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Research article

Two-exponential estimators for estimating population mean

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Abstract: We introduce two-exponential shrinkage estimator using two stage two phase sampling for estimating population mean of study variable. Some properties of the proposed two-exponential shrinkage estimator are presented. The mathematical comparison in terms of the mean square error is done in order to demonstrate some conditions for which the proposed shrinkage estimators is more efficient than the already existing estimators in literature. A real life application is provided to show the performance of the proposed shrinkage estimator.

Keywords: auxiliary variable; mean square error; two stage sampling; two phase sampling; first stage sampling unit; second stage sampling unit
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1. Introduction

In large scale surveys, it is a usual practice to prefer multi-stage sampling to estimate the population characteristics over single-stage sampling. The main purpose to use multi-stage sampling is the clear reduction in the cost of survey operations even if estimates derived from multi-stage sampling are likely to be less efficient than those of the single-stage sampling. Sukhatme et al. [1] advised some ratio and regression type estimators in two-stage sampling using single auxiliary variable when first stage units are of unequal or equal sizes. The suitable use of auxiliary information in estimation stage results as a considerable reduction in the mean square error of the estimator. By making use of auxiliary information, Srivastava and Garg [2] proposed separate-type estimator for the estimation of population mean in two-stage sampling design. Taking inspiration from Srivastava and Garg [2], Koyuncu and Kadilar [3] and Jabeen et al. [4] proposed separate type estimator under two-stage sampling. In the literature, the use of two-phase sampling under two-stage sampling design is not well documented. Saini and Bahl [5] proposed estimator under two-stage sampling design using double sampling for stratification and multi-auxiliary information. A generalized ratio cum product estimator for population mean in simple random sampling was developed by Singh et al. [6]. Shabbir [7] produces the estimators of population mean under stratified two phase sampling.

In the literature ([8–11]), the use of two phase sampling under two stage sampling design is not well documented. Also the shrinkage estimators have been discussed several times in literature by considering the unbiasedness of the estimators but no one has discussed the situation when the property of unbiasedness is not fulfill that is very common in real life applications. In order to fill this gap, we are motivated to produce two-exponential estimators under two stage two phase sampling that is discussed in section 2. Also we will discuss general shrinkage estimator in section 3. We will compare both estimators mathematically and by using real population data in section 4 and 5. Finally the conclusion will be discussed in section 6.

Let a population consists of N first stage units, each containing M_i second-stage units where i=$1, 2,\dots,N$. Let a first-stage sample of size n $(\subset N)$ is selected and subsequently a second-stage sample of $m_{i(1)}(\subset M_i)$ units is selected and information on some auxiliary variables say $x_{ij(1)}$ is taken. Here it is assumed that each first stage unit/cluster is of different size so each cluster is assigned a weight $\eta_i = \frac{M_i}{\bar{M}}$ to it. Let a sub-sample (second-phase sample) of $m_{i(2)}$ units is selected from $m_{i(1)}$ (first-phase sample) such that $m_{i(2)} \subset M_i$. Let $m_{i(2)}$ units are observed so as to collect information regarding study variable $y_{ij(2)}$ and auxiliary variables $x_{ij(2)}$. Let $\bar{X}_{ts} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\bar{M}}M_i\bar{X}_{i.}$ be the mean in the population and $\bar{X}_i = \frac{1}{M_i}\sum_{j=1}^{M_i}x_{ij}$ be the mean of i^th^ first stage unit in the population. Let $\bar{x}_{\mathrm{ts}(1)} = \frac{1}{n}\sum_{i=1}^{n}\eta_i\bar{x}_{i(1)}$ and $\bar{x}_{i(2)} = \frac{1}{m_{i(2)}}\sum_{j=1}^{m_{i(2)}}x_{ij(2)}$ respectively are the means of first-phase and second-phase sample in two-stage sampling where $\bar{x}_{i(1)}$ and $\bar{x}_{i(2)}$ be the sample means of first phase and second phase in ith stage. Let $S_{xb}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(\eta_i\bar{X}_i - \bar{X}_{ts})^2$ be the population variance between fsu's and $S_{xwi}^2 = \frac{1}{M_i-1}\sum_{i=1}^{n}(x_{ij} - \bar{X}_{i.})^2$ be the population variance within first stage units. Similarly these notations can be defined for other variables. Further we consider that the selection of units at each stage (or phase) has been made by simple random sampling without replacement.

2. Materials and method

2.1. Proposed generalized estimator

We propose a generalized estimator by considering the exponential relationship as:

$$t^G = \bar{y}_{ts}\exp\left(\alpha\left(1-\frac{a\,\bar{x}_{\mathrm{ts}(1)}}{\left(\bar{X}+(a-1)\,\bar{x}_{\mathrm{ts}(1)}\right)}\right)\right)\exp\left(\beta\left(1-\frac{b\,\bar{z}_{ts(2)}}{\left(\bar{Z}+(b-1)\,\bar{z}_{ts(2)}\right)}\right)\right), \tag{1}$$

Where (a , b) are constants to be determined such that the mean square error is minimum, (α , β) are known constants takes the value (0, 1, -1) to produce different ratio-type and product-type estimators as presented in Table B1 (Appendix B).

2.2. Bias and mean square error of generalized estimator

To derive the bias and mean square error, we proceed as follows:

$$\frac{\bar{y}_{ts(2)}-\bar{Y}_{ts}}{\bar{Y}_{ts}}=e_{0(2)}\,,\ \frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{X}_{ts}}=e_{1(1)},\ \frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}}=e_{2(2)},\ \frac{\bar{x}_{ts(2)}-\bar{X}_{ts}}{\bar{X}_{ts}}=e_{1(2)},\ \frac{\bar{z}_{ts(1)}-\bar{Z}_{ts}}{\bar{Z}_{ts}}=e_{2(1)}$$

Further we assume that $E\left(e_{0(2)}\right)=E\left(e_{1(2)}\right)=E\left(e_{2(2)}\right)=E\left(e_{2(1)}\right)=E\left(e_{1(1)}\right)=0$, and some expectations under two-stage sampling design are obtained in order to obtain the bias and mean square error as,

$$\left.\begin{array}{l}
E(e_{0(2)}{}^2)=C_{020(2)},E(e_{1(2)}{}^2)=C_{200(2)},\\
E(e_{2(2)}{}^2)=C_{002(2)},E(e_{1(1)}{}^2)=C_{200(1)},\\
E(e_{2(1)}{}^2)=C_{002(1)},E(e_{1(2)}e_{1(1)})=C_{200(1)}\\
E(e_{0(2)}e_{1(2)})=C_{110(2)},E(e_{1(2)}e_{2(2)})=C_{101(2)},\\
E(e_{0(2)}e_{2(2)})=C_{011(2)},E(e_{0(2)}e_{1(1)})=C_{110(1)},\\
E(e_{1(2)}e_{2(1)})=C_{101(1)},E(e_{0(2)}e_{2(1)})=C_{011(1)}\\
\text{Where}\\
C_{020(2)}=\frac{1}{\bar{Y}_{ts}^2}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{yb}^2+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{M_i}\right)S_{ywi}^2,\right\},\\
C_{200(1)}=\frac{1}{\bar{X}_{ts}^2}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{xb}^2+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)S_{xwi}^2\right\}\\
C_{200(2)}=\frac{1}{\bar{X}_{ts}^2}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{xb}^2+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{M_i}\right)S_{xwi}^2\right\},\\
C_{002(1)}=\frac{1}{\bar{Z}_{ts}^2}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{zb}^2+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)S_{zwi}^2\right\},\\
C_{002(2)}=\frac{1}{\bar{Z}_{ts}^2}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{zb}^2+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{M_i}\right)S_{zwi}^2\right\},\\
C_{110(1)}=\frac{1}{\bar{Y}_{ts}\bar{X}_{ts}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xyb}S_{yb}S_{xb}+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)\rho_{xywi}S_{ywi}S_{xwi}\right\},\\
C_{110(2)}=\frac{1}{\bar{Y}_{s}\bar{X}_{s}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xyb}S_{yb}S_{xb}+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{M_i}\right)\rho_{xywi}S_{ywi}S_{xwi}\right\},\\
C_{101(1)}=\frac{1}{\bar{Z}_{ts}\bar{X}_{ts}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xzb}S_{zb}S_{xb}+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)\rho_{xzwi}S_{zwi}S_{xwi}\right\},\\
C_{101(2)}=\frac{1}{\bar{Z}_{ts}\bar{X}_{ts}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xzb}S_{zb}S_{xb}+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{M_i}\right)\rho_{xzwi}S_{zwi}S_{xwi}\right\},C_{200(1)}=\frac{1}{\bar{Z}_{ts}\bar{X}_{ts}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{\mathrm{xb}}S_{xb}+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)S_{xwi}S_{xwi}\right\}\\
C_{011(2)}=\frac{1}{\bar{Z}_{ts}\bar{Y}_{ts}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yzb}S_{zb}S_{yb}+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(2)}}-\frac{1}{M_i}\right)\rho_{yzwi}S_{zwi}S_{ywi}\right\},C_{011(1)}=\frac{1}{\bar{Z}_{ts}\bar{Y}_{ts}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yzb}S_{zb}S_{yb}+\frac{1}{nN}\sum_{i=1}^{N}\eta_i^2\left(\frac{1}{m_{i(1)}}-\frac{1}{M_i}\right)\rho_{yzwi}S_{zwi}S_{ywi}\right\}
\end{array}\right\} \quad (2)$$

Using (2) we can express (1) to derive the bias and mean square error as,

$$t^G=\bar{Y}_{ts}\left(1+e_{0(2)}\right)\exp\left[-\frac{\alpha}{a}e_{1(1)}\left(1+\frac{(a-1)}{a}e_{1(1)}\right)^{-1}\right]\exp\left[-\frac{\beta}{b}e_{2(2)}\left(1+\frac{(b-1)}{b}e_{2(2)}\right)^{-1}\right], \quad (3)$$

If $\left|e_{1(1)}\right|<1$, we expand the series, $\left(1+\frac{(a-1)}{a}e_{1(1)}\right)^{-1}$ and $\left(1+\frac{(b-1)}{b}e_{2(2)}\right)^{-1}$ up to the order n^{-1}, we get,

$$t^G=\bar{Y}_{ts}\left(1+e_{0(2)}\right)\exp\left[-\frac{\alpha}{a}e_{1(1)}\left(1-\frac{(a-1)}{a}e_{1(1)}+\frac{(a-1)^2}{a^2}e_{1(1)}{}^2+\ldots\right)\right]$$
$$\exp\left[-\frac{\beta}{b}e_{2(2)}\left(1-\frac{(b-1)}{b}e_{2(2)}+\frac{(b-1)^2}{b^2}e_{2(2)}{}^2+\ldots\right)\right], \quad (4)$$

If the contribution of terms involving powers in $e_{0(2)}$, $e_{1(1)}$ and $e_{2(2)}$ higher than two is negligible, we have;

$$t^G=\bar{Y}_{ts}\left(1+e_{0(2)}\right)\left[1-\frac{\alpha}{a}e_{1(1)}+\frac{\alpha(a-1)^2}{a^2}e^2{}_{1(1)}+\frac{1}{2}\frac{\alpha^2}{a^2}e^2{}_{1(1)}\right]\left[1-\frac{\beta}{b}e_{2(2)}+\frac{\beta(b-1)^2}{b^2}e^2{}_{2(2)}+\frac{1}{2}\frac{\beta^2}{b^2}e^2{}_{2(2)}\right], \quad (5)$$

or

$$t^G-\bar{Y}_{ts}=\bar{Y}_{ts}\begin{bmatrix}e_{0(2)}-\frac{\alpha}{a}e_{1(1)}+\frac{\alpha(a-1)}{a^2}e^2{}_{1(1)}+\frac{\alpha\,\beta}{a\,b}e_{1(1)}\,e_{2(2)}+\frac{\alpha^2}{2a^2}e^2{}_{1(1)}-\frac{\beta}{b}e_{2(2)}+\frac{\beta^2}{2b^2}e_{2(2)}{}^2+\frac{\beta\,(b-1)}{b^2}e_{2(2)}{}^2\\-\frac{\beta}{b}e_{2(2)}\,e_{0(2)}-\frac{\alpha}{a}e_{1(1)}\,e_{0(2)}\end{bmatrix}, \quad (6)$$

In order to get the bias, we take expectation on (6) and get,

$$Bias\,(t^G)=\bar{Y}_{ts}\left[\frac{\alpha^2}{2a^2}C_{200(1)}+\frac{\alpha(a-1)}{a^2}C_{200(1)}+\frac{\beta^2}{2b^2}C_{002(2)}+\frac{\beta\,(b-1)}{b^2}C_{002(2)}-\frac{\beta}{b}C_{011(2)}-\frac{\alpha}{a}C_{110(1)}+\frac{\alpha}{a}\frac{\beta}{b}C_{101(1)}\right], \quad (7)$$

To get the mean square error of the estimator, we take square and retain terms up to first order of e's then we take expectation of (6) and we obtain,

$$\text{MSE}\left(t^G\right)=\bar{Y}_{ts}^2\left(C_{020(2)}+\left(\frac{\alpha}{a}\right)^2C_{200(1)}+\left(\frac{\beta}{b}\right)^2C_{002(2)}-2\frac{\alpha}{a}C_{110(1)}-2\frac{\beta}{b}C_{011(2)}+2\frac{\alpha}{a}\frac{\beta}{b}C_{101(1)}\right), \quad (8)$$

For the following optimal value of the constants a and b, we achieve the minimum Variance among the class of proposed generalized estimator,

$$a=\frac{\alpha\left(C_{200(1)}C_{002(2)}-C_{101(2)}{}^2\right)}{C_{110(1)}C_{002(2)}-C_{011(1)}C_{101(2)}}, \text{ and } \quad b=\frac{\beta\left(C_{200(1)}\,C_{002(2)}-C_{101(2)}{}^2\right)}{C_{200(1)}\,C_{011(1)}-C_{110(1)}C_{101(2)}}, \quad (9)$$

We obtain minimum mean square error as,

$$\min MSE\left(t^G\right)=\bar{Y}_{ts}^2\left(C_{020(2)}-\frac{C_{110(1)}{}^2C_{002(2)}+C_{011(1)}{}^2C_{200(1)}-2C_{110(1)}C_{101(2)}C_{011(1)}}{C_{200(1)}C_{002(2)}-C_{101(2)}{}^2}\right), \quad (10)$$

We observe from (10) that proposed generalized estimator gives us more precise results under the optimal conditions, as compare to its class of the estimators.

From (7)–(10), we get expressions of the bias, mean square error, optimal values and minimum mean square error for the exponential-type estimators presented in Table B1 as class of estimators. Some of the results are discussed as,

i). For $\alpha=1,\beta=1$, we get a class of exponential-type ratio-cum-ratio estimators given in Table B1 and expressions of the mean square error and bias for these estimators are given as,

$$\left.\begin{array}{l} \text{MSE}(t^{j-err}) = \left\{\bar{Y}_{ts}^2\left(C_{020(2)}+\frac{1}{a^2}C_{200(1)}+\frac{1}{b^2}C_{002(2)}-2\frac{1}{a}C_{110(1)}-2\frac{1}{b}C_{011(2)}+2\frac{1}{ab}C_{101(1)}\right) \quad j(\in G)=1,3,5\right\}, \\ \text{and} \\ \text{Bias}\ (t^{j-err}) = \left\{\bar{Y}_{ts}\left(\frac{a-1}{a^2}C_{200(1)}+\frac{1}{2a^2}C_{200(1)}+\frac{b-1}{b^2}C_{002(2)}+\frac{1}{2b^2}C_{002(2)}-\frac{1}{b}C_{101(2)}-\frac{1}{a}C_{110(1)}\right) \quad j(\in G)=1,3,5\right\}, \end{array}\right) \tag{11}$$

Substitution of the different values of *a* and *b* yield the mean square error and bias for the estimators belongs to the class of exponential-type ratio-cum-ratio estimators. The optimal values which lead to minimum mean square error for the class of exponential ratio-cum-ratio estimators is obtained as,

$$a=\frac{C_{200(1)}C_{002(2)}-C_{101(2)}{}^2}{C_{110(1)}C_{002(2)}-C_{011(1)}C_{101(2)}} \text{ and } b=\frac{C_{200(1)}C_{002(2)}-C_{101(2)}{}^2}{C_{200(1)}C_{011(1)}-C_{110(1)}C_{101(2)}}.$$

ii). For $\alpha=-1,\beta=-1$, we get exponential-type product-cum-product estimators given in Table B1.The mean square error of t^G are expressed using as,

$$\left.\begin{array}{l} \text{MSE}(t^{j-epp}) = \left\{\bar{Y}_{ts}^2\left(C_{020(2)}+\frac{1}{a^2}C_{200(1)}+\frac{1}{b^2}C_{002(2)}+2\frac{1}{a}C_{110(1)}+2\frac{1}{b}C_{011(2)}+2\frac{1}{ab}C_{101(1)}\right), \quad j(\in G)=2,4,6\right\} \\ \text{and} \\ \text{Bias}\,t^{j-epp}) = \left\{\bar{Y}_{ts}\left(\frac{a-1}{a^2}C_{200(1)}+\frac{1}{2a^2}C_{200(1)}+\frac{b-1}{b^2}C_{002(2)}+\frac{1}{2b^2}C_{002(2)}-\frac{1}{b}C_{101(2)}+\frac{1}{a}C_{110(1)}\right), \quad j(\in G)=2,4,6\right\} \end{array}\right) \tag{12}$$

Substitution of the different values of "*a*" *and* "*b*" yield the Mean Square Error and bias for the estimators belongs to the class of exponential-type ratio-cum-ratio estimators. The optimal values which lead to minimum mean square error for the class of exponential product-cum-product estimators is obtained as,

$$a=\frac{C_{101(2)}{}^2-C_{200(1)}C_{002(2)}}{C_{110(1)}C_{002(2)}-C_{011(1)}C_{101(2)}} \text{ and } b=\frac{C_{101(2)}{}^2-C_{200(1)}C_{002(2)}}{C_{200(1)}C_{011(1)}-C_{110(1)}C_{101(2)}}.$$

iii). For $\alpha=-1,\beta=1$, we get exponential-type product-cum-ratio estimators given in Table B1.The mean square error of t^G is expressed as,

$$\left.\begin{array}{l}\text{MSE}\,(t^{j-erp})=\left\{\bar{Y}_{ts}^{2}\left(C_{020(2)}+\frac{1}{a}C_{200(1)}+\frac{1}{b}C_{002(2)}+2\frac{1}{a}C_{110(1)}-2\frac{1}{b}C_{011(2)}-2\frac{1}{ab}C_{101(1)}\right), \qquad j(\in G)=7,9,11\right\} \\ \text{and} \\ \text{Bias}\,(t^{j-erp})=\left\{\bar{Y}_{ts}\left(\frac{a-1}{a^{2}}C_{200(1)}+\frac{1}{2a^{2}}C_{200(1)}+\frac{b-1}{b^{2}}C_{002(2)}+\frac{1}{2b^{2}}C_{002(2)}-\frac{1}{b}C_{101(2)}+\frac{1}{a}C_{110(1)}\right),\ j(\in G)=7,9,11\right\}\end{array}\right) \quad (13)$$

Substitution of the different values of “*a”* and “*b”* yield the mean square error and bias for the estimators belongs to the class of exponential-type ratio-cum-ratio estimators. The optimal values which lead to minimum mean square error for the class of exponential product-cum-ratio estimators are obtained as,

$$a=\frac{C_{101(2)}{}^{2}-C_{200(1)}C_{002(2)}}{C_{110(1)}C_{002(2)}-C_{011(1)}C_{101(2)}} \text{ and } b=\frac{C_{200(1)}C_{011(1)}-C_{110(1)}C_{101(2)}}{C_{200(1)}C_{002(2)}-C_{101(2)}{}^{2}}.$$

iv). For $\alpha=1,\beta=-1$, we get exponential-type ratio-cum-product estimators given in Table A1. The mean square error of t^{G} is expressed as,

$$\left.\begin{array}{l}\text{MSE}\,(t^{j-epr})=\left\{\bar{Y}_{ts}^{2}\left(C_{020(2)}+\frac{1}{a}C_{200(1)}+\frac{1}{b}C_{002(2)}-2\frac{1}{a}C_{110(1)}+2\frac{1}{b}C_{011(2)}-2\frac{1}{a}\frac{1}{b}C_{101(1)}\right), \qquad j(\in G)=8,10,12\right\} \\ \text{and} \\ \text{Bias}\,(t^{j-epr})=\left\{\bar{Y}_{ts}\left(\frac{a-1}{a^{2}}C_{200(1)}+\frac{1}{2a^{2}}C_{200(1)}+\frac{b-1}{b^{2}}C_{002(2)}+\frac{1}{2b^{2}}C_{002(2)}+\frac{1}{b}C_{101(2)}-\frac{1}{a}C_{110(1)}\right), \qquad j(\in G)=8,10,12\right\}\end{array}\right) \quad (14)$$

Substitution of the different values of “*a”* and “*b”* yield the mean square error and bias for the estimators belongs to the class of exponential-type ratio-cum-ratio estimators. The optimal values which lead to minimum mean square error for the class of exponential ratio-cum-product estimators, are obtained as,

$$a=\frac{C_{200(1)}C_{002(2)}-C_{101(2)}{}^{2}}{C_{110(1)}C_{002(2)}-C_{011(1)}C_{101(2)}} \text{ and } b=\frac{C_{101}{}^{2}-C_{200(1)}C_{002(2)}}{C_{200(1)}C_{011(1)}-C_{110(1)}C_{101(2)}}.$$

It is to mention that minimum mean square error of t^{j-epr}, t^{j-epr}, t^{j-epr}, and t^{j-epr} in (11)–(14) will be same but the conditions under which these attain minimum value are not unique.

3. Proposed two-exponential general Shrinkage estimator

In literature, the shrinkage estimators have been discussed several times considering the property that the estimators are unbiased. In many practical situations the property of unbiasedness is not met so consequently the existing literature will not be helpful. Such deviations from the said property have motivated us to consider the shrinkage estimators following the property of biasedness. Important results are presented in theorem and subsequently proof is provided in Appendix A.

3.1. Theorem for estimating any population characteristic

For any general estimator t, let t_s be a general shrinkage estimator for estimating any population characteristic T (e.g., population mean) which is unknown to us,

$$t_s = \lambda t \tag{15}$$

A general form for the bias and mean square error of a general shrinkage estimator up to the first order of approximation may be given as,

$$Bias(t_s) = (\lambda - 1)T + Bias(t) \tag{16}$$

and

$$MSE(t_s) = T^2(\lambda - 1)^2 + \lambda^2 MSE(t) + 2T\lambda(\lambda - 1)Bias(t) \tag{17}$$

And form of the optimal shrinkage estimator in (15) along with minimum bias and minimum mean square error will be as,

$$t_s = \frac{\left(T^2 + MSE(t) + 2TBias(t)\right)}{T(T + Bias(t)}t \tag{18}$$

$$Bias\left(t_s^{opt}\right) = \left(\frac{\left(T^2 + MSE(t) + 2TBias(t)\right)}{T(T + Bias(t)} - 1\right)T + Bias(t) \tag{19}$$

and

$$\min MSE(t_s) = T^2 - \frac{T^2(T + Bias(t)}{T^2 + MSE(t) + 2TBias(t)} \tag{20}$$

Proof: For details see Appendix A.

By using (15), generalized form of a shrinkage class of estimator is,

$$t_s^G = \lambda t_G$$

where

$$t_G = \bar{y}_{ts} \exp\left(\alpha\left(1 - \frac{a\,\bar{x}_{ts(1)}}{\left(\bar{X}_{ts} + (a-1)\bar{x}_{ts(1)}\right)}\right)\right) \exp\left(\beta\left(1 - \frac{b\,\bar{z}_{ts(2)}}{\left(\bar{Z}_{ts} + (b-1)\bar{z}_{ts(2)}\right)}\right)\right) \tag{21}$$

For different choices of the values of the constants α, β, a, b, we get different family of shrinkage estimators as given in Table B2 (See Appendix).

Following the Theorem, we can directly write the mean square error and bias expressions of t_s^G using (7) and (8) respectively as,

$$Bias\,(t_s^G)=\lambda\,\bar{Y}_{ts}\left[\frac{\alpha\,(a-1)}{a^2}C_{200(1)}+\frac{\alpha^2}{2a^2}C_{200(1)}+\frac{\beta\,(b-1)}{b^2}C_{002(2)}+\frac{\beta^2}{2b^2}C_{002(2)}-\frac{\beta}{b}C_{101(2)}-\frac{\alpha}{a}C_{110(1)}\right]+\bar{Y}_{ts}\,(\lambda-1),$$ or

$$Bias\,(t_s^G)=\lambda\,Bias(t^{G-e})+\bar{Y}_{ts}\,(\lambda-1), \tag{22}$$

and

$$MSE\left(t_s^G\right)=\lambda^2\,\bar{Y}_{ts}^2\left(C_{020(2)}+\left(\frac{\alpha}{a}\right)^2C_{200(1)}+\left(\frac{\beta}{b}\right)^2C_{002(2)}-2\frac{\alpha}{a}C_{110(1)}-2\frac{\beta}{b}C_{011(2)}+2\frac{\alpha}{a}\frac{\beta}{b}C_{101(1)}\right)+\left(\lambda-1\right)^2\bar{Y}_{ts}^{\,2}$$
$$+2\lambda\left(\lambda-1\right)\bar{Y}_{ts}\left[\frac{\alpha\,(a-1)}{a^2}C_{200(1)}+\frac{\alpha^2}{2\,a^2}C_{200(1)}+\frac{\beta\,(b-1)}{b^2}C_{002(2)}+\frac{\beta^2}{2b^2}C_{002(2)}-\frac{\beta}{b}C_{101(2)}-\frac{\alpha}{a}C_{110(1)}\right]$$

or

$$MSE\left(t_s^G\right)=\lambda^2MSE\left(t^{G-e}\right)+\left(\lambda-1\right)^2\bar{Y}_{ts}^{\,2}+2\bar{Y}_{ts}\,\lambda\left(\lambda-1\right)MSE\left(t^{G-e}\right) \tag{23}$$

The optimal value of λ , which minimize the expression (23) is obtained as,

$$\lambda=\frac{\left(T^2+MSE\,(t^{G-e})+2\,T\;Bias(t^{G-e})\right)}{T\,(T+Bias(t^{G-e})} \tag{24}$$

Using (23), the optimal estimator for the shrinkage family of estimator (t_s^G) is written as

$$t_s^G=\bar{y}_{ts}\frac{\left(\bar{Y}_{ts}^{\,2}+MSE(t^{G-e})+2\bar{Y}_{ts}Bias(t^{G-e})\right)}{\bar{Y}_{ts}(\bar{Y}_{ts}+Bias(t^{G-e})}\exp\left(\alpha\left(1-\frac{a\,\bar{x}_{ts(1)}}{\left(\bar{X}_{ts}+(a-1)\,\bar{x}_{ts(1)}\right)}\right)\right)\exp\left(\beta\left(1-\frac{b\,\bar{z}_{ts(2)}}{\left(\bar{Z}_{ts}+(b-1)\,\bar{z}_{ts(2)}\right)}\right)\right)\Bigg\} \tag{25}$$

The expression for minimum value of min $MSE\left(t_s^G\right)$ is obtained as,

$$\left.\min MSE\left(t_s^G\right)=\bar{Y}_{ts}^2-\frac{\bar{Y}_{ts}^2\left(\bar{Y}_{ts}+Bias\left(t^{G-e}\right)\right)^2}{\left[\bar{Y}_{ts}^2+MSE\left(t^{G-e}\right)+2\bar{Y}_{ts}Bias\left(t^{G-e}\right)\right]}\right\}$$
$$\min MSE\left(t_s^G\right)=\bar{Y}_{ts}^2\left[1-\frac{\left(1+\dfrac{Bias\left(t_a\right)}{\bar{Y}_{ts}}\right)^2}{\left[1+\dfrac{MSE\left(t_a\right)}{\bar{Y}_{ts}^2}+2\,\dfrac{Bias\left(t_a\right)}{\bar{Y}_{ts}}\right]}\right] \tag{26}$$

From (22)–(26), we get expressions of the bias, mean square error, optimal values and minimum mean square error for the exponential-type estimators presented in Table B2 (See Appendix B) as class of estimators. Some of the results are discussed as,

i). For $\alpha=1,\beta=1$, we get a class of exponential-type ratio-cum-ratio estimators given in Table B2 (See Appendix) and expressions of the mean square error and bias for these estimators are given as,

$$\text{MSE}(t_s^{j-err}) = \left\{ \bar{Y}_{ts}^2 \left[1 - \frac{\left(1 + \frac{Bias\left(t^{i-err}\right)}{\bar{Y}_{ts}}\right)^2}{\left[1 + \frac{MSE\left(t^{i-err}\right)}{\bar{Y}_{ts}^2} + 2\frac{Bias\left(t^{i-err}\right)}{\bar{Y}_{ts}^2}\right]} \right] \quad j(\in G) = 1,3,5 \right\}$$

and (27)

$$\text{Bias}(t_s^{j-err}) = \left\{ \bar{Y}_{ts} \left(\lambda \left(\frac{\text{Bias}(t^{j-err})}{\bar{Y}_{ts}} \right) + (\lambda - 1) \right) \quad j(\in G) = 1,3,5 \right\}$$

Substitution of the different values of *a* and *b* produce the mean square error and bias for the estimators belongs to the class of exponential-type ratio-cum-ratio estimators. The optimal values which lead to minimum mean square error for the class of exponential product-cum-ratio estimators are obtained as,

$$a = \frac{C_{200(1)}C_{002(2)} - C_{101(2)}{}^2}{C_{110(1)}C_{002(2)} - C_{011(1)}C_{101(2)}} \; and \; b = \frac{C_{200(1)}C_{002(2)} - C_{101(2)}{}^2}{C_{200(1)}C_{011(1)} - C_{110(1)}C_{101(2)}}, \; \lambda = \frac{\left(T^2 + MSE\,(t^{G-e}) + 2T\,Bias\,(t^{G-e})\right)}{T(T + Bias\,(t^{G-e})},$$

$$where\; MSE\left(t^G\right) = \bar{Y}_{ts}{}^2 \left[C_{020(2)} + \frac{1}{a^2}C_{200(1)} + \frac{1}{b^2}C_{002(2)} - 2\frac{1}{a}C_{110(1)} - 2\frac{1}{b}C_{011(2)} + 2\frac{1}{a}\frac{1}{b}C_{101(1)} \right].$$

$$\text{Bias}(t^G) = \bar{Y}_{ts} \left(\frac{a-1}{a^2}C_{200(1)} + \frac{1}{2a^2}C_{200(1)} + \frac{b-1}{b^2}C_{002(2)} + \frac{1}{2b^2}C_{002(2)} - \frac{1}{b}C_{101(2)} - \frac{1}{a}C_{110(1)} \right)$$

ii). For $\alpha = -1, \beta = -1$, we get exponential-type product-cum-product estimators given in Table B2 (See Appendix B).The mean square error of t_s^G is expressed using as,

$$\text{MSE}(t_s^{j-epp}) = \left\{ \bar{Y}_{ts}^2 \left[1 - \frac{\left(1 + \frac{Bias\left(t^{i-err}\right)}{\bar{Y}_{ts}}\right)^2}{\left[1 + \frac{MSE\left(t^{i-err}\right)}{\bar{Y}_{ts}^2} + 2\frac{Bias\left(t^{i-err}\right)}{\bar{Y}_{ts}^2}\right]} \right] \quad j(\in G) = 2,4,6 \right\}$$

and (28)

$$\text{Bias}(t_s^{j-epp}) = \left\{ \bar{Y}_{ts} \left(\lambda \frac{\text{Bias}(t_s^{j-epp})}{\bar{Y}_{ts}} + (\lambda - 1) \right) \quad j(\in G) = 2,4,6 \right\}$$

Substitution of the different values of *a* and *b* yield the mean square error and bias for the estimators belongs to the class of exponential-type product-cum-product estimators. The optimal values which lead to minimum mean square error for the class of exponential product-cum-ratio estimators are obtained as,

$$a=\frac{C_{200(1)}C_{002(2)}-C_{101(2)}{}^2}{C_{110(1)}C_{002(2)}-C_{011(1)}C_{101(2)}}\ and\ b=\frac{C_{200(1)}C_{002(2)}-C_{101(2)}{}^2}{C_{200(1)}C_{011(1)}-C_{110(1)}C_{101(2)}},\lambda=\frac{\left(T^2+MSE(t^{G-e})+2TBias(t^{G-e})\right)}{T(T+Bias(t^{G-e})},$$

$$where\ MSE\left(t^G\right)=\bar{Y}_{ts}{}^2\left[C_{020(2)}+\frac{1}{a^2}C_{200(1)}+\frac{1}{b^2}C_{002(2)}-2\frac{1}{a}C_{110(1)}-2\frac{1}{b}C_{011(2)}+2\frac{1}{a}\frac{1}{b}C_{101(1)}\right].$$

$$\text{Bias}\,t^G)=\bar{Y}_{ts}\left(\frac{a-1}{a^2}C_{200(1)}+\frac{1}{2a^2}C_{200(1)}+\frac{b-1}{b^2}C_{002(2)}+\frac{1}{2b^2}C_{002(2)}-\frac{1}{b}C_{101(2)}-\frac{1}{a}C_{110(1)}\right)$$

iii). For $\alpha=-1,\beta=1$, we get exponential-type product-cum-ratio estimators given in Table B2. The mean square error of t_s^G is expressed as,

$$\text{MSE}\,(t_s^{j-erp})=\left\{\bar{Y}_{ts}^2\left[1-\frac{\left(1+\frac{Bias\left(t^{i-err}\right)}{\bar{Y}_{ts}}\right)^2}{\left[1+\frac{MSE\left(t^{i-err}\right)}{\bar{Y}_{ts}^2}+2\frac{Bias\left(t^{i-err}\right)}{\bar{Y}_{ts}^2}\right]}\right]\quad j(\in G)=7,9,11\right\}$$

and (29)

$$\text{Bias}\,(t_s^{j-erp})=\left\{\bar{Y}_{ts}\left(\lambda\frac{\text{Bias}\,(t_s^{j-erp})}{\bar{Y}_{ts}}+(\lambda-1)\right)j(\in G)=7,9,11\right\}$$

Substitution of the different values of *a* and *b* produce the mean square error and bias for the estimators belongs to the class of exponential-type ratio-cum-product estimators. The optimal values which lead to minimum mean square error for the class of exponential ratio-cum-product estimators, are obtained as,

$$a=\frac{C_{200(1)}C_{002(2)}-C_{101(2)}{}^2}{C_{110(1)}C_{002(2)}-C_{011(1)}C_{101(2)}}\ and\ b=\frac{C_{200(1)}C_{002(2)}-C_{101(2)}{}^2}{C_{200(1)}C_{011(1)}-C_{110(1)}C_{101(2)}},\lambda=\frac{\left(T^2+MSE(t^{G-e})+2TBias(t^{G-e})\right)}{T(T+Bias(t^{G-e})},$$

$$where\ MSE\left(t^{G-e}\right)=\bar{Y}_{ts}{}^2\left[C_{020(2)}+\frac{1}{a^2}C_{200(1)}+\frac{1}{b^2}C_{002(2)}-2\frac{1}{a}C_{110(1)}-2\frac{1}{b}C_{011(2)}+2\frac{1}{a}\frac{1}{b}C_{101(1)}\right].$$

$$\text{Bias}\,t^{G-e})=\bar{Y}_{ts}\left(\frac{a-1}{a^2}C_{200(1)}+\frac{1}{2a^2}C_{200(1)}+\frac{b-1}{b^2}C_{002(2)}+\frac{1}{2b^2}C_{002(2)}-\frac{1}{b}C_{101(2)}-\frac{1}{a}C_{110(1)}\right)$$

iv). For $\alpha=1,\beta=-1$, we get exponential-type ratio-cum-product estimators given in Table B2. The mean square error of t_s^G is expressed as,

$$\text{MSE}\,(t_s^{j-epr})=\left\{\bar{Y}_{ts}^2\left[1-\frac{\left(1+\frac{Bias\left(t^{i-err}\right)}{\bar{Y}_{ts}}\right)^2}{\left[1+\frac{MSE\left(t^{i-err}\right)}{\bar{Y}_{ts}^2}+2\frac{Bias\left(t^{i-err}\right)}{\bar{Y}_{ts}^2}\right]}\right]\quad j(\in G)=8,10,12\right\}$$

and (30)

$$\text{Bias}\,(t_s^{j-epr})=\left\{\bar{Y}_{ts}\left(\lambda\frac{\text{Bias}\,(t_s^{j-epr})}{\bar{Y}_{ts}}+(\lambda-1)\right)\quad j(\in G)=8,10,12\right\}$$

Substitution of the different values of *a* and *b* yield the mean square error and bias for the estimators belongs to the class of exponential-type product-cum-ratio estimators. The optimal values which lead to minimum mean square error for the class of exponential product-cum-ratio estimators, are obtained as,

$$a=\frac{C_{200(1)}C_{002(2)}-C_{101(2)}{}^{2}}{C_{110(1)}C_{002(2)}-C_{011(1)}C_{101(2)}}\ and\ b=\frac{C_{200(1)}C_{002(2)}-C_{101(2)}{}^{2}}{C_{200(1)}C_{011(1)}-C_{110(1)}C_{101(2)}},\lambda=\frac{\left(T^{2}+MSE\,(t^{G-e})+2T\,Bias(t^{G-e})\right)}{T(T+Bias(t^{G-e})},$$

$$where\ MSE\left(t^{G}\right)=\bar{Y}_{ts}{}^{2}\left[C_{020(2)}+\frac{1}{a^{2}}C_{200(1)}+\frac{1}{b^{2}}C_{002(2)}-2\frac{1}{a}C_{110(1)}-2\frac{1}{b}C_{011(2)}+2\frac{1}{a}\frac{1}{b}C_{101(1)}\right].$$

$$\text{Bias }(t^{G})=\bar{Y}_{ts}\left(\frac{a-1}{a^{2}}C_{200(1)}+\frac{1}{2a^{2}}C_{200(1)}+\frac{b-1}{b^{2}}C_{002(2)}+\frac{1}{2b^{2}}C_{002(2)}-\frac{1}{b}C_{101(2)}-\frac{1}{a}C_{110(1)}\right)$$

It is to mention that minimum mean square error of t_s^{j-epr}, t_s^{j-epr}, t_s^{j-epr}, and t_s^{j-epr} in (27)–(30) will be same but the conditions under which these attain minimum value are not unique.

4. Efficiency comparison

In this section, some efficiency conditions have been derived in terms of mean square error and bias. The efficiency comparison of proposed two-exponential general shrinkage estimator (see section 3) and two-exponential estimator (see section 2) with unbiased estimator in two-stage sampling. In order to derive the efficiency conditions, consider the following notations,

$$A_1=C_{020(2)}+\left(\frac{\alpha}{a}\right)^{2}C_{200(1)}+\left(\frac{\beta}{b}\right)^{2}C_{002(2)}-2\left(\frac{\alpha}{a}\right)C_{110(1)}-2\left(\frac{\beta}{b}\right)C_{011(2)}+2\left(\frac{\alpha}{a}\right)\left(\frac{\beta}{b}\right)C_{101(1)}$$

$$A_2=\left[\frac{\alpha(a-1)}{a^{2}}C_{200(1)}+\frac{\alpha^{2}}{2a^{2}}C_{200(1)}+\frac{\beta(b-1)}{b^{2}}C_{002(2)}+\frac{\beta^{2}}{2b^{2}}C_{002(2)}-\frac{\beta}{b}C_{101(2)}-\frac{\alpha}{a}C_{110(1)}\right]$$

$$A_3=C_{020(2)}+C_{200(1)}+C_{002(2)}-2C_{110(1)}-2C_{011(2)}+2C_{101(1)}$$

$$A_4=\left(C_{020(2)}-\frac{C_{110(1)}{}^{2}C_{002(2)}+C_{011(1)}{}^{2}C_{200(1)}-2C_{110(1)}C_{101(2)}C_{011(1)}}{C_{200(1)}C_{002(2)}-C_{101(2)}{}^{2}}\right)$$

$$A_5=\left[\lambda^{2}+2\lambda(\lambda-1)\left(\frac{1}{\alpha}C_{200(1)}+\frac{1}{2}C_{200(1)}\right)\right]$$

$$A_6=\left[\lambda^{2}\left(C_{110(1)}-\frac{\beta}{b}C_{101(1)}\right)+\lambda(\lambda-1)\left(C_{200(1)}-C_{110(1)}\right)\right]$$

$$A_7=\left[\lambda^{2}\left(\left(\frac{\beta}{b}\right)^{2}C_{002(2)}-2\frac{\beta}{b}C_{011(2)}\right)+(\lambda-1)^{2}+2\lambda(\lambda-1)\left(\frac{\beta(b-1)}{b^{2}}C_{002(2)}+\left(\frac{\beta}{b}\right)^{2}C_{002(2)}-\frac{\beta}{b}C_{101(1)}\right)\right]$$

4.1. When a,b is known and λ is unknown

The efficiency conditions may be written as:

i. $MSE\,(t_s^{G})-MSE\,(\bar{y}_{ts})\le 0$ if

$$\min\left(\frac{(1-A_2)\pm\sqrt{\left((1-A_2)^2-(1+A_1+2A_2)(C_{020(2)})\right)}}{(1-A_2)}\right)\le\lambda\le\max\left(\frac{(1-A_2)\pm\sqrt{\left((1-A_2)^2-(1+A_1+2A_2)(C_{020(2)})\right)}}{(1-A_2)}\right), \quad (31)$$

ii. $MSE(t_s^G)-MSE(t_r)\le 0$

$$\min\left(\frac{(1-A_2)\pm\sqrt{\left((1-A_2)^2-(1+A_1+2A_2)(A_3)\right)}}{(1-A_2)}\right)\le\lambda\le\max\left(\frac{(1-A_2)\pm\sqrt{\left((1-A_2)^2-(1+A_1+2A_2)(A_3)\right)}}{(1-A_2)}\right) \quad (32)$$

iii. $MSE(t_s^G)-MSE(t^G)\le 0$

$$\min\left(\frac{(1-A_2)\pm\sqrt{\left((1-A_2)^2-(1+A_1+2A_2)(A_1)\right)}}{(1-A_2)}\right)\le\lambda\le\max\left(\frac{(1-A_2)\pm\sqrt{\left((1-A_2)^2-(1+A_1+2A_2)(A_1)\right)}}{(1-A_2)}\right) \quad (33)$$

iv. $MSE(t_s^G)-\min MSE(t^G)\le 0$

$$\min\left(\frac{(1-A_2)\pm\sqrt{\left((1-A_2)^2-(1+A_1+2A_2)(A_4)\right)}}{(1-A_2)}\right)\le\lambda\le\max\left(\frac{(1-A_2)\pm\sqrt{\left((1-A_2)^2-(1+A_1+2A_2)(A_4)\right)}}{(1-A_2)}\right) \quad (34)$$

4.2. If a is unknown λ, b is known

i. $MSE(t_s^G)-MSE(\bar{y}_{ts})\le 0$ if

$$\min\left(\frac{A_6\pm\sqrt{A_6^2-A_5\left(A_7-C_{020(2)}\right)}}{\alpha A_5}\right)\le\frac{1}{a}\le\max\left(\frac{A_6\pm\sqrt{A_6^2-A_5\left(A_7-C_{020(2)}\right)}}{\alpha A_5}\right) \quad (35)$$

ii. $MSE(t_s^G)-MSE(t_r)\le 0$

$$\min\left(\frac{A_6\pm\sqrt{A_6^2-A_5\left(A_7-A_3\right)}}{\alpha A_5}\right)\le\frac{1}{a}\le\max\left(\frac{A_6\pm\sqrt{A_6^2-A_5\left(A_7-A_3\right)}}{\alpha A_5}\right) \quad (36)$$

Substituting different values of (α,β,a,b) in above equations, we get different mathematical comparisons for the estimators given in Table B2 (See Appendix B). Also similar comparison is obtained if we assume b as unknown and λ, a is known.

5. Numerical study

For the demonstration of the performance of proposed two-exponential general shrinkage estimator, we take real population consists of four clusters with unequal first stage units. The description about the populations is given in Table B1 in Appendix B. The mean square error and percentage relative efficiency values for each of the estimators are given in Table C2.For this population, correlations within the clusters are positive (see Table C1). It is therefore the population is applicable only for ratio/ratio-type estimators.

The performance of the proposed two-exponential general shrinkage estimator and two-exponential estimator under two stage two phase sampling has been expressed in form of their mean square error and percentage relative efficiency values in Table C2. From Table C2 we see that the minimum mean square error of two-exponential general shrinkage estimator t_s^G that is equal to the mean square error of t_s^5. The percentage relative efficiency demonstrate the same result as the percentage relative efficiency of the proposed two-exponential general shrinkage is high among the family of proposed two-exponential general shrinkage and t_s^5 is the same efficient as t_s^G.

We also see that the two-exponential general shrinkage estimator (section 3) is better to be used as compare to two-exponential estimator (section 2). The performance of t_s^1 is better than t^1, similarly t_s^3 performs better than t^3. So it is concluded that for this particular population the use of two-exponential general shrinkage estimator in two stage two phase sampling produce better results.

6. Conclusion

Finally from the above empirical results, it is concluded that the performance of the two-exponential general shrinkage estimator (section 3) is higher as compare to proposed two-exponential estimator in two stage two phase sampling. So the two-exponential general shrinkage estimator (section 3) is acceptable for the real life application in two stage two phase sampling design.

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Conflict of interest

The authors declare no conflict of interest.

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Appendix A

Proof of Theorem. Let t be any general estimator of some population parameter T in two stage sampling for which a general shrinkage estimator t_s is defined as,

$$t_s = \lambda t, \tag{37}$$

By definition the bias of t_s is

$$Bias(t_s) = \lambda E(t) - T$$

or
$$= \lambda E(t-T) + (\lambda - 1)T$$

or
$$= \lambda Bias(t) + (\lambda - 1)T \tag{38}$$

By definition the *MSE* of t_s may be defined as,

$$MSE(t_s) = E(t_s - T)^2$$

or
$$= E(\lambda t - T)^2$$

consider $t = T(1+e_T)$, with $E(e_T) = 0$ and $E(e_T{}^2) = \frac{Var(t)}{T^2} = \frac{MSE(t)}{T^2}$.

$$= E\{\lambda(\mathrm{T}(1+e_T)) - T\}^2$$

or
$$= \mathrm{T}^2 E\left\{\lambda^2 \left(1+e_T\right)^2 + 1 - 2\lambda\left(1+e_T\right)\right\}$$

or
$$MSE\left(t_s\right) = \mathrm{T}^2 \left\{\lambda^2 \left(1+\frac{MSE(t)}{\mathrm{T}^2}\right) + 1 - 2\lambda\right\} \tag{39}$$

We may also write (B-3) as
$$MSE\left(t_s\right) = \lambda^2 MSE(t) + (\lambda - 1)^2 \mathrm{T}^2$$

or
$$MSE\left(t_s\right) = \lambda^2 \mathrm{T}^2 C_t^2 + (\lambda - 1)^2 \mathrm{T}^2 \qquad \text{Where} \quad C_t^2 = \frac{MSE(t)}{T^2} \tag{40}$$

Where C_t is a coefficient of variation of an estimator *t*.

Partially differentiating (40) w.r.t λ and simplifying, we get:
$$\lambda = \frac{1}{1+T^{-2} MSE(t)}, \tag{41}$$

or
$$\lambda = \frac{1}{1+C_t^2}, \tag{42}$$

Now using (41) in (15), (16) and (17) respectively, we may get,
$$t_s = \frac{t}{1+T^{-2} MSE(t)} \tag{43}$$

$$Bias\left(t_s\right) = \frac{1}{1+T^{-2} MSE(t)}\left[Bias(t) + T^{-2}\, T\, MSE(t)\right] \tag{44}$$

$$MSE_{\min}\left(t_s\right) = \frac{MSE(t)}{1+T^{-2} MSE(t)} \tag{45}$$

Alternatively we may also produce the expressions in (44) to (46) as,
$$t_s = \frac{t}{1+C_t^2} \tag{46}$$

$$Bias\left(t_s\right) = \frac{1}{1+C_t^2}\left(Bias(t) + C_t^2\, \mathrm{T}\right) \tag{47}$$

$$MSE_{\min}\left(t_s\right) = \frac{T^2 C_t^2}{1+C_t^2} \tag{48}$$

Appendix B

Table B1. Some members of the proposed two exponential estimator (t^{G}).

Ratio-cum-Ratio estimator $\alpha=1,\beta=1$	Product -cum-product estimator $\alpha=-1,\beta=-1$	a	b
$t^{1-err}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}+\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{Z}_{ts}-\bar{z}_{ts(2)}}{\bar{Z}_{ts}+\bar{z}_{ts(2)}}\right)$	$t^{2-epp}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{x}_{ts(1)}+\bar{X}_{ts}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{z}_{ts(2)}+\bar{Z}_{ts}}\right)$	2	2
$t^{3-err}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}}\right)\exp\left(\frac{\bar{Z}_{ts}-\bar{z}_{ts(2)}}{\bar{Z}_{ts}}\right)$	$t^{4-epp}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{X}_{ts}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}}\right)$	1	1
$t^{5-err}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}+(a-1)\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{Z}_{ts}-\bar{z}_{ts(2)}}{\bar{Z}_{ts}+(b-1)\bar{z}_{ts(2)}}\right)$	$t^{6-epp}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{X}_{ts}+(a-1)\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}+(b-1)\bar{z}_{ts(2)}}\right)$	a	b
Product -cum-ratio estimator $\alpha=-1,\beta=1$	**Ratio-cum-product estimator** $\alpha=1,\beta=-1$	a	b
$t^{7-epr}=\bar{y}_{ts}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}+\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}+\bar{z}_{ts(2)}}\right)$	$t^{8-erp}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{x}_{ts(1)}+\bar{X}_{ts}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{z}_{ts(2)}+\bar{Z}_{ts}}\right)$	2	2
$t^{9-epr}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}}\right)\exp\left(\frac{\bar{z}_{ts(1)}-\bar{Z}_{ts}}{\bar{Z}_{ts}}\right)$	$t^{10-erp}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{X}_{ts}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}}\right)$	1	1
$t^{11-epr}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}+(a-1)\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}+(b-1)\bar{z}_{ts(2)}}\right)$	$t^{12-erp}=\bar{y}_{ts(2)}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{X}_{ts}+(a-1)\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}+(b-1)\bar{z}_{ts(2)}}\right)$	a	b

Table B2. Some members of the generalized shrinkage estimator (t_s^{G}).

Ratio-cum-product estimator $\alpha=1,\beta=1$	Product -cum-product estimator $\alpha=-1,\beta=-1$	a	λ	b
$t_s^{1-err}=\lambda\bar{y}_{ts}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}+\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{Z}_{ts}-\bar{z}_{ts(2)}}{\bar{Z}_{ts}+\bar{z}_{ts(2)}}\right)$	$t_s^{2-epp}=\lambda\bar{y}_{ts}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{X}_{ts}+\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}+\bar{z}_{ts(2)}}\right)$	2	λ	2
$t_s^{3-err}=\lambda\bar{y}_{ts}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}}\right)\exp\left(\frac{\bar{Z}_{ts}-\bar{z}_{ts(2)}}{\bar{Z}_{ts}}\right)$	$t_s^{4-epp}=\lambda\bar{y}_{ts}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{X}_{ts}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}}\right)$	1	λ	1
$t_s^{5-erp}=\lambda\bar{y}_{ts}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}+(a-1)\bar{x}_{ts}}\right)\exp\left(\frac{\bar{Z}_{ts}-\bar{z}_{ts(2)}}{\bar{Z}_{ts}+(b-1)\bar{z}_{ts(2)}}\right)$	$t_s^{6-epp}=\lambda\bar{y}_{ts}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{X}_{ts}+(a-1)\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}+(b-1)\bar{z}_{ts(2)}}\right)$	a	λ	b
Product -cum-ratio estimator $\alpha=-1,\beta=1$	**Ratio-cum-product estimator** $\alpha=1,\beta=-1$	a	λ	b
$t_s^{7-epr}=\lambda\bar{y}_{ts}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}+\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}+\bar{z}_{ts(2)}}\right)$	$t_s^{8-erp}=\lambda\bar{y}_{ts(2)}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{x}_{ts(1)}+\bar{X}_{ts}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{z}_{ts(2)}+\bar{Z}_{ts}}\right)$	2	λ	2
$t_s^{9-epr}=\lambda\bar{y}_{ts(2)}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}}\right)\exp\left(\frac{\bar{z}_{ts(1)}-\bar{Z}_{ts}}{\bar{Z}_{ts}}\right)$	$t_s^{10-erp}=\lambda\bar{y}_{ts(2)}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{X}_{ts}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}}\right)$	1	λ	1
$t_s^{11-epr}=\lambda\bar{y}_{ts(2)}\exp\left(\frac{\bar{X}_{ts}-\bar{x}_{ts(1)}}{\bar{X}_{ts}+(a-1)\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}+(b-1)\bar{z}_{ts(2)}}\right)$	$t_s^{12-erp}=\lambda\bar{y}_{ts(2)}\exp\left(\frac{\bar{x}_{ts(1)}-\bar{X}_{ts}}{\bar{X}_{ts}+(a-1)\bar{x}_{ts(1)}}\right)\exp\left(\frac{\bar{z}_{ts(2)}-\bar{Z}_{ts}}{\bar{Z}_{ts}+(b-1)\bar{z}_{ts(2)}}\right)$	a	λ	b

Appendix C

Table C1. Data Statistics for Population-I.

	Population-I (unequal fsu's)			
Cluster	1	2	3	4
M_i	18	14	12	20
m_i	9	7	6	10
$\bar{Y}_i$	25.77722	22.79286	28.43500	23.0905
$\bar{X}_i$	51.06389	46.49700	67.00217	57.11855
$C_{y_i}{}^2$	0.58025	0.39297	0.34783	0.31545
$C_{x_i}{}^2$	0.43322	0.29984	0.41947	0.40689
ρ_{i1}	0.88373	0.83895	0.82425	0.82113

Table C2. *MSEs* and PREs of the adapted and suggested class of estimators.

Estimators	MSE	PRE With new fraction
$\bar{y}_s$	24.44649	100
t^{1-err}	11.66857	209.51
t^{3-err}	20.27969	120.5467
t^{5-err}	11.46787	213.1738
t_s^{1-err}	11.58499	211.0229
t_s^{3-err}	18.92219	129.1974
t_s^{5-err}	11.25906	217.1317
t_s^{G-opt}	11.25906	217.1317