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Research article

Embedding of Q_p spaces into tent spaces and Volterra integral operator

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Abstract: In this paper, the boundedness and compactness of the inclusion mapping from Q_p spaces into tent spaces $\mathcal{T}^q_{\frac{qp}{2},s}$ are completely characterized when $q > 2$. As an application, the boundedness of the Volterra integral operator T_g from Q_p to the space $\mathcal{LF}(q, q-2, \frac{qp}{2})$ is obtained. Moreover, the essential norm and compactness of T_g are also investigated.

Keywords: Q_p space; tent space; Carleson measure; Volterra integral operator
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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of all functions analytic in $\mathbb{D}$. Let $0 < p < \infty$ and $-1 < \alpha < \infty$. The Dirichlet type space $\mathcal{D}^p_\alpha$ is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{D}^p_\alpha} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure in $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. When $p = 2$ and $\alpha = 0$, it gives the classic Dirichlet space $\mathcal{D}$. When $p = 2$ and $\alpha = 1$, it gives the Hardy space H^2. When $\alpha = p$, $\mathcal{D}^p_\alpha$ is just the classical Bergman space A^p.

Let $0 < p < \infty$. The Q_p space is the space of all $f \in H(\mathbb{D})$ such that (see, e.g., [23])

$$\|f\|^2_{Q_p} = |f(0)|^2 + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_a(z)|^2)^p dA(z) < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. When $p > 1$, Q_p is the Bloch space $\mathcal{B}$ (see [24, 25]), which denote the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

When $p = 1$, $Q_1 = BMOA$, the space of analytic functions in the Hardy space $H^1(\mathbb{D})$ whose boundary functions have bounded mean oscillation (see, e.g., [25]).

Let $0 < p, s < \infty$, $-2 < q < \infty$. A function $f \in H(\mathbb{D})$ is said to belong to $\mathcal{F}(p, q, s)$ if

$$\|f\|^p_{\mathcal{F}(p,q,s)} = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dA(z) < \infty.$$

An $f \in \mathcal{F}_0(p, q, s)$ if $f \in H(\mathbb{D})$ and

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dA(z) = 0.$$

$\mathcal{F}(p, q, s)$ is a Banach space under the norm $\|\cdot\|_{\mathcal{F}(p,q,s)}$ when $p \geq 1$. This space was first introduced by Zhao in [24] and called general function space because it can get many function spaces if it takes special parameters of p, q, s. From [24] we see that $\mathcal{F}(p, p - 2, s)$ is just the Bloch space when $s > 1$.

For $0 < q, s < \infty$, let $\mathcal{LF}(q, q - 2, s)$ denote the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|^q_{\mathcal{LF}(q,q-2,s)} = |f(0)|^q + \sup_{a\in\mathbb{D}} \frac{1}{\left(\log \frac{2}{1-|a|^2}\right)^q} \int_{\mathbb{D}} |f'(z)|^q (1 - |z|^2)^{q-2} (1 - |\sigma_a(z)|^2)^s dA(z) < \infty.$$

It is easy to check that $\mathcal{LF}(q, q - 2, s)$ is a Banach space under the norm $\|\cdot\|_{\mathcal{LF}(q,q-2,s)}$ when $q \geq 1$.

Let $g \in H(\mathbb{D})$. The Volterra integral operator T_g, which introduced by Pommerenke in [13], was defined by

$$T_g f(z) = \int_0^z f(w)g'(w)dw, \qquad f \in H(\mathbb{D}).$$

The importance of the operator T_g comes from the fact that $T_g f + I_g f = M_g f - f(0)g(0)$, where the operators M_g and I_g are defined by

$$(M_g f)(z) = g(z)f(z), \ \ I_g f(z) = \int_0^z f'(w)g(w)dw \ \ f \in H(\mathbb{D}), \ \ z \in \mathbb{D},$$

respectively. Note that the integral form of the classical Cesàro operator C is

$$C(f)(z) = \frac{1}{z} \int_0^z f(\zeta)\frac{1}{1-\zeta}d\zeta = \frac{1}{z} \int_0^z f(\zeta)\left(\ln \frac{1}{1-\zeta}\right)' d\zeta.$$

Hence the operator T_g can also be seen as the generalization of the Cesàro operator C. In [13], Pommerenke showed that T_g is bounded on H^2 if and only if $g \in BMOA$. In [2], Aleman and Siskakis showed that T_g is bounded (compact) on A^p if and only if $g \in \mathcal{B}(g \in \mathcal{B}_0)$. Recently, the operator T_g has been received many attention. See [1, 2, 4–8, 12, 14, 15, 18, 19, 22, 24] and the references therein for more study of the operator T_g.

For an arc $I \subseteq \partial\mathbb{D}$, let $|I| = \frac{1}{2\pi}\int_I |d\zeta|$ be the normalized length of I. Let $0 < \alpha < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. As usual, we say that μ is a α-Carleson measure if

$$\|\mu\|_\alpha := \sup_{I\subseteq\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^\alpha} < \infty,$$

where $S(I)=\{z\in\mathbb{D}: 1-|I|\le |z|<1, \quad \frac{z}{|z|}\in I\}$ is the Carleson box based on I. When $\alpha=1$, it gives the classical Carleson measure. μ is said to be a vanishing α-Carleson measure if $\lim_{|I|\to 0}\frac{\mu(S(I))}{|I|^{\alpha}}=0$.

Let $0<\lambda, q<\infty$, $s\ge 0$ and μ be a positive Borel measure on $\mathbb{D}$. The tent space $\mathcal{T}^q_{\lambda,s}(\mu)$ consists of all $f\in H(\mathbb{D})$ satisfied

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{1}{|I|^{\lambda}(\log\frac{1}{|I|})^{s}}\int_{S(I)}|f(z)|^q d\mu(z)<\infty.$$

The tent space $\mathcal{T}^q_{\lambda,s}(\mu)$ was introduced by Liu, Lou and Zhu in [10]. When $q=2$ and $s=0$, $\mathcal{T}^2_{\lambda,0}(\mu)=\mathcal{T}^{\infty}_{\lambda}$ was first introduced by Xiao in [22].

In [22], Xiao studied the inclusion mapping $i: Q_p\to\mathcal{T}^{\infty}_p(\mu)$. He showed that the inclusion mapping $i: Q_p\to\mathcal{T}^{\infty}_p(\mu)$ is bounded (resp. compact) if and only if

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{(\log\frac{2}{|I|})^2\mu(S(I))}{|I|^p}<\infty \qquad (\text{ resp. } \lim_{|I|\to 0}\frac{(\log\frac{2}{|I|})^2\mu(S(I))}{|I|^p}=0).$$

As an application, he proved that the operator $T_g: Q_p\to Q_p$ is bounded if and only if

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{(\log\frac{2}{|I|})^2}{|I|^p}\int_{S(I)}|f'(z)|^2(1-|z|^2)^p dA(z)<\infty.$$

In [10], Liu, Lou and Zhu studied the embedding of some Möbius invariant spaces, such as the Bloch space and the Q_p space, into $\mathcal{T}^2_{\lambda,s}$. Among others, they proved the following Theorem A. See [6, 9, 12, 14–17, 21] and the references therein for more study of analytic function spaces embedding into various tent spaces.

Theorem A. *Let $0<p<1$ and μ be a positive Borel measure on $\mathbb{D}$. If Q_p is continuously contained in $\mathcal{T}^2_{p,2}$, then μ is a p-Carleson measure. If $\mathcal{D}^2_p$ is continuously contained in $L^2(\mathbb{D}, d\mu)$, then Q_p is continuously contained in $\mathcal{T}^2_{p,2}$.*

By [22, Lemma 2.1 (ii)], we see that μ is a p-Carleson measure if $\mathcal{D}^2_p$ is continuously contained in $L^2(\mathbb{D}, d\mu)$. But the converse is not clear. The nature question then arise, what can one say if we change $\mathcal{T}^2_{p,2}$ into $\mathcal{T}^q_{\lambda,s}$ when $q>2$?

In this paper, we give an answer by using a new method, which was different to [10, 22]. We study the boundedness and compactness of the inclusion mapping from Q_p spaces into tent spaces $\mathcal{T}^q_{\frac{qp}{2},s}$. As an application, we study the boundedness of Volterra integral operator T_g acting from Q_p to $\mathcal{LF}(q, q-2, \frac{qp}{2})$. Meanwhile, the compactness and essential norm of the operator T_g acting from Q_p to $\mathcal{LF}(q, q-2, \frac{qp}{2})$ are also investigated.

Throughout this paper, we say that $A\lesssim B$ if there exists a constant C such that $A\le CB$. The symbol $A\approx B$ means that $A\lesssim B\lesssim A$.

2. Embedding of Q_p spaces into tent spaces

In this section, we study the embedding from Q_p to tent spaces. We give a complete characterization for the boundedness and compactness of the inclusion mapping $i: Q_p\to T^q_{\lambda,s}(\mu)$. We say that the inclusion mapping $i : Q_p \to \mathcal{T}^q_{\lambda,s}(\mu)$ is compact if

$$\lim_{n\to\infty} \frac{1}{|I|^{\lambda}(\log \frac{1}{|I|})^s} \int_{S(I)} |f_n(z)|^q d\mu(z) = 0$$

whenever $I \subseteq \partial\mathbb{D}$ and $\{f_n\}$ is a bounded sequence in Q_p that converges to 0 uniformly on compact subsets of $\mathbb{D}$.

The following result is one of the main results in this paper.

Theorem 1. *Let $0 < p < 1$ and μ be a positive Borel measure. If $2 < q < \infty$ and $0 < s \le q < \infty$, then the following statements hold.*

(i) The inclusion mapping $i : \ Q_p \to \mathcal{T}^q_{\frac{qp}{2},s}(\mu)$ is bounded if and only if

$$\|\mu\|_{LCM_{q-s,\frac{qp}{2}}} = \sup_{I\subseteq\partial\mathbb{D}} \frac{\left(\log \frac{2}{|I|}\right)^{q-s} \mu(S(I))}{|I|^{\frac{qp}{2}}} < \infty. \tag{2.1}$$

(ii) The inclusion mapping $i : \ Q_p \to \mathcal{T}^q_{\frac{qp}{2},s}(\mu)$ is compact if and only if

$$\lim_{|I|\to 0} \frac{\left(\log \frac{2}{|I|}\right)^{q-s} \mu(S(I))}{|I|^{\frac{qp}{2}}} = 0. \tag{2.2}$$

Proof. (i) Assume that the inclusion mapping $i : \ Q_p \to \mathcal{T}^q_{\frac{qp}{2},s}(\mu)$ is bounded. For any fixed arc $I \subseteq \partial\mathbb{D}$, let $e^{i\theta}$ be the center of I and $a = (1 - |I|)e^{i\theta}$. Set $f_a(z) = \log \frac{2}{(1-\overline{a}z)}$. Then $f_a \in Q_p$ and

$$|1 - \overline{a}z| \approx 1 - |a| = |I|, \quad |f_a(z)| \approx \log \frac{2}{|I|},$$

whenever $z \in S(I)$. By the boundedness of i, we have

$$\frac{1}{|I|^{\frac{qp}{2}}(\log \frac{2}{|I|})^s} \int_{S(I)} |f_a(z)|^q d\mu(z) \lesssim \|f_a\|^q_{Q_p} < \infty,$$

which implies (1), as desired.

Conversely, assume that (1) holds. Let $f \in Q_p$. For any fixed arc $I \subseteq \partial\mathbb{D}$, let $e^{i\theta}$ be the center of I and $a = (1 - |I|)e^{i\theta}$. We have

$$\frac{1}{|I|^{\frac{qp}{2}}(\log \frac{2}{|I|})^s} \int_{S(I)} |f(z)|^q d\mu(z) \lesssim A + B,$$

where

$$A = \frac{1}{|I|^{\frac{qp}{2}}(\log \frac{2}{|I|})^s} \int_{S(I)} |f(z) - f(a)|^q d\mu(z), \quad B = \frac{|f(a)|^q \mu(S(I))}{|I|^{\frac{qp}{2}}(\log \frac{2}{|I|})^s}.$$

Since $f \in Q_p \subseteq \mathcal{B}$, we obtain

$$|f(a)|^q \lesssim \|f\|^q_{\mathcal{B}}(\log \frac{2}{1-|a|^2})^q \lesssim \|f\|^q_{Q_p}\Big(\log \frac{2}{1-|a|^2}\Big)^q,$$

which implies that for any $I \subseteq \partial\mathbb{D}$,

$$B \lesssim \sup_{I\subseteq\partial\mathbb{D}} \frac{(\log \frac{2}{|I|})^{q-s}\mu(S(I))}{|I|^{\frac{qp}{2}}} \|f\|_{Q_p}^q \lesssim \|f\|_{Q_p}^q.$$

Since $0 < s \le q < \infty$, we get

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^{\frac{qp}{2}}} \lesssim \sup_{I\subseteq\partial\mathbb{D}} \frac{\left(\log \frac{2}{|I|}\right)^{q-s} \mu(S(I))}{|I|^{\frac{qp}{2}}} < \infty,$$

which implies that $\mathcal{D}_p^2 \subseteq L^q(d\mu)$ by [4, Theorem 1]. Therefore,

$$\begin{aligned}
A \lesssim & \frac{1}{|I|^{\frac{qp}{2}}} \int_{S(I)} |f(z) - f(a)|^q d\mu(z) \\
\lesssim & (1 - |a|^2)^{qp} \int_{S(I)} \left| \frac{f(z) - f(a)}{(1 - \overline{a}z)^{\frac{3p}{2}}} \right|^q d\mu(z) \\
\lesssim & (1 - |a|^2)^{qp} \int_{\mathbb{D}} \left| \frac{f(z) - f(a)}{(1 - \overline{a}z)^{\frac{3p}{2}}} \right|^q d\mu(z) \\
\lesssim & (1 - |a|^2)^{qp} \left(|f(0) - f(a)|^2 + \int_{\mathbb{D}} \left| \frac{d}{dz} \frac{f(z) - f(a)}{(1 - \overline{a}z)^{\frac{3p}{2}}} \right|^2 (1 - |z|^2)^p dA(z) \right)^{\frac{q}{2}}.
\end{aligned}$$

By the growth of functions in Q_p and

$$x^\alpha \left(\log \frac{2}{x} \right)^\beta \lesssim 1, \quad 0 < x < 1, \quad 0 < \alpha, \beta < \infty,$$

we deduce that

$$(1 - |a|^2)^{pq} |f(0) - f(a)|^q \lesssim \|f\|_{Q_p}^q.$$

Thus, we only need to prove that

$$E = (1 - |a|^2)^{2p} \int_{\mathbb{D}} \left| \frac{d}{dz} \frac{f(z) - f(a)}{(1 - \overline{a}z)^{\frac{3p}{2}}} \right|^2 (1 - |z|^2)^p dA(z) \lesssim \|f\|_{Q_p}^2.$$

Since

$$\frac{d}{dz} \frac{f(z) - f(a)}{(1 - \overline{a}z)^{\frac{3p}{2}}} = \frac{f'(z)(1 - \overline{a}z)^{\frac{3p}{2}} + \overline{a}(\frac{3p}{2})(f(z) - f(a))(1 - \overline{a}z)^{\frac{3p}{2}-1}}{(1 - \overline{a}z)^{3p}},$$

we obtain $E \lesssim E_1 + E_2$, where

$$E_1 = (1 - |a|^2)^{2p} \int_{\mathbb{D}} \frac{|f'(z)|^2}{|1 - \overline{a}z|^{3p}} (1 - |z|^2)^p dA(z)$$

and

$$E_2 = (1 - |a|^2)^{2p} \int_{\mathbb{D}} \frac{|f(z) - f(a)|^2}{|1 - \overline{a}z|^{3p+2}} (1 - |z|^2)^p dA(z).$$

Noting that

$$1-|\varphi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2},\quad a,z\in\mathbb{D},$$

we have

$$E_1=\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|a|^2)^{p+p}(1-|z|^2)^p}{|1-\overline{a}z|^{3p}}dA(z)\lesssim\|f\|_{Q_p}^2.$$

By [11], we deduce that

$$\begin{aligned}E_2=&(1-|a|^2)^{2p}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^2}{|1-\overline{a}z|^{3p+2}}(1-|z|^2)^p dA(z)\\ =&\int_{\mathbb{D}}\frac{|f(z)-f(a)|^2}{|1-\overline{a}z|^2}\frac{(1-|a|^2)^{p+p}(1-|z|^2)^p}{|1-\overline{a}z|^{3p}}dA(z)\\ \lesssim&\int_{\mathbb{D}}\left|\frac{f(z)-f(a)}{1-\overline{a}z}\right|^2(1-|\varphi_a(z)|^2)^p dA(z)\lesssim\|f\|_{Q_p}^2.\end{aligned}$$

Therefore, $E\lesssim\|f\|_{Q_p}^2$, as desired.

(*ii*) Suppose that the inclusion mapping $i:\ Q_p\to\mathcal{T}^q_{\frac{qp}{2},s}(\mu)$ is compact. Let $\{I_n\}\subseteq\partial\mathbb{D}$ and $|I_n|\to 0$ as $n\to\infty$. Suppose $e^{i\theta_n}$ is the center of I_n and $a_n=(1-|I_n|)e^{i\theta_n}$. Set $f_{a_n}(z)=\log\frac{2}{(1-\overline{a_n}z)}$. Then $f_{a_n}\in Q_p$ and $\log\frac{2}{(1-\overline{a_n}z)}\approx\log\frac{2}{|I_n|}$. Therefore

$$\frac{\left(\log\frac{2}{|I_n|}\right)^{q-s}\mu(S(I_n))}{|I_n|^{\frac{qp}{2}}}\lesssim\frac{1}{|I_n|^{\frac{qp}{2}}\left(\log\frac{2}{|I_n|}\right)^s}\int_{S(I_n)}|f_{a_n}(z)|^q d\mu(z)\to 0,\quad n\to\infty,$$

which implies that (2) holds.

Conversely, assume that (2) holds. Then it is clear that

$$\|\mu\|_{LCM_{q-s,\frac{qp}{2}}}=\sup_{I\subseteq\partial\mathbb{D}}\frac{\left(\log\frac{2}{|I|}\right)^{q-s}\mu(S(I))}{|I|^{\frac{qp}{2}}}<\infty\quad\text{and}\quad\sup_{I\subseteq\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^{\frac{qp}{2}}}<\infty.$$

Let $\{f_n\}$ be a bounded sequence in Q_p such that $\{f_n\}$ converges to zero uniformly on each compact subset of $\mathbb{D}$. From [12] we have

$$\begin{aligned}&\frac{1}{|I|^{\frac{qp}{2}}(\log\frac{2}{|I|})^s}\int_{S(I)}|f_n(z)|^q d\mu(z)\\ \lesssim&\frac{1}{|I|^{\frac{qp}{2}}(\log\frac{2}{|I|})^s}\int_{S(I)}|f_n(z)|^q d\mu_r(z)+\frac{1}{|I|^{\frac{qp}{2}}(\log\frac{2}{|I|})^s}\int_{S(I)}|f_n(z)|^q d(\mu-\mu_r)(z)\\ \lesssim&\frac{1}{|I|^{\frac{qp}{2}}(\log\frac{2}{|I|})^s}\int_{S(I)}|f_n(z)|^q d\mu_r(z)+\|\mu-\mu_r\|_{LCM_{q-s,\frac{qp}{2}}}\|f_n\|_{Q_p}^q\\ \lesssim&\frac{1}{|I|^{\frac{qp}{2}}(\log\frac{2}{|I|})^s}\int_{S(I)}|f_n(z)|^q d\mu_r(z)+\|\mu-\mu_r\|_{LCM_{q-s,\frac{qp}{2}}}\\ \lesssim&\frac{1}{|I|^{\frac{qp}{2}}}\int_{S(I)}|f_n(z)|^q d\mu_r(z)+\|\mu-\mu_r\|_{LCM_{q-s,\frac{qp}{2}}}\to 0\end{aligned}$$

as $r \to 1^-$ and $n \to \infty$. Therefore, $\lim_{n\to\infty} \|f_n\|_{\mathcal{T}^q_{\frac{qp}{2},s}(\mu)} = 0$. This shows that the inclusion mapping $i : Q_p \to \mathcal{T}^q_{\frac{qp}{2},s}(\mu)$ is compact. □

In particular, let $s = q$, we get the following result.

Corollary 1. *Let* $0 < p < 1$, $2 < q < \infty$ *and* μ *be a positive Borel measure. Then the inclusion mapping* $i : Q_p \to \mathcal{T}^q_{\frac{qp}{2},q}(\mu)$ *is bounded (resp., compact) if and only if*

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^{\frac{qp}{2}}} < \infty \quad \left(resp., \ \lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^{\frac{qp}{2}}} = 0 \right).$$

3. Volterra integral operator $T_g : Q_p \to \mathcal{LF}(q, q-2, \frac{qp}{2})$

In this section, we study the boundednss, compactness and the essential norm of Volterra integral operator $T_g : Q_p \to \mathcal{LF}(q, q-2, \frac{qp}{2})$. We need the following equivalent characterization of functions in $\mathcal{LF}(q, q-2, s)$.

Proposition 1. *Let* $1 < q < \infty$ *and* $0 < s < \infty$. *Then* $f \in \mathcal{LF}(q, q-2, s)$ *if and only if*

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^s(\log\frac{2}{|I|})^q} \int_{S(I)} |f'(z)|^q(1-|z|^2)^{q-2+s}dA(z) < \infty. \tag{3.1}$$

Proof. Let $f \in \mathcal{LF}(q, q-2, s)$. For any $I \in \partial\mathbb{D}$, let $a = (1-|I|)\zeta \in \mathbb{D}$, where ζ is the center of I. Then

$$1-|a| \approx |1-\overline{a}z| \approx |I|, \quad z \in S(I).$$

Combining with $1-|\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}$, we have

$$\begin{aligned}
&\frac{1}{|I|^s(\log\frac{2}{|I|})^q} \int_{S(I)} |f'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\
\approx&\frac{1}{\left(\log\frac{2}{1-|a|^2}\right)^q} \int_{S(I)} |f'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^s dA(z)\\
\lesssim&\sup_{b\in\mathbb{D}} \frac{1}{\left(\log\frac{2}{1-|b|^2}\right)^q} \int_{\mathbb{D}} |f'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_b(z)|^2)^s dA(z) < \infty,
\end{aligned}$$

as desired.

Conversely, assume that (3) holds. For any given nonzero $a \in \mathbb{D}$, let I_a be the subarc of $\partial\mathbb{D}$ with midpoint $a/|a|$ and length $1-|a|$; and for $a = 0$, let $I_a = \partial\mathbb{D}$. Moreover, let $J_n = 2^n I_a$ for $n = 0, 1, \cdots, N-1$, where N is the smallest positive integer such that $2^N|I_a| \geq 1$. Then we have the following estimate:

$$\frac{1-|a|}{|1-\overline{a}z|^2} \approx \frac{1}{|I_a|}, \quad z \in I_a \tag{3.2}$$

and

$$\frac{1-|a|}{|1-\overline{a}z|^2} \approx \frac{1}{2^{2n}|I_a|}, \quad z \in J_{n+1}\backslash J_n. \tag{3.3}$$

Without loss of generality, we may assume $|a| > 3/4$. By (4) and (5) we have

$$\begin{aligned}
&\frac{1}{\left(\log\frac{2}{1-|a|^2}\right)^q}\int_{\mathbb{D}}|f'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^s dA(z)\\
&\lesssim \sum_{n=0}^{N-1}\frac{1}{|2^{2n}I_a|^s\left(\log\frac{2}{|I_a|}\right)^q}\int_{S(J_{n+1})\backslash S(J_n)}|f'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\
&\qquad\qquad + \frac{1}{|I_a|^s\left(\log\frac{2}{|I_a|}\right)^q}\int_{S(J_0)}|f'(z)|^q(1-|z|^2)^{q-2+s}dA(z)\\
&\lesssim \sum_{n=0}^{N-1}\frac{1}{|2^{2n}I_a|^s\left(\log\frac{2}{|I_a|}\right)^q}\int_{S(J_{n+1})}|f'(z)|^q(1-|z|^2)^{q-2+s}dA(z)+C\\
&\lesssim \sum_{n=0}^{N-1}\frac{1}{|2^{2n}I_a|^s\left(\log\frac{2}{|I_a|}\right)^q}\times|2^{n+1}I_a|^s\left(\log\frac{2}{|2^{n+1}I_a|}\right)^q+C\\
&\lesssim \sum_{n=0}^{\infty}\frac{1}{2^{ns}}\frac{\left(\log\frac{2}{|2^{n+1}I_a|}\right)^q}{\left(\log\frac{2}{|I_a|}\right)^q}+C\\
&\lesssim \sum_{n=0}^{\infty}\frac{1}{2^{ns}}+C<\infty.
\end{aligned}$$

The proof is complete. □

Theorem 2. *Let $0 < p < 1$, $2 < q < \infty$ and $g \in H(\mathbb{D})$. Then $T_g : \ Q_p \to \mathcal{LF}(q, q-2, \frac{qp}{2})$ is bounded if and only if $g \in \mathcal{F}(q, q-2, \frac{qp}{2})$.*

Proof. Suppose that $g \in \mathcal{F}(q, q-2, \frac{qp}{2})$. By [24] we have

$$\|g\|_{\mathcal{F}(q,q-2,\frac{qp}{2})} \approx \sup_{I\subseteq\partial\mathbb{D}}\frac{1}{|I|^{\frac{qp}{2}}}\int_{S(I)}|g'(z)|^q(1-|z|^2)^{q-2+\frac{qp}{2}}dA(z),$$

which means that $d\mu_g(z) = |g'(z)|^q(1-|z|^2)^{q-2+\frac{qp}{2}}dA(z)$ is a $\frac{qp}{2}$-Carleson measure. Let $f \in Q_p$. By Corollary 1, we see that $i : \ Q_p \to \mathcal{T}^q_{\frac{qp}{2},q}(\mu_g)$ is bounded, i.e.,

$$\begin{aligned}
&\sup_{I\subseteq\partial\mathbb{D}}\frac{1}{|I|^{\frac{qp}{2}}(\log\frac{2}{|I|})^q}\int_{S(I)}|(T_gf)'(z)|^q(1-|z|^2)^{q-2+\frac{qp}{2}}dA(z)\\
&=\sup_{I\subseteq\partial\mathbb{D}}\frac{1}{|I|^{\frac{qp}{2}}(\log\frac{2}{|I|})^q}\int_{S(I)}|f(z)|^q d\mu_g(z)<\infty,
\end{aligned}$$

which together with Proposition 1 imply that

$$\sup_{a\in\mathbb{D}}\frac{1}{\left(\log\frac{2}{1-|a|^2}\right)^q}\int_{\mathbb{D}}|(T_gf)'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{\frac{qp}{2}}dA(z)<\infty.$$

Therefore $T_g: Q_p \to \mathcal{LF}(q, q-2, \frac{qp}{2})$ is bounded.

Conversely, assume that $T_g: Q_p \to \mathcal{LF}(q, q-2, \frac{qp}{2})$ is bounded. For any fixed arc $I \subseteq \partial\mathbb{D}$ and let $e^{i\theta}$ be the center of I and $a = (1-|I|)e^{i\theta}$. Set $f_a(z) = \log \frac{2}{(1-\bar{a}z)}$. Then $f_a \in Q_p$ for $0 < p < \infty$. Since

$$|1-\bar{a}z| \approx 1-|a| = |I|, \quad |f_a(z)| \approx \log \frac{2}{|I|},$$

when $z \in S(I)$, we get

$$\begin{aligned}\infty >&\|T_g f_a\|^q_{\mathcal{LF}(q,q-2,\frac{qp}{2})}\\ \geq&\frac{1}{|I|^{\frac{qp}{2}}(\log\frac{2}{|I|})^q}\int_{S(I)}|f_a(z)|^q|g'(z)|^q(1-|z|^2)^{q-2+\frac{qp}{2}}dA(z)\\ \approx&\frac{1}{|I|^{\frac{qp}{2}}}\int_{S(I)}|g'(z)|^q(1-|z|^2)^{q-2+\frac{qp}{2}}dA(z),\end{aligned}$$

which implies that $g \in \mathcal{F}(q, q-2, \frac{qp}{2})$ by [24]. The proof is complete. □

Next, we give an estimation for the essential norm of T_g. First, we recall some definitions. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T: X \to Y$ be a bounded linear operator. The essential norm of $T: X \to Y$, denoted by $\|T\|_{e,X\to Y}$, is defined by

$$\|T\|_{e,X\to Y} = \inf_K\{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

It is easy to see that $T: X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$. Let A be a closed subspace of X. Given $f \in X$, the distance from f to A, denoted by $\text{dist}_X(f, A)$, is defined by $\text{dist}_X(f, A) = \inf_{g\in A}\|f-g\|_X$.

Lemma 1. *Let* $2 < q < \infty$ *and* $0 < \lambda < \infty$. *If* $g \in \mathcal{F}(q, q-2, \lambda)$, *then*

$$\begin{aligned}\text{dist}_{\mathcal{F}(q,q-2,\lambda)}(g, \mathcal{F}_0(q, q-2, \lambda)) &\approx \limsup_{r\to 1^-}\|g-g_r\|_{\mathcal{F}(q,q-2,\lambda)}\\ &\approx \limsup_{|a|\to 1}\left(\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^\lambda dA(z)\right)^{1/q}.\end{aligned}$$

Here $g_r(z) = g(rz)$, $0 < r < 1, z \in \mathbb{D}$.

Proof. For any given $g \in \mathcal{F}(q, q-2, \lambda)$, then $g_r \in \mathcal{F}_0(q, q-2, \lambda)$ and $\|g_r\|_{\mathcal{F}(q,q-2,\lambda)} \lesssim \|g\|_{\mathcal{F}(q,q-2,\lambda)}$. Let $\delta \in (0, 1)$. We choose $a \in (0, \delta)$. Then $\sigma_a(z)$ lies in a compact subset of $\mathbb{D}$. So

$$\lim_{r\to 1}\sup_{z\in\mathbb{D}}|g'(\sigma_a(z)) - rg'(r\sigma_a(z))| = 0.$$

Making a change of variables, we have

$$\begin{aligned}&\lim_{r\to 1}\sup_{|a|\leq\delta}\int_{\mathbb{D}}|g'(z)-g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^\lambda dA(z)\\ =&\lim_{r\to 1}\sup_{|a|\leq\delta}\int_{\mathbb{D}}|g'(\sigma_a(z))-g_r'(\sigma_a(z))|^q(1-|z|^2)^{q+\lambda-2}|\sigma_a'(z)|^q dA(z)\\ =&\lim_{r\to 1}\sup_{|a|\leq\delta}\sup_{z\in\mathbb{D}}|g'(\sigma_a(z))-g_r'(\sigma_a(z))|^q\int_{\mathbb{D}}(1-|z|^2)^{q+\lambda-2}|\sigma_a'(z)|^q dA(z)\\ =&0.\end{aligned}$$

By the definition of distance, we obtain

$$\begin{aligned}
&\mathrm{dist}_{\mathcal{F}(q,q-2,\lambda)}(g,\mathcal{F}_0(q,q-2,\lambda)) = \inf_{f\in\mathcal{F}_0(q,q-2,\lambda)} \|g-f\|_{\mathcal{F}(q,q-2,\lambda)}\\
\leq\ & \lim_{r\to 1}\|g-g_r\|_{\mathcal{F}(q,q-2,\lambda)}\\
\leq\ & \lim_{r\to 1}\left(\sup_{|a|>\delta}\int_{\mathbb{D}}|g'(z)-g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{\lambda}dA(z)\right)^{1/q}\\
&+\lim_{r\to 1}\left(\sup_{|a|\leq\delta}\int_{\mathbb{D}}|g'(z)-g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{\lambda}dA(z)\right)^{1/q}\\
\lesssim\ & \left(\sup_{|a|>\delta}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{\lambda}dA(z)\right)^{1/q}\\
&+\lim_{r\to 1}\left(\sup_{|a|>\delta}\int_{\mathbb{D}}|g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{\lambda}dA(z)\right)^{1/q}.
\end{aligned}$$

Denote by $\psi_{r,a}(z) = \sigma_{ra}\circ r\sigma_a(z)$. Then $\psi_{r,a}$ is an analytic self-map of $\mathbb{D}$ and $\psi_{r,a}(0) = 0$. Making a change variable of $z = \sigma_a(z)$ and applying the Littlewood's subordination theorem (see Theorem 1.7 of [3]), we have

$$\begin{aligned}
&\int_{\mathbb{D}}|g_r'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{\lambda}dA(z)\\
=\ & \int_{\mathbb{D}}|g_r'(\sigma_a(z))|^q(1-|\sigma_a(z)|^2)^{q}(1-|z|^2)^{\lambda-2}dA(z)\\
\leq\ & \int_{\mathbb{D}}|g'\circ\sigma_{ra}\circ\psi_{r,a}(z)|^q(1-|\sigma_{ra}\circ\psi_{r,a}(z)|^2)^{q}(1-|z|^2)^{\lambda-2}dA(z)\\
\leq\ & \int_{\mathbb{D}}|g'\circ\sigma_{ra}\circ\psi_{r,a}(z)|^q(1-|\sigma_{ra}\circ\psi_{r,a}(z)|^2)^{q}(1-|z|^2)^{\lambda-2}dA(z)\\
\leq\ & \int_{\mathbb{D}}|g'\circ\sigma_{ra}(z)|^q(1-|\sigma_{ra}(z)|^2)^{q}(1-|z|^2)^{\lambda-2}dA(z)\\
\leq\ & \int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_{ra}(z)|^2)^{\lambda}dA(z).
\end{aligned}$$

Since δ is arbitrary, we get

$$\begin{aligned}
&\mathrm{dist}_{\mathcal{F}(q,q-2,\lambda)}(g,\mathcal{F}_0(q,q-2,\lambda))\\
\lesssim\ & \limsup_{|a|\to 1}\left(\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{\lambda}dA(z)\right)^{1/q}.
\end{aligned}\tag{3.4}$$

On the other hand, for any $g\in\mathcal{F}(q,q-2,\lambda)$,

$$\begin{aligned}
&\mathrm{dist}_{\mathcal{F}(q,q-2,\lambda)}(g,\mathcal{F}_0(q,q-2,\lambda)) = \inf_{f\in\mathcal{F}_0(q,q-2,\lambda)} \|g-f\|_{\mathcal{F}(q,q-2,\lambda)}\\
\gtrsim\ & \limsup_{|a|\to 1}\left(\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{\lambda}dA(z)\right)^{1/q},
\end{aligned}$$

which, together with (3.4), implies the desired result. The proof is complete. □

Lemma 2. *Let* $0 < p < 1$ *and* $2 < q < \infty$*. If* $0 < r < 1$ *and* $g \in \mathcal{F}(q, q-2, \frac{qp}{2})$*, then* $T_{g_r} : Q_p \to \mathcal{LF}(q, q-2, \frac{qp}{2})$ *is compact.*

Proof. Given $\{f_n\} \subset Q_p$ such that $\{f_n\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$ and $\sup_n \|f_n\|_{Q_p} \le 1$. Then by the following well-known inequality

$$|h(z)| \lesssim \|h\|_{\mathcal{B}} \log \frac{2}{1-|z|^2}, \qquad h \in \mathcal{B},$$

we get

$$\begin{aligned}
&\|T_{g_r} f_n\|^q_{\mathcal{LF}(q,q-2,\frac{qp}{2})}\\
=&\sup_{a\in\mathbb{D}} \frac{1}{\left(\log \frac{2}{1-|a|^2}\right)^q} \int_{\mathbb{D}} |f_n(z)|^q |g_r'(z)|^q (1-|z|^2)^{q-2} (1-|\sigma_a(z)|^2)^{\frac{qp}{2}} dA(z)\\
\lesssim& \frac{\|g\|^q_{\mathcal{B}}}{(1-r^2)^q} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)|^q (1-|z|^2)^{q-2} (1-|\sigma_a(z)|^2)^{\frac{qp}{2}} dA(z)\\
\lesssim& \frac{\|g\|^q_{\mathcal{F}(q,q-2,\frac{qp}{2})} \|f_n\|^{q-2}_{\mathcal{B}}}{(1-r^2)^q} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)|^2 \left(\log \frac{2}{1-|z|^2}\right)^{q-2} (1-|z|^2)^{q-2} (1-|\sigma_a(z)|^2)^{\frac{qp}{2}} dA(z)\\
\lesssim& \frac{\|g\|^q_{\mathcal{F}(q,q-2,\frac{qp}{2})} \|f_n\|^{q-2}_{Q_p}}{(1-r^2)^q} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)|^2 (1-|\sigma_a(z)|^2)^p dA(z)\\
\lesssim& \frac{\|g\|^q_{\mathcal{F}(q,q-2,\frac{qp}{2})} \|f_n\|^{q-2}_{Q_p}}{(1-r^2)^q} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n'(z)|^2 (1-|\sigma_a(z)|^2)^p dA(z)\\
\lesssim& \frac{\|g\|^q_{\mathcal{F}(q,q-2,\frac{qp}{2})} \|f_n\|^{q}_{Q_p}}{(1-r^2)^q}.
\end{aligned}$$

By the dominated convergence theorem, we get the desire result. The proof is complete. □

The following result is an important tool to study the essential norm and compactness of operators on some analytic function spaces, see [20].

Lemma 3. *Let* X, Y *be two Banach spaces of analytic functions on* $\mathbb{D}$*. Suppose that*

(1) The point evaluation functionals on Y *are continuous.*
(2) The closed unit ball of X *is a compact subset of* X *in the topology of uniform convergence on compact sets.*
(3) $T : X \to Y$ *is continuous when* X *and* Y *are given the topology of uniform convergence on compact sets.*

Then, T *is a compact operator if and only if for any bounded sequence* $\{f_n\}$ *in* X *such that* $\{f_n\}$ *converges to zero uniformly on every compact set of* $\mathbb{D}$*, then the sequence* $\{Tf_n\}$ *converges to zero in the norm of* Y*.*

Theorem 3. *Let* $0 < p < 1$, $2 < q < \infty$ *and* $g \in H(\mathbb{D})$*. If* $T_g : Q_p \to \mathcal{LF}(q, q-2, \frac{qp}{2})$ *is bounded, then*

$$\|T_g\|_{e, Q_p \to \mathcal{LF}(q,q-2,\frac{qp}{2})} \approx \mathrm{dist}_{\mathcal{F}(q,q-2,\frac{qp}{2})}(g, \mathcal{F}_0(q, q-2, \frac{qp}{2})).$$

Proof. Let $\{I_n\} \subseteq \partial\mathbb{D}$ and $|I_n| \to 0$ as $n \to \infty$. Suppose $e^{i\theta_n}$ is the center of I_n and $w_n = (1 - |I_n|)e^{i\theta_n}$. For each n, let

$$f_{w_n}(z) = \frac{1}{\log \frac{2}{1-|w_n|^2}} \left(\log \frac{2}{1 - \overline{w_n} z}\right)^2.$$

Then $|f_{w_n}(z)| \approx \log \frac{2}{|I_n|}$ when $z \in S(I_n)$ and $\{f_{w_n}\}$ is bounded in Q_p. Furthermore, $\{f_{w_n}\}$ converges to zero uniformly on every compact subset of $\mathbb{D}$. Given a compact operator $K : Q_p \to \mathcal{LF}(q, q-2, \frac{qp}{2})$, by Lemma 3 we have $\lim_{n\to\infty} \|Kf_{w_n}\|_{\mathcal{LF}(q,q-2,\frac{qp}{2})} = 0$. So

$$\begin{aligned}
\|T_g - K\| &\gtrsim \limsup_{n\to\infty} \|(T_g - K)f_{w_n}\|_{\mathcal{LF}(q,q-2,\frac{qp}{2})} \\
&\gtrsim \limsup_{n\to\infty} \left(\|T_g f_{w_n}\|_{\mathcal{LF}(q,q-2,\frac{qp}{2})} - \|Kf_{w_n}\|_{\mathcal{LF}(q,q-2,\frac{qp}{2})}\right) \\
&= \limsup_{n\to\infty} \|T_g f_{w_n}\|_{\mathcal{LF}(q,q-2,\frac{qp}{2})} \\
&\gtrsim \limsup_{n\to\infty} \left(\frac{1}{\left(\log \frac{2}{1-|w_n|^2}\right)^q} \int_{\mathbb{D}} |f_{w_n}(z)|^q |g'(z)|^q (1-|z|^2)^{q-2} (1 - |\sigma_{w_n}(z)|^2)^{\frac{qp}{2}} dA(z) \right)^{\frac{1}{q}} \\
&\gtrsim \limsup_{n\to\infty} \left(\frac{1}{\left(\log \frac{2}{1-|w_n|^2}\right)^q} \int_{S(I_n)} |f_{w_n}(z)|^q |g'(z)|^q (1-|z|^2)^{q-2} (1 - |\sigma_{w_n}(z)|^2)^{\frac{qp}{2}} dA(z) \right)^{\frac{1}{q}} \\
&\gtrsim \limsup_{n\to\infty} \left(\frac{1}{|I_n|^{\frac{qp}{2}}} \int_{S(I_n)} |g'(z)|^q (1-|z|^2)^{q-2+\frac{qp}{2}} dA(z) \right)^{\frac{1}{q}},
\end{aligned}$$

which implies that

$$\|T_g\|_{e,Q_p\to\mathcal{LF}(q,q-2,\frac{qp}{2})} \gtrsim \limsup_{n\to\infty} \left(\int_{\mathbb{D}} |g'(z)|^q (1-|z|^2)^{q-2} (1 - |\sigma_{w_n}(z)|^2)^{\frac{qp}{2}} dA(z) \right)^{\frac{1}{q}}.$$

It follows from Lemma 1 that

$$\|T_g\|_{e,Q_p\to\mathcal{LF}(q,q-2,\frac{qp}{2})} \gtrsim \mathrm{dist}_{\mathcal{F}(q,q-2,\frac{qp}{2})}(g, \mathcal{F}_0(q, q-2, \frac{qp}{2})).$$

On the other hand, by Lemma 2, $T_{g_r} : Q_p \to \mathcal{LF}(q, q-2, \frac{qp}{2})$ is compact. Then

$$\|T_g\|_{e,Q_p\to\mathcal{LF}(q,q-2,\frac{qp}{2})} \leq \|T_g - T_{g_r}\| = \|T_{g-g_r}\| \approx \|g - g_r\|_{\mathcal{F}(q,q-2,\frac{qp}{2})}.$$

Using Lemma 1 again, we have

$$\|T_g\|_{e,Q_p\to\mathcal{LF}(q,q-2,\frac{qp}{2})} \lesssim \limsup_{r\to1^-} \|g - g_r\|_{\mathcal{F}(q,q-2,\frac{qp}{2})} \approx \mathrm{dist}_{\mathcal{F}(q,q-2,\frac{qp}{2})}(g, \mathcal{F}_0(q, q-2, \frac{qp}{2})).$$

The proof is complete. □

The following result can be deduced by Theorem 3 directly.

Corollary 2. *Let* $0 < p < 1$, $2 < q < \infty$ *and* $g \in H(\mathbb{D})$. *Then* $T_g : Q_p \to \mathcal{LF}(q, q-2, \frac{qp}{2})$ *is compact if and only if*

$$g \in \mathcal{F}_0(q, q-2, \frac{qp}{2}).$$

4. Conclusions

In this paper, we mainly prove that inclusion mapping $i:\ Q_p \to \mathcal{T}^q_{\frac{qp}{2},s}(\mu)$ is bounded if and only if $\sup_{I\subseteq\partial\mathbb{D}} \frac{\left(\log\frac{2}{|I|}\right)^{q-s}\mu(S(I))}{|I|^{\frac{qp}{2}}} < \infty$, when $0 < p < 1$, $2 < q < \infty$ and $0 < s \leq q < \infty$. As an application, we prove that Volterra integral operator T_g from Q_p to the space $\mathcal{LF}(q, q-2, \frac{qp}{2})$ is bounded if and only if $g \in \mathcal{F}(q, q-2, \frac{qp}{2})$.

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Conflict of interest

We declare that we have no conflict of interest.

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