Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(1): 772–793.
DOI:10.3934/math.2021047
Received: 14 May 2020
Accepted: 09 July 2020
Published: 01 November 2020

Research article

Error estimates of variational discretization for semilinear parabolic optimal control problems

Chunjuan Hou[1], **Zuliang Lu**[2,3,*], **Xuejiao Chen**[1] **and Fei Huang**[2]

[1] Huashang College Guangdong University of Finance and Economics, Guangzhou, 511300, P. R. China
[2] Key Laboratory for Nonlinear Science and System Structure, Chongqing Three Gorges University, Chongqing, 404000, P. R. China
[3] Research Center for Mathematics and Economics, Tianjin University of Finance and Economics, Tianjin, 300222, P. R. China

* **Correspondence:** Email: zulianglu@sanxiau.edu.cn.

Abstract: In this paper, variational discretization directed against the optimal control problem governed by nonlinear parabolic equations with control constraints is studied. It is known that the a priori error estimates is $|||u - u_h|||_{L^\infty(J;L^2(\Omega))} = O(h + k)$ using backward Euler method for standard finite element. In this paper, the better result $|||u - u_h|||_{L^\infty(J;L^2(\Omega))} = O(h^2 + k)$ is gained. Beyond that, we get a posteriori error estimates of residual type.

Keywords: optimal control problems; semilinear parabolic equations; error estimates; finite element methods
Mathematics Subject Classification: 49J20, 65N30

1. Introduction

It is generally known that optimal control has an important role in this ever-changing society, such as engineering numerical simulation, economic, atmosphere, petroleum, architecture and scientific, and so on. Meanwhile, optimal control is an important knowledge point not only for science and engineering students, but also for non science and engineering students. for example, it is widely used in economic management disciplines, such as management accounting, dynamic optimization, actuarial, pricing analysis, etc, even in artistic modeling structure analysis. An effective numerical method is necessary for the successful application of optimal control, for optimal control problems (OCP), finite element approximation should be a powerful numerical method in the calculation, and there are many related literatures. Optimal control problems using finite element methods (FEM) for partial differential equations (PDEs) are introduced systematically, which can be found in [1–4] for elliptic optimal control problems in (see e.g., [2, 5–10]), for parabolic optimal control problems in [1, 3, 11–13] and for Stokes optimal control problem in [14, 15]. A priori error estimates of finite element method was established in [16–18], a posteriori error estimates of residual type has been established in [12, 15, 19], and a posteriori error estimates based on recovery techniques has been derived in [14, 20, 21]. Some error estimates and superconvergence results of mixed finite element method for optimal control problems can be found in [5, 9].

For the optimal control problem with control constraints, the regularity of the general control method is lower than that of the state and the co-state. Therefore, the state and the co-state variables are approached by piecewise linear finite element functions, and the control variable is approached by piecewise constant function, and a projection gradient method with preset conditions is constructed. see [1, 3, 12, 14, 15, 19, 20]. In year 2012, Tang and Chen [22] studied variational discretization for the optimal control problem governed by parabolic equations with control constraints.

In this paper, we discuss a variational discretization for optimal control problems governed by nonlinear parabolic equations with control constraints, and we derive a priori error estimates and a posteriori error estimates of residual type. Some of techniques directly relevant to our work can be found in [16, 23, 24].

In this paper, the model problem that we shall investigate is the following two dimensional optimal control problem:

$$\begin{cases} \min\limits_{u(x,t)\in D}\left\{\dfrac{1}{2}\displaystyle\int_0^T\left(\|y(x,t)-y_d(x,t)\|^2_{L^2(\Omega)}+\|u(x,t)\|^2_{L^2(\Omega)}\right)dt\right\}, \\ y_t(x,t)-\operatorname{div}(A(x)\nabla y(x,t))+\phi(y(x,t))=f(x,t)+u(x,t), \quad x\in\Omega,\ t\in J, \\ y(x,t)=0, \quad x\in\partial\Omega,\ t\in J, \\ y(x,0)=y_0(x), \qquad x\in\Omega, \end{cases} \tag{1.1}$$

where D is defined by

$$D=\left\{\, g(x,t)\in L^2(J;L^2(\Omega)) : c\le g(x,t)\le d, \quad a.e.\ \ x\in\Omega,\ t\in J\,\right\},$$

where c and d are two constants.

Ω in $\mathbb{R}^n (n\le 3)$ is a bounded domain and $\partial\Omega$ is a Lipschitz boundary, $J=[0,T]$, $0<T<+\infty$, n-matrix $A(x)=(a_{ij}(x))\in(W^{1,\infty}(\bar{\Omega}))^{n\times n}$, and $(A(x)\xi)\cdot\xi\ge c\mid\xi\mid^2$, $\forall\,\xi\in\mathbb{R}^n$. The function $\phi(\cdot)\in W^{2,\infty}$ for any $R>0$, $\phi'(y)\in L^2(\Omega)$ for any $y\in H^1(\Omega)$, and $\phi'(y)\ge 0$. Here $y_t(x,t)$ denotes the partial derivative of y in time, $y_d(x,t), f(x,t)\in C(J;L^2(\Omega))$, $y_0(x)\in H_0^1(\Omega)$.

We use the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{Wm,p(\Omega)}$, when $p=2$, $W^{m,2}(\Omega)$ can be expressed as $H^m(\Omega)$, $W_0^{m,2}(\Omega)$ can be expressed as $H_0^m(\Omega)$. Further more, $H_0^1(\Omega):=\left\{\, g\in H^1(\Omega) : g|_{\partial\Omega}=0\,\right\}$. We denote by $L^k(J;W^{m,p}(\Omega))$ the Banach space of all L^k integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|g\|_{L^k(J;W^{m,p}(\Omega))}=\left(\int_0^T\|g\|^k_{W^{m,p}(\Omega)}dt\right)^{\frac{1}{k}}$ for $k\in[1,\infty)$ and the standard modification for $k=\infty$. Similarly, the spaces $H^l(J;W^{m,p}(\Omega))$ and $C^l(J;W^{m,p}(\Omega))$ also can be defined, the details can be found in [17]. In addition, c or C denotes a ordinary positive constant.

The research ideas and concrete design of the paper is as follows: we give the backward Euler approximation and variational discretization approximation for our model in Section 2, then gain a prior error estimate in Section 3. After that, we get a posterior error estimate in Section 4. In Section 5, we give conclusion and future works.

2. Approximation for the model problem of variational discretization

In this section, in the light of the model (1.1), we give variational discretization approximation using backward Euler method. We define $W = L^2(J; V)$ with $V = H_0^1(\Omega)$, $\|\cdot\|_V = \|\cdot\|_{H_0^1(\Omega)}$, and the control space $X = L^2(J; U)$ with $U = L^2(\Omega)$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. Throughout the paper, $(\cdot\,,\cdot)$ denotes the inner product in $L^2(\Omega)$.

Let

$$a(u, v) = \int_\Omega (A(x)\nabla u) \cdot \nabla v, \qquad \forall\, u,\ v \in V,$$
$$(p, q) = \int_\Omega p \cdot q, \qquad \forall\, p,\ q \in L^2(\Omega).$$

From Friedriechs' inequality, it produces

$$a(u, u) \geq c\|u\|_V^2, \qquad \forall\, u \in V,$$
$$|a(u, v)| \leq C\|u\|_V\|v\|_V, \quad \forall\, u, v \in V.$$

In order to keep things simple, we suppose the domain Ω is in R^2 and is a convex polygon.

Next, we introduce the co-state equation

$$-p_t(x, t) - \mathrm{div}(A(x)\nabla p(x, t)) + \phi'(y(x, t))p(x, t) = y(x, t) - y_d, \quad x \in \Omega,\ t \in [0, T), \tag{2.1}$$
$$p(x, t) = 0, \qquad x \in \partial\Omega,\ t \in [0, T), \tag{2.2}$$
$$p(x, T) = 0, \qquad x \in \Omega. \tag{2.3}$$

Then a possible weak formula for the state equation reads: For given u, y_0, find $y(u)$ such that

$$(y_t, w) + a(y, w) + (\phi(y), v) = (f + u, w), \quad t \in (0, T],\ \forall\, w \in V,$$
$$y(x, 0) = y_0(x), \quad x \in \Omega.$$

It is well known (see [16]) that the above problem has a unique solution y. It follows from embedding that $y \in C(0, T; L^2(\Omega))$. So the above model control problem (1.1) can be rewritten as (QCP):

$$\min_{u\in D}\left\{\frac{1}{2}\int_0^{\mathrm{T}} (\|\, y - y_d\,\|^2 + \|\, u\,\|^2)dt\right\}, \tag{2.4}$$
$$(y_t, w) + a(y, v) + (\phi(y), w) = (f + u, w), \quad t \in (0, T],\ \forall\, w \in V, \tag{2.5}$$
$$y(x, 0) = y_0(x), \quad x \in \Omega, \tag{2.6}$$

where $y \in H^1(0, T; U) \bigcap W$.

It is well known (see, e.g., [1]) that the convex control problem (QCP) has a unique solution (y, u), and that a doublet (y, u) is the solution of (QCP) if and only if there exists a co-state $p \in H^1(J; U) \bigcap W$, such that the triplet (y, p, u) satisfies the following optimal conditions (QCP-OPT) for $t \in (0, T]$

$$(y_t, w) + a(y, w) + (\phi(y), w) = (f + u, w), \quad \forall\, w \in V, \tag{2.7}$$

$$y(x,0) = y_0(x), \quad x \in \Omega,$$
$$-(p_t, q) + a(q,p) + (\phi'(y)p) = (y - y_d, q), \quad \forall\, q \in V, \tag{2.8}$$
$$p(x,T) = 0, \quad x \in \Omega,$$
$$(u + p, \tilde{u} - u) \geq 0, \quad \forall\, \tilde{u} \in D. \tag{2.9}$$

Now, we introduce the pointwise projection operator as followed:

$$\Pi_{[c,d]}(h(x,t)) = \max(c, \min(d, -h(x,t))). \tag{2.10}$$

As in [16], it is easy to see that (2.9) can be equivalently read as:

$$u(x,t) = \Pi_{[c,d]}(p(x,t)). \tag{2.11}$$

Set Γ^h be a regular triangulations of the polygonal domain Ω, such that $\bar{\Omega} = \bigcup_{\lambda\in\Gamma^h} \bar{\lambda}$. Let $h = \max_{\lambda\in\Gamma^h}\{h_\lambda\}$, where h_λ denotes the diameter of the element λ. Associated with Γ^h is a finite dimensional subspace W^h of $C(\bar{\Omega})$, such that $w_h|_\lambda$ is the polynomial of total degree no more than n ($n \geq 1$), $\forall\, w_h \in W^h$. Let $V^h = W^h \cap V$. Then it is easy to see that $V^h \subset V$.

Suppose N is a positive integer, and we set $0 = t_0 < t_1 < \cdots < t_N = T$, $k_l = t_l - t_{l-1}$, $l = 1, 2, \cdots, N$, $k = \max_{l\in[1,N]}\{k_l\}$. Set $q^l = q(x, t_l)$, we give the discrete time-dependent norms for $1 \leq p < \infty$:

$$|||q(x,t)|||_{L^p(J;H^l(\Omega))} := \left(\sum_{i=1-s}^{N-s} k_{i+s} \left\|q^i\right\|_l^p\right)^{\frac{1}{p}},$$

when $s = 0$ be the control variable and the state variable $u(x,t)$ and $y(x,t)$, when $s = 1$ be the co-state variable $p(x,t)$, with the standard modification for $p = \infty$.

A possible finite element approximation of (QCP) is to find $(y_h, u_h) \in U_h \times V_h$, which we shall label $(\mathrm{QCP})^h$

$$\min_{u_h\in D_h}\left\{\frac{1}{2}\int_0^{\mathrm{T}}(\| y_h - y_d \|^2 + \| u_h \|^2)dt\right\} \tag{2.12}$$
$$(y_{h,t}, w_h) + a(y_h, w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), \quad \forall\, w_h \in V_h, \tag{2.13}$$
$$y_h(x,0) = y_0^h(x), \quad x \in \Omega, \tag{2.14}$$

where $y_h \in H^1(0;T;V_h)$, $y_0^h \in V_h$ is an approximation of y_0.

The control problem $(\mathrm{QCP})^h$ again has a unique solution (y_h, u_h) and a doublet $(y_h, u_h) \in V_h \times U_h$ is the solution of $(\mathrm{QCP})^h$ if and only if there is a co-state p_h, such that the triplet (y_h, p_h, u_h) satisfies the following optimality conditions $(\mathrm{QCP\text{-}OPT})^h$ for $t \in (0,T]$:

$$(y_{h,t}, w_h) + a(y_h, w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), \quad \forall\, w_h \in V_h, \tag{2.15}$$
$$y_h(x,0) = y_0^h(x), \quad x \in \Omega,$$
$$-(p_{h,t}, q_h) + a(q_h, p_h) + (\phi'(y_h)p_h, q_h) = (y_h - y_d, q_h), \quad \forall\, q_h \in V_h, \tag{2.16}$$
$$p_h(x,T) = 0, \quad x \in \Omega,$$
$$(u_h + p_h, \tilde{u}_h - u_h) \geq 0, \quad \forall\, \tilde{u}_h \in D_h. \tag{2.17}$$

Therefore, as we defined, the exact state solution and its approximation can be read as:

$$(y, p) = (y(u), p(u)),$$
$$(y_h, p_h) = (y_h(u_h), p_h(u_h)).$$

Next we consider the above discrete approximation using the backward Euler scheme to approximate the partial derivative in time t. Then the discrete approximation of (2.4)-(2.6) is the following form:

$$\begin{cases} \min\limits_{u_h^l \in K} \left\{ \dfrac{1}{2} \sum\limits_{i=1}^{N} k_l \left(\left\| y_h^l - y_d^l \right\|^2 + \left\| u_h^l \right\|^2 \right) \right\}, \\ \left(\dfrac{y_h^l - y_h^{l-1}}{k_i}, v_h \right) + a\left(y_h^l, v_h \right) + (\phi(y_h^l), v_h) = \left(f^l + u_h^l, w_h \right), \\ \forall w_h \in V^h, \quad l = 1, 2, \cdots, N, \\ y_h^0(x, 0) = y_0^h(x), \quad x \in \Omega. \end{cases} \tag{2.18}$$

It follows that the control problem (2.18) has a unique solution $\left(y_h^l, u_h^l\right)$, $l = 1, 2, \cdots, N$, and $\left(y_h^l, u_h^l\right) \in V^h \times D$, $l = 1, 2, \cdots, N$, is the solution of (2.18) if and only if there is a co-state $p_h^{l-1} \in V^h$, $l = 1, 2, \cdots, N$, such that the triplet $\left(y_h^l, p_h^{l-1}, u_h^l\right) \in V^h \times V^h \times D$, $l = 1, 2, \cdots, N$, satisfies the following variational discretization optimality conditions:

$$\left(\frac{y_h^l - y_h^{l-1}}{k_l}, w_h \right) + a\left(y_h^l, w_h \right) + (\phi(y_h^l), w_h) = \left(f^l + u_h^l, w_h \right), \tag{2.19}$$
$$y_h^0(x) = y_0^h(x), \; x \in \Omega, \; l = 1, 2, \cdots, N,$$
$$\left(\frac{p_h^{l-1} - p_h^l}{k_l}, q_h \right) + a\left(q_h, p_h^{l-1} \right) + (\phi'(y_h^l) p_h^{l-1}, q_h) = \left(y_h^l - y_d^l, q_h \right), \tag{2.20}$$
$$p_h^N(x) = 0, \; x \in \Omega, \; l = N, N-1, \cdots, 1,$$
$$\left(u_h^l + p_h^{l-1}, \tilde{u}_h - u_h^l \right) \geq 0, \; l = 1, 2, \cdots, N. \tag{2.21}$$

where $w_h \in V^h$, $q_h \in V^h$ and $\tilde{u}_h \in D_h$. It is obvious that the variational inequality (2.21) satisfies the result as follows:

$$u_h^l = \Pi\left(p_h^{l-1} \right), \quad l = 1, 2, \cdots, N. \tag{2.22}$$

According to (2.22), we should solve the numerical solution of the control variable u_h after the co-state variable p_h.

3. A priori error estimates for OCP

In this section, we shall derive a priori error estimates for the backward Euler variational discretization approximation scheme. We will start with errors estimation of state and co-state variable. Now, we recall the elliptic projection R_h: $V \to V_h$, for any $v \in V$, which satisfies:

$$a\left(v - R_h v, v_h\right) = 0, \quad \forall v_h \in V_h. \tag{3.1}$$

We have the approximation property (see [23])

$$\|v - R_h v\| \le Ch^s \|v\|_s, \qquad s = 1, 2. \tag{3.2}$$

Lemma 3.1. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (2.7)-(2.9) and (2.19)-(2.21), respectively. Assume $y \in H^1(J; H^2(\Omega)) \cap H^2(J; U)$ and $y(x, 0) \in H^2(\Omega)$. Then there is a constant C independent of h and k such that*

$$|||y_h - y|||_{L^\infty(J;U)} \le C\left(h^2 + k + |||u_h - u|||_X^2\right). \tag{3.3}$$

Proof. We set

$$y_h^l - y^l = y_h^l - R_h y^l + R_h y^l - y^l := \theta^l + \eta^l, \tag{3.4}$$

$a(R_h v, v_h) = a(v, v_h), \ \ \forall v_h \in V^h$. It follows from Lemma 1.1 in [25], we have

$$\begin{aligned} \left\|\eta^l\right\| = \left\|R_h y^l - y^l\right\| \le Ch^2 \left\|y^l\right\|_2 \\ \le Ch^2 \left(\|y^0\|_2 + \int_0^{t_l} \|y_t(s)\|_2 ds\right) \\ \le Ch^2. \end{aligned} \tag{3.5}$$

It follows from (2.7) and (2.19), noting that $''I''$ denotes the unit operator, so we have

$$\begin{aligned} &\left(\frac{\theta^l - \theta^{l-1}}{k_l}, w_h\right) + a\left(\theta^l, w_h\right) \\ =& -\left(\frac{R_h y^l - R_h y^{l-1}}{k_l}, w_h\right) - a\left(R_h y^l, w_h\right) - (\phi(y_h^l), w_h) + \left(f^l + u_h^l, w_h\right) \\ =& -\left(\frac{R_h y^l - R y^{l-1}}{k_l}, w_h\right) - a\left(y^l, w_h\right) + \left(f^l + u^l, w_h\right) - (\phi(y_h^l), w_h) + \left(u_h^l - u^l, w_h\right) \\ =& -\left(\frac{R_h y^l - R_h y^{l-1}}{k_l}, w_h\right) + \left(y_t^l, w_h\right) + (\phi(y^l), w_h) - (\phi(y_h^l), w_h) + \left(u_h^l - u^l, w_h\right) \\ =& -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, w_h\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, w_h\right) - (\phi(y_h^l) - \phi(y^l), w_h) + \left(U_h^l - u^l, w_h\right) \\ =& -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, w_h\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, w_h\right) - \left(\phi'(y_h^l)(y_h^l - y^l), w_h\right) + \left(u_h^l - u^l, w_h\right) \\ =& -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, w_h\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, w_h\right) - \left(\phi'(y_h^l)(y_h^l - R_h y^l), w_h\right) \\ & - \left(\phi'(y_h^l)(R_h y^l - y^l), w_h\right) + \left(u_h^l - u^l, w_h\right) \\ \le& -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, w_h\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, w_h\right) - \left(\phi'(y_h^i)(y_h^i - R_h y^l), w_h\right) \\ & - \left(\phi'(y_h^l)(R y^l - y^l), w_h\right) + \left(u_h^l - u^l, w_h\right) \end{aligned}$$

We select $w_h = \theta^l$,

$$
\begin{aligned}
&\left(\frac{\theta^l - \theta^{l-1}}{k_l}, \theta^l\right) + a\left(\theta^l, \theta^l\right) \\
&\leq -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, \theta^l\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, \theta^l\right) - \left(\phi'(y_h^l)\theta^l, \theta^l\right) \\
&- \left(\phi'(y_h^l)\eta^l, \theta^l\right) + \left(u_h^l - u^l, \theta^l\right) \\
&\leq -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, \theta^l\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, \theta^l\right) - \left(\phi'(y_h^l)\eta^l, \theta^l\right) + \left(u_h^l - u^l, \theta^l\right)
\end{aligned} \tag{3.6}
$$

Note that $0 \leq c\left\|\theta^l\right\|_V^2 \leq a\left(\theta^l, \theta^l\right)$, according to Hölder inequality, we obtain

$$
\left\|\theta^l\right\| \leq \left\|\theta^{l-1}\right\| + \left\|(R_h - I)\left(y^l - y^{l-1}\right)\right\| + \left\|y^l - y^{l-1} - k_l y_t^l\right\| + k_l\left\|\phi' y_h^l)\eta^l\right\| + k_l\left\|u_h^l - u^l\right\|.
$$

Summing l from 1 to N^* $(1 \leq N^* \leq N)$, note that $y \in H^1(J; H^2(\Omega)) \cap H^2(J; U)$ and $y(x, 0) \in H^2(\Omega)$, we get

$$
\begin{aligned}
\left\|\theta^{N^*}\right\| \leq& \left\|\theta^0\right\| + \sum_{l=1}^{N^*}\left\|(R_h - I)\left(y^l - y^{l-1}\right)\right\| + \sum_{l=1}^{N^*}\left\|y^l - y^{l-1} - k_l y_t^l\right\| \\
&+ \sum_{l=1}^{N^*} k_l\left\|\phi'(y_h^l)\eta^l\right\| + \sum_{l=1}^{N^*} k_l\left\|u_h^l - u^l\right\| \\
\leq& \left\|\theta^0\right\| + \sum_{l=1}^{N^*}\int_{t_{l-1}}^{t_l}(R_h - I)y_t(s)ds + \sum_{l=1}^{N^*}\left\|\int_{t_{l-1}}^{t_l}(t_{l-1} - s)y_{tt}(s)ds\right\| \\
&+ \sum_{l=1}^{N^*} k_l\left\|\phi'(y_h^l)\right\|_{W^{1,\infty}}\left\|\eta^l\right\| + C|||u_h - u|||_X^2 \\
\leq& \left\|\theta^0\right\| + \sum_{l=1}^{N^*}\int_{t_{l-1}}^{t_l} Ch^2\|y_t(s)\|_2 ds + \sum_{l=1}^{N^*}\int_{t_{l-1}}^{t_l}\|(t_{l-1} - s)y_{tt}(s)\|ds \\
&+ Ch^2\sum_{l=1}^{N^*} k_l\left\|\phi'(y_h^i)\right\|_{W^{2,\infty}} + C|||u_h - u|||_X^2 \\
\leq& \left\|\theta^0\right\| + Ch^2\int_0^{t_{N^*}}\|y_t(s)\|_2 ds + k\int_0^{t_{N^*}}\|y_{tt}(s)\|ds + Ch^2 + C|||u_h - u|||_X^2 \\
\leq& Ch^2\|y^0\|_2 + Ch^2\int_0^T\|y_t(s)\|_2 ds + k\int_0^T\|y_{tt}(s)\|ds + Ch^2 + C|||u_h - u|||_X^2 \\
\leq& C(h^2 + k + |||u_h - u|||_X^2).
\end{aligned} \tag{3.7}
$$

Then (3.3) follows from (3.5)-(3.7) and Triangle inequality. □

Lemma 3.2. *Suppose (y, p, u) and (y_h, p_h, u_h) be the solutions of (2.7)-(2.9) and (2.19)-(2.21), respectively. Let $p \in H^1(J; H^2(\Omega)) \cap H^2(J; U)$, assume $y_d, y \in H^1(J; U)$, exist a constant C independent of k and h, so that*

$$|||p_h - p|||_{L^\infty(J;U)} \le C\left(h^2 + k + |||y_h - y|||_X\right). \tag{3.8}$$

Proof. Similarly, we set

$$p_h^l - p^l = p_h^l - R_h p^l + R_h p^l - p^l := \zeta^l + \xi^l. \tag{3.9}$$

Note that $p^N = 0$, then

$$\begin{aligned}\left\|\xi^l\right\| &= \left\|R_h p^l - p^l\right\| \le Ch^2 \left\|p^l\right\|_2 \\ &\le Ch^2 \int_T^{t_l} \|p_t(s)\|_2 ds \le Ch^2.\end{aligned} \tag{3.10}$$

It follows from (2.8) and (2.20), we have

$$\begin{aligned}
&\left(\frac{\zeta^{l-1} - \zeta^l}{k_l}, q_h\right) + a\left(q_h, \zeta^{l-1}\right) + (\phi'(y_h^i)(p_h^{l-1} - R_h p^{l-1}), q_h) \\
=& -\left(\frac{R_h p^{l-1} - R p^l}{k_l}, q_h\right) - a\left(q_h, R_h p^{l-1}\right) + \left(Y_h^l - y_d^l, q_h\right) - (\phi'(y_h^l)(R_h p^{l-1}), q_h) \\
=& -\left(\frac{R_h p^{l-1} - R_h p^l}{k_l}, q_h\right) - a\left(q_h, p^{l-1}\right) + \left(y_h^l - y_d^l, q_h\right) - \left(\phi'\left(y_h^l\right)\left(R_h p^{l-1}\right), q_h\right) \\
=& -\left(\frac{R_h p^{l-1} - R_h p^l}{k_l}, q_h\right) - a\left(q_h, p^{l-1}\right) + \left(y^{l-1} - y_d^{l-1}, q_h\right) \\
&+ \left(y_h^l - y^l + y^l - y^{l-1} + y_d^{l-1} - y_d^l, q_h\right) - (\phi'(y_h^l)(R_h p^{l-1}), q_h) \\
=& -\left(\frac{R_h p^{l-1} - R_h p^l}{k_l}, q_h\right) - \left(p_t^{l-1}, q_h\right) + \left(y_h^l - y^l + y^l - y^{l-1} + y_d^{l-1} - y_d^l, q_h\right) \\
&+ \left(\phi'\left(y^{l-1}\right) p^{l-1}, q_h\right) - \left(\phi'\left(y_h^l\right)\left(R_h p^{l-1}\right), q_h\right) \\
=& -\left(\frac{(R_h - I)\left(p^{l-1} - p^l\right)}{k_l}, q_h\right) - \left(\frac{p^{l-1} - p^l}{k_l} + p_t^{l-1}, q_h\right) \\
&+ \left(y_h^l - y^l + y^l - y^{l-1} + y_d^{l-1} - y_d^l, q_h\right) + (\phi'(y^{l-1})p^{l-1}, q_h) - (\phi'(y_h^l)(R_h p^{l-1}), q_h) \\
=& -\left(\frac{(R_h - I)\left(p^{l-1} - p^l\right)}{k_l}, q_h\right) - \left(\frac{p^{l-1} - p^l}{k_l} + p_t^{l-1}, q_h\right) \\
&+ \left(y_h^l - y^l + y^l - y^{l-1} + y_d^{l-1} - y_d^l, q_h\right) + \left(\phi'\left(y^{l-1}\right)\left(p^{l-1} - R_h p^{l-1}\right), q_h\right) \\
&+ \left(\left(\phi'\left(y^{l-1}\right) - \phi'\left(y_h^l\right)\right)\left(R_h p^{l-1}\right), q_h\right) \\
=& -\left(\frac{(R_h - I)\left(p^{l-1} - p^l\right)}{k_l}, q_h\right) - \left(\frac{p^{l-1} - p^l}{k_l} + p_t^{l-1}, q_h\right) \\
&+ \left(y_h^l - y^l + y^l - y^{l-1} + y_d^{l-1} - y_d^l, q_h\right) + \left(\left(\phi'\left(y_h^l\right) - \phi'\left(y^{l-1}\right)\right)\left(p^l - p^{l-1}\right), q_h\right)
\end{aligned}$$

$$-\left(\phi'\left(y_h^l\right)\left(R_h p^{l-1}-p^{l-1}\right),q_h\right)-\left(\left(\phi'\left(y_h^l\right)-\phi'\left(y^{l-1}\right)\right)p^l,q_h\right). \tag{3.11}$$

We select $q_h=\zeta^{l-1}$, note that $0\le c\left\|\zeta^{l-1}\right\|_V^2\le a\left(\zeta^{l-1},\zeta^{l-1}\right)$, by using Hölder inequality, we obtain

$$\begin{aligned}\left\|\zeta^{l-1}\right\|\le&\left\|\zeta^l\right\|+\left\|(R_h-I)\left(p^{l-1}-p^l\right)\right\|+\left\|p^{l-1}-p^l+k_lp_t^{l-1}\right\|\\&+k_l\left(\left\|y_h^l-y^l\right\|+\left\|y^l-y^{l-1}\right\|+\left\|y_d^{l-1}-y_d^l\right\|\right)\\&+k^l\Big(\left\|\left(\phi'\left(y_h^l\right)-\phi'\left(y^{l-1}\right)\right)p^l\right\|+\left\|\phi'\left(y_h^l\right)\xi^{l-1}\right\|\\&+\left\|\left(\phi'\left(y_h^l\right)-\phi'\left(y^{l-1}\right)\right)\left(p^l-p^{l-1}\right)\right\|\Big).\end{aligned} \tag{3.12}$$

Note that $\zeta^N=0$ and $p\in H^1(J;H^2(\Omega))\cap H^2(J;U)$, summing l from M^* $(0\le M^*\le N)$ to N, we get

$$\begin{aligned}&\left\|\zeta^{M^*}\right\|\le\sum_{l=M^*}^N\left\|(R_h-I)\left(p^{l-1}-p^l\right)\right\|+\sum_{l=M^*}^N\left\|p^{l-1}-p^l+k_lp_t^{l-1}\right\|\\&\quad+\sum_{l=M^*}^N k_l\left(\left\|y_h^l-y^l\right\|+\left\|y^l-y^{l-1}\right\|+\left\|y_d^{l-1}-y_d^l\right\|\right)\\&\quad+\sum_{l=M^*}^N k^l\Big(\left\|\left(\phi'\left(y_h^l\right)-\phi'\left(y^{l-1}\right)\right)p^l\right\|+\left\|\phi'\left(y_h^l\right)\xi^{l-1}\right\|+\left\|\left(\phi'\left(y_h^l\right)-\phi'\left(y^{l-1}\right)\right)\left(p^l-p^{l-1}\right)\right\|\Big)\\&\le\sum_{l=M^*}^N\int_{t_l}^{t_{l-l}}(R_h-I)p_t(s)ds+\sum_{l=M^*}^N\left\|\int_{t_l}^{t_{l-1}}(t_l-s)p_{tt}(s)ds\right\|\\&\quad+\sum_{l=M^*}^N k_l\left(\left\|y_h^l-y^l\right\|+\left\|y^l-y^{l-1}\right\|+\left\|y_d^{l-1}-y_d^l\right\|+\left\|\phi\left(y_h^l\right)\right\|_{W^{2,\infty}}\left\|y_h^l-y^{l-1}\right\|\left\|p^l\right\|\right)\\&\quad+\sum_{l=M^*}^N k_l\left(\left\|\phi\left(y_h^l\right)\right\|_{W^{1,\infty}}\left\|\xi^{l-1}\right\|+\left\|\phi\left(y_h^l\right)\right\|_{W^{2,\infty}}\int_{t_l}^{t_{l-1}}p_t(s)ds\right)\\&\le\sum_{l=M^*}^N\int_{t_l}^{t_{l-1}}Ch^2\|p_t(s)\|_2ds+\sum_{l=M^*}^N\int_{t_l}^{t_{l-1}}\|(t_l-s)p_{tt}(s)\|\,ds\\&\quad+\sum_{l=M^*}^N k_l\left(\left\|y_h^l-y^l\right\|+\left\|y^l-y^{l-1}\right\|+\left\|y_d^{l-1}-y_d^l\right\|+\left\|\phi\left(y_h^l\right)\right\|_{W^{1,\infty}}\left\|\xi^{l-1}\right\|\right)+Ch^2\\&\le Ch^2\int_{t_{M^*-1}}^T\|p_t(s)\|_2ds+k\int_{t_{M^*-1}}^T\|p_{tt}(s)\|\,ds\\&\quad+\sum_{l=M^*}^N k_l\left(\left\|y_h^l-y^l\right\|+\left\|y^l-y^{l-1}\right\|+\left\|y_d^{l-1}-y_d^l\right\|\right)+Ch^2\\&\le Ch^2\int_0^T\|p_t(s)\|_2ds+k\int_0^T\|p_{tt}(s)\|\,ds+|||y_h-y|||_X^2\\&\quad+Ck^2\left(\|y_t\|_X+\|y_{dt}\|_X\right)\\&\le C\left(h^2+k+|||Y_h-y|||_X^2\right).\end{aligned} \tag{3.13}$$

Then (3.8) follows from (3.9)-(3.10), (3.13) and the triangle inequality. □

Now, we estimate the error of the control variable. We need give two intermediate variables $(y_h^l(u), p_h^l(u)) \in V^h \times V^h,\ l = 1, 2, \cdots, N$, satisfies the following system:

$$\left(\frac{y_h^l(u) - y_h^{l-1}(u)}{k_l}, w_h\right) + a(y_h^l(u), w_h) + (\phi(y_h^l(u)), w_h) = \left(f^l + u^l, w_h\right), \quad \forall w_h \in V^h, \tag{3.14}$$
$$y_h(u)^0(x) = y_0(x), \qquad x \in \Omega,$$

$$\left(\frac{p_h^{l-1}(u) - p_h^l(u)}{k_l}, q_h\right) + a(q_h, p_h^{l-1}(u)) + (\phi'(y_h^l(u))p_h^{l-1}(u), q_h) = \left(y_h^l(u) - y_d^l, q_h\right), \quad \forall q_h \in V^h, \tag{3.15}$$
$$p_h^N(u)(x) = 0, \qquad x \in \Omega.$$

For ease of exposition, we set

$$\varphi^l = y_h^l - y_h^l(u), \qquad \zeta^l = p_h^l - p_h^l(u), \qquad l = 0, 1, \cdots, N.$$

It is clear that $\varphi^0 = 0$ and $\zeta^N = 0$.

Lemma 3.3. *Let (y, p, u) and $(y_h(u), p_h(u))$ be the solutions of (2.7)-(2.9) and (3.14)-(3.15), respectively. Assume that $u, p \in H^1(J; U)$. Then there exists a positive constant C independent of k and h, so the following conclusion holds*

$$|||u - u_h|||_X \leq C\left(|||p_h(u) - p|||_X + |||p_h - p|||_X + k\right). \tag{3.16}$$

Proof. It follows from variational inequalities (2.9), (2.21) and Hölder inequality that

$$\begin{aligned}
|||u - u_h|||_X^2 &= \sum_{l=1}^{N} k_l\left(u^l - u_h^l, u^l - u_h^l\right)\\
&= \sum_{l=1}^{N} k_l\left(u^l + p^l, u^l - U_h^l\right) - \sum_{l=1}^{N} k_l\left(u_h^l + p_h^{l-1}(u), u^l - u_h^l\right)\\
&\quad + \sum_{l=1}^{N} k_l\left(p_h^{l-1}(u) - p^l, u^l - u_h^l\right)\\
&\leq \sum_{l=1}^{N} k_l\left(u_h^l + p_h^{l-1}(u), u_h^l - u^l\right) + \sum_{l=1}^{N} k_l\left(p_h^{l-1}(u) - p^l, u^l - u_h^l\right)\\
&= \sum_{l=1}^{N} k_l\left(u_h^l + p_h^{l-1}, u_h^l - u^l\right) + \sum_{l=1}^{N} k_l\left(p_h^{l-1}(u) - p_h^{l-1}, u_h^l - u^l\right)\\
&\quad + \sum_{l=1}^{N} k_l\left(p_h^{l-1}(u) - p^l, u^l - u_h^l\right)\\
&\leq \sum_{l=1}^{N} k_l\left(p_h^{l-1}(u) - p_h^{l-1}, u_h^l - u^l\right) + \sum_{l=1}^{N} k_l\left(p_h^{l-1}(u) - p^l, u^l - u_h^l\right)
\end{aligned}$$

$$:=I_1 + I_2. \tag{3.17}$$

Note that $\varphi^0 = 0$ and $\zeta^N = 0$, then it follows from (2.19)-(2.20) and (3.14)-(3.15) that

$$\begin{aligned}
I_1 =& \sum_{l=1}^{N} k_l \left(p_h^{l-1}(u) - p_h^{l-1}, u_h^l - u^l\right) \\
\leq& \sum_{l=1}^{N} k_l \left(p_h^{l-1}(u) - p^{l-1}, u_h^l - u^l\right) + \sum_{l=1}^{N} k_l \left(p^{l-1} - p_h^{l-1}, u_h^l - u^l\right) \\
\leq& C(\delta_1) \sum_{l=1}^{N} k_l \left\|p_h^{l-1}(u) - p^{l-1}\right\|^2 + C(\delta_1) \sum_{l=1}^{N} k_l \left\|p_h^{l-1} - p^{l-1}\right\|^2 + \delta_1 \sum_{l=1}^{N} k_l \left\|u^l - u_h^l\right\|^2. \\
\leq& C(\delta_1)|||p_h(u) - p|||_X^2 + C(\delta_1)|||p_h - p|||_X^2 \\
&+ C(\delta_1)|||u - u_h|||_X^2
\end{aligned} \tag{3.18}$$

For the second term, note that $p \in H^1(J; U)$, according to Hölder inequality, we have

$$\begin{aligned}
I_2 =& \sum_{l=1}^{N} k_l \left(p_h^{l-1}(u) - p^l, u^l - u_h^l\right) ber \\
=& \sum_{l=1}^{N} k_l \left(p_h^{l-1}(u) - p^{l-1}, u^l - u_h^l\right) + \sum_{l=1}^{N} k_l \left(p^{l-1} - p^l, u^l - u_h^l\right) \\
\leq& C(\delta) \sum_{l=1}^{N} k_l \left\|p_h^{l-1}(u) - p^{l-1}\right\|^2 + C(\delta) \sum_{l=1}^{N} k_l \left\|p^{l-1} - p^l\right\|^2 + \delta \sum_{l=1}^{N} k_l \left\|u^l - U_h^l\right\|^2 \\
\leq& C(\delta)|||p_h(u) - p|||_X^2 + C(\delta)k^2\|p_t\|_X + \delta|||u - u_h|||_X^2 \\
\leq& C(\delta)|||p_h(u) - p|||_X^2 + Ck^2 + \delta|||u - u_h|||_X^2.
\end{aligned} \tag{3.19}$$

Let δ be small enough, then (3.16) follows from (3.17)-(3.19). □

Lemma 3.4. *Let (y, p, u) be the solution of (2.7)-(2.9), $(y_h(u), p_h(u))$ is defined in (3.14)-(3.15), the conditions of obedience are the same as the three Lemmas 3.1-3.3, then*

$$|||y_h(u) - y|||_{L^\infty(J;L^2(\Omega))} \leq C\left(h^2 + k\right). \tag{3.20}$$

Proof. We set

$$y_h^l(u) - y^l = y_h^l(u) - R_h y^l + R_h y^l - y^l := \theta_1^l + \eta^l, \tag{3.21}$$

where

$$\begin{aligned}
\left\|\eta^l\right\| =& \left\|Ry^l - y^l\right\| \leq Ch^2 \left\|y^l\right\|_2 \\
\leq& Ch^2 \left(\|y^0\|_2 + \int_0^{t_l} \|y_t(s)\|_2 ds\right) \\
\leq& Ch^2.
\end{aligned} \tag{3.22}$$

From (2.7) and (2.19), for $\forall w_h \in V^h$ we have

$$\begin{aligned}
&\left(\frac{\theta_1^l - \theta_1^{l-1}}{k_l}, w_h\right) + a\left(\theta^l, w_h\right) \\
=& -\left(\frac{R_h y^l - R_h y^{l-1}}{k_l}, w_h\right) - a\left(R_h y^l, w_h\right) - (\phi(y_h^l), w_h) + \left(f^l + u^l, w_h\right) \\
=& -\left(\frac{R_h y^l - R_h y^{l-1}}{k_l}, w_h\right) - a\left(y^l, w_h\right) + \left(f^l + u^l, w_h\right) - (\phi(y_h^l), w_h) \\
=& -\left(\frac{R_h y^l - R_h y^{l-1}}{k_l}, w_h\right) + \left(y_t^l, w_h\right) + (\phi(y^l), w_h) - (\phi(y_h^l), w_h) \\
=& -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, w_h\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, w_h\right) - (\phi(y_h^l) - \phi(y^l), w_h) \\
=& -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, w_h\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, w_h\right) - \left(\phi'(y_h^l)(y_h^l - y^l), w_h\right) \\
=& -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, w_h\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, w_h\right) - \left(\phi'(y_h^l)(y_h^l - R_h y^l), w_h\right) \\
& - \left(\phi'(y_h^l)(R_h y^l - y^l), w_h\right) \\
\le& -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, w_h\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, w_h\right) - \left(\phi'(y_h^l)(y_h^l - R_h y^l), w_h\right) \\
& - \left(\phi'(y_h^l)(R_h y^l - y^l), w_h\right).
\end{aligned} \tag{3.23}$$

We set $w_h = \theta_1^l$, so

$$\begin{aligned}
&\left(\frac{\theta_1^l - \theta_1^{l-1}}{k_l}, \theta_1^l\right) + a\left(\theta_1^l, \theta_1^l\right) \\
\le& -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, \theta_1^l\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, \theta_1^l\right) \\
& - \left(\phi'(y_h^l)\theta_1^l, \theta_1^l\right) - \left(\phi'(y_h^l)\eta^l, \theta_1^l\right) + \left(u_h^l - u^l, \theta_1^l\right) \\
\le& -\left(\frac{(R_h - I)\left(y^l - y^{l-1}\right)}{k_l}, \theta_1^l\right) - \left(\frac{y^l - y^{l-1}}{k_l} - y_t^l, \theta_1^l\right) - \left(\phi'(y_h^l)\eta^l, \theta_1^l\right).
\end{aligned} \tag{3.24}$$

Note that $0 \le c\left\|\theta_1^l\right\|_V^2 \le a\left(\theta_1^l, \theta_1^l\right)$, according to Hölder inequality, we obtain

$$\left\|\theta_1^l\right\| \le \left\|\theta^{l-1}\right\| + \left\|(R_h - I)\left(y^l - y^{l-1}\right)\right\| + \left\|y^l - y^{l-1} - k_l y_t^l\right\| + k_l \left\|\phi'(y_h^l)\eta^l\right\|. \tag{3.25}$$

Summing i from 1 to N^* $(1 \le N^* \le N)$, noting $y \in H^1(J; H^2(\Omega)) \cap H^2(J; U)$ and $y(x, 0) \in H^2(\Omega)$, we get

$$
\begin{aligned}
&\left\|\theta^{N^*}\right\| \\
\le&\left\|\theta^0\right\| + \sum_{l=1}^{N^*}\left\|(R_h - I)\left(y^l - y^{l-1}\right)\right\| + \sum_{l=1}^{N^*}\left\|y^l - y^{l-1} - k_l y_t^l\right\| + \sum_{l=1}^{N^*} k_l \left\|\phi'(y_h^l)\eta^l\right\| \\
\le&\left\|\theta^0\right\| + \sum_{l=1}^{N^*}\int_{t_{l-1}}^{t_l}(R_h - I)y_t(s)ds + \sum_{l=1}^{N^*}\left\|\int_{t_{l-1}}^{t_l}(t_{l-1} - s)y_{tt}(s)ds\right\| \sum_{l=1}^{N^*} k_l \left\|\phi'(y_h^l)\right\|_{W^{1,\infty}}\left\|\eta^l\right\| \\
\le&\left\|\theta^0\right\| + \sum_{l=1}^{N^*}\int_{t_{l-1}}^{t_l} Ch^2\|y_t(s)\|_2 ds + \sum_{l=1}^{N^*}\int_{t_{l-1}}^{t_l}\|(t_{l-1} - s)y_{tt}(s)\|ds \\
&+ Ch^2\sum_{l=1}^{N^*} k_l \left\|\phi'(y_h^l)\right\|_{W^{2,\infty}} \\
\le&\left\|\theta^0\right\| + Ch^2\int_0^{t_{N^*}}\|y_t(s)\|_2 ds + k\int_0^{t_{N^*}}\|y_{tt}(s)\|ds + Ch^2 \\
\le& Ch^2\|y^0\|_2 + Ch^2\int_0^T\|y_t(s)\|_2 ds + k\int_0^T\|y_{tt}(s)\|ds + Ch^2 \\
\le& C(h^2 + k).
\end{aligned} \tag{3.26}
$$

Then (3.20) follows from (3.21)-(3.22), (3.26) and Triangle inequality. □

Lemma 3.5. *Let (y, p, u) be the solution of (2.7)-(2.9), $(y_h(u), y_h(u))$ is defined in (3.14)-(3.15). Then*

$$
|||p_h(u) - p|||_X \le C\left(h^2 + k\right). \tag{3.27}
$$

Proof. By using Lemma 3.2 and Lemma 3.4, we get that

$$
|||y_h(u) - y|||_X \le C\left(h^2 + k\right), \tag{3.28}
$$

and

$$
|||p_h(u) - p|||_{L^\infty(J;U)} \le C\left(h^2 + k + |||y_h(u) - y|||_X^2\right). \tag{3.29}
$$

Then from (3.28)-(3.29) and embedding theorem, we have that

$$
|||p_h(u) - p|||_{L^\infty(J;U)} \le C\left(h^2 + k\right). \tag{3.30}
$$

Thus,

$$
|||p_h(u) - p|||_X \le C|||p_h(u) - p|||_{L^\infty(J;U)} \le C\left(h^2 + k\right). \tag{3.31}
$$

We obtain (3.27). □

Now we combine Lemmas 3.1-3.5 to come up with the following main result.

Theorem 3.1. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (2.7)-(2.9) and (2.19)-(2.21), respectively. Suppose all the conditions in Lemma 3.1-3.5 are valid. Then*

$$
|||p_h - p|||_{L^\infty(J;U)} + |||u_h - u|||_X \le C\left(h^2 + k + |||y_h - y|||_X\right). \tag{3.32}
$$

Proof. It is easy to see that (3.32) follows from (3.3), (3.8), (3.16) and (3.27). □

4. A posteriori error estimates for OCP

In this section, we shall derive a posteriori error estimates for the backward Euler variational discretization approximation scheme.

First of all, we recall two important results:

Lemma 4.1. *[6] Let π_h be the standard Lagrange interpolation operator. For $m = 0$ or 1, $q > \frac{n}{2}$ and $\forall v \in W^{2,q}(\Omega)$, the conclusion is as follows*

$$|v - \pi_h v|_{W^{m,q}(\Omega)} \leq Ch^{2-m}|v|_{W^{2,q}(\Omega)}.$$

Lemma 4.2. *[26] For all $v \in W^{1,q}(\Omega^h)$, $1 \leq q < \infty$,*

$$\|v\|_{W^{0,q}(\partial\tau)} \leq C\left(h_\tau^{-\frac{1}{q}}\|v\|_{W^{0,q}(\tau)} + h_\tau^{1-\frac{1}{q}}|v|_{W^{1,q}(\tau)}\right).$$

If we set

$$J(u) = \frac{1}{2}\int_0^T (\|y - y_d\|^2 + \|u\|^2)dt.$$

It can be shown that

$$(J'(u), v) = \int_0^T (u + p, v)dt,$$
$$(J'(u_h), v) = \int_0^T (u_h + p(u_h), v)dt,$$

where $y(u_h)$, $p(u_h)$ satisfies the system as follows:

$$(y_t(u_h), w) + a(y(u_h), w) + (\phi(y(u_h)), w) = (f + u_h, w), \qquad t \in J,\ \forall w \in V, \tag{4.1}$$
$$y(u_h)(x, 0) = y_0(x), \qquad x \in \Omega,$$
$$-(p_t(u_h), q) + a(q, p(u_h)) + (\phi'(y(u_h))\, p(u_h), q) = (y(u_h) - y_d, q), \qquad t \in J,\ \forall q \in V, \tag{4.2}$$
$$p(u_h)(x, T) = 0, \qquad x \in \Omega.$$

For the needs of the paper, now we give two Lemmas and prove them.

Lemma 4.3. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (2.7)-(2.9) and (2.19)-(2.21), respectively. Then there is a constant C independent of k and h, it reduces that*

$$\|u - u_h\|_X^2 \leq C\,\|\tilde{p}_h - p(u_h)\|_X^2. \tag{4.3}$$

Proof. It follows from δ-Cauchy inequality, the variational inequality in (2.9) and (2.21), we have that

$$\begin{aligned}
\|u - u_h\|_X^2 &\leq (J'(u) - J'(u_h), u - u_h) \\
&= \int_0^T (u + p, u - u_h)\,dt - \int_0^T (u_h + p(u_h), u - u_h)\,dt \\
&\leq \int_0^T \left[-(u_h + \tilde{p}_h, u - u_h) + (\tilde{p}_h - p(u_h), u - u_h)\right]dt \\
&\leq C(\delta)\,\|\tilde{p}_h - p(u_h)\|_X^2 + \delta\|u - u_h\|.
\end{aligned} \tag{4.4}$$

Let δ be small enough, then (4.3) follows from (4.4). □

For any function $w \in C(J; L^2(\Omega))$, we let

$$\hat{w}(x,t)|_{(t_{l-1},t_l]} = w(x,t_l),\ \tilde{w}(x,t)|_{(t_{l-1},t_l]} = w(x,t_{l-1}).$$

For $l = 1, 2, \cdots, N$, we set

$$\begin{aligned} y_h|_{(t_{l-1},t_l]} &= \left((t_l - t)y_h^{l-1} + (t - t_{l-1})y_h^l\right)/k_l, \\ p_h|_{(t_{l-1},t_l]} &= \left((t_l - t)p_h^{l-1} + (t - t_{l-1})p_h^l\right)/k_l, \\ u_h|_{(t_{l-1},t_l]} &= u_h^l. \end{aligned}$$

Then the optimality conditions (2.19)-(2.21) can be rewritten as:

$$(y_{ht}, w_h) + a(\hat{y}_h, w_h) + (\phi(\hat{y}_h), w_h) = \left(\hat{f} + u_h, w_h\right), \qquad \forall w_h \in V^h,\ \ t \in (t_{l-1}, t_l], \tag{4.5}$$

$$y_h(x,0) = y_0^h(x), \quad x \in \Omega,$$

$$-(p_{ht}, q_h) + a(q_h, \tilde{p}_h) + (\phi'(\hat{y}_h)\tilde{p}_h, q_h) = (\hat{y}_h - \hat{y}_d, q_h), \qquad \forall q_h \in V^h,\ \ t \in (t_{l-1}, t_l], \tag{4.6}$$

$$p_h(x,T) = 0, \qquad x \in \Omega,$$

$$(u_h + \tilde{p}_h, \tilde{u}_h - u_h) \geq 0, \qquad \forall \tilde{u}_h \in D_h. \tag{4.7}$$

Lemma 4.4. *Let (y_h, p_h, u_h) be the solution of (4.5)-(4.7), and $(y(u_h), p(u_h))$ as defined in (4.1)-(4.2), so there is a constant C independent of k and h, it holds that*

$$\|y(u_h) - y_h\|_X^2 + \|p(u_h) - p_h\|_X^2 \leq C\sum_{l=1}^{8}\eta_l^2, \tag{4.8}$$

where

$$\eta_1^2 = \sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\sum_{\lambda}h_\lambda^2\int_\lambda\left(\hat{y}_h - \hat{y}_d + div\left(A^*\nabla\tilde{p}_h\right) + p_{ht}\right)^2 dxdt,$$

$$\eta_2^2 = \sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\sum_{l\cap\partial\Omega=\emptyset}h_l\int_l\left[A^*\nabla\tilde{p}_h\cdot n\right]^2 dsdt,$$

$$\eta_3^2 = \sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\int_\Omega\left|A^*\nabla\left(\tilde{p}_h - p_h\right)\right|^2 dxdt,$$

$$\eta_4^2 = \sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\sum_{\lambda}h_\lambda^2\int_\lambda\left(\hat{f} + u_h + div\left(A\nabla\hat{y}_h\right) - y_{ht}\right)^2 dxdt,$$

$$\eta_5^2 = \sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\sum_{s\cap\partial\Omega=\emptyset}h_s\int_s\left[A\nabla\hat{y}_h\cdot n\right]^2 drdt,$$

$$\eta_6^2 = \sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\left\|f - \hat{f}\right\|^2 dt,$$

$$\eta_7^2 = \sum_{l=1}^{N}\int_{t_{l-1}}^{t_l}\int_\Omega\left|A\nabla\left(\hat{y}_h - y_h\right)\right|^2 dxdt,$$

$$\eta_8^2 = \|y_0(x) - y_h(x,0)\|^2,$$

where h_s is the size of the face $s = \bar{\lambda}_s^1 \cap \bar{\lambda}_s^2$, with λ_s^1, λ_s^2 are two neighboring elements in Γ^h, $[A\nabla\hat{y}_h \cdot n]_s$ and $[A^*\nabla\tilde{p}_h \cdot n]_s$ are the $A-$ normal and A^*- normal derivative jumps over the interior face s, respectively, defined by

$$[A\nabla\hat{y}_h \cdot n]_s = \left(A\nabla\hat{y}_h|_{\lambda_s^1} - A\nabla\hat{y}_h|_{\lambda_s^2}\right)\cdot n,$$
$$[A^*\nabla\tilde{p}_h \cdot n]_s = \left(A^*\nabla\tilde{p}_h|_{\lambda_s^1} - A^*\nabla\tilde{p}_h|_{\lambda_s^2}\right)\cdot n,$$

where n is the normal vector on $s = \lambda_s^1 \cap \lambda_s^2$ outwards λ_s^1. Then we define $[A\nabla\hat{y}_h \cdot n]_s = 0$ and $[A^*\nabla\tilde{p}_h \cdot n]_s = 0$ when $s \subset \partial\Omega$.

Proof. Let $e^p = p(u_h) - u_h$, and $e_I^p = \hat{\pi}_h e^p$, where $\hat{\pi}_h$ is the interpolation defined in Lemma 4.1. Note that $p(u_h)(x,T) = 0$ and $p_h(x,T) = 0$, using integration by parts, we have

$$\begin{aligned}\int_0^T -(p_t(u_h) - p_{ht}, e^p)dt =& -\int_\Omega\int_0^T (p_t(u_h) - p_{ht})\cdot e^p dtdx\\ =&\frac{1}{2}\int_\Omega [(p(u_h) - p_h)(x,0)]^2 dx\\ \geq& 0.\end{aligned}$$

It follows from $a(u,u) \geq c\|u\|_V^2$, $\forall v \in V$, combining equations (4.2) and (4.6), we have

$$\begin{aligned}
& c\|p(u_h) - p_h\|^2_{L^2(J;H^1(\Omega))}\\
\leq& \int_0^T \left[(A\nabla e^p, \nabla(p(u_h) - p_h)) - (p_t(u_h) - p_{ht}, e^p) + (\phi'(y(u_h))(p(u_h) - p_h), e^p)\right]dt\\
=& \int_0^T \left(\nabla(e^p - e_I^p), A^*\nabla(p(u_h) - \tilde{p}_h)\right)dt - \int_0^T \left(p_t(u_h) - p_{ht}, e^p - e_I^p\right)dt\\
&+ \int_0^T (\nabla e^p, A^*\nabla(\tilde{p}_h - p_h))dt + \int_0^T \left[\left(\nabla e_I^p, A^*\nabla(p(u_h) - \tilde{p}_h)\right) - \left(p_t(u_h) - p_{ht}, e_I^p\right)\right]dt\\
&+ \int_0^T \left[\left(\phi'(y(u_h))p(u_h) - \phi'(\hat{y}_h)\tilde{p}_h, e_I^p\right) + \left(\phi'(y(u_h))p(u_h) - \phi'(\hat{y}_h)\tilde{p}_h, e^p - e_I^p\right)\right]dt\\
&+ \int_0^T \left((\phi'(\hat{y}_h)\tilde{p}_h - \phi'(y(u_h))p_h), e^p\right)dt\\
=& \int_0^T \left(y(u_h) - \hat{y}_d + \operatorname{div}(A^*\nabla\tilde{p}_h) + p_{ht}, e^p - e_I^p\right)dt + \int_0^T \sum_\lambda \int_{\partial\lambda} (A^*\nabla\tilde{p}_h \cdot n)\left(e^p - e_I^p\right)dsdt\\
&+ \int_0^T \left[\left(y(u_h) - \hat{y}_h, e_I^p\right) + (\nabla e^p, A^*\nabla(\tilde{p}_h - p_h))\right]dt\\
&+ \int_0^T \left[\left(\phi'(\hat{y}_h)\tilde{p}_h, e_I^p\right) - (\phi'(y(u_h))p_h, e^p)\right]dt\\
=& \int_0^T \left(\hat{y}_h - \hat{y}_d + \operatorname{div}(A^*\nabla\tilde{p}_h) + p_{ht}, e^p - e_I^p\right)dt + \int_0^T \sum_\lambda \int_{\partial\lambda} (A^*\nabla\tilde{p}_h \cdot n)\left(e^p - e_I^p\right)dsdt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \left(y(U_h)-\hat{Y}_h, e^p\right) dt+\int_0^T \left(\nabla e^p, A^*\nabla\left(\tilde{p}_h-p_h\right)\right) dt\\
&+\int_0^T \left[\left(\phi'\left(\hat{y}_h\right)\tilde{p}_h, e_I^p\right)-\left(\phi'\left(y\left(u_h\right)\right)p_h, e^p\right)\right] dt\\
&:=I_1+I_2+I_3+I_4+I_5. \qquad (4.9)
\end{aligned}
$$

For the first term, by using δ-Cauchy inequality, Lemma 4.1 and Lemma 4.2, we get

$$
\begin{aligned}
I_1 &=\int_0^T \left(\hat{y}_h-\hat{y}_d+\operatorname{div}\left(A^*\nabla\tilde{p}_h\right)+p_{ht}, e^p-e_I^p\right) dt\\
&=\int_0^T \int_\Omega \left(\hat{y}_h-\hat{y}_d+\operatorname{div}\left(A^*\nabla\tilde{p}_h\right)+p_{ht}\right)\left(e^p-e_I^p\right) dxdt\\
&=\int_0^T \sum_\lambda \int_\lambda \left(\hat{y}_h-\hat{y}_d+\operatorname{div}\left(A^*\nabla\tilde{p}_h\right)+p_{ht}\right)\left(e^p-e_I^p\right) dxdt\\
&\le C(\delta)\int_0^T \sum_\lambda h_\lambda^2\int_\lambda \left(\hat{y}_h-\hat{y}_d+\operatorname{div}\left(A^*\nabla\tilde{p}_h\right)+p_{ht}\right)^2 dxdt+\delta\int_0^T h_\lambda^{-2}\sum_\lambda\int_\lambda \left|e^p-e_I^p\right|^2 dxdt\\
&\le C(\delta)\int_0^T \sum_\lambda h_\lambda^2\int_\lambda \left(\hat{y}_h-\hat{y}_d+\operatorname{div}\left(A^*\nabla\tilde{p}_h\right)+p_{ht}\right)^2 dxdt+\delta\int_0^T \|e^p\|_{H^1(\Omega)}^2 dt\\
&\le C(\delta)\eta_1^2+\delta\|p(u_h)-p_h\|_{L^2(J;H^1(\Omega))}^2. \qquad (4.10)
\end{aligned}
$$

From δ-Cauchy inequality, Lemma 4.1 and Lemma 4.2, we obtain

$$
\begin{aligned}
I_2 &=\int_0^T \sum_\lambda \int_{\partial\lambda} \left(A^*\nabla\tilde{p}_h\cdot n\right)\left(e^p-e_I^p\right) dsdt\\
&\le C(\delta)\int_0^T \sum_{s\cap\partial\Omega=\emptyset} h_s\int_s \left[A^*\nabla\tilde{p}_h\cdot n\right]^2 drdt+\delta\int_0^T \|e^p\|_{H^1(\Omega)}^2 dt\\
&=C(\delta)\eta_2^2+\delta\|p(u_h)-p_h\|_{L^2(J;H^1(\Omega))}^2. \qquad (4.11)
\end{aligned}
$$

Using δ-Cauchy inequality, we get

$$
\begin{aligned}
I_3 &=\int_0^T \left(y(u_h)-\hat{y}_h, e^p\right) dt\\
&=\int_0^T \left(y(u_h)-Y_h+y_h-\hat{y}_h, e^p\right) dt\\
&\le C(\delta)\,\|\hat{y}_h-y_h\|_{L^2(J;L^2(\Omega))}^2+C(\delta)\,\|y(u_h)-y_h\|_X^2+\delta\|e^p\|_{L^2(J;H^1(\Omega))}^2\\
&\le C(\delta)\,\|\hat{y}_h-y_h\|_X^2+C(\delta)\|y(u_h)-y_h\|_X^2+\delta\|e^p\|_X^2. \qquad (4.12)
\end{aligned}
$$

It yields from Friedriechs inequality and δ-Cauchy inequality, so we have

$$
I_4=\int_0^T \left(\nabla e^p, A^*\left(\nabla\tilde{p}_h-p_h\right)\right) dt
$$

$$\le C(\delta)\int_0^T\int_\Omega |A^*\nabla(\tilde p_h - p_h)|^2\,dxdt + \delta\int_0^T\int_\Omega |\nabla e^p|^2\,dxdt$$
$$\le C(\delta)\eta_3^2 + C\delta\|p(u_h)-p_h\|^2_{L^2(J;H^1(\Omega))}. \tag{4.13}$$

$$\begin{aligned}
I_5 =& \int_0^T \Big[\Big(\phi'(\hat y_h)\,\tilde p_h, e_I^p\Big) - (\phi'(y(u_h))\,p_h, e^p)\Big]dt\\
=& \int_0^T \Big[(\phi'(\hat y_h)\,\tilde p_h - \phi'(y(u_h))\,\tilde p_h, e^p) + \Big(\phi'(\hat y_h)\,\tilde p_h, e^p - e_I^p\Big) \qquad (4.14)\\
&+ (\phi'(y(u_h))\,\tilde p_h - \phi'(y(u_h))\,p_h, e^p)\Big]dt\\
=& \int_0^T \Big[((\phi'(\hat y_h) - \phi'(y(u_h)))\,\tilde p_h, e^p) + \Big(\phi'(\hat y_h)\,\tilde p_h, e^p - e_I^p\Big)\\
&+ (\phi'(y(u_h))(\tilde p_h - p_h), e^p)\Big]dt\\
\le& C(\delta)\int_0^T \|\tilde p_h\|_{0,4}\,\|\phi'(\hat y_h) - \phi'(y(u_h))\|\cdot\|e^p\|_{0,4}\,dt\\
&+ C(\delta)\int_0^T \Big[\|\tilde p_h\|_{0,4}\,\|\phi'(\hat y_h)\|\cdot\Big\|e^p - e_I^p\Big\|_{0,4} + \|\tilde p_h - p_h\|_{0,4}\,\|\phi'(y(u_h))\|\cdot\|e^p\|_{0,4}\Big]dt\\
\le& C(\delta)\int_0^T \|\tilde p_h\|_1\,\|\phi\|_{2,\infty}\,\|\hat y_h - (y(u_h))\|\cdot\|e^p\|_1\,dt\\
&+ C(\delta)\int_0^T \Big[\|\tilde p_h\|_1\,\|\phi(\hat y_h)\|_{1,\infty}\cdot\Big\|e^p - e_I^p\Big\|_1 + \|\tilde p_h - p_h\|_1\,\|\phi\|_{1,\infty}\cdot\|e^p\|_1\Big]dt\\
\le& C(\delta)\,\|\hat y_h - (y(u_h))\|_X^2\\
&+ C(\delta)\int_0^T \Big[\|\tilde p_h\|_1\,\|\phi(\hat y_h)\|_{1,\infty}\cdot\Big\|e^p - e_I^p\Big\|_1 + \|\tilde p_h - p_h\|_1\,\|\phi(y(u_h))\|_{1,\infty}\cdot\|e^p\|_1\Big]dt. \qquad (4.15)
\end{aligned}$$

Let δ be small enough, we obtain

$$\|p(u_h)-p_h\|_W^2 \le C\sum_{l=1}^{3}\eta_l^2 + C(\delta)\,\|\hat y_h - y_h\|_X^2 + C(\delta)\|y(u_h)-y_h\|_X^2.$$

Similarly, assume $e^y = y(u_h) - y_h$, and its average interpolation is e_I^y. It holds from integration by parts that

$$\begin{aligned}
\int_0^T (y_t(u_h) - y_{ht}, e^y)dt =& \int_\Omega\int_0^T e^y\cdot(y_t(u_h) - y_{ht})dtdx\\
=&\frac{1}{2}\int_\Omega [(y(u_h)-y_h)(x,T)]^2\,dx - \frac{1}{2}\|y_0(x) - y_h(x,0)\|^2.
\end{aligned}$$

It follows from δ-Cauchy inequality, Lemmas 4.1, Lemma 4.2, (4.1) and (4.5), we obtain

$$\begin{aligned}
&c\|y(u_h)-y_h\|^2_{L^2(J;H^1(\Omega))}\\
\le& \int_0^T [(A\nabla(y(u_h)-y_h), \nabla e^y) + (y_t(u_h) - y_{ht}, e^y)]\,dt + \frac{1}{2}\|y_0(x) - y_h(x,0)\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \left[\left(A\nabla\left(y(u_h) - \hat{y}_h\right), \nabla\left(e^y - e_I^y\right)\right) + \left(y_t(u_h) - y_{ht}, e^y - e_I^y\right)\right] dt \\
&\quad + \int_0^T \left[\left(A\nabla\left(\hat{y}_h - y_h\right), \nabla e^y\right) + \left(y_t(u_h) - y_{ht}, e_I^y\right)\right] dt \\
&\quad + \int_0^T \left(A\nabla\left(y(u_h) - \hat{y}_h\right), \nabla e_I^y\right) dt + \frac{1}{2}\|y_0(x) - y_h(x,0)\|^2 \\
&= \int_0^T \left(\hat{f} + u_h + \operatorname{div}\left(A\nabla\hat{y}_h\right) - y_{ht}, e^y - e_I^y\right) dt \\
&\quad + \int_0^T \sum_\lambda \int_{\partial\lambda} \left(A\nabla\hat{y}_h \cdot n\right)\left(e^y - e_I^y\right) ds dt \\
&\quad + \int_0^T \left[\left(f - \hat{f}, e^y\right) + \left(A\nabla\left(\hat{y}_h - y_h\right), \nabla e^y\right)\right] dt + \frac{1}{2}\|y_0(x) - y_h(x,0)\|^2 \\
&\le C(\delta) \int_0^T \sum_\lambda h_\lambda^2 \int_\lambda \left(\hat{f} + u_h + \operatorname{div}\left(A\nabla\hat{y}_h\right) - y_{ht}\right)^2 dx dt \\
&\quad + C(\delta) \int_0^T \sum_{s\cap\partial\Omega=\emptyset} h_s \int_s \left[A\nabla\hat{y}_h \cdot n\right]^2 dr dt + C(\delta)\left\|f - \hat{f}\right\|_X^2 \\
&\quad + C(\delta) \int_0^T \int_\Omega \left|A\nabla(\hat{y}_h - y_h)\right|^2 dx dt + \frac{1}{2}\|y_0(x) - y_h(x,0)\|^2 \\
&\quad + C(\delta)\|y_h - \hat{y}_h\|^2_{L^2(J;H^1(\Omega))} + \delta\|e^y\|^2_{L^2(J;H^1(\Omega))} \\
&= C(\delta) \sum_{l=4}^{8} \eta_l^2 + C(\delta)\|y_h - \hat{y}_h\|_X^2 + \delta\|e^y\|^2_{L^2(J;H^1(\Omega))}.
\end{aligned}
$$

Let δ be small enough, we have

$$
\|y(u_h) - y_h\|^2_{L^2(J;H^1(\Omega))} \le C(\delta) \sum_{l=4}^{8} \eta_l^2 + C(\delta)\|y_h - \hat{y}_h\|_X^2 . \tag{4.16}
$$

It follows from the assumptions on $A(x)$ and Friedriechs inequality, we obtain

$$
\|\hat{y}_h - y_h\|_X^2 \le \|\hat{y}_h - y_h\|_W^2 \le C \int_0^T \int_\Omega \left|A\nabla(\hat{y}_h - y_h)\right|^2 dx dt.
$$

From (4.9)-(4.16), (4.8) is derived. □

Theorem 4.1. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of (2.7)-(2.9) and (2.19)-(2.21), respectively. Assume that all the conditions in Lemmas 4.1-4.4 are valid. Then there exists a constant C independent h and k , it yields the result as follows*

$$
\|y_h - y\|^2_{L^2(J;H^1(\Omega))} + \|p_h - p\|^2_{L^2(J;H^1(\Omega))} + \|u_h - u\|_X^2 \le C \sum_{i=1}^{8} \eta_l^2, \tag{4.17}
$$

where $\eta_1,\ \eta_2\ \cdots,\ \eta_8$ are defined in Lemma 4.4.

Proof. It follows from Friedriechs inequality and the conditions on $A(x)$, we have

$$\|\tilde{p}_h - p_h\|^2_{L^2(J;H^1(\Omega))} \leq C \int_0^T \int_\Omega |A^*\nabla(\tilde{p}_h - p_h)|^2 \, dxdt. \tag{4.18}$$

Note that

$$\|y_h - y\|^2_{L^2(J;H^1(\Omega))} \leq \|y_h - y(u_h)\|^2_{L^2(J;H^1(\Omega))} + \|y(u_h) - y\|^2_{L^2(J;H^1(\Omega))}, \tag{4.19}$$

$$\|p_h - p\|^2_{L^2(J;H^1(\Omega))} \leq \|p_h - p(u_h)\|^2_{L^2(J;H^1(\Omega))} + \|p(u_h) - p\|^2_{L^2(J;H^1(\Omega))}. \tag{4.20}$$

From the regularity estimation of (2.7)-(2.8) minus (4.1)-(4.2), we have

$$\|p(u_h) - p\|^2_{L^2(J;H^1(\Omega))} \leq \|y(u_h) - y\|^2_{L^2(J;H^1(\Omega))} \leq C\|u_h - u\|^2_X. \tag{4.21}$$

Then (4.17) follows from (4.3), (4.8) and (4.18)-(4.21). □

5. Conclusion and future works

In this paper we discuss the variational discretization for the nonlinear parabolic OCP. We derive a priori error estimates where $|||u - u_h|||_{L^\infty(J;L^2(\Omega))} = O(h^2 + k)$ and a posteriori error estimates of residual type. The results for these error estimates by variational discretization be an extension of the linear parabolic problems.

In our future work, we shall use this method to deal with fourth order parabolic optimal control problems, including linear and nonlinear styles.

Acknowledgments

This work is supported by National Science Foundation of China (11201510), China Postdoctoral Science Foundation (2017T100155, 2015M580197), Youth Innovative Talents Project (Natural Science) of Research on Humanities and Social Sciences in Guangdong Normal University (2017KQNCX265), General Scientific Research Project of "Innovation and Strengthening School Engineering" of Guangdong Education Department (2016GXJK227), Innovation Team Building at Institutions of Higher Education in Chongqing (CXTDX201601035), and Chongqing Research Program of Basic Research and Frontier Technology (cstc2019jcyj-msxmX0280), and School projects of Huashang College Guangdong University of Finance and Economics(2020HSDS02).

Conflict of interest

The authors declare no conflict of interest in this paper.

References

1. P. Neittaanmaki, D. Tiba, *Optimal Control of Nonlinear Parabolic Systems: Theory, Algorithms and Applications*, Dekker, New Nork, 1994.

2. D. Tiba, *Lectures on The Optimal Control of Elliptic Equations*, University of Jyvaskyla Press, Finland, 1995.
3. W. Liu, N. Yan, *Adaptive Finite Element Methods For Optimal Control Governed by PDEs*, Science Press, Beijing, 2008.
4. Y. Chen, Z. Lu, W. Liu, *Numerical solution of partial differential equation*, Science Press, Beijing, 2015.
5. T. Hou, C. Liu, Y. Yang, Error estimates and superconvergence of a mixed finite element method for elliptic optimal control problems, *Comput. Math. Appl.*, **74** (2017), 714–726.
6. P. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amstterdam, 1978.
7. Y. Chen, N. Yi, W. Liu, A legendre galerkin spectral method for optimal control problems governed by elliptic equations, *SIAM J. Numer. Anal.*, **46** (2008), 2254–2275.
8. Y. Chen, Y. Dai, Superconvergence for optimal control problems gonverned by semi-linear elliptic equations, *J. Sci. Comput.*, **39** (2009), 206–221.
9. X. Xing, Y. Chen, L^{∞}-error estimates for general optimal control problem by mixed finite element methods, *Int. J. Numer. Anal. Model.*, **5** (2008), 441–456.
10. R. Li, W. Liu, H. Ma, T. Tang, Adaptive finite element approximation for distributed elliptic optimal control problems, *SIAM J. Control Optim.*, **41** (2002), 1321–1349.
11. P. Philip, Optimal control of partial differential equations, *SIAM J. Control Optim.*, **50** (2012), 943–963.
12. W. Liu, N. Yan, A posteriori error estimates for optimal control problems governed by parabolic equations, *Numer. Math.*, **93** (2003), 497–521.
13. B. I. Ananyev, *A control problem for parabolic systems with incomplete information*, Mathematical Optimization Theory and Operations Research, Springer, Cham, 2019.
14. H. Liu, N. Yan, Recovery type superconvergence and a posteriori error estimates for control problems governed by Stokes equations, *J. Comput. Appl. Math.*, **209** (2007), 187–207.
15. W. Liu, N. Yan, A posteriori error estimates for control problems governed by Stokes equations, *SIAM J. Numer. Anal.*, **40** (2002), 1850–1869.
16. J. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, 1971.
17. J. Lions, E. Magenes, *Non Homogeneous Boundary Value Problems and Applications*, Springer-verlag, Berlin, 1972.
18. H. Fu, H. Rui, A priori error estimates for optimal problems governed by transient advection-diffusion equations, *J. Sci. Comput.*, **38** (2009), 290–315.
19. W. Liu, N. Yan, A posteriori error estimates for distributed convex optimal control problems, *Adv. Comput. Math.*, **15** (2001), 285–309.
20. R. Li, W. B. Liu, N. N. Yan, A posteriori error estimates of recovery type for distributed convex optimal control problems, *J. Sci. Comput.*, **33** (2007), 155–182.
21. N. Yan, postriori error estimators of gradient recovery type for FEM of a model optimal control problem, *Adv. Comp. Math.*, **19** (2003), 323–336.

22. Y. Tang, Y. Chen, Variational discretization for parabolic optimal control problems with control constraints, *J. Systems Sci. Compl.*, **25** (2012), 880–895.

23. M. Hinze, A variational discretization concept in cotrol constrained optimization: the linear-quadratic case, *Comput. Optim. Appl.*, **30** (2005), 45–63.

24. M. Hinze, N. Yan, Z. Zhou, Variational discretization for optimal control governed by convection dominated diffusion equations, *J. Comput. Math.*, **27** (2009), 237–253.

25. M. Huang, *Numerical Methods for Evelution Equations*, Secience Press, Beijing, 2004.

26. A. Kufner, O. John, S. Fuck, *Function Spaces*, Nordhoff, Leiden, The Netherlands, 1997.

AIMS Press