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Research article

On inequalities of Hermite-Hadamard-Mercer type involving Riemann-Liouville fractional integrals

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Abstract: The goal of this article is to establish many inequalities of Hermite-Hadamard-Mercer type involving Riemann-Liouville fractional operators. We also establish some related fractional integral inequalities connected to the left side of Hermite-Hadamard-Mercer type inequality for differentiable convex functions. Further remarks and observations for these results are given. Finally, we see the efficiency of our inequalities via some applications on special means.

Keywords: Riemann-Liouville fractional integral; Hermite-Hadamard-Mercer inequality; Jensen-Mercer inequality
Mathematics Subject Classification: 26D07, 26D10, 26D15

1. Introduction

For a convex function $\sigma : \mathrm{I} \subseteq \mathbf{R} \to \mathbf{R}$ on I with $\nu_1, \nu_2 \in \mathrm{I}$ and $\nu_1 < \nu_2$, the Hermite-Hadamard inequality is defined by [1]:

$$\sigma\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\sigma(\eta)d\eta \leq \frac{\sigma(\nu_1)+\sigma(\nu_2)}{2}. \tag{1.1}$$

The Hermite-Hadamard integral inequality (1.1) is one of the most famous and commonly used inequalities. The recently published papers [2–4] are focused on extending and generalizing the convexity and Hermite-Hadamard inequality.

The situation of the fractional calculus (integrals and derivatives) has won vast popularity and significance throughout the previous five decades or so, due generally to its demonstrated applications in numerous seemingly numerous and great fields of science and engineering [5–7].

Now, we recall the definitions of Riemann-Liouville fractional integrals.

Definition 1.1 ([5–7]). *Let $\sigma \in L_1[\nu_1, \nu_2]$. The Riemann-Liouville integrals $J^{\vartheta}_{\nu_1+}\sigma$ and $J^{\vartheta}_{\nu_2-}\sigma$ of order $\vartheta > 0$ with $\nu_1 \geq 0$ are defined by*

$$J^{\vartheta}_{\nu_1+}\sigma(x) = \frac{1}{\Gamma(\vartheta)}\int_{\nu_1}^{x}(x-\eta)^{\vartheta-1}\sigma(\eta)d\eta, \quad \nu_1 < x \tag{1.2}$$

and

$$J^{\vartheta}_{\nu_2-}\sigma(x) = \frac{1}{\Gamma(\vartheta)}\int_{x}^{\nu_2}(\eta-x)^{\vartheta-1}\sigma(\eta)d\eta, \quad x < \nu_2, \tag{1.3}$$

respectively. Here $\Gamma(\vartheta)$ is the well-known Gamma function and $J^{0}_{\nu_1+}\sigma(x) = J^{0}_{\nu_2-}\sigma(x) = \sigma(x)$.

With a huge application of fractional integration and Hermite-Hadamard inequality, many researchers in the field of fractional calculus extended their research to the Hermite-Hadamard inequality, including fractional integration rather than ordinary integration; for example see [8–21].

In this paper, we consider the integral inequality of Hermite-Hadamard-Mercer type that relies on the Hermite-Hadamard and Jensen-Mercer inequalities. For this purpose, we recall the Jensen-Mercer inequality: Let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ nonnegative weights such that $\sum_{i=1}^{n}\mu_i = 1$. Then, the Jensen inequality [22, 23] is as follows, for a convex function σ on the interval $[\nu_1, \nu_2]$, we have

$$\sigma\left(\sum_{i=1}^{n}\mu_i x_i\right) \leq \sum_{i=1}^{n}\mu_i\sigma(x_i), \tag{1.4}$$

where for all $x_i \in [\nu_1, \nu_2]$ and $\mu_i \in [0, 1]$, $(i = \overline{1, n})$.

Theorem 1.1 ([2, 23]). *If σ is convex function on $[\nu_1, \nu_2]$, then*

$$\sigma\left(\nu_1+\nu_2-\sum_{i=1}^{n}\mu_i x_i\right) \leq \sigma(\nu_1)+\sigma(\nu_2)-\sum_{i=1}^{n}\mu_i\sigma(x_i), \tag{1.5}$$

for each $x_i \in [\nu_1, \nu_2]$ and $\mu_i \in [0, 1]$, $(i = \overline{1, n})$ with $\sum_{i=1}^{n}\mu_i = 1$. For some results related with Jensen-Mercer inequality, see [24–26].

In view of above indices, we establish new integral inequalities of Hermite-Hadamard-Mercer type for convex functions via the Riemann-Liouville fractional integrals in the current project. Particularly, we see that our results can cover the previous researches.

2. Main results

Theorem 2.1. *For a convex function $\sigma : [\nu_1, \nu_2] \subseteq \mathbf{R} \to \mathbf{R}$ with $x, y \in [\nu_1, \nu_2]$, we have:*

$$\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right) \leq \frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right.$$
$$\left. + \; J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right] \leq \sigma(\nu_1)+\sigma(\nu_2)-\frac{\sigma(x)+\sigma(y)}{2}. \quad (2.1)$$

Proof. By using the convexity of σ, we have

$$\sigma\left(\nu_1+\nu_2-\frac{u+v}{2}\right) \leq \frac{1}{2}\left[\sigma(\nu_1+\nu_2-u)+\sigma(\nu_1+\nu_2-v)\right], \quad (2.2)$$

and above with $u = \frac{1-\eta}{2}x + \frac{1+\eta}{2}y$, $v = \frac{1+\eta}{2}x + \frac{1-\eta}{2}y$, where $x, y \in [\nu_1, \nu_2]$ and $\eta \in [0, 1]$, leads to

$$\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right) \leq \frac{1}{2}\left[\sigma\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)\right.$$
$$\left. + \sigma\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)\right]. \quad (2.3)$$

Multiplying both sides of (2.3) by $\eta^{\vartheta-1}$ and then integrating with respect to η over $[0, 1]$, we get

$$\begin{aligned}
\frac{1}{\vartheta}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right) &\leq \frac{1}{2}\left[\int_0^1 \eta^{\vartheta-1}\sigma\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)d\eta\right. \\
&\quad \left. + \int_0^1 \eta^{\vartheta-1}\sigma\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)d\eta\right] \\
&= \frac{1}{2}\left[\frac{2^{\vartheta}}{(y-x)^{\vartheta}}\int_{\nu_1+\nu_2-y}^{\nu_1+\nu_2-\frac{x+y}{2}}\left(\left(\nu_1+\nu_2-\frac{x+y}{2}\right)-w\right)^{\vartheta-1}\sigma(w)\,dw\right. \\
&\quad \left. + \frac{2^{\vartheta}}{(y-x)^{\vartheta}}\int_{\nu_1+\nu_2-\frac{x+y}{2}}^{\nu_1+\nu_2-x}\left(w-\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right)^{\vartheta-1}\sigma(w)\,dw\right] \\
&= \frac{2^{\vartheta-1}\Gamma(\vartheta)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)+J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right],
\end{aligned}$$

and thus the proof of first inequality in (2.1) is completed.

On the other hand, we have by using the Jensen-Mercer inequality:

$$\sigma\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right) \leq \sigma(\nu_1)+\sigma(\nu_2)-\left(\frac{1-\eta}{2}\sigma(x)+\frac{1+\eta}{2}\sigma(y)\right) \quad (2.4)$$

$$\sigma\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right) \leq \sigma(\nu_1)+\sigma(\nu_2)-\left(\frac{1+\eta}{2}\sigma(x)+\frac{1-\eta}{2}\sigma(y)\right). \quad (2.5)$$

Adding inequalities (2.4) and (2.5) to get

$$\sigma\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)+\sigma\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)$$
$$\leq 2\left[\sigma(\nu_1)+\sigma(\nu_2)\right]-\left[\sigma(x)+\sigma(y)\right]. \quad (2.6)$$

Multiplying both sides of (2.6) by $\eta^{\vartheta-1}$ and then integrating with respect to η over $[0,1]$ to obtain

$$\int_0^1 \eta^{\vartheta-1}\sigma\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)d\eta+\int_0^1 \eta^{\vartheta-1}\sigma\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)d\eta$$
$$\leq \frac{2}{\vartheta}\left[\sigma(\nu_1)+\sigma(\nu_2)\right]-\frac{1}{\vartheta}\left[\sigma(x)+\sigma(y)\right].$$

By making use of change of variables and then multiplying by $\frac{\vartheta}{2}$, we get the second inequality in (2.1). □

Remark 2.1. *If we choose $\vartheta=1$, $x=\nu_1$ and $y=\nu_2$ in Theorem 2.1, then the inequality (2.1) reduces to (1.1).*

Corollary 2.1. *Theorem 2.1 with*

- $\vartheta=1$ *becomes [24, Theorem 2.1].*
- $x=\nu_1$ *and* $y=\nu_2$ *becomes:*

$$\sigma\left(\frac{\nu_1+\nu_2}{2}\right)\leq\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(\nu_2-\nu_1)^{\vartheta}}\left[J^{\vartheta}_{\nu_1+}\sigma\left(\frac{\nu_1+\nu_2}{2}\right)+J^{\vartheta}_{\nu_2-}\sigma\left(\frac{\nu_1+\nu_2}{2}\right)\right]\leq\frac{\sigma(\nu_1)+\sigma(\nu_2)}{2},$$

which is obtained by Mohammed and Brevik in [10].

The following Lemma linked with the left inequality of (2.1) is useful to obtain our next results.

Lemma 2.1. *Let $\sigma:[\nu_1,\nu_2]\subseteq\mathbf{R}\to\mathbf{R}$ be a differentiable function on (ν_1,ν_2) and $\sigma'\in L[\nu_1,\nu_2]$ with $\nu_1\leq\nu_2$ and $x,y\in[\nu_1,\nu_2]$. Then, we have:*

$$\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)+J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right]$$
$$-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)=\frac{(y-x)}{4}\int_0^1\eta^{\vartheta}\Big[\sigma'\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)$$
$$-\sigma'\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)\Big]d\eta. \quad (2.7)$$

Proof. From right hand side of (2.7), we set

$$\begin{aligned}\varpi_1-\varpi_2 &:= \int_0^1 \eta^{\vartheta}\left[\sigma'\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)-\sigma'\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)\right]d\eta\\ &= \int_0^1 \eta^{\vartheta}\sigma'\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)d\eta\\ &-\int_0^1 \eta^{\vartheta}\sigma'\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)d\eta. \end{aligned} \tag{2.8}$$

By integrating by parts with $w=\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)$, we can deduce:

$$\begin{aligned}\varpi_1 &= -\frac{2}{(y-x)}\sigma(\nu_1+\nu_2-y)+\frac{2\vartheta}{(y-x)}\int_0^1 \eta^{\vartheta-1}\sigma\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)d\eta\\ &= -\frac{2}{(y-x)}\sigma(\nu_1+\nu_2-y)+\frac{2^{\vartheta+1}\vartheta}{(y-x)^{\vartheta+1}}\int_{\nu_1+\nu_2-y}^{\nu_1+\nu_2-\frac{x+y}{2}}\sigma\left(\left(\nu_1+\nu_2-\frac{x+y}{2}\right)-w\right)^{\vartheta-1}\sigma(w)\,dw\\ &= -\frac{2}{(y-x)}\sigma(\nu_1+\nu_2-y)+\frac{2^{\vartheta+1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta+1}}J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right).\end{aligned}$$

Similarly, we can deduce:

$$\varpi_2=\frac{2}{y-x}\sigma(\nu_1+\nu_2-x)-\frac{2^{\vartheta+1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta+1}}J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right).$$

By substituting ϖ_1 and ϖ_2 in (2.8) and then multiplying by $\frac{(y-x)}{4}$, we obtain required identity (2.7). □

Corollary 2.2. *Lemma 2.1 with*

- $\vartheta=1$ *becomes:*

$$\begin{aligned}\frac{1}{y-x}\int_{\nu_1+\nu_2-y}^{\nu_1+\nu_2-x}\sigma(w)\,dw-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)=\frac{(y-x)}{4}\int_0^1\eta\Bigg[\sigma'\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)\\ -\sigma'\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)\Bigg]d\eta.\end{aligned}$$

- $\vartheta=1$*,* $x=\nu_1$ *and* $y=\nu_2$ *becomes:*

$$\begin{aligned}\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\sigma(w)\,dw-\sigma\left(\frac{\nu_1+\nu_2}{2}\right)=\frac{(\nu_2-\nu_1)}{4}\int_0^1\eta\Bigg[\sigma'\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}\nu_1+\frac{1+\eta}{2}\nu_2\right)\right)\\ -\sigma'\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}\nu_1+\frac{1-\eta}{2}\nu_2\right)\right)\Bigg]d\eta.\end{aligned}$$

- $x=\nu_1$ *and* $y=\nu_2$ *becomes:*

$$\begin{aligned}&\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(\nu_2-\nu_1)^{\vartheta}}\left[J^{\vartheta}_{\nu_1+}\sigma\left(\frac{\nu_1+\nu_2}{2}\right)+J^{\vartheta}_{\nu_2-}\sigma\left(\frac{\nu_1+\nu_2}{2}\right)\right]-\sigma\left(\frac{\nu_1+\nu_2}{2}\right)\\ &=\frac{(\nu_2-\nu_1)}{4}\int_0^1\eta^{\vartheta}\left[\sigma'\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}\nu_1+\frac{1+\eta}{2}\nu_2\right)\right)-\sigma'\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}\nu_1+\frac{1-\eta}{2}\nu_2\right)\right)\right]d\eta.\end{aligned}$$

Theorem 2.2. *Let $\sigma : [\nu_1, \nu_2] \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable function on (ν_1, ν_2) and $|\sigma'|$ is convex on $[\nu_1, \nu_2]$ with $\nu_1 \leq \nu_2$ and $x, y \in [\nu_1, \nu_2]$. Then, we have:*

$$\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)+J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right]\right.$$
$$\left.-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right| \leq \frac{(y-x)}{2(1+\vartheta)}\left[|\sigma'(\nu_1)|+|\sigma'(\nu_2)|-\frac{|\sigma'(x)|+|\sigma'(y)|}{2}\right]. \quad (2.9)$$

Proof. By taking modulus of identity (2.7), we get

$$\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)+J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right]\right.$$
$$\left.-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right| \leq \frac{(y-x)}{4}\left[\int_0^1 \eta^{\vartheta}\left|\sigma'\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)\right|d\eta\right.$$
$$\left.+\int_0^1 \eta^{\vartheta}\left|\sigma'\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)\right|d\eta\right].$$

Then, by applying the convexity of $|\sigma'|$ and the Jensen-Mercer inequality on above inequality, we get

$$\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)+J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right]\right.$$
$$\left.-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right| \leq \frac{(y-x)}{4}\left[\int_0^1 \eta^{\vartheta}\left[|\sigma'(\nu_1)|+|\sigma'(\nu_2)|-\left(\frac{1+\eta}{2}|\sigma'(x)|+\frac{1-\eta}{2}\right)|\sigma'(y)|\right]d\eta\right.$$
$$\left.+\int_0^1 \eta^{\vartheta}\left[|\sigma'(\nu_1)|+|\sigma'(\nu_2)|-\left(\frac{1-\eta}{2}|\sigma'(x)|+\frac{1+\eta}{2}\right)|\sigma'(y)|\right]d\eta\right]$$
$$=\frac{(y-x)}{2(1+\vartheta)}\left[|\sigma'(\nu_1)|+|\sigma'(\nu_2)|-\frac{|\sigma'(x)|+|\sigma'(y)|}{2}\right],$$

which completes the proof of Theorem 2.2. □

Corollary 2.3. *Theorem 2.2 with*

- $\vartheta = 1$ *becomes:*

$$\left|\frac{1}{y-x}\int_{\nu_1+\nu_2-y}^{\nu_1+\nu_2-x}\sigma(w)dw-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right| \leq \frac{(y-x)}{4}\left[|\sigma'(\nu_1)|+|\sigma'(\nu_2)|-\frac{|\sigma'(x)|+|\sigma'(y)|}{2}\right].$$

- $\vartheta = 1$, $x = \nu_1$ *and* $y = \nu_2$ *becomes [27, Theorem 2.2].*
- $x = \nu_1$ *and* $y = \nu_2$ *becomes:*

$$\left|\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\sigma(w)\,dw-\sigma\left(\frac{\nu_1+\nu_2}{2}\right)\right| \leq \frac{(\nu_2-\nu_1)}{4}\left[\frac{|\sigma'(\nu_1)|+|\sigma'(\nu_2)|}{2}\right].$$

Theorem 2.3. *Let $\sigma : [\nu_1, \nu_2] \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable function on (ν_1, ν_2) and $|\sigma'|^q, q > 1$ is convex on $[\nu_1, \nu_2]$ with $\nu_1 \leq \nu_2$ and $x, y \in [\nu_1, \nu_2]$. Then, we have:*

$$
\begin{aligned}
&\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right.\right.\\
&\qquad\left.\left.+J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right]-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right|\\
&\qquad\leq\frac{(y-x)}{4\sqrt[p]{\vartheta p+1}}\left[\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{|\sigma'(x)|^q+3|\sigma'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right.\\
&\qquad\qquad\left.+\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{3|\sigma'(x)|^q+|\sigma'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right], \quad (2.10)
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By taking modulus of identity (2.7) and using Hölder's inequality, we get

$$
\begin{aligned}
&\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right.\right.\\
&\qquad\left.\left.+J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right]-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right|\\
&\qquad\leq\frac{(y-x)}{4}\left(\int_0^1\eta^{\vartheta p}\right)^{\frac{1}{p}}\left\{\left(\int_0^1\left|\sigma'\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)\right|^q d\eta\right)^{\frac{1}{q}}\right.\\
&\qquad\qquad\left.+\left(\int_0^1\left|\sigma'\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)\right|^q d\eta\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Then, by applying the Jensen-Mercer inequality with the convexity of $|\sigma'|^q$, we can deduce

$$
\begin{aligned}
&\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right.\right.\\
&\left.\left.+J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right]-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right|\\
&\leq\frac{(y-x)}{4}\left(\int_0^1\eta^{\vartheta p}\right)^{\frac{1}{p}}\left\{\left(\int_0^1|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{1-\eta}{2}|\sigma'(x)|^q+\frac{1+\eta}{2}|\sigma'(y)|^q\right)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{1+\eta}{2}|\sigma'(x)|^q+\frac{1-\eta}{2}|\sigma'(y)|^q\right)\right)^{\frac{1}{q}}\right\}\\
&=\frac{(y-x)}{4\sqrt[p]{\vartheta p+1}}\left[\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{|\sigma'(x)|^q+3|\sigma'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{3|\sigma'(x)|^q+|\sigma'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

which completes the proof of Theorem 2.3. □

Corollary 2.4. *Theorem 2.3 with*

- $\vartheta = 1$ *becomes:*

$$\left|\frac{1}{y-x}\int_{\nu_1+\nu_2-y}^{\nu_1+\nu_2-x}\sigma(w)\,dw-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right|$$
$$\leq \frac{(y-x)}{4\sqrt[p]{p+1}}\left[\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{|\sigma'(x)|^q+3|\sigma'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{3|\sigma'(x)|^q+|\sigma'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right].$$

- $\vartheta = 1$, $x=\nu_1$ *and* $y=\nu_2$ *becomes:*

$$\left|\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\sigma(w)\,dw-\sigma\left(\frac{\nu_1+\nu_2}{2}\right)\right|\leq\frac{(\nu_2-\nu_1)}{2^{\frac{2}{p}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[|\sigma'(\nu_1)|+|\sigma'(\nu_2)|\right].$$

- $x=\nu_1$ *and* $y=\nu_2$ *becomes:*

$$\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(\nu_2-\nu_1)^{\vartheta}}\left[J_{\nu_1+}^{\vartheta}\sigma\left(\frac{\nu_1+\nu_2}{2}\right)+J_{\nu_2-}^{\vartheta}\sigma\left(\frac{\nu_1+\nu_2}{2}\right)\right]-\sigma\left(\frac{\nu_1+\nu_2}{2}\right)\right|$$
$$\leq\frac{2^{\vartheta-1-\frac{2}{q}}}{\nu_2-\nu_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[|\sigma'(\nu_1)|+|\sigma'(\nu_2)|\right].$$

Theorem 2.4. *Let* $\sigma:[\nu_1,\nu_2]\subseteq\mathbf{R}\to\mathbf{R}$ *be a differentiable function on* (ν_1,ν_2) *and* $|\sigma'|^q, q\geq 1$ *is convex on* $[\nu_1,\nu_2]$ *with* $\nu_1\leq\nu_2$ *and* $x,y\in[\nu_1,\nu_2]$. *Then, we have:*

$$\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J_{(\nu_1+\nu_2-y)+}^{\vartheta}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right.\right.$$
$$\left.\left.+J_{(\nu_1+\nu_2-x)-}^{\vartheta}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right]-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right|$$
$$\leq\frac{(y-x)}{4(\vartheta+1)}\left[\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{|\sigma'(x)|^q+(2\vartheta+3)|\sigma'(y)|^q}{2(\vartheta+2)}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{(2\vartheta+3)|\sigma'(x)|^q+|\sigma'(y)|^q}{2(\vartheta+2)}\right)\right)^{\frac{1}{q}}\right].\quad(2.11)$$

Proof. By taking modulus of identity (2.7) with the well-known power mean inequality, we can deduce

$$\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right.\right.$$
$$\left.\left.+J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right]-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right|$$
$$\leq\frac{(y-x)}{4}\left(\int_0^1\eta^{\vartheta}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1\eta^{\vartheta}\left|\sigma'\left(\nu_1+\nu_2-\left(\frac{1-\eta}{2}x+\frac{1+\eta}{2}y\right)\right)\right|^q d\eta\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\eta^{\vartheta}\left|\sigma'\left(\nu_1+\nu_2-\left(\frac{1+\eta}{2}x+\frac{1-\eta}{2}y\right)\right)\right|^q d\eta\right)^{\frac{1}{q}}\right\}.$$

By applying the Jensen-Mercer inequality with the convexity of $|\sigma'|^q$, we can deduce

$$\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(y-x)^{\vartheta}}\left[J^{\vartheta}_{(\nu_1+\nu_2-y)+}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right.\right.$$
$$\left.\left.+J^{\vartheta}_{(\nu_1+\nu_2-x)-}\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right]-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right|$$
$$\leq\frac{(y-x)}{4}\left(\int_0^1\eta^{\vartheta}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1\eta^{\vartheta}\left[|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{1-\eta}{2}|\sigma'(x)|^q+\frac{1+\eta}{2}|\sigma'(y)|^q\right)\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\eta^{\vartheta}\left[|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{1+\eta}{2}|\sigma'(x)|^q+\frac{1-\eta}{2}|\sigma'(y)|^q\right)\right]\right)^{\frac{1}{q}}\right\}$$
$$=\frac{(y-x)}{4(\vartheta+1)}\left[\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{|\sigma'(x)|^q+(2\vartheta+3)|\sigma'(y)|^q}{2(\vartheta+2)}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{(2\vartheta+3)|\sigma'(x)|^q+|\sigma'(y)|^q}{2(\vartheta+2)}\right)\right)^{\frac{1}{q}}\right],$$

which completes the proof of Theorem 2.4. □

Corollary 2.5. *Theorem 2.4 with*

- $q=1$ *becomes Theorem 2.2.*
- $\vartheta=1$ *becomes:*

$$\left|\frac{1}{y-x}\int_{\nu_1+\nu_2-y}^{\nu_1+\nu_2-x}\sigma(w)\,dw-\sigma\left(\nu_1+\nu_2-\frac{x+y}{2}\right)\right|$$
$$\leq\frac{(y-x)}{8}\left[\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{|\sigma'(x)|^q+5|\sigma'(y)|^q}{6}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|\sigma'(\nu_1)|^q+|\sigma'(\nu_2)|^q-\left(\frac{5|\sigma'(x)|^q+|\sigma'(y)|^q}{6}\right)\right)^{\frac{1}{q}}\right].$$

- $\vartheta = 1$, $x = \nu_1$ *and* $y = \nu_2$ *becomes:*

$$\left|\frac{1}{\nu_2 - \nu_1}\int_{\nu_1}^{\nu_2}\sigma(w)\,dw - \sigma\left(\frac{\nu_1+\nu_2}{2}\right)\right| \le \frac{(y-x)}{8}\left[\left(\frac{5\left|\sigma'(\nu_1)\right|^q + \left|\sigma'(\nu_2)\right|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{\left|\sigma'(\nu_1)\right|^q + 5\left|\sigma'(\nu_2)\right|^q}{6}\right)^{\frac{1}{q}}\right].$$

- $x = \nu_1$ *and* $y = \nu_2$ *becomes:*

$$\left|\frac{2^{\vartheta-1}\Gamma(\vartheta+1)}{(\nu_2-\nu_1)^{\vartheta}}\left[J_{\nu_1+}^{\vartheta}\sigma\left(\frac{\nu_1+\nu_2}{2}\right) + J_{\nu_2-}^{\vartheta}\sigma\left(\frac{\nu_1+\nu_2}{2}\right)\right] - \sigma\left(\frac{\nu_1+\nu_2}{2}\right)\right| \le \frac{(\nu_2-\nu_1)}{4(\vartheta+1)}\left[\left(\frac{(2\vartheta+3)\left|\sigma'(\nu_1)\right|^q + \left|\sigma'(\nu_2)\right|^q}{2(\vartheta+2)}\right)^{\frac{1}{q}} + \left(\frac{\left|\sigma'(\nu_1)\right|^q + (2\vartheta+3)\left|\sigma'(\nu_2)\right|^q}{2(\vartheta+2)}\right)^{\frac{1}{q}}\right].$$

3. Applications to special means

Here, we consider the following special means:

- The arithmetic mean:

$$\mathcal{A}(\nu_1,\nu_2) = \frac{\nu_1+\nu_2}{2}, \quad \nu_1,\nu_2 \ge 0.$$

- The harmonic mean:

$$\mathcal{H}(\nu_1,\nu_2) = \frac{2\nu_1\nu_2}{\nu_1+\nu_2}, \quad \nu_1,\nu_2 > 0.$$

- The logarithmic mean:

$$\mathcal{L}(\nu_1,\nu_2) = \begin{cases} \frac{\nu_2-\nu_1}{\ln\nu_2-\ln\nu_1}, & \text{if } \nu_1 \ne \nu_2,\\ \nu_1, & \text{if } \nu_1 = \nu_2, \end{cases} \quad \nu_1,\nu_2 > 0.$$

- The generalized logarithmic mean:

$$\mathcal{L}_n(\nu_1,\nu_2) = \begin{cases} \left[\frac{\nu_2^{n+1}-\nu_1^{n+1}}{(n+1)(\nu_2-\nu_1)}\right]^{\frac{1}{n}}, & \text{if } \nu_1 \ne \nu_2\\ \nu_1, & \text{if } \nu_1 = \nu_2, \end{cases} \quad \nu_1,\nu_2 > 0;\ n \in \mathbf{Z}\setminus\{-1,0\}.$$

Proposition 3.1. *Let* $0 < \nu_1 < \nu_2$ *and* $n \in \mathbf{N}$, $n \ge 2$. *Then, for all* $x, y \in [\nu_1, \nu_2]$, *we have:*

$$\left|\mathcal{L}_n^n(\nu_1+\nu_2-y, \nu_1+\nu_2-x) - (2\mathcal{A}(\nu_1,\nu_2) - \mathcal{A}(x,y))^n\right| \le \frac{n(y-x)}{4}\left[2\mathcal{A}\left(\nu_1^{n-1},\nu_2^{n-1}\right) - \mathcal{A}\left(x^{n-1},y^{n-1}\right)\right]. \quad (3.1)$$

Proof. By applying Corollary 2.3 (first item) for the convex function $\sigma(x) = x^n,\ x > 0$, one can obtain the result directly. □

Proposition 3.2. *Let* $0 < \nu_1 < \nu_2$. *Then, for all* $x, y \in [\nu_1, \nu_2]$, *we have:*

$$\left|\mathcal{L}^{-1}(\nu_1+\nu_2-y, \nu_1+\nu_2-x) - (2\mathcal{A}(\nu_1,\nu_2)-\mathcal{A}(x,y))^{-1}\right| \leq \frac{(y-x)}{4}\left[2\mathcal{H}^{-1}\left(\nu_1^2,\nu_2^2\right)-\mathcal{H}^{-1}\left(x^2,y^2\right)\right]. \quad (3.2)$$

Proof. By applying Corollary 2.3 (first item) for the convex function $\sigma(x) = \frac{1}{x}$, $x > 0$, one can obtain the result directly. □

Proposition 3.3. *Let* $0 < \nu_1 < \nu_2$ *and* $n \in \mathbf{N}$, $n \geq 2$. *Then, we have:*

$$\left|\mathcal{L}_n^n(\nu_1,\nu_2) - \mathcal{A}^n(\nu_1,\nu_2)\right| \leq \frac{n(\nu_2-\nu_1)}{4}\left[\mathcal{A}\left(\nu_1^{n-1},\nu_2^{n-1}\right)\right], \quad (3.3)$$

and

$$\left|\mathcal{L}^{-1}(\nu_1,\nu_2) - \mathcal{A}^{-1}(\nu_1,\nu_2)\right| \leq \frac{(\nu_2-\nu_1)}{4}\mathcal{H}^{-1}\left(\nu_1^2,\nu_2^2\right). \quad (3.4)$$

Proof. By setting $x = \nu_1$ and $y = \nu_2$ in results of Proposition 3.1 and Proposition 3.2, one can obtain the Proposition 3.3. □

Proposition 3.4. *Let* $0 < \nu_1 < \nu_2$ *and* $n \in \mathbf{N}$, $n \geq 2$. *Then, for* $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ *and for all* $x, y \in [\nu_1, \nu_2]$, *we have:*

$$\begin{aligned}\left|\mathcal{L}_n^n(\nu_1+\nu_2-y, \nu_1+\nu_2-x) - (2\mathcal{A}(\nu_1,\nu_2)-\mathcal{A}(x,y))^n\right| &\leq \frac{n(y-x)}{4\sqrt[p]{p+1}}\left\{\left[2\mathcal{A}\left(\nu_1^{q(n-1)},\nu_2^{q(n-1)}\right)-\frac{1}{2}\mathcal{A}\left(x^{q(n-1)},3y^{q(n-1)}\right)\right]^{\frac{1}{q}}\right.\\ &\quad\left.+\left[2\mathcal{A}\left(\nu_1^{q(n-1)},\nu_2^{q(n-1)}\right)-\frac{1}{2}\mathcal{A}\left(3x^{q(n-1)},y^{q(n-1)}\right)\right]^{\frac{1}{q}}\right\}. \quad (3.5)\end{aligned}$$

Proof. By applying Corollary 2.4 (first item) for convex function $\sigma(x) = x^n$, $x > 0$, one can obtain the result directly. □

Proposition 3.5. *Let* $0 < \nu_1 < \nu_2$. *Then, for* $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ *and for all* $x, y \in [\nu_1, \nu_2]$, *we have:*

$$\begin{aligned}\left|\mathcal{L}^{-1}(\nu_1+\nu_2-y, \nu_1+\nu_2-x) - (2\mathcal{A}(\nu_1,\nu_2)-\mathcal{A}(x,y))^{-1}\right| &\leq \frac{\sqrt[q]{2}(y-x)}{4\sqrt[p]{p+1}}\left\{\left[\mathcal{H}^{-1}\left(\nu_1^{2q},\nu_2^{2q}\right)-\frac{3}{4}\mathcal{H}^{-1}\left(x^{2q},3y^{2q}\right)\right]^{\frac{1}{q}}\right.\\ &\quad\left.+\left[\mathcal{H}^{-1}\left(\nu_1^{2q},\nu_2^{2q}\right)-\frac{3}{4}\mathcal{H}^{-1}\left(3x^{2q},y^{2q}\right)\right]^{\frac{1}{q}}\right\}. \quad (3.6)\end{aligned}$$

Proof. By applying Corollary 2.4 (first item) for the convex function $\sigma(x) = \frac{1}{x}$, $x > 0$, one can obtain the result directly. □

Proposition 3.6. *Let* $0 < \nu_1 < \nu_2$ *and* $n \in \mathbf{N}$, $n \geq 2$. *Then, for* $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *we have:*

$$\left|\mathcal{L}_n^n(\nu_1, \nu_2) - \mathcal{A}^n(\nu_1, \nu_2)\right| \leq \frac{n(\nu_2 - \nu_1)}{4\sqrt[p]{p+1}} \left\{ \left[2\mathcal{A}\left(\nu_1^{q(n-1)}, \nu_2^{q(n-1)}\right) - \frac{1}{2}\mathcal{A}\left(\nu_1^{q(n-1)}, 3\nu_2^{q(n-1)}\right) \right]^{\frac{1}{q}} + \left[2\mathcal{A}\left(\nu_1^{q(n-1)}, \nu_2^{q(n-1)}\right) - \frac{1}{2}\mathcal{A}\left(3\nu_1^{q(n-1)}, \nu_2^{q(n-1)}\right) \right]^{\frac{1}{q}} \right\}, \quad (3.7)$$

and

$$\left|\mathcal{L}^{-1}(\nu_1, \nu_2) - \mathcal{A}^{-1}(\nu_1, \nu_2)\right| \leq \frac{\sqrt[q]{2}(\nu_2 - \nu_1)}{4\sqrt[p]{p+1}} \left\{ \left[\mathcal{H}^{-1}\left(\nu_1^{2q}, \nu_2^{2q}\right) - \frac{3}{4}\mathcal{H}^{-1}\left(\nu_1^{2q}, 3\nu_2^{2q}\right) \right]^{\frac{1}{q}} + \left[\mathcal{H}^{-1}\left(\nu_1^{2q}, \nu_2^{2q}\right) - \frac{3}{4}\mathcal{H}^{-1}\left(3\nu_1^{2q}, \nu_2^{2q}\right) \right]^{\frac{1}{q}} \right\}. \quad (3.8)$$

Proof. By setting $x = \nu_1$ and $y = \nu_2$ in results of Proposition 3.4 and Proposition 3.5, one can obtain the Proposition 3.6. □

4. Conclusions

As we emphasized in the introduction, integral inequality is the most important field of mathematical analysis and fractional calculus. By using the well-known Jensen-Mercer and power mean inequalities, we have proved new inequalities of Hermite-Hadamard-Mercer type involving Riemann-Liouville fractional operators. In the last section, we have considered some propositions in the context of special functions; these confirm the efficiency of our results.

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Conflict of interest

The authors declare no conflict of interest.

References

1. J. Hadamard, Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann, *J. Math. Pures Appl.,* **58** (1893), 171–215.
2. S. S. Dragomir, C. E. M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000.

3. M. Z. Sarikaya, M. E. Kiris, Some new inequalities of Hermite-Hadamard type for s-convex functions, *Miskolc Math. Notes,* **16** (2015), 491–501.
4. P. O. Mohammed, Some new Hermite-Hadamard type inequalities for MT-convex functions on differentiable coordinates, *J. King Saud Univ. Sci.,* **30** (2018), 258–262.
5. K. S. Miller, B. Ross, *An Introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.
6. K. B. Oldham, J. Spanier, *The fractional calculus*, Academic Press, San Diego, 1974.
7. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Taylor & Francis, London, 2002.
8. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Model.,* **57** (2013), 2403–2407.
9. M. Z. Sarikaya, H. Yildirim, On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals, *Miskolc Math. Notes,* **17** (2017), 1049–1059.
10. P. O. Mohammed, I. Brevik, A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals, *Symmetry,* **12** (2020), 610, Doi:10.3390/sym12040610.
11. A. Fernandez, P. Mohammed, Hermite-Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels, *Math. Meth. Appl. Sci.,* (2020), 1–18. Available from: `https://doi.org/10.1002/mma.6188.`
12. J. Han, P. O. Mohammed, H. Zeng, Generalized fractional integral inequalities of Hermite-Hadamard-type for a convex function, *Open Math.,* **18** (2020), 794–806.
13. D. Baleanu, P. O. Mohammed, S. Zeng, Inequalities of trapezoidal type involving generalized fractional integrals, *Alex. Eng. J.,* **59** (2020), 2975–2984.
14. P. O. Mohammed, Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals of a convex function with respect to a monotone function, *Math. Meth. Appl. Sci.,* (2019), 1–11. Available from: `https://doi.org/10.1002/mma.5784.`
15. P. O. Mohammed, M. Z. Sarikaya, Hermite-Hadamard type inequalities for F-convex function involving fractional integrals, *J. Inequal. Appl.,* **2018** (2018), 1–33.
16. P. O. Mohammed, M. Z. Sarikaya, On generalized fractional integral inequalities for twice differentiable convex functions, *J. Comput. Appl. Math.,* **372** (2020), 112740.
17. P. O. Mohammed, M. Z. Sarikaya, D. Baleanu, On the generalized Hermite-Hadamard inequalities via the tempered fractional integrals, *Symmetry,* **12** (2020), 595.
18. P. O. Mohammed, T. Abdeljawad, A. Kashuri, Fractional Hermite-Hadamard-Fejér inequalities for a convex function with respect to an increasing function involving a positive weighted symmetric function, *Symmetry,* **12** (2020), 1503.
19. P. O. Mohammed, T. Abdeljawad, S. Zeng, A. Kashuri, Fractional Hermite-Hadamard integral inequalities for a new class of convex functions, *Symmetry,* **12** (2020), 1485.
20. P. O. Mohammed, M. Vivas-Cortez, T. Abdeljawad, Y. Rangel-Oliveros, Integral inequalities of Hermite-Hadamard type for quasi-convex functions with applications, *AIMS Math.,* **5** (2020), 7316–7331.

21. D. Baleanu, P. O. Mohammed, M. Vivas-Cortez, Y. Rangel-Oliveros, Some modifications in conformable fractional integral inequalities, *Adv. Differ. Equ.,* **2020** (2020), 1–25.
22. D. S. Mitrinović, J. E. Pečarić, A. M. Fink, On the Jensen inequality, *Univ. Beograd. Publ. Elektrotehn Fak. Ser. Mat. Fis.,* (1979), 50–54.
23. A. Matković, J. Pečarić, I. Perić, A variant of Jensens inequality of Mercers type for operators with applications, *Linear Algebra Appl.,* **418** (2006), 551–564.
24. M. Kian, M. S. Moslehian, Refinements of the operator Jensen-Mercer inequality, *Electron J. Linear Algebra,* **26** (2013), 50.
25. A. M. Mercer, A variant of jensens inequality, *J. Inequal. Pure Appl. Math.,* **4** (2003), 73.
26. A. M. Fink, M. Jodeit Jr, Jensen inequalities for functions with higher monotonicities, *Aequations Math.,* **40** (1990), 26–43.
27. U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comput.,* **147** (2004), 137–146.

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