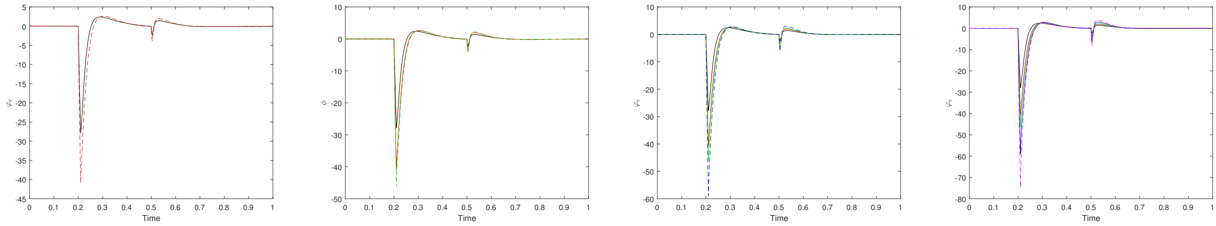

Figure 8. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the extracellular potential φ_e at midpoint of heart with $\beta = 1$ and $\alpha =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85, ▬ 0.8.

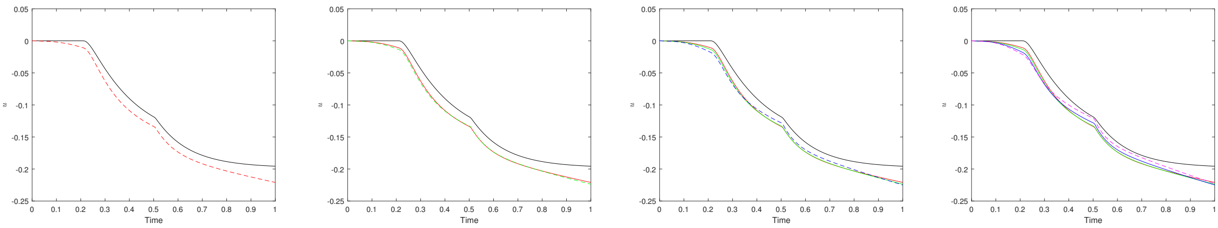

Figure 9. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the ionic variable u at midpoint of heart with $\beta = 1$ and $\alpha =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85, ▬ 0.8.

5.2. *Mitchell-Schaeffer ionical model*

The second ionic model corresponds to Mitchell-Shaeffer biophysical ionic model (see [56]). This two-variable model can be defined with operators $\mathcal{I}$ and $\mathcal{G}$ as:

$$\mathcal{I}(\phi, u) = -\frac{u}{\tau_{in}} \frac{(\phi - \phi_{min})^2(\phi_{max} - \phi)}{(\phi_{max} - \phi_{min})} - \frac{1}{\tau_{out}} \frac{\phi - \phi_{min}}{(\phi_{max} - \phi_{min})},$$
$$\mathcal{G}(\phi, u) = \begin{cases} \dfrac{u}{\tau_{open}} - \dfrac{1}{\tau_{open}(\phi_{max} - \phi_{min})^2} & \text{if } \phi < \phi_{gate}, \\ \dfrac{u}{\tau_{close}} & \text{if } \phi \geq \phi_{gate}. \end{cases} \tag{5.11}$$

These operators depend on the change-over voltage ϕ_{gate}, the resting potential ϕ_{min}, the maximum potential ϕ_{max}, and on times constants τ_{in}, τ_{out}, τ_{open} and τ_{close}, the two times τ_{open} and τ_{close}, respectively controlling the durations of the action potential and of the recovery phase, and the two times τ_{in} and τ_{out}, respectively controlling the length of depolarization and repolarization phases. These constants are such that $\tau_{in} < \tau_{out} < \min(\tau_{open}, \tau_{close})$.

First, we investigate the influence of membrane capacitive memory on action potential properties in absence of ionic variable memory (i.e., we fix β to 1 and we vary α). In Figures 10-13 (response to a single stimulus) and in Figures 14-17 (response to two stimuli in sequence), we plot the time evolution of ϕ, φ_e and u, at midpoint of heart and φ_s at at midpoint of torso, for different values of α with $\beta = 1$. We find first that following the stimuli cessation, spontaneous action potentials are triggered and persist for the duration of simulation. Second, the spontaneous action potential cycle length decreases as fractional-order α decreases, and we notice therefore an increase in the rate of spontaneous activity (when α decreases). We observe the same kind of behaviors for the other state variables than for ϕ, with an acceleration of spontaneous activity when α decreases.

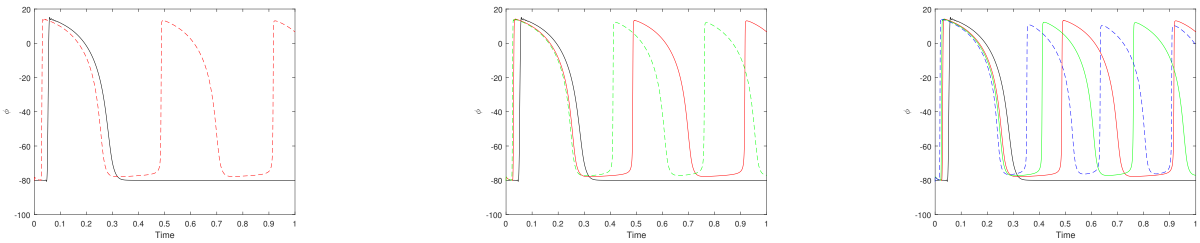

Figure 10. Response to a single stimulus. Time evolution of the transmembrane potential ϕ at midpoint of heart with $\beta = 1$ and $\alpha =$ ▬ (black) 1, ▬ (red) 0.925, ▬ (green) 0.9, ▬ (blue) 0.85.

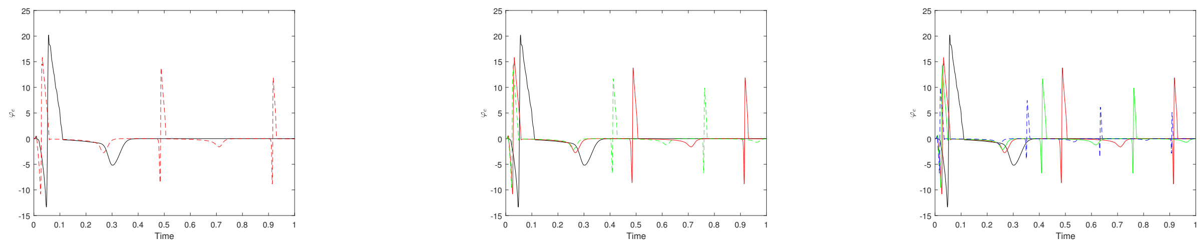

Figure 11. Response to a single stimulus. Time evolution of the extracellular potential φ_e at midpoint of heart with $\beta = 1$ and $\alpha =$ ▬ (black) 1, ▬ (red) 0.925, ▬ (green) 0.9, ▬ (blue) 0.85.

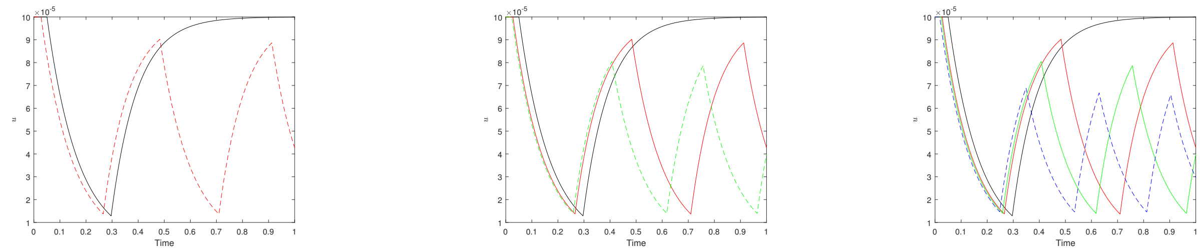

Figure 12. Response to a single stimulus. Time evolution of the ionic variable u at midpoint of heart $\beta = 1$ and $\alpha =$ ▬ (black) 1, ▬ (red) 0.925, ▬ (green) 0.9, ▬ (blue) 0.85.

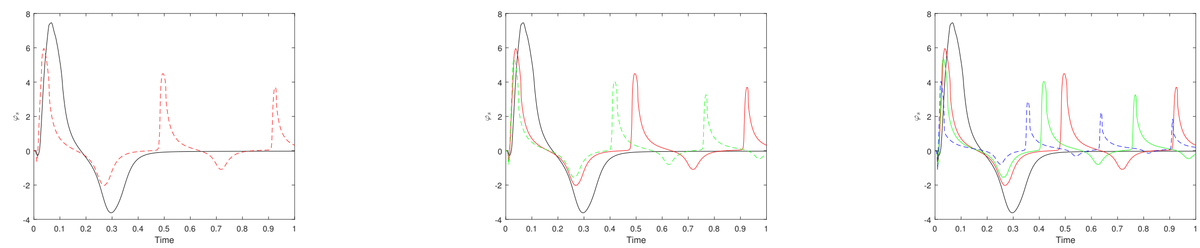

Figure 13. Response to a single stimulus. Time evolution of the torso potential φ_s at midpoint of torso with $\beta = 1$ and $\alpha =$ ▬ (black) 1, ▬ (red) 0.925, ▬ (green) 0.9, ▬ (blue) 0.85.

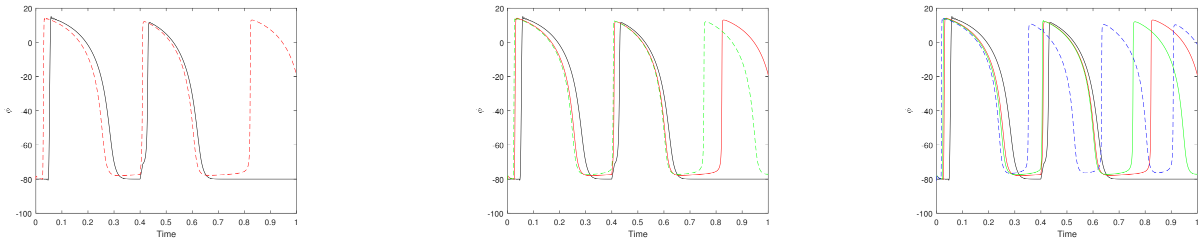

Figure 14. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the transmembrane potential ϕ at midpoint of heart with $\beta = 1$ and $\alpha =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85.

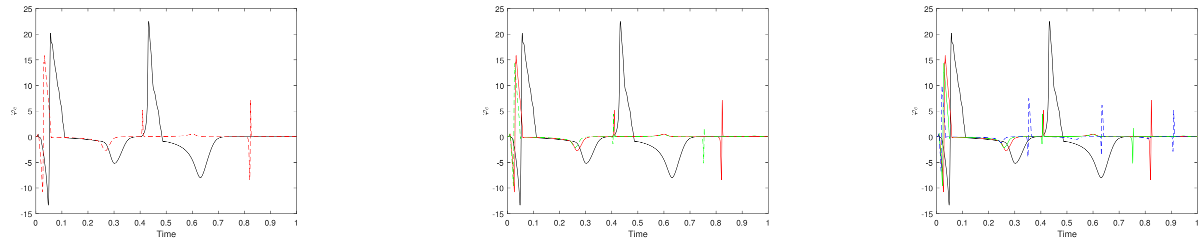

Figure 15. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the extracellular potential φ_e at midpoint of heart with $\beta = 1$ and $\alpha =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85.

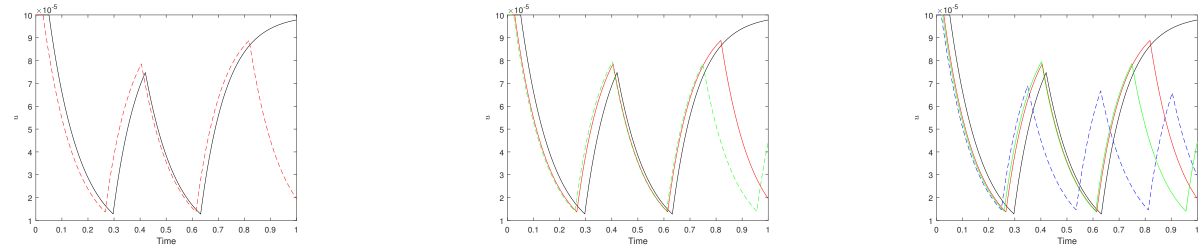

Figure 16. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the ionic variable u at midpoint of heart with $\beta = 1$ and $\alpha =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85.

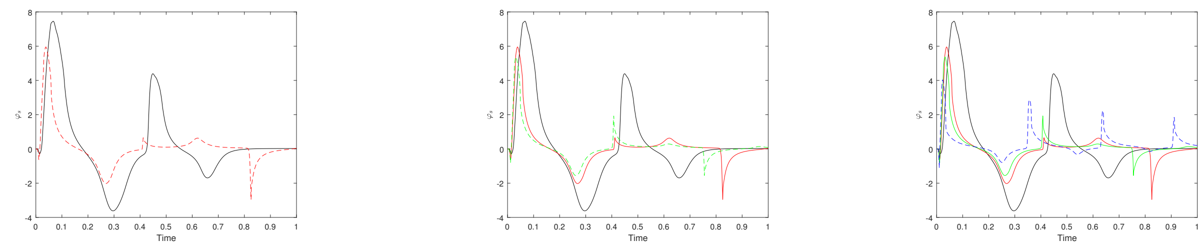

Figure 17. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the torso potential φ_s at midpoint of torso with $\beta = 1$ and $\alpha =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85.

We next investigate the influence of ionic variable memory on action potential properties in absence of membrane capacitive memory. In Figures 18-21 (single stimulus) and in Figures 22-25 (two stimuli in sequence), we plot the time evolution of ϕ, φ_e and u, at midpoint of heart and φ_s at at midpoint of torso, for different values of β with $\alpha = 1$. We observe that the smaller the value of β becomes, the more APD is shortened. Therefore the ionic current memory alters the action potential duration. Moreover from some value of APD we observe the triggering of spontaneous chain activities.

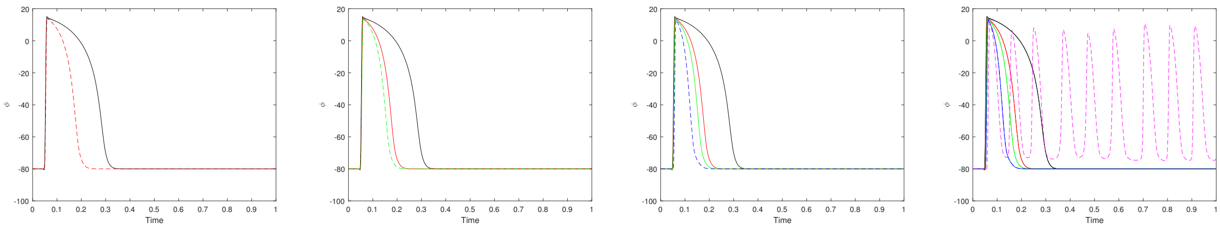

Figure 18. Response to a single stimulus. Time evolution of the transmembrane potential ϕ at midpoint of heart with $\alpha = 1$ and $\beta =$ — 1, — 0.925, — 0.9, — 0.85, — 0.8.

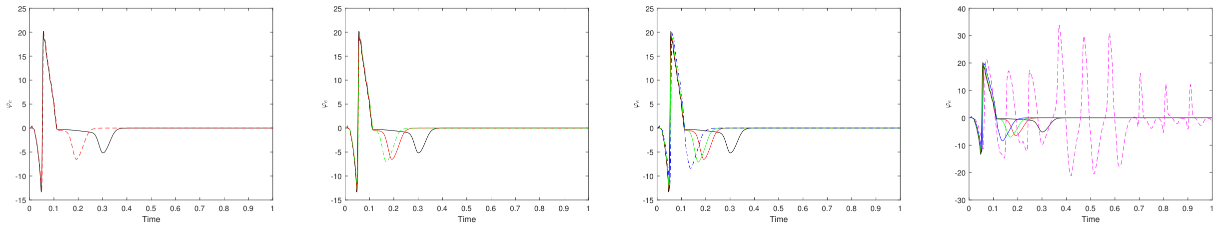

Figure 19. Response to a single stimulus. Time evolution of the extracellular potential φ_e at midpoint of heart with $\alpha = 1$ and $\beta =$ — 1, — 0.925, — 0.9, — 0.85, — 0.8.

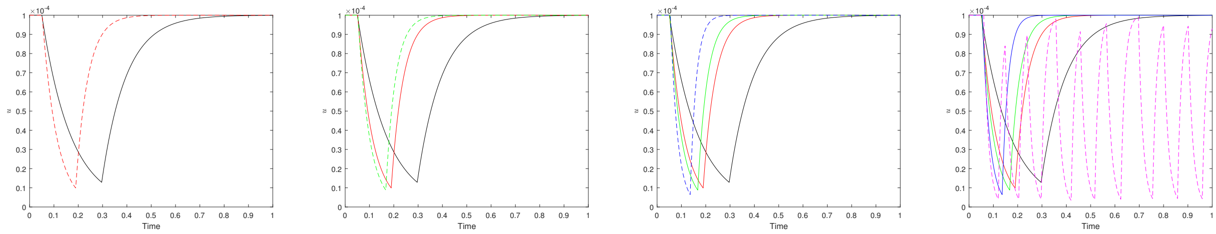

Figure 20. Response to a single stimulus. Time evolution of the ionic variable u at midpoint of heart with $\alpha = 1$ and $\beta =$ — 1, — 0.925, — 0.9, — 0.85, — 0.8.

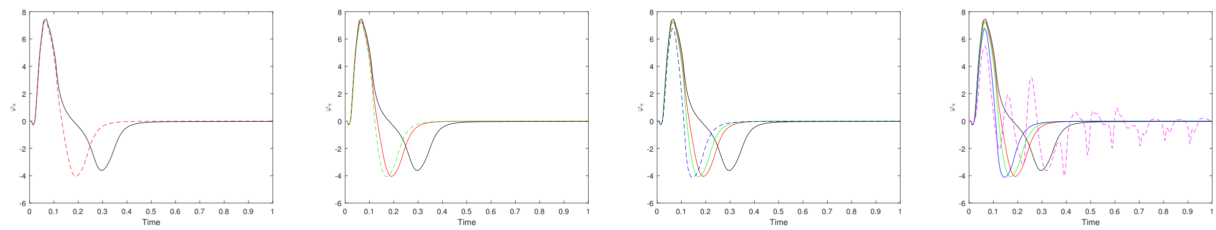

Figure 21. Response to a single stimulus. Time evolution of the torso potential φ_s at midpoint of torso with $\alpha = 1$ and $\beta =$ — 1, — 0.925, — 0.9, — 0.85, — 0.8.

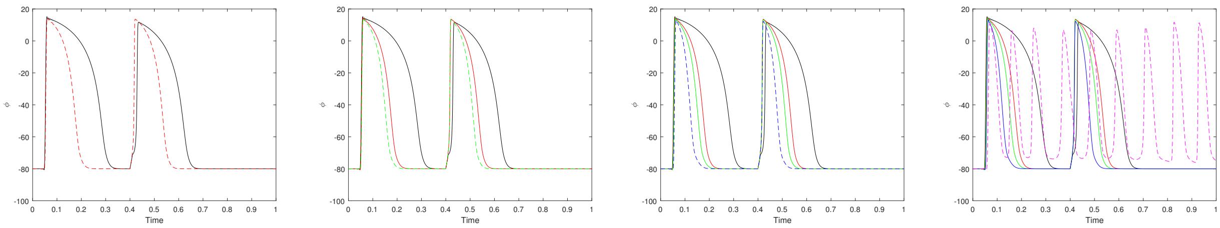

Figure 22. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the transmembrane potential ϕ at midpoint of heart with $\alpha = 1$ and $\beta =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85, ▬ 0.8.

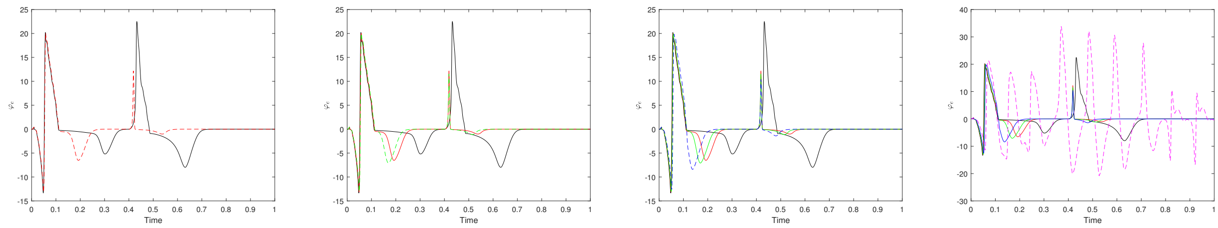

Figure 23. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the extracellular potential φ_e at midpoint of heart with $\alpha = 1$ and $\beta =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85, ▬ 0.8.

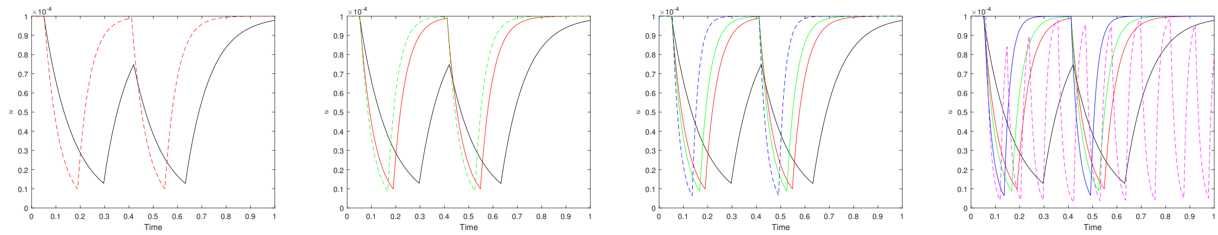

Figure 24. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the ionic variable u at midpoint of heart with $\alpha = 1$ and $\beta =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85, ▬ 0.8.

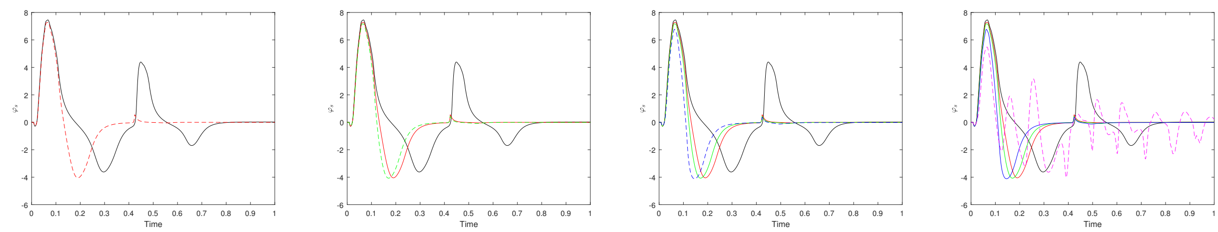

Figure 25. Response to a multiple stimuli (two stimuli in sequence). Time evolution of the torso potential φ_s at midpoint of torso with $\alpha = 1$ and $\beta =$ ▬ 1, ▬ 0.925, ▬ 0.9, ▬ 0.85, ▬ 0.8.

6. Conclusions

Modeling and control of electrical cardiac activity represent nowadays a very valuable tool to facilitate diagnosis and maximize the efficiency and safety of treatment for cardiac disorders. In this work, we have developed a new bidomain model of the cardiac electrophysiology which takes into account cardiac memory phenomena, named "memory bidomain model". Cardiac memory is a non-anecdotal phenomena, the impact of which in clinical practice is significant, particularly at the level of diagnostic and decision-making challenges. The derived model is a degenerate nonlinear coupled system of reaction-diffusion equations in shape of a fractional-order ODE coupled with a set of time fractional-order PDEs. Cardiac memory is represented via fractional-order capacitor and fractional-order cellular membrane dynamics. The existence of a weak solution as well as regularity and Lipschitz continuity of map solution results are established, with an abstract class of ionic models, including some classical models as Rogers-McCulloch, Fitz-Hugh-Nagumo and Aliev-Panfilov. This theoretical analysis is completed by numerical results that validate the interest of developed model. For this, some preliminary numerical simulations are performed to illustrate our results by comparing dynamic restitution curves obtained by numerically solving the first-order model and time fractional-order models (which reproduce critical memory effects). FitzHugh-Nagumo model and Mitchell-Shaeffer model are considered in these numerical applications. We observe that memory can alter the key dependent electrical properties including APD, action potential morphology and spontaneous activity.

These first observations demonstrate that memory effects can play a significant role in cardiac electrical dynamics (whether normal or disease states) and then motivates the continuation and deepening of these studies.

For predicting and acting on phenomena and processes occurring inside and surrounding cardiac medium, it would be interesting to study control and regulation problems in order to determine[§] the best optimal prognostic values of sources in presence of disturbances and pacing history, using the approach developed in [8,10,13]. The resolution of this type of problem makes it possible in practice to improve the stimuli applied during e.g., defibrillation sessions. This stimulus must be adjusted according to the patient's metabolism and type of cardiac problem (cardiac arrest rhythms, bradycardia, tachycardia, etc.).

Conflict of interest

The author declares there is no conflicts of interest in this paper.

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[§]from some observations, cardiac MRI-Measurement or desired target

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