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Research article

Some properties of η-convex stochastic processes

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Abstract: The stochastic processes is a significant branch of probability theory, treating probabilistic models that develop in time. It is a part of mathematics, beginning with the axioms of probability and containing a rich and captivating arrangement of results following from those axioms. In probability, a convex function applied to the expected value of an random variable is always bounded above by the expected value of the convex function of the random variable. The definition of η-convex stochastic process is introduced in this paper. Moreover some basic properties of η-convex stochastic process are derived. We also derived Jensen, Hermite–Hadamard and Ostrowski type inequalities for η-convex stochastic process.

Keywords: stochastic process; η-convex function; η-convex stochastic processes; Hermite Hadmard type inequality; Ostrowski type inequality and Jensen inequality
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1. Introduction

In probability theory and other related fields, a stochastic process is a mathematical tool generally characterized as a group of random variables. Verifiably, the random variables were related with or listed by a lot of numbers, normally saw as focuses in time, giving the translation of a stochastic process speaking to numerical estimations some system randomly changing over time, for example, the development of a bacterial populace, an electrical flow fluctuating because of thermal noise, or the development of a gas molecule. Stochastic processes are broadly utilized as scientific models of systems that seem to shift in an arbitrary way. They have applications in numerous areas including sciences, for example, biology, chemistry, ecology, neuroscience, and physics as well as technology and engineering fields, for example, picture preparing, signal processing, data theory, PC science, cryptography and telecommunications. Furthermore, apparently arbitrary changes in money related markets have inspired the broad utilization of stochastic processes in fund. For the detailed survey about convex functions, inequality theory and applications, we refer [1–4] and references therein.

The study of convex stochastic process was initiated by Nikodem in 1980 [5]. He also investigated some regularity properties of convex stochastic process. Later on, some further results on convex stochastic process are derived in 1992 by Skowronski [6]. In recent developments on convex stochastic process, Kotrys [7] investigated Hermite–Hadamard type inequality for convex stochastic process and gave results for strongly convex stochastic process. In [8], the inequality for h-convex stochastic process were derived. The interesting work on stochastic process are [17–21].

The aim of this paper is to introduce the notion of η-convex stochastic process and derive Hermite–Hadamard and Jensen Type inequality for η-convex stochastic process. The main motivation for this paper is the idea of ϕ-convex function and η-convex function [9, 10], respectively. For other interesting generalizations, we refer [13–16, 22–25] to the readers and references therein.

The mapping ξ defined for a σ field Ω to $\mathbb{R}$ is F-measurable for each Borel set $B \in \beta(\mathbb{R})$, if

$$\{\omega \in \Omega / \xi(\omega) \in B\} \in F.$$

For a probability space (Ω, F, P), the mapping ξ is said to be random variable. The random variable ξ becomes integrable if

$$\int_{\Omega} |\xi| dp < \infty.$$

If the random variable ξ is integrable, then $E(\xi) = \int_{\Omega} \xi dp$ exists and is called expectation of ξ. The family of integrable random variables $\xi : \Omega \to \mathbb{R}$ is denoted by $L^{'}(\Omega, F, P)$.

Now we present the definition and basic properties of mean-square integral [11].

Suppose that $\xi_1 : I \times \Omega \to \mathbb{R}$ is a stochastic process with $E[\xi_1(t)^2] < \infty$ for all $t \in I$ and $[a, b] \in I$, $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ is a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, \cdots, n$. Further, suppose that $\xi_2 : I \times \Omega \to \mathbb{R}$ be a random variable. Then, it is said to be mean-square integral of the process ξ_1 on $[a, b]$, if for each normal sequence of partitions of the interval $[a, b]$ and for each $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, \cdots, n$, we have

$$\lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} \xi_1(\Theta_k)\Delta(t_k - t_{k-1}) - \xi_2\right)^2\right] = 0.$$

Then, we can write

$$\xi_2(.) = \int_a^b \xi_1(s, .)ds \quad (a.e.). \tag{1.1}$$

The monotonicity of the mean square integral will be used frequently throughout the paper. If $\xi_1(t, .) \leq \xi_2(t, .)$ (a.e.) for the interval $[a, b]$, then

$$\int_a^b \xi_1(t, .)dt \leq \int_a^b \xi_2(t, .)dt \quad (a.e.) \tag{1.2}$$

The inequality (1.2) is the immediate consequence of the definition of the mean-square integral.

Lemma 1.1. *If $X : I \times \Omega \to \mathbb{R}$ is a stochastic process of the form $X(t, .) = A(.)t + B(.)$, where $A, B : \Omega \to \mathbb{R}$ are random variables such that $E[A^2] < \infty, E[B^2] < \infty$ and $[a, b] \subset I$, then*

$$\int_a^b X(t, .)dt = A(.)\frac{b^2 - a^2}{2} + B(.)(b - a) \quad (a.e.). \tag{1.3}$$

Now, we present the definition of η-convex stochastic process.

Definition 1.2. Let (Ω, A, P) be a probability space and $I \subseteq \mathbb{R}$ be an interval, then $\xi : I \times \Omega \to \mathbb{R}$ is an η-convex stochastic process, if

$$\xi(\lambda b_1 + (1 - \lambda)b_2, .) \leq \xi(b_2, .) + \lambda\eta(\xi(b_1, .), \xi(b_2, .)) \quad (a.e) \tag{1.4}$$

for all $b_1, b_2 \in I$ and $\lambda \in [0, 1]$.

In (1.4), if we take $\eta(b_1, b_2) = b_1 - b_2$, we obtain convex stochastic process. By taking $\xi(b_1, .) = \xi(b_2, .)$ in (1.4) we get

$$\lambda\eta(\xi(b_1, .), \xi(b_1, .)) \geq 0$$

for any $b_1 \in I$ and $t \in [0, 1]$. Which implies that

$$\eta(\xi(b_1, .), \xi(b_1, .)) \geq 0$$

for any $b_1 \in I$.

Also, if we take $\lambda = 1$ in (1.4), we get

$$\xi(b_1, .) - \xi(b_2, .) \leq \eta(\xi(b_1, .), \xi(b_2, .))$$

for any $b_1, b_2 \in I$. The second condition implies the first one, so if we want to define η convex stochastic process on an interval I of real numbers, we should assume that

$$\eta(b_1, b_2) \geq b_1 - b_2 \tag{1.5}$$

for any $b_1, b_2 \in I$.

One can observe that, if $\xi : I \to \mathbb{R}$ is convex stochastic process and $\eta : \xi(I) \times \xi(I) \to \mathbb{R}$ is an arbitrary bi-function that satisfies the condition (1.5), then for any $b_1, b_2 \in I$ and $t \in [0, 1]$, we have

$$\begin{aligned} \xi(tb_1 + (1 - t)b_2, .) &\leq \xi(b_2, .) + \lambda(\xi(b_1, .) - \xi(b_2, .)) \\ &\leq \xi(b_2, .) + \lambda\eta(\eta(b_1, .), \eta(b_2, .)), \end{aligned}$$

which tells that ξ is η convex stochastic process.

Definition 1.3. (η Quasi-convex stochastic process) A stochastic Process $\xi : I \times \Omega \to \mathbb{R}$ is said to be η quasi-convex stochastic process if

$$\xi(tb_1 + (1 - t)b_2, .) \leq max\{\xi(b_2, .), \xi(b_2, .) + \eta(\xi(b_1, .), \xi(b_2, .)) \quad (a.e.)$$

Definition 1.4. (η-affine) A stochastic process $\xi : I \times \Omega \to \mathbb{R}$ is said to be η-affine if

$$\xi(tb_1 + (1 - t)b_2, .) = \xi(b_2, .) + t\eta(\xi(b_1, .), \xi(b_2, .)) \quad (a.e.)$$

for all $b_1, b_2 \in I$ and $t \in [0, 1]$

Definition 1.5. (Non-Negatively Homogeneous)A function $\eta : A \times B \to \mathbb{R}$ is said to be non-negatively homogenous if

$$\eta(\gamma b_1, \gamma b_2) = \gamma\eta(b_1, b_2) \tag{1.6}$$

for all $b_1, b_2 \in \mathbb{R}$ and $\gamma \geq 0$.

Definition 1.6. (Additive)A function η is said to be additive if

$$\eta(x_1, y_1) + \eta(x_2, y_2) = \eta(x_1 + x_2, y_1 + y_2) \tag{1.7}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Definition 1.7. (Non-negatively linear function)A function η is said to be non-negatively linear, if it satisfy (1.6) and (1.7).

Definition 1.8. (Non-decreasing in first variable)A function η is said to be non-decreasing in first variable if $b_1 \leq b_2$ implies $\eta(b_1, b_3) \leq \eta(b_2, b_3)$ for all $b_1, b_2, b_3 \in \mathbb{R}$.

Definition 1.9. (Non-negatively sub-linear in first variable)A function η is said to be non-negatively sub-linear in first variable if

$$\eta(\gamma(b_1 + b_2), b_3) \leq \gamma\eta(b_1, b_3) + \gamma\eta(b_2, b_3)$$

for all $b_1, b_2, b_3 \in \mathbb{R}$ and $\gamma \geq 0$.

We shall begin with few preliminary proposition for η-convex function.

Proposition 1. *Consider two η convex stochastic process $\xi_1, \xi_2 : I \times \Omega \to \mathbb{R}$, such that*

1. *If η is additive then $\xi_1 + \xi_2 : I \to \mathbb{R}$ is η convex stochastic process.*
2. *If η is non-negatively homogenous, then for any $\gamma \geq 0$, $\gamma\xi_1 : I \times \Omega \to \mathbb{R}$ is η- convex stochastic process.*

Proof. The proof of the proposition is straight forward. □

Proposition 2. *If $\xi : [b_1, b_2] \to \mathbb{R}$ is η convex stochastic process, then*

$$\max_{x\in[b_1,b_2]} \xi(x, .) \leq max\{\xi(b_2, .), \xi(b_2, .) + \eta(\xi(b_1, .), \xi(b_2, .))\}.$$

Proof. Consider $x = \alpha b_1 + (1 - \alpha)b_2$ for arbitrarily $x \in [b_1, b_2]$ and some $\alpha \in [0, 1]$. We can write

$$\xi(x, .) = \xi(\alpha b_1 + (1 - \alpha)b_2, .)\,.$$

Since ξ is η convex stochastic process, so by definition

$$\xi(x, .) \leq \xi(b_2, .) + \alpha\eta(\xi(b_1, .), \xi(b_2, .)) \tag{1.8}$$

and

$$\xi(b_2, .) + \alpha\eta(\xi(b_1, .), \xi(b_2, .)) \leq max\{\xi(b_2, .), \xi(b_2, .) + \eta(\xi(b_1, .), \xi(b_2, .))\,. \tag{1.9}$$

Since x is arbitrary, so from (1.8) and (1.9), we get our desired result. □

Theorem 1.10. *A random variable $\xi : I \times \Omega \to \mathbb{R}$ is η convex stochastic process if and only if for any $c_1, c_2, c_3 \in I$ with $c_1 \le c_2 \le c_3$, we have*

$$det\begin{pmatrix} (c_3 - c_2) & \xi(c_2, .) - \xi(c_3, .) \\ (c_3 - c_1) & \eta(\xi(c_1, .), \xi(c_3, .)) \end{pmatrix} \ge 0\,.$$

Proof. Suppose that ξ is an η-convex stochastic process and $c_1, c_2, c_3 \in I$ such that $c_1 \le c_2 \le c_3$. Then, their exits $\alpha_1 \in (0, 1)$, such that

$$c_2 = \alpha_1 c_1 + (1 - \alpha_1)c_3$$

where $\alpha_1 = \frac{c_2 - c_3}{c_1 - c_3}$.

By definition of η convex stochastic process, we have

$$\begin{aligned} \xi(c_2, .) &= \xi(\alpha_1 c_1 + (1 - \alpha_1)c_3, .) \\ &\le \xi(c_3, .) + \left(\frac{c_2 - c_3}{c_1 - c_3}\right)\eta(\xi(c_1, .), \xi(c_3, .)) \end{aligned}$$

so

$$0 \le \xi(c_3, .) - \xi(c_2, .) + \frac{(c_2 - c_3)}{(c_1 - c_3)}\eta(\xi(c_1, .), \xi(c_3, .))$$

$$0 \le (\xi(c_3, .) - \xi(c_2, .))(c_3 - c_1) + (c_3 - c_2)\eta(\xi(c_1, .), \xi(c_3, .))\ .$$

Hence

$$det\begin{pmatrix} (c_3 - c_2) & \xi(c_2, .) - \xi(c_3, .) \\ (c_3 - c_1) & \eta(\xi(c_1, .), \xi(c_3, .)) \end{pmatrix} \ge 0\,.$$

For the reverse inequality, take $y_1, y_2 \in I$ with $y_1 \le y_2$. Choose any $\alpha_1 \in (0, 1)$, then, we have

$$y_1 \le \alpha_1 y_1 + (1 - \alpha_1)y_2 \le y_2.$$

So, the above determinant is;

$$0 \le [y_2 - [\alpha_1 y_1 + (1 - \alpha_1)y_2]]\,\eta(\xi(y_1, .), \xi(y_2, .)) - (y_2 - y_1)(\xi(\alpha_1 y_1 + (1 - \alpha_1)y_2, .) - \xi(y_2, .)$$

implies

$$\begin{aligned} (\xi(\alpha_1 y_1 + (1 - \alpha_1)y_2, .) &\le \xi(y_2, .) + \alpha_1\frac{(y_2 - y_1)}{(y_2 - y_1)}\eta(\xi(y_1, .), \xi(y_2, .)) \\ &\le \xi(y_2, .) + \alpha_1\eta(\xi(y_1, .), \xi(y_2, .)). \end{aligned}$$

Which is as required. □

2. Jensen type inequality

We will use the following relation to prove the Jesen type inequality for η-convex stochastic process. Let $\xi : I \times \Omega \to \mathbb{R}$ be an η-convex stochastic process. For $x_1, x_2 \in$ I and $\alpha_1 + \alpha_2$=1, we have

$$\xi(\alpha_1 x_1 + \alpha_2 x_2, .) \le \xi(x_2, .) + \alpha_1\eta(\xi(x_1, .), \xi(x_2, .))\,.$$

Also, when $n > 2$ for $x_1, x_2, ..., x_n \in I$, $\sum\limits_{i=1}^{n} \alpha_i = 1$ and $T_i = \sum\limits_{j=1}^{i} \alpha_j$, we have

$$\begin{aligned}\xi\left(\sum_{i=1}^{n} \alpha_i x_i, .\right) &= \xi\left(T_{n-1} \sum_{i=1}^{n-1} \frac{\alpha_i}{T_{n-1}} x_i + \alpha_n x_n, .\right) \\ &\leq \xi(x_n, .) + T_{n-1}\eta\left(\xi\left(\sum_{i=1}^{n-1} \frac{\alpha_i}{T_{n-1}} x_i, .\right), \xi(x_n, .)\right) \end{aligned} \tag{2.1}$$

Theorem 2.1. *Let $\xi : I \times \Omega \to \mathbb{R}$ be an η-convex stochastic process and $\eta : A \times B \to \mathbb{R}$ be the non-decreasing non-negatively sub-linear in first variable. If $T_i = \sum\limits_{j=1}^{i} \alpha_j$ for $i = 1, 2, ..., n$ such that $T_n = 1$, then*

$$\xi\left(\sum_{i=1}^{n} \alpha_i x_i, .\right) \leq \xi(x_n, .) + \sum_{i=1}^{n-1} T_i \eta_\xi(x_i, x_{i+1}, ..., x_n) \tag{2.2}$$

where $\eta_\xi(x_i, x_{i+1}, ..., x_n) = \eta(\eta_\xi(x_i, x_{i+1}, ..., x_{n-1}, .), \xi(x_n, .))$ and $\eta_\xi(x, .) = \xi(x, .)$ for all $x \in I$.

Proof. Since η is non-decreasing, non-negatively, sub-linear in first variable, so from (2.1)

$$\begin{aligned}\xi(\sum_{i=1}^{n} \alpha_i x_i, .) &\leq \xi(x_n, .) + T_{n-1}\eta\left(\xi\left(\sum_{i=1}^{n-1} \frac{\alpha_i}{T_{n-1}} x_i\right), f(x_n)\right) \\ &= \xi(x_n, .) + T_{n-1}\eta\left(\xi\left(\frac{T_{n-2}}{T_{n-1}} \sum_{i=1}^{n-2} \frac{\alpha_i}{T_{n-2}} x_i, \quad + \frac{\alpha_{n-1}}{T_{n-1}} x_{n-1}, .\right), \xi(x_n, .)\right) \\ &\leq \xi(x_n, .) + T_{n-1}\eta(\xi(x_{n-1}, .) + \left(\frac{T_{n-2}}{T_{n-1}}\right)\eta\left(\xi\left(\sum_{i=1}^{n-2} \frac{\alpha_i}{T_{n-2}} x_i, \xi(x_{n-1}, .)\right), \xi(x_n, .)\right) \\ &\leq \xi(x_n, .) + T_{n-1}\eta(\xi(x_{n-1}, x.), \xi(x_n, .)) + T_{n-2}\eta(\eta\left(\xi\left(\sum_{i=1}^{n-2} \frac{\alpha_i}{T_{n-2}} x_i, .\right), \xi(x_{n-1}, .)), \xi(x_n, .)\right) \\ &\leq \dots \\ &\leq (x_n) + T_{n-1}\eta(\xi(xn-1, .), \xi(x_n, .)) + T_{n-2}\eta(\xi(x_{n-2}, .), \xi(x_{n-1}, .)), \xi(x_n, .)) \\ &\qquad + \dots + T_1(\eta(xi(x_1, .), \xi(x_2, .)), \xi(x_3, .)) \dots, \xi(x_{n-1}, .)), \xi(x_n, .)) \\ &= \xi(x_n, .) + \sum_{i=1}^{n-1} T_i \eta_\xi(x_i, x_{i+1}, \dots, x_n, .).\end{aligned}$$

Hence the proof is complete. □

3. Hermite–Hadamard inequality

Now, we established new inequality for the η-convex stochastic process that is connected with the Hermite–Hadamard inequality.

Theorem 3.1. *Suppose that $\xi : [c_1, c_2] \times \Omega \to \mathbb{R}$ is an η convex stochastic process such that η is bounded above $\xi[c_1, c_2] \times \xi[c_1, c_2]$, then*

$$\begin{aligned}
\xi\left(\frac{c_1+c_2}{2}\right) - \frac{1}{2}M_\eta &\le \frac{1}{c_2-c_1}\int_{c_1}^{c_2} \xi(y,.)dy \\
&\le \frac{1}{2}[\xi(c_1,.)+\xi(c_2,.)] + \frac{1}{2}\left[\frac{\eta(\xi(c_1,.),\xi(c_2,.)) + \eta(\xi(c_2,.),\xi(c_1,.))}{2}\right] \\
&\le \frac{1}{2}[\xi(c_1,.)+\xi(c_2,.)] + \frac{1}{2}M_\eta \qquad (3.1)
\end{aligned}$$

where M_η is upper bound of η.

Proof. For the right side of inequality, consider an arbitrary point $y = \alpha_1 c_1 + (1-\alpha_1)c_2$ with $\alpha_1 \in [0,1]$. We can write as

$$\xi(y,.) = \xi((\alpha_1 c_1 + (1-\alpha_1)c_2),.)\,.$$

Since ξ is η convex stochastic process, so by definition

$$\xi(y,.) \le \xi(c_2,.) + \alpha_1 \eta(\xi(c_1,.),\xi(c_2,.))$$

with $\alpha_1 = \frac{y-c_2}{c_1-c_2}$. It follows that

$$\xi(y,.) \le \xi(c_2,.) + \left(\frac{y-c_2}{c_1-c_2}\right)\eta(\xi(c_1,.),\xi(c_2,.))\,.$$

Now, using Lemma 1.1, we get

$$\begin{aligned}
\xi(y,.) &\le \frac{1}{c_2-c_1}\left[\left(\xi(c_2,.)(c_2-c_1) + \frac{(c_2-c_1)}{2}\eta(\xi(c_1,.), \quad \xi(c_2,.))\right)\right]\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\xi(y,.)dy \\
&\le \xi(c_2,.) + \frac{1}{2}\eta(\xi(c_1,.),\xi(c_2,.))\,.
\end{aligned}$$

Also, we have

$$\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\xi(y,.)dy \le \xi(c_1,.) + \frac{1}{2}\eta(\xi(c_2,.),\xi(c_1,.))\,.$$

Therefore, we get

$$\begin{aligned}
\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\xi(y,.)dy &\le \min\left\{\xi(c_2,.) + \frac{1}{2}\eta(\xi(c_1,.),\xi(c_2,.)), \xi(c_1,.) \ + \frac{1}{2}\eta(\xi(c_2,.),\xi(c_1,.))\right\} \\
&\le \frac{1}{2}[\xi(c_1,.)+\xi(c_2,.)] + [\eta(\xi(c_1,.),\xi(c_2,.)) \ + \eta(\xi(c_2,.),\xi(c_1,.))] \\
&\le \frac{1}{2}[\xi(c_1,.)+\xi(c_2,.)] + M_\eta\,,
\end{aligned}$$

where $M_\eta = [\eta(\xi(c_1,.),\xi(c_2,.)) + \eta(\xi(c_2,.),\xi(c_1,.))]\,.$

For the left side of inequality, the definition of η-convex stochastic process of ξ implies that

$$\begin{aligned}
\xi\left(\frac{c_1+c_2}{2},.\right) &= \xi\left(\frac{c_1+c_2}{4}-\frac{\alpha_1(c_2-c_1)}{4}+\frac{c_1+c_2}{4}+\frac{\alpha_1(c_2-c_1)}{4},.\right)\\
&= \xi\left(\frac{1}{2}\frac{c_1+c_2-\alpha_1(c_2-c_1)}{2}+\frac{1}{2}\frac{c_1+c_2+\alpha_1(c_2-c_1)}{2},.\right)\\
&\leq \xi\left(\frac{c_1+c_2+\alpha_1(c_2-c_1)}{2},.\right)+\left(\frac{1}{2}\right)\eta\left(\xi\left(\frac{c_1+c_2-\alpha_1(c_2-c_1)}{2}\right),\ \xi\left(\frac{1}{2}\frac{c_1+c_2-\alpha_1(c_2-c_1)}{2},.\right)\right)\\
&\leq \xi\left(\frac{c_1+c_2+\alpha_1(c_2-c_1)}{2},.\right)+\frac{1}{2}M_\eta \ \ \forall\ \ \alpha_1\in[0,1]\,.
\end{aligned}$$

Here

$$\left(\frac{c_1+c_2+\alpha_1(c_2-c_1)}{2},.\right) \qquad \geq \xi\left(\frac{c_1+c_2}{2},.\right)-\frac{1}{2}M_\eta \quad (a.e.) \tag{3.2}$$

and

$$\xi\left(\frac{c_1+c_2-\alpha_1(c_2-c_1)}{2},.\right)\geq \xi\left(\frac{c_1+c_2}{2},.\right)-\frac{1}{2}M_\eta \quad (a.e.)\,. \tag{3.3}$$

Finally, using change of variable, we have

$$\begin{aligned}
\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\xi(y,.)dy &= \frac{1}{c_2-c_1}\left[\int_{c_1}^{\frac{c_1+c_2}{2}}\xi(y,.)dy+\int_{\frac{c_1+c_2}{2}}^{c_2}\xi(y,.)dy\right]\\
&= \frac{1}{2}\int_0^1\left[\xi\left(\frac{c_1+c_2-\alpha_1(c_2-c_1)}{2},.\right)+\xi\left(\frac{c_1+c_2+\alpha_1(c_2-c_1)}{2},.\right)\right]d\alpha_1\,.
\end{aligned}$$

From (3.2) and (3.3), we get

$$\begin{aligned}
\frac{1}{c_2-c_1}\int_{c_1}^{c_2}\xi(y,.)dy &\geq \frac{1}{2}\int_0^1\left[\xi\left(\frac{c_1+c_2}{2},.\right)-\frac{1}{2}M_\eta+\xi\left(\frac{c_1+c_2}{2},.)-\frac{1}{2}M_\eta\right)\right]d\alpha_1\\
&\geq \frac{1}{2}\int_0^1\left[2\xi\left(\frac{c_1+c_2}{2},.\right)-\frac{2}{2}M_\eta\right]d\alpha_1\\
&\geq \frac{1}{2}\left[2\xi\left(\frac{c_1+c_2}{2},.\right)-M_\eta\right]\\
&\geq \xi\left(\frac{c_1+c_2}{2},.\right)-\frac{1}{2}M_\eta\,.
\end{aligned}$$

Hence, the proof is completed. □

Remark 1. By taking $\eta(x,y)=x-y$ in (3.1), we get the classical Hermite–Hadamard inequality for convex stochastic process [7].

4. Ostrowski type inequality

In order to prove Ostrowski type inequality for η-convex stochastic process, the following Lemma is required.

Lemma 4.1. *[12] Let $\xi : I \times \Omega \to \mathbb{R}$ be a stochastic process which is mean square differentiable on I°. If $\xi^{'}$ is mean square integrable on $[c_1, c_2]$, where $c_1, c_2 \in I$ with $c_1 < c_2$, then the following equality holds*

$$\xi(t,.) - \frac{1}{c_2 - c_1}\int_{c_1}^{c_2} \xi(u,.)du = \frac{(x-c_1)^2}{c_2-c_1}\int_0^1 t\xi^{'}(tx + (1-t)c_1, .)dt - \frac{(c_2-x)^2}{c_2-c_1}\int_0^1 t\xi^{'}(tx+(1-t)c_2, .)dt, (a.e.), \tag{4.1}$$

for each $x \in [c_1, c_2]$.

Theorem 4.2. *Let $\xi : I \times \Omega \to \mathbb{R}$ be a mean square stochastic process such that $\xi^{'}$ is mean square integrable on $[c_1, c_2]$, where $c_1, c_2 \in I$ with $c_1 < c_2$. If $|\xi^{'}|$ is an η- convex stochastic process on I and $|\xi^{'}(t,.)| \le M$ for every t, then*

$$\begin{aligned}
&\left|\xi(t,.) - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\xi(u,.)du\right| \\
&\le \frac{M}{2}\left[\frac{(t-c_1)^2 + (c_2-t)^2}{c_2-c_1}\right] + \frac{(t-c_1)^2}{3(c_2-c_1)}\eta(|\xi^{'}(t,.)|, |\xi^{'}(c_1,.)|) + \frac{(c_2-t)^2}{3(c_2-c_1)}\eta(|\xi^{'}(t,.)|, |\xi^{'}(c_2,.)|), (a.e.).
\end{aligned}$$

Proof. Since $|\xi^{'}|$ is an η–convex stochastic process, so by (4.1), we have

$$\begin{aligned}
&\left|\xi(t,.) - \frac{1}{c_2-c_1}\int_{c_1}^{c_2}\xi(u,.)du\right| \\
&\le \frac{(t-c_1)^2}{c_2-c_1}\int_0^1 y|\xi^{'}(yt+(1-y)c_1,.)|dy + \frac{(c_2-t)^2}{c_2-c_1}\int_0^1 y|\xi^{'}(yt+(1-y)c_2,.)|dy \\
&\le \frac{(t-c_1)^2}{c_2-c_1}\int_0^1 y\left[|\xi^{'}(c_1,.)| + y\eta(|\xi^{'}(t,.)|, |\xi^{'}(c_1,.)|)\right]dy + \frac{(c_2-t)^2}{c_2-c_1}\int_0^1 y\left[|\xi^{'}(c_2)| + y\eta(|\xi^{'}(t,.)|, |\xi^{'}(c_2,.)|)\right]dy \\
&\le M\left[\frac{(t-c_1)^2+(c_2-t)^2}{c_2-c_1}\right]\int_0^1 ydy + \frac{(t-c_1)^2}{c_2-c_1}\int_0^1 y^2\eta(|\xi^{'}(t,.)|, |\xi^{'}(c_1,.)|)dt + \frac{(c_2-t)^2}{c_2-c_1}\int_0^1 y^2\eta(|\xi^{'}(t,.)|, |\xi^{'}(c_2,.)|)dy \\
&\le \frac{M}{2}\left[\frac{(t-c_1)^2+(c_2-t)^2}{c_2-c_1}\right] + \frac{(t-c_1)^2}{3(c_2-c_1)}\eta(|\xi^{'}(t,.)|, |\xi^{'}(c_1,.)|) + \frac{(c_2-t)^2}{3(c_2-c_1)}\eta(|\xi^{'}(t,.)|, |\xi^{'}(c_2,.)|).
\end{aligned}$$

Hence proof is completed. □

5. Conclusions

There are many applications of Stochastic-processes, for example, Kolmogorov-Smirnoff test on equality of distributions [26–28]. The other application includes Sequential Analysis [29, 30] and Quickest Detection [31, 32]. In this paper, we introduced η-convex Stochastic processes and proved Jensen, Hermite-Hadamard and Fejr type inequalities. Our results are applicable, because the expected value of a random variable is always bounded above by the expected value of the convex function of the random variable.

Conflict of interest

The authors declare that no competing interests exist.

References

1. Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, S. M. Kang, Generalized Riemann-Liouville k-Fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities. *IEEE Access*, **6** (2018), 64946–64953.
2. Y. C. Kwun, G. Farid, G. S. Ullah, W. Nazeer, K. Mahreen, S. M. Kang, Inequalities for a unified integral operator and associated results in fractional calculus, *IEEE Access*, **7** (2019), 126283–126292.
3. S. M. Kang, G. Farid, W. Nazeer, B. Tariq, Hadamard and FejrHadamard inequalities for extended generalized fractional integrals involving special functions, *J. Inequalities Appl.*, **2018** (2018), 119.
4. S. M. Kang, G. Abbas, G. Farid, W. Nazeer, A generalized FejrHadamard inequality for harmonically convex functions via generalized fractional integral operator and related results, *Mathematics*, 6 (2018), 122.
5. K. Nikodem, On convex stochastic processes, *Aequationes Math.*, **20** (1980), 184–197.
6. A. Skowronski, On some properties ofj-convex stochastic processes. *Aequationes Math.*, **44** (1992), 249–258.
7. D. Kotrys, Hermite-Hadamard inequality for convex stochastic processes, *Aequationes Math.*, **83** (2012), 143–151.
8. D. Barrez, L. Gonzlez, N. Merentes, A. Moros, On h-convex stochastic processes, *Math. Aeterna*, **5** (2015), 571–581.
9. M. R. Delavar, S. S. Dragomir, On η-convexity, *Math. Inequal. Appl.*, **20** (2017), 203–216.
10. K. Sobczyk, Stochastic differential equations: With applications to physics and engineering (Vol. 40), *Springer Science Business Media*, (2013).
11. M. E. Gordji, M. R. Delavar, M. De La Sen, On Φ-convex functions, *J. Math. Inequal.*, **10** (2016), 173–183.
12. L. Gonzales, J. Materano, M. V. Lopez, Ostrowski-Type inequalities via hconvex stochastic processes, *JP J. Math. Sci.*, **16** (2016), 15–29.
13. M. Alomari, M. Darus, U. S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, *Comput. Math. Appl.*, **59** (2010), 225–232.
14. G. A. Anastassiou, Ostrowski type inequalities, *Proc. Am. Math. Society*, **123** (1995), 3775–3781.
15. P. Cerone, S. S. Dragomir, J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *RGMIA Res. Rep. Collect.*, **1** (1998).
16. S. S. Dragomir, W. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules. *Appl. Math. Lett.*, **11** (1998), 105–109.

17. J. Materano, N. Merentes, M. Valera-Lopez, Some estimates on the Simpsons type inequalities through s- convex and quasiconvex stochastic processes, *Math. Aeterna*, **5** (2015), 673–705.
18. E. Set, M. Z. Sarikaya, M. Tomar, Hermite-Hadamard type inequalities for coordinates convex stochastic processes, *Math. Aeterna*, **5** (2015), 363–382.
19. K. Nikodem, On quadratic stochastic processes. *Aequationes Math.*, **21** (1980), 192–199.
20. L. Gonzlez, N. Merentes, M. Valera-Lpez, Some estimates on the Hermite-Hadamard inequality through convex and quasi-convex stochastic processes, *Math. Aeterna.*, **5** (2015), 745–767.
21. B. P. Rao, Identifiability in Stochastic Models, *Acad. Press, Boston*, (1992).
22. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermite-Hadamard inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Modell.*, **57** (2013), 2403–2407.
23. S. S. Dragomir, A. McAndrew, Refinements of the Hermite-Hadamard inequality for convex functions, *J. Inequal. Pure. Appl. Math.*, **6** (2005).
24. F. C. Mitroi, C. I. Spiridon, Refinements of Hermite-Hadamard inequality on simplices, *arXiv preprint arXiv:1105.5043*, (2011).
25. M. Rassouli, S. S. Dragomir, Refining recursively the Hermite-Hadamard inequality on a simplex, *Bull. Aust. Math. Soc.*, **92** (2015), 57–67.
26. S. Engmann, D. Cousineau, Comparing distributions: The two-sample Anderson-Darling test as an alternative to the Kolmogorov-Smirnoff test, *J. Appl. Quant. Methods*, **6** (2011), 1–17.
27. B. B. Bhattacharya, Two-Sample Tests Based on Geometric Graphs: Asymptotic Distribution and Detection Thresholds, *arXiv preprint arXiv:1512.00384*, (2015).
28. R. Dey, Hypothesis tests with precedence probabilities and precedence-type tests, *Wiley Interdiscip. Rev: Comput. Stat.*, **10** (2018), e1417.
29. D. Siegmund, Sequential analysis: Tests and confidence intervals, *Springer Sci. Bus. Media*, (2013).
30. P. K. Andersen, O. Borgan, R. D. Gill, N. Keiding, Statistical models based on counting processes, *Springer Science Bus. Media*, (2012).
31. A. Delorme, T. Sejnowski, S. Makeig, Enhanced detection of artifacts in EEG data using higher-order statistics and independent component analysis, *Neuroimage*, **34** (2007), 1443–1449.
32. P. Dollr, R. Appel, S. Belongie, P. Perona, Fast feature pyramids for object detection, *IEEE Trans. Pattern Anal. Mach. Intell.*, **36** (2014), 1532–1545.

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