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*Research article*

# Weighted boundedness for Toeplitz type operator related to singular integral transform with variable Calderón-Zygmund kernel

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**Abstract:** In the paper, some weighted maximal inequalities for the Toeplitz operator related to the singular integral transform with variable Calderón-Zygmund kernel are proved. As an application, the boundedness of the operator on weighted Lebesgue space are obtained.

**Keywords:** Toeplitz operator; variable Calderón-Zygmund kernel; singular integral transform; weighted $BMO$ function; weighted Lipschitz function
**Mathematics Subject Classification:** 42B20, 42B25

## 1. Introduction

Suppose $b$ is a locally integrable function on $R^n$ and $T$ is an integral operator. The principle model of commutator generated by $b$ and $T$ is Calderón commutator $[T, M_b](f) = T(bf) - bT(f)$(see [5]). The boundedness of commutator characterizes some function spaces (see [2, 10, 21]). In the mid seventies, Coifman, Rochberg and Weiss showed that the commutator is bounded on Lebesgue space. In fact they even proved that this property characterizes $BMO$ functions. As the development of singular integrals (see [7, 21]), the commutator has been well studied. In [5, 19, 20], the authors proved that the commutators of $BMO$ functions and the singular integral are bounded on Lebesgue space. In [3], the author proved a similar result where singular integral is replaced by fractional integral. In [10, 18], the boundedness of the commutator of the Lipschitz function and singular integral on Triebel-Lizorkin and Lebesgue spaces are gained. In [1, 9], the boundedness for the commutator by the weighted $BMO$ and Lipschitz functions and singular integral on Lebesgue spaces are gained (also see [8]). In [2], the authors introduced certain singular integral operator with variable kernel and obtained its boundedness. In [13–15], the boundedness for the commutator by the $BMO$ function and operator is obtained. In [17], the authors proved the boundedness of the multilinear oscillatory singular integral by $BMO$ function and the operator. In [11, 12, 16], certain Toeplitz operator related to the strongly singular integral is studied.

Motivated by these, in the paper, certain Toeplitz operator of the weighted $BMO$ and Lipschitz functions with the singular integral transform with variable Calderón-Zygmund kernel are studied.

## 2. Preliminaries and notations

In the paper, we will study following singular integral transforms (see [2])

**Definition.** Let $K(x,\cdot)$ be a variable Calderón-Zygmund kernel for a.e. $x \in R^n$ as [2] and for a locally integrable function $b$ on $R^n$ and the singular integral transform $T$ with variable Calderón-Zygmund kernel as

$$T(f)(x) = \int_{R^n} \Omega(x, x-y) f(y) dy.$$

The Toeplitz operator relater to $T$ is defined as

$$T_b = \sum_{k=1}^{m} T^{k,1} M_b T^{k,2},$$

where $T^{k,1}$ are the $\pm I$(the identity operator) or singular integral transform with variable Calderón-Zygmund kernel, and $T^{k,2}$ are the linear operators for $k = 1, ..., m$, $M_b(f) = bf$.

Now, we introduce some notations. In the paper, $Q$ will denote a cube of $R^n$. For a weight function $\omega$ (i.e. $\omega$ is a nonnegative locally integrable function), let $\omega(Q) = \int_Q \omega(x)dx$ and $\omega_Q = |Q|^{-1}\int_Q \omega(x)dx$.

For a locally integrable function $b$, the maximal sharp function of $b$ is defined by

$$M^{\#}(b)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy.$$

We know that (see [7])

$$M^{\#}(b)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |b(y) - c| dy.$$

Let

$$M(b)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(y)| dy.$$

For $\eta > 0$, let $M^{\#}_{\eta}(b)(x) = M^{\#}(|b|^{\eta})^{1/\eta}(x)$ and $M_{\eta}(b)(x) = M(|b|^{\eta})^{1/\eta}(x)$.

For $0 < \eta < n$, $1 \le p < \infty$ and weight function $v$, set

$$M_{\eta,p,v}(b)(x) = \sup_{Q \ni x} \left( \frac{1}{v(Q)^{1-p\eta/n}} \int_Q |b(y)|^p v(y) dy \right)^{1/p}$$

and

$$M_v(b)(x) = \sup_{Q \ni x} \frac{1}{v(Q)} \int_Q |b(y)| v(y) dy.$$

The $A_p$ weight is defined by (see [7])

$$A_p = \left\{ 0 < v \in L^1_{loc}(R^n) : \sup_Q \left( \frac{1}{|Q|} \int_Q v(x) dx \right) \left( \frac{1}{|Q|} \int_Q v(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty,$$

and

$$A_1 = \{0 < v \in L^p_{loc}(R^n) : M(v)(x) \le Cv(x), a.e.\}.$$

Given a weight function $v$, the weighted Lebesgue space $L^p(R^n, v)$ is the space of functions $b$ such that, for $1 \le p < \infty$,

$$\|b\|_{L^p(v)} = \left(\int_{R^n} |b(x)|^p v(x)dx\right)^{1/p} < \infty.$$

The weighted $BMO$ space $BMO(v)$ is the space of functions $f$ such that

$$\|f\|_{BMO(v)} = \sup_Q \frac{1}{v(Q)} \int_Q |f(y) - f_Q| dy < \infty.$$

For $0 < \beta < 1$, the weighted Lipschitz space $Lip_\beta(v)$ is the space of functions $f$ such that

$$\|f\|_{Lip_\beta(v)} = \sup_Q \frac{1}{v(Q)^{\beta/n}} \left(\frac{1}{v(Q)} \int_Q |f(y) - f_Q|^p v(x)^{1-p} dy\right)^{1/p} < \infty.$$

**Remark.**(1). We know that (see [6]), for $f \in Lip_\beta(v)$, $v \in A_1$ and $x \in Q$,

$$|f_Q - f_{2^kQ}| \le Ck\|f\|_{Lip_\beta(v)} v(x) v(2^kQ)^{\beta/n}.$$

(2). Given $f \in Lip_\beta(v)$ and $v \in A_1$. By [5], It is known that spaces $Lip_\beta(v)$ coincide and the norms $\|f\|_{Lip_\beta(v)}$ are equivalent for different values $1 \le p < \infty$.

The following preliminary lemma needs.

**Lemma 1.**([7, p.485]) Suppose $0 < p < q < \infty$ and any positive function $f$. It is defined that, for $1/r = 1/p - 1/q$,

$$\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, \qquad N_{p,q}(f) = \sup_Q \|f\chi_Q\|_{L^p} / \|\chi_Q\|_{L^r},$$

where the sup is taken for all measurable sets $Q$ with $0 < |Q| < \infty$. Then

$$\|f\|_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p} \|f\|_{WL^q}.$$

**Lemma 2.**(see [2]) Suppose $T$ is the singular integral transform as **Definition 2**. Then $T$ is bounded on $L^p(R^n, v)$ for $v \in A_p$ with $1 < p < \infty$, and weak $(L^1, L^1)$ bounded.

**Lemma 3.**(see [1]) Suppose $b \in BMO(v)$. Then

$$|b_Q - b_{2^jQ}| \le Cj\|b\|_{BMO(v)} v_{Q_j},$$

where $v_{Q_j} = \max_{1 \le i \le j} |2^iQ|^{-1} \int_{2^iQ} v(x)dx$.

**Lemma 4.**(see [1]) Suppose $v \in A_p$, $1 < p < \infty$. Then there exists $\varepsilon > 0$ such that $v^{-r/p} \in A_r$ for any $p' \le r \le p' + \varepsilon$.

**Lemma 5.**(see [1]) Suppose $v \in BMO(v)$, $v = (\mu v^{-1})^{1/p}$, $\mu, v \in A_p$ and $p > 1$. Then there exists $\varepsilon > 0$ such that for $p' \le r \le p' + \varepsilon$,

$$\int_Q |f(x) - f_Q|^r \mu(x)^{-r/p} dx \le C\|f\|^r_{BMO(v)} \int_Q v(x)^{-r/p} dx.$$

**Lemma 6.**(see [1]) Suppose $\nu \in A_p$, $1 < p < \infty$. Then there exists $0 < \delta < 1$ such that $\nu^{1-r'/p} \in A_{p/r'}(d\mu)$ for any $p' < r < p'(1+\delta)$, where $d\mu = \nu^{r'/p}dx$.

**Lemma 7.**(see [1]) Suppose $\mu, \nu \in A_p$, $\nu = (\mu\nu^{-1})^{1/p}$, $1 < p < \infty$. Then there exists $1 < q < p$ such that

$$\omega_Q(\nu_Q)^{1/q}\left(\frac{1}{|Q|}\int_Q \nu(x)^{-q'}\nu(x)^{-q'/q}dx\right)^{1/q'} \le C.$$

**Lemma 8.**(see [5, 6]) Suppose $0 \le \eta < n$, $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $\nu \in A_1$. Then

$$\|M_{\eta,s,\nu}(f)\|_{L^q(\nu)} \le C\|f\|_{L^p(\nu)}.$$

**Lemma 9.**(see [7]). Suppose $0 < p, \eta < \infty$ and $\nu \in \cup_{1\le r<\infty}A_r$. Then, for any smooth function $f$,

$$\int_{R^n} M_\eta(f)(x)^p\nu(x)dx \le C\int_{R^n} M^\#_\eta(f)(x)^p\nu(x)dx.$$

## 3. Theorems and Proofs

We can prove the following theorems.

**Theorem 1.** Suppose $T$ is the singular integral transform as **Definition 2**, $1 < p < \infty$, $\mu, \nu \in A_p$, $\nu = (\mu\nu^{-1})^{1/p}$, $0 < \eta < 1$ and $b \in BMO(\nu)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$, $\varepsilon > 0$, $0 < \delta < 1$, $1 < q < p$ and $p' < r < \min(p'+\varepsilon, p'(1+\delta))$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^\#_\eta(T_b(f))(\tilde{x}) \le C\|b\|_{BMO(\nu)}\sum_{k=1}^m\left([M_{\nu^{r'/p}}(|\nu T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'} + [M_\nu(|\nu T^{k,2}(f)|^q)(\tilde{x})]^{1/q}\right).$$

**Theorem 2.** Suppose $T$ is the singular integral transform as **Definition 2**, $\nu \in A_1$, $0 < \eta < 1$, $1 < s < \infty$, $0 < \beta < 1$ and $b \in Lip_\beta(\nu)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^\#_\eta(T_b(f))(\tilde{x}) \le C\|b\|_{Lip_\beta(\nu)}\nu(\tilde{x})\sum_{k=1}^m M_{\beta,s,\nu}(T^{k,2}(f))(\tilde{x}).$$

**Theorem 3.** Suppose $T$ is the singular integral transform as **Definition 2**, $1 < p < \infty$, $\mu, \nu \in A_p$, $\nu = (\mu\nu^{-1})^{1/p}$ and $b \in BMO(\nu)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ are the bounded operators on $L^p(R^n, \nu)$ for $1 < p < \infty$ and $\nu \in A_p(1 \le k \le m)$, then $T_b$ is bounded from $L^p(R^n, \mu)$ to $L^p(R^n, \nu)$.

**Theorem 4.** Suppose $T$ is the singular integral transform as **Definition 2**, $\nu \in A_1$, $0 < \beta < 1$, $b \in Lip_\beta(\nu)$, $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ are the bounded linear operators on $L^p(R^n, \nu)$ for $1 < p < \infty$ and $\nu \in A_1(1 \le k \le m)$, then $T_b$ is bounded from $L^p(R^n, \nu)$ to $L^q(R^n, \nu^{1-q})$.

**Proof of Theorem 1.** It is only to prove the following inequality holds, for $f \in C_0^\infty(R^n)$ and some constant $C_0$:

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|^\eta\, dx\right)^{1/\eta}\\ \leq\;& C\|b\|_{BMO(\nu)} \sum_{k=1}^m \left([M_{\nu^{r'/p}}(|\nu T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'} + [M_\nu(|\nu T^{k,2}(f)|^q)(\tilde{x})]^{1/q}\right).\end{aligned}$$

We assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We write, by $T_1(g) = 0$,

$$T_b(f)(x) = T_{b-b_{2Q}}(f)(x) = T_{(b-b_{2Q})\chi_{2Q}}(f)(x) + T_{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - f_2(x_0)|^\eta\, dx\right)^{1/\eta}\\ \leq\;& C\left(\frac{1}{|Q|}\int_Q |f_1(x)|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|}\int_Q |f_2(x) - f_2(x_0)|^\eta dx\right)^{1/\eta} = I_1 + I_2.\end{aligned}$$

For $I_1$, we know $\nu^{-r/p} \in A_r$ by Lemma 4, we get

$$\left(\frac{1}{|Q|}\int_Q \nu(x)^{-r/p} dx\right)^{1/r} \leq C\left(\frac{1}{|Q|}\int_Q \nu(x)^{r'/p} dx\right)^{-1/r'},$$

then, by Lemmas 1, 2 and 5, we obtain

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^\eta dx\right)^{1/\eta}\\
=\;& \frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}} \frac{\|T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)\chi_Q\|_{L^\eta}}{\|\chi_Q\|_{L^{\eta/(1-\eta)}}}\\
\leq\;& \frac{C}{|Q|}\|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)\|_{WL^1}\\
\leq\;& \frac{C}{|Q|}\int_{R^n} |M_{(b-b_{2Q})\chi_{2Q}}T^{k,2}(f)(x)| dx\\
=\;& \frac{C}{|Q|}\int_{2Q} |b(x) - b_{2Q}|\mu(x)^{-1/p}|T^{k,2}(f)(x)|\nu(x)\nu(x)^{1/p} dx\\
\leq\;& C\left(\frac{1}{|Q|}\int_{2Q} |b(x) - b_{2Q}|^r\mu(x)^{-r/p} dx\right)^{1/r}\left(\frac{1}{|Q|}\int_{2Q} |T^{k,2}(f)(x)|^{r'}\nu(x)^{r'}\nu(x)^{r'/p} dx\right)^{1/r'}\\
\leq\;& C\|b\|_{BMO(\nu)}\left(\frac{1}{|2Q|}\int_{2Q} \nu(x)^{-r/p} dx\right)^{1/r}\left(\frac{1}{|Q|}\int_{2Q} |T^{k,2}(f)(x)\nu(x)|^{r'}\nu(x)^{r'/p} dx\right)^{1/r'}\\
\leq\;& C\|b\|_{BMO(\nu)}\left(\frac{1}{|2Q|}\int_{2Q} \nu(x)^{r'/p} dx\right)^{-1/r'}\left(\frac{1}{|Q|}\int_{2Q} |T^{k,2}(f)(x)\nu(x)|^{r'}\nu(x)^{r'/p} dx\right)^{1/r'}\\
\leq\;& C\|b\|_{BMO(\nu)}\left(\frac{1}{\nu(2Q)^{r'/p}}\int_{2Q} |T^{k,2}(f)(x)\nu(x)|^{r'}\nu(x)^{r'/p} dx\right)^{1/r'}\\
\leq\;& C\|b\|_{BMO(\nu)}[M_{\nu^{r'/p}}(|\nu T^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'},\end{aligned}$$

thus

$$\begin{aligned} I_1 &\leq C\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^{\eta}dx\right)^{1/\eta} \\ &\leq C\|b\|_{BMO(v)}\sum_{k=1}^{m}[M_{v^{r'/p}}(|vT^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'}. \end{aligned}$$

For $I_2$, by [2], we know that

$$T(f)(x) = \sum_{k=1}^{\infty}\sum_{l=1}^{g_k} a_{kl}(x)\int_{R^n}\frac{Y_{kl}(x-y)}{|x-y|^n}f(y)dy,$$

and for $|x-y| > 2|x_0-x| > 0$,

$$\left|\frac{Y_{kl}(x-y)}{|x-y|^n} - \frac{Y_{kl}(x_0-y)}{|x_0-y|^n}\right| \leq Ck^{n/2}|x-x_0|/|x_0-y|^{n+1}$$

Thus, by the same argument of proof in [4], for $x \in Q$, we get

$$\begin{aligned} &|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)| \\ \leq\ & C\|b\|_{BMO(v)}\sum_{j=1}^{\infty}2^{-j}\left(\frac{1}{v(2^{j+1}Q)^{r'/p}}\int_{2^{j+1}Q}|T^{k,2}(f)(y)v(y)|^{r'}v(y)^{r'/p}dy\right)^{1/r'} \\ +\ & C\|b\|_{BMO(v)}[M_v(|vT^{k,2}(f)|^q)(\tilde{x})]^{1/q}\sum_{j=1}^{\infty}j2^{-j} \\ &\times v_{2^jQ}(v_{2^jQ})^{1/q}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}v(y)^{-q'}v(y)^{-q'/q}dy\right)^{1/q'} \\ \leq\ & C\|b\|_{BMO(v)}[M_{v^{r'/p}}(|vT^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'} \\ +\ & C\|b\|_{BMO(v)}[M_v(|vT^{k,2}(f)|^q)(\tilde{x})]^{1/q}. \end{aligned}$$

Thus

$$\begin{aligned} I_2 &\leq \frac{1}{|Q|}\int_Q\sum_{k=1}^{m}|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x) - T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|dx \\ &\leq C\|b\|_{BMO(v)}\sum_{k=1}^{m}\left([M_{v^{r'/p}}(|vT^{k,2}(f)|^{r'})(\tilde{x})]^{1/r'} + [M_v(|vT^{k,2}(f)|^q)(\tilde{x})]^{1/q}\right). \end{aligned}$$

Theorem 1 is proved.

**Proof of Theorem 2.** It only to prove the following inequality holds, for $f \in C_0^\infty(R^n)$ and some constant $C_0$:

$$\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|^\eta \, dx\right)^{1/\eta} \le C\|b\|_{Lip_\beta(v)}\omega(\tilde{x})\sum_{k=1}^m M_{\beta,s,\omega}(T^{k,2}(f))(\tilde{x}).$$

We assume $T^{k,1}$ are $T(k=1,...,m)$ and similar to Theorem 1, for a cube $Q = Q(x_0,d)$ and $\tilde{x} \in Q$, we get

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |T_b(f)(x) - f_2(x_0)|^\eta \, dx\right)^{1/\eta} \\
\le\ & C\left(\frac{1}{|Q|}\int_Q |f_1(x)|^\eta dx\right)^{1/\eta} + C\left(\frac{1}{|Q|}\int_Q |f_2(x) - f_2(x_0)|^\eta dx\right)^{1/\eta} = I_3 + I_4.
\end{aligned}$$

For $I_3$, we have

$$\begin{aligned}
I_3 &\le \frac{C}{|Q|}\int_{2Q} |b(x) - b_{2Q}| v(x)^{-1/s} |T^{k,2}(f)(x)| v(x)^{1/s} dx \\
&\le \frac{C}{|Q|}\left(\int_{2Q} |b(x) - b_{2Q}|^{s'} v(x)^{1-s'} dx\right)^{1/s'} \left(\int_{2Q} |T^{k,2}(f)(x)|^s v(x) dx\right)^{1/s} \\
&\le \frac{C}{|Q|}\|b\|_{Lip_\beta(v)} v(2Q)^{1/s'+\beta/n} v(2Q)^{1/s-\beta/n} M_{\beta,s,v}(T^{k,2}(f))(\tilde{x}) \\
&\le C\|b\|_{Lip_\beta(v)} \frac{v(2Q)}{|2Q|} M_{\beta,s,v}(T^{k,2}(f))(\tilde{x}) \\
&\le C\|b\|_{Lip_\beta(v)} v(\tilde{x}) M_{\beta,s,v}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_3 &\le C\sum_{k=1}^m \left(\frac{1}{|Q|}\int_Q |T^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^\eta dx\right)^{1/\eta} \\
&\le C\|b\|_{Lip_\beta(v)} v(\tilde{x}) \sum_{k=1}^m M_{\beta,s,v}(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

For $I_4$, by using the same argument as in the proof of $I_2$, we have, for $x \in Q$,

$$\begin{aligned}
&|T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| \\
\le\ & C\sum_{u=1}^\infty u^{-n/2-2} \sum_{j=1}^\infty \int_{2^j d \le |y-x_0| < 2^{j+1} d} |b(y) - b_{2Q}| \frac{|x - x_0|}{|x_0 - y|^{n+1}} |T^{k,2}(f)(y)| dy \\
\le\ & C\sum_{j=1}^\infty \frac{d}{(2^{j+1}d)^{n+1}} \int_{2^{j+1}Q} |b(y) - b_{2^{j+1}Q} + b_{2^{j+1}Q} - b_{2Q}| v(y)^{-1/s} |T^{k,2}(f)(y)| v(y)^{1/s} dy \\
\le\ & C\sum_{j=1}^\infty \frac{d}{(2^{j+1}d)^{n+1}} \left(\int_{2^{j+1}Q} |b(y) - b_{2^{j+1}Q}|^{s'} v(y)^{1-s'} dy\right)^{1/s'} \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s v(y) dy\right)^{1/s}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{\infty}\frac{d}{(2^{j+1}d)^{n+1}}|b_{2^{j+1}Q}-b_{2Q}|\left(\int_{2^{j+1}Q}v(y)^{-1/(s-1)}dy\right)^{1/s'}\left(\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^{s}v(y)dy\right)^{1/s}\\
\leq\ & C\sum_{j=1}^{\infty}\frac{d}{(2^{j+1}d)^{n+1}}||b||_{Lip_\beta(v)}v(2^{j+1}Q)^{1/s'+\beta/n}v(2^{j+1}Q)^{1/s-\beta/n}M_{\beta,s,v}(T^{k,2}(f))(\tilde{x})\\
&+\sum_{j=1}^{\infty}\frac{d}{(2^{j+1}d)^{n+1}}||b||_{Lip_\beta(v)}v(\tilde{x})jv(2^{j+1}Q)^{\beta/n}v(2^{j+1}Q)^{1/s-\beta/n}M_{\beta,s,v}(T^{k,2}(f))(\tilde{x})\\
&\times\frac{|2^{j+1}Q|}{v(2^{j+1}Q)^{1/s}}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}v(y)dy\right)^{1/s}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}v(y)^{-1/(s-1)}dy\right)^{(s-1)/s}\\
\leq\ & C||b||_{Lip_\beta(v)}v(\tilde{x})M_{\beta,s,v}(T^{k,2}(f))(\tilde{x})\sum_{j=1}^{\infty}j2^{-j}\\
\leq\ & C||b||_{Lip_\beta(v)}v(\tilde{x})M_{\beta,s,v}(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

thus

$$
\begin{aligned}
I_4\ &\leq\ \frac{1}{|Q|}\int_Q\sum_{k=1}^{m}|T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x)-T^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{k,2}(f)(x_0)|dx\\
&\leq\ C||b||_{Lip_\beta(v)}v(\tilde{x})\sum_{k=1}^{m}M_{\beta,s,v}(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

Theorem 2 is proved.

**Proof of Theorem 3.** It is noticed $v^{r'/p}\in A_{r'+1-r'/p}\subset A_p$ and $v(x)dx\in A_{p/r'}(v(x)^{r'/p}dx)$, we have, by Theorem 1 and Lemma 9,

$$
\begin{aligned}
&\int_{R^n}|T_b(f)(x)|^p v(x)dx\leq\int_{R^n}|M_\eta(T_b(f))(x)|^p v(x)dx\leq C\int_{R^n}|M_\eta^{\#}(T_b(f))(x)|^p v(x)dx\\
\leq\ & C||b||_{BMO(v)}\sum_{k=1}^{m}\int_{R^n}\left([M_{v^{r'/p}}(|vT^{k,2}(f)|^{r'})(x)]^{p/r'}+[M_v(|vT^{k,2}(f)|^{q})(x)]^{p/q}\right)v(x)dx\\
\leq\ & C||b||_{BMO(v)}\sum_{k=1}^{m}\int_{R^n}|v(x)T^{k,2}(f)(x)|^p v(x)dx\\
=\ & C||b||_{BMO(v)}\sum_{k=1}^{m}\int_{R^n}|T^{k,2}(f)(x)|^p\mu(x)dx\\
\leq\ & C||b||_{BMO(v)}\int_{R^n}|f(x)|^p\mu(x)dx.
\end{aligned}
$$

Theorem 3 is proved.

**Proof of Theorem 4.** In Theorem 2 we choose $1 < s < p$ and by $v^{1-q} \in A_\infty$, we get, by Lemmas 8 and 9,

$$
\begin{aligned}
&\|T_b(f)\|_{L^q(v^{1-q})} \le \|M_\eta(T_b(f))\|_{L^q(v^{1-q})} \le C\|M_\eta^\#(T_b(f))\|_{L^q(v^{1-q})} \\
\le\ & C\|b\|_{Lip_\beta(v)} \sum_{k=1}^m \|vM_{\beta,s,v}(T^{k,2}(f))\|_{L^q(v^{1-q})} \\
=\ & C\|b\|_{Lip_\beta(v)} \sum_{k=1}^m \|M_{\beta,s,v}(T^{k,2}(f))\|_{L^q(v)} \\
\le\ & C\|b\|_{Lip_\beta(v)} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p(v)} \\
\le\ & C\|b\|_{Lip_\beta(v)}\|f\|_{L^p(v)}.
\end{aligned}
$$

Theorem 4 is proved.

## 4. Conclusions

Some new weighted maximal inequalities for the Toeplitz operator related to the singular integral transform with variable Calderón-Zygmund kernel are proved. As an application, the boundedness of the operator on weighted Lebesgue space are obtained.

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## Conflict of interest

The author declare that he has no conflict of interest.

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