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Research article

Reducibility and quasi-periodic solutions for a two dimensional beam equation with quasi-periodic in time potential

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Abstract: This article is devoted to the study of a two-dimensional ($2D$) quasi-periodically forced beam equation

$$u_{tt} + \Delta^2 u + \varepsilon\phi(t)(u + u^3) = 0, \quad x \in \mathbb{T}^2, \quad t \in \mathbb{R}$$

under periodic boundary conditions, where ε is a small positive parameter, $\phi(t)$ is a real analytic quasi-periodic function in t with frequency vector $\omega = (\omega_1, \omega_2 \ldots, \omega_m)$. We prove that the equation possesses a Whitney smooth family of small-amplitude quasi-periodic solutions corresponding to finite dimensional invariant tori of an associated infinite dimensional Hamiltonian system. The proof is based on an infinite dimensional KAM theorem and Birkhoff normal form. By solving the measure estimation of infinitely many small divisors, we construct a symplectic coordinate transformation which can reduce the linear part of Hamiltonian system to constant coefficients. And we construct some conversion of coordinates which can change the Hamiltonian of the equation into some Birkhoff normal form depending sparse angle-dependent terms, which can be achieved by choosing the appropriate tangential sites. Lastly, we prove that there are many quasi-periodic solutions for the above equation via an abstract KAM theorem.

Keywords: two dimensional beam equation; quasi-periodic in time potentials; reducibility; normal form
Mathematics Subject Classification: 37K55, 70K40

1. Introduction and main result

In this paper, we will are concerned with existence of quasi-periodic solutions for a two-dimensional ($2D$) quasi-periodically forced beam equation

$$u_{tt} + \Delta^2 u + \varepsilon\phi(t)(u + u^3) = 0, \quad x \in \mathbb{T}^2, \quad t \in \mathbb{R} \tag{1.1}$$

with periodic boundary conditions

$$u(t, x_1, x_2) = u(t, x_1 + 2\pi, x_2) = u(t, x_1, x_2 + 2\pi) \tag{1.2}$$

where ε is a small positive parameter, $\phi(t)$ is a real analytic quasi-periodic function in t with frequency vector $\omega = (\omega_1, \omega_2 \ldots, \omega_m) \subset [\varrho, 2\varrho]^m$ for some constant $\varrho > 0$. Such quasi-periodic functions can be written in the form

$$\phi(t) = \varphi(\omega_1 t, \ldots, \omega_m t),$$

where $\omega_1, \ldots, \omega_m$ are rationally independent real numbers, the "basic frequencies" of ϕ, and φ is a continuous function of period 2π in all arguments, called the hull of ϕ. Thus ϕ admits a Fourier series expansion

$$\phi(t) = \sum_{k\in\mathbb{Z}^m} \varphi_k e^{\mathrm{i}k\cdot\omega t},$$

where $k \cdot \omega = \sum_{\hat{j}=1}^{m} k_{\hat{j}} \cdot \omega_{\hat{j}}$. We think of this equation as an infinite dimensional Hamiltonian system and we study it through an infinite-dimensional KAM theory. The KAM method is a composite of Birkhoff normal form and KAM iterative techniques, and the pioneering works were given by Wayne [25], Kuksin [15] and Pöschel [19]. Over the last years the method has been well developed in one dimensional Hamiltonian PDEs. However, it is difficult to apply to higher dimensional Hamiltonian PDEs. Actually, it is difficult to draw a nice result because of complicated small divisor conditions and measure estimates between the corresponding eigenvalues when the space dimension is greater than one. In [11, 12] the authors obtained quasi-periodic solutions for higher dimensional Hamiltonian PDEs by means of an infinite dimensional KAM theory, where Geng and You proved that the higher dimensional nonlinear beam equations and nonlocal Schrödinger equations possess small-amplitude linearly-stable quasi-periodic solutions. In this aspect, Eliasson-Kuksin [9], C.Procesi and M.Procesi [20], Eliasson-Grebert-Kuksin [5] made the breakthrough of obtaining quasi-periodic solutions for more interesting higher dimensional Schrödinger equations and beam equations. However, all of the work mentioned above require artificial parameters, and therefore it cannot be used for classical equations with physical background such as the higher dimensional cubic Schrödinger equation and the higher dimensional cubic beam equation. These equations with physical background have many special properties, readers can refer to [4, 16, 22–24] and references therein.

Fortunately, Geng-Xu-You [10], in 2011, used an infinite dimensional KAM theory to study the two dimensional nonlinear cubic Schrödinger equation on $\mathbb{T}^2$. The main approach they use is to pick the appropriate tangential frequencies, to make the non-integrable terms in normal form as sparse as possible such that the homological equations in KAM iteration is easy to solve. More recently, by the same approach, Geng and Zhou [13] looked at the two dimensional completely resonant beam equation with cubic nonlinearity

$$u_{tt} + \Delta^2 u + u^3 = 0, \quad x \in \mathbb{T}^2, \quad t \in \mathbb{R}. \tag{1.3}$$

All works mentioned above do not conclude the case with forced terms. The present paper study the problem of existence of quasi-periodic solutions of the equation (1.1)+(1.2). Let's look at this problem through the infinite-dimensional KAM theory as developed by Geng-Zhou [13]. So the main step is to convert the equation into a form that the KAM theory for PDE can be applied. This requires reducing the linear part of Hamiltonian system to constant coefficients. A large part of the present paper will be devoted to proving the reducibility of infinite-dimensional linear quasi-periodic systems. In fact, the question of reducibility of infinite-dimensional linear quasi-periodic systems is also interesting itself.

In 1960s, Bogoliubov-Mitropolsky-Samoilenko [3] found that KAM technique can be applied to study reducibility of non-autonomous finite-dimensional linear systems to constant coefficient equations. Subsequently, the technique is well developed for the reducibility of finite-dimensional systems, and we don't want to repeat describing these developments here. Comparing with the finite-dimensional systems, the reducibility results in infinite dimensional Hamiltonian systems are relatively few. Such kind of reducibility result for PDE using KAM technique was first obtained by Bambusi and Graffi [1] for Schrödinger equation on $\mathbb{R}$. About the reducibility results in one dimensional PDEs and its applications, readers refer to [2,7,17,18,21] and references therein.

Recently there have been some interesting results in the case of systems in higher space dimensions. Eliasson and Kuksin [6] obtained the reducibility for the linear d-dimensional Schrödinger equation

$$\dot{u} = -\mathrm{i}(\Delta u - \epsilon V(\phi_0 + t\omega, x; \omega)u), \quad x \in \mathbb{T}^d.$$

Grébert and Paturel [14] proved that a linear d-dimensional Schrödinger equation on $\mathbb{R}^d$ with harmonic potential $|x|^2$ and small t-quasiperiodic potential

$$\mathrm{i}\partial_t u - \Delta u + |x|^2 u + \varepsilon V(t\omega, x)u = 0, \quad x \in \mathbb{R}^d$$

reduced to an autonomous system for most values of the frequency vector $\omega \in \mathbb{R}^n$. For recent development for high dimensional wave equations, Eliasson-Grébert-Kuksin [8] , in 2014, studied reducibility of linear quasi-periodic wave equation.

However, the reducibility results in higher dimension are still very few. The author Min Zhang of the present paper has studied the two dimensional Schrödinger equations with Quasi-periodic forcing in [27]. However, it would seem that the result cannot be directly applied to our problems because of the difference in the linear part of Hamiltonian systems and the Birkhoff normal forms. As far as we know, the reducibility for the linear part of the beam equation (1.1) is still open. In this paper, by utilizing the measure estimation of infinitely many small divisors, we construct a symplectic change of coordinates which can reduce the linear part of Hamiltonian system to constant coefficients. Subsequently, we construct a symplectic change of coordinates which can transform the Hamiltonian into some Birkhoff normal form depending sparse angle-dependent terms, which can be achieved by choosing the appropriate tangential sites. Lastly, we show that there are many quasi-periodic solutions for the equation (1.1) via KAM theory.

Remark 1.1. Similar to [13], we introduced a special subset of $\mathbb{Z}^2$

$$\mathbb{Z}^2_{odd} = \{n = (n_1, n_2), \quad n_1 \in 2\mathbb{Z} - 1, n_2 \in 2\mathbb{Z}\}, \tag{1.4}$$

for the small divisor problem could be simplified. Then we define subspace $\mathcal{U}$ in $L^2(\mathbb{T}^2)$ as follows

$$\mathcal{U} = \{u = \sum_{j \in \mathbb{Z}^2_{odd}} u_j \phi_j, \quad \phi_j(x) = e^{\mathrm{i}<j,x>}\}.$$

We only prove the existence of quasi-periodic solutions of the equation (1.1) in $\mathcal{U}$.

The following definition quantifies the conditions the tangential sites satisfy. It acquired from Geng-Xu-You [10].

Definition 1.1. A finite set $S = \{i_1^* = (\tilde{x}_1, \tilde{y}_1), \cdots, i_n^* = (\tilde{x}_n, \tilde{y}_n)\} \subset \mathbb{Z}_{odd}^2 (n \geq 2)$ is called admissible if
(i). Any three different points of them are not vertices of a rectangle (if $n > 2$) or $n = 2$.
(ii). For any $d \in \mathbb{Z}_{odd}^2 \setminus S$, there exists at most one triplet $\{i, j, l\}$ with $i, j \in S, l \in \mathbb{Z}_{odd}^2 \setminus S$ such that $d - l + i - j = 0$ and $|i|^2 - |j|^2 + |d|^2 - |l|^2 = 0$. If such triplet exists, we say that d, l are resonant in the first type and denote all such d by $\mathcal{L}_1$.
(iii). For any $d \in \mathbb{Z}_{odd}^2 \setminus S$, there exists at most one triplet $\{i, j, l\}$ with $i, j \in S, l \in \mathbb{Z}_{odd}^2 \setminus S$ such that $d + l - i - j = 0$ and $|d|^2 + |l|^2 - |i|^2 - |j|^2 = 0$. If such triplet exists, we say that d, l are resonant in the second type and denote all such d by $\mathcal{L}_2$.
(iv). Any $d \in \mathbb{Z}_{odd}^2 \setminus S$ should not be in $\mathcal{L}_1$ and $\mathcal{L}_2$ at the same time. It means that $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$.

Remark 1.2. We can give an example to show the admissible set S above is non-empty. For example, for any given positive integer $n \geq 2$, the first point $(\tilde{x}_1, \tilde{y}_1) \in \mathbb{Z}_{odd}^2$ is chosen as $\tilde{x}_1 > n^2, \tilde{y}_1 = 2\tilde{x}_1^{5^n}$, and the second one is chosen as $\tilde{x}_2 = \tilde{x}_1^5, \tilde{y}_2 = 2\tilde{x}_2^{5^n}$, the others are defined inductively by

$$\tilde{x}_{\hat{j}+1} = \tilde{x}_{\hat{j}}^5 \prod_{2 \leq \hat{m} \leq \hat{j}, 1 \leq \hat{l} < \hat{m}} ((\tilde{x}_{\hat{m}} - \tilde{x}_{\hat{l}})^2 + (\tilde{y}_{\hat{m}} - \tilde{y}_{\hat{l}})^2 + 1), \quad 2 \leq \hat{j} \leq n - 1,$$

$$\tilde{y}_{\hat{j}+1} = 2\tilde{x}_{\hat{j}+1}^{5^n}, \quad 2 \leq \hat{j} \leq n - 1.$$

The choice of the admissible set is same to that in [13], where the proof of such admissible set is given.

In this paper, we assume that
(H) $\phi(t)$ is a real analytic quasi-periodic function in t with frequency vector ω, and $[\phi] \neq 0$ where $[\phi]$ denotes the time average of ϕ, coinciding with the space average.

The main result of this paper in the following. The proof is based on an infinite dimensional KAM theorem inspired by Geng-Zhou [13].

Theorem 1.1. (Main Theorem) Given ϱ, $\phi(t)$ as above. Then for arbitrary admissible set $S \subset \mathbb{Z}_{odd}^2$ and for any $0 < \gamma < 1, 0 < \rho < 1$ and $\gamma' > 0$ be small enough, there exists $\varepsilon^*(\rho, \gamma, \gamma') > 0$ so that for all $0 < \varepsilon < \varepsilon^*$, there exists $R \subset [\varrho, 2\varrho]^m$ with meas $R > (1 - \gamma)\varrho^m$ and there exists $\Sigma_{\gamma'} \subset \Sigma := R \times [0, 1]^n$ with meas $(\Sigma \setminus \Sigma_{\gamma'}) = O(\sqrt[4]{\gamma'})$, so that for $(\omega, \tilde{\xi}_{i_1^*}, \ldots, \tilde{\xi}_{i_n^*}) \in \Sigma'_\gamma$, the beam equation (1.1)+(1.2) admits a quasi-periodic solution in the following

$$u(t, x) = \sum_{j \in S} \left(1 + g_j(\omega t, \omega, \varepsilon)\right) \sqrt{\frac{3\tilde{\xi}_j}{16|j|^2\pi^2}} (e^{i\tilde{\omega}_j t} e^{i<j,x>} + e^{-i\tilde{\omega}_j t} e^{-i<j,x>}) + O(|\tilde{\xi}|^{3/2}),$$

where $g_j(\vartheta, \omega, \varepsilon) = \varepsilon^\rho g_j^*(\vartheta, \omega, \varepsilon)$ is of period 2π in each component of ϑ and for $j \in S, \vartheta \in \Theta(\sigma_0/2), \omega \in \Omega$, we have $|g_j^*(\vartheta, \omega, \varepsilon)| \leq C$. And the solution $u(t, x)$ is quasi-periodic in terms of t with the frequency vector $\tilde{\omega} = (\omega, (\tilde{\omega}_j)_{j \in S})$, and $\tilde{\omega}_j = \varepsilon^{-4}|j|^2 + O(|\tilde{\xi}|) + O(\varepsilon)$.

2. The Hamiltonian setting

Let's rewrite the beam equation (1.1) as follows

$$u_{tt} + \Delta^2 u + \varepsilon\phi(t)(u + u^3) = 0, \quad x \in \mathbb{T}^2, \quad t \in \mathbb{R}. \tag{2.1}$$

Introduce a variable $v = u_t$, the equation (2.1) is transformed into

$$\begin{cases} u_t = v, \\ v_t = -\Delta^2 u - \varepsilon\phi(t)(u + u^3). \end{cases} \tag{2.2}$$

Introducing $q = \frac{1}{\sqrt{2}}((-\Delta)^{\frac{1}{2}} u - \mathrm{i}(-\Delta)^{-\frac{1}{2}} v)$ and (2.2) is transformed into

$$-\mathrm{i}q_t = -\Delta q + \frac{1}{\sqrt{2}}\varepsilon\phi(t)(-\Delta)^{-\frac{1}{2}}\left((-\Delta)^{-\frac{1}{2}}(\frac{q+\bar{q}}{\sqrt{2}}) + ((-\Delta)^{-\frac{1}{2}}(\frac{q+\bar{q}}{\sqrt{2}}))^3\right). \tag{2.3}$$

The equation can be written as the Hamiltonian equation $\dot{q} = \mathrm{i}\frac{\partial H}{\partial \bar{q}}$ and the corresponding Hamiltonian functions is

$$H = \int_{\mathbb{T}^2} ((-\Delta)q)\bar{q}dx + \frac{1}{2}\varepsilon\phi(t)\int_{\mathbb{T}^2}\left((-\Delta)^{-\frac{1}{2}}(\frac{q+\bar{q}}{\sqrt{2}})\right)^2 dx + \frac{1}{4}\varepsilon\phi(t)\int_{\mathbb{T}^2}\left((-\Delta)^{-\frac{1}{2}}(\frac{q+\bar{q}}{\sqrt{2}})\right)^4 dx. \tag{2.4}$$

The eigenvalues and eigenfunctions of the linear operator $-\Delta$ with the periodic boundary conditions are respectively $\lambda_j = |j|^2$ and $\phi_j(x) = \frac{1}{2\pi}e^{\mathrm{i}<j,x>}$. Now let's expand q into a Fourier series

$$q = \sum_{j\in\mathbb{Z}^2_{odd}} q_j\phi_j, \tag{2.5}$$

the coordinates belong to some Hilbert space $l^{a,s}$ of sequences $q = (\cdots, q_j, \cdots)_{j\in\mathbb{Z}^2_{odd}}$ that has the finite norm

$$\|q\|_{a,s} = \sum_{j\in\mathbb{Z}^2_{odd}} |q_j||j|^s e^{|j|a} \quad (a > 0, s > 0).$$

The corresponding symplectic structure is $\mathrm{i}\sum_{j\in\mathbb{Z}^2_{odd}} dq_j \wedge d\bar{q}_j$. In the coordinates, the Hamiltonian equation (2.3) can be written as

$$\dot{q}_j = \mathrm{i}\frac{\partial H}{\partial \bar{q}_j}, \quad \forall j \in \mathbb{Z}^2_{odd} \tag{2.6}$$

with

$$H = \Lambda + G$$

where

$$\Lambda = \sum_{j\in\mathbb{Z}^2_{odd}} (\lambda_j|q_j|^2 + \frac{\varepsilon}{4\lambda_j}\phi(t)(q_jq_{-j} + 2|q_j|^2 + \bar{q}_j\bar{q}_{-j}))$$

$$\begin{aligned} G = &\frac{1}{64\pi^2}\varepsilon\phi(t)\sum_{\substack{i+j+d+l=0\\ i,j,d,l\in\mathbb{Z}^2_{odd}}} \frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}(q_iq_jq_dq_l + \bar{q}_i\bar{q}_j\bar{q}_d\bar{q}_l) \\ &+\frac{3}{32\pi^2}\varepsilon\phi(t)\sum_{\substack{i-j+d-l=0\\ i,j,d,l\in\mathbb{Z}^2_{odd}}} \frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}q_i\bar{q}_jq_d\bar{q}_l \\ &+\frac{1}{16\pi^2}\varepsilon\phi(t)\sum_{\substack{i+j+d-l=0\\ i,j,d,l\in\mathbb{Z}^2_{odd}}} \frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}(q_iq_jq_d\bar{q}_l + \bar{q}_i\bar{q}_j\bar{q}_dq_l). \end{aligned}$$

Denote $\varphi(\vartheta)$ be the shell of $\phi(t)$, we introduce the action-angle variable $(J, \vartheta) \in \mathbb{R}^m \times \mathbb{T}^m$, then (2.6) can be written as follows

$$\dot{\vartheta} = \omega, \quad \dot{J} = -\frac{\partial H}{\partial \vartheta}, \quad \dot{q}_j = \mathrm{i}\frac{\partial H}{\partial \bar{q}_j}, \quad j \in \mathbb{Z}^2_{odd}$$

and the corresponding Hamiltonian function is

$$H = \bar{H} + \varepsilon G^4, \tag{2.7}$$

where

$$\bar{H} =< \omega, J > + \sum_{j \in \mathbb{Z}^2_{odd}} (\lambda_j |q_j|^2 + \frac{\varepsilon}{4\lambda_j} \varphi(\vartheta)(q_j q_{-j} + 2|q_j|^2 + \bar{q}_j \bar{q}_{-j})), \tag{2.8}$$

$$\begin{aligned} G^4 = & \frac{1}{64\pi^2} \sum_{\substack{i+j+d+l=0 \\ i,j,d,l \in \mathbb{Z}^2_{odd}}} \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_d \lambda_l}} (G^{4,0}_{ijdl}(\vartheta) q_i q_j q_d q_l + G^{0,4}_{ijdl}(\vartheta) \bar{q}_i \bar{q}_j \bar{q}_d \bar{q}_l) \\ & + \frac{3}{32\pi^2} \sum_{\substack{i-j+d-l=0 \\ i,j,d,l \in \mathbb{Z}^2_{odd}}} \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_d \lambda_l}} G^{2,2}_{ijdl}(\vartheta) q_i \bar{q}_j q_d \bar{q}_l \\ & + \frac{1}{16\pi^2} \sum_{\substack{i+j+d-l=0 \\ i,j,d,l \in \mathbb{Z}^2_{odd}}} \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_d \lambda_l}} (G^{3,1}_{ijdl}(\vartheta) q_i q_j q_d \bar{q}_l + G^{1,3}_{ijdl}(\vartheta) \bar{q}_i \bar{q}_j \bar{q}_d q_l) \end{aligned} \tag{2.9}$$

and

$$G^{4,0}_{ijdl}(\vartheta) = G^{0,4}_{ijdl}(\vartheta) = \begin{cases} \varphi(\vartheta), & i+j+d+l = 0, \\ 0, & i+j+d+l \neq 0, \end{cases} \tag{2.10}$$

$$G^{2,2}_{ijdl}(\vartheta) = \begin{cases} \varphi(\vartheta), & i-j+d-l = 0, \\ 0, & i-j+d-l \neq 0, \end{cases} \tag{2.11}$$

$$G^{3,1}_{ijdl}(\vartheta) = G^{1,3}_{ijdl}(\vartheta) = \begin{cases} \varphi(\vartheta), & i+j+d-l = 0, \\ 0, & i+j+d-l \neq 0. \end{cases} \tag{2.12}$$

3. Reducibility via KAM theory

Now We are going to study the reducibility of the Hamiltonian (2.8). To make this reducibility, we introduce the notations and spaces as follows.

For given $\sigma_0 > 0, \Gamma > 0, 0 < \rho < 1$, define

$$\sigma_\nu = \sigma_0 \left(1 - \frac{\sum_{\hat{j}=1}^{\nu} \hat{j}^{-2}}{2\sum_{\hat{j}=1}^{\infty} \hat{j}^{-2}}\right), \quad \nu = 1, 2, \ldots$$

$$\Gamma_\nu = \Gamma \left(1 + C \sum_{\hat{j}=\nu}^{+\infty} \varepsilon^{\rho}_{\hat{j}}\right), \quad \nu = 0, 1, \ldots$$

where C is a constant. Let

$$\varepsilon_0 = \varepsilon, \quad \varepsilon_\nu = \varepsilon^{(1+\rho)^\nu}, \quad \nu = 1, 2, \ldots$$

$$\Theta(\sigma_\nu) = \left\{\vartheta = (\vartheta_1, \ldots, \vartheta_m) \in \mathbb{C}^m/2\pi\mathbb{Z}^m : |\mathrm{Im}\vartheta_{\hat{j}}| < \sigma_\nu, \hat{j} = 1, 2, \ldots, m\right\}, \nu = 0, 1, 2, \ldots.$$

and denote

$$D_\nu^{a,s} = \{(\vartheta, J, q, \bar{q}) \in \mathbb{C}^m/2\pi\mathbb{Z}^m \times \mathbb{C}^m \times l^{a,s} \times l^{a,s} : |\mathrm{Im}\vartheta| < \sigma_\nu, |J| < \Gamma_\nu^2, \\ \|q\|_{a,s} < \Gamma_\nu, \|\bar{q}\|_{a,s} < \Gamma_\nu\} \qquad \nu = 0, 1, 2, \ldots,$$

$$D_\infty^{a,s} = \{(\vartheta, J, q, \bar{q}) \in \mathbb{C}^m/2\pi\mathbb{Z}^m \times \mathbb{C}^m \times l^{a,s} \times l^{a,s} : |\mathrm{Im}\vartheta| < \sigma_0/2, |J| < \Gamma^2, \\ \|q\|_{a,s} < \Gamma, \|\bar{q}\|_{a,s} < \Gamma\},$$

where $|\cdot|$ stands for the sup-norm of complex vectors and $l^{a,s}$ stands for complex Hilbert space. For arbitrary four order Whitney smooth function $F(\omega)$ on closed bounded set R^*, let

$$\|F\|_{R^*}^* = \sup_{\omega \in R^*} \sum_{0 \le \hat{j} \le 4} |\partial_\omega^{\hat{j}} F|.$$

Let $F(\omega)$ is a vector function from R^* to $l^{a,s}(or\mathbb{R}^{m_1 \times m_2})$ which is four order whitney smooth on R^*, we denote

$$\|F\|_{a,s,R^*}^* = \|(\|F_i(\omega)\|_{R^*}^*)_i\|_{a,s} \quad \left(or\|F\|_{R^*}^* = \max_{1 \le i_1 \le m_1} \sum_{1 \le i_2 \le m_2} (\|F_{i_1 i_2}(\omega)\|_{R^*}^*)\right).$$

Given $\sigma_{D^{a,s}} > 0, \Gamma_{D^{a,s}} > 0$, we define

$$D^{a,s} = \{(\vartheta, J, q, \bar{q}) \in \mathbb{C}^m/2\pi\mathbb{Z}^m \times \mathbb{C}^m \times l^{a,s} \times l^{a,s} : |\mathrm{Im}\vartheta| < \sigma_{D^{a,s}}, |J| < \Gamma_{D^{a,s}}^2, \\ \|q\|_{a,s} < \Gamma_{D^{a,s}}, \|\bar{q}\|_{a,s} < \Gamma_{D^{a,s}}\}.$$

If $\tilde{w} = (\vartheta, J, q, \bar{q}) \in D^{a,s}$, we define the weighted norm for $\tilde{w}$ by

$$|\tilde{w}|_{a,s} = |\vartheta| + \frac{1}{\Gamma_{D^{a,s}}^2}|J| + \frac{1}{\Gamma_{D^{a,s}}}\|q\|_{a,s} + \frac{1}{\Gamma_{D^{a,s}}}\|\bar{q}\|_{a,s}.$$

Let $F(\eta; \omega)$ is a function from $D^{a,s} \times R^*$ to $l^{a,s}(or\mathbb{R}^{m_1 \times m_2})$ which is four order whitney smooth on ω, we denote

$$\|F\|_{a,s,D^{a,s}\times R^*}^* = \sup_{\eta \in D^{a,s}} \|F\|_{a,s,R^*}^* \quad \left(or\|F\|_{D^{a,s}\times R^*}^* = \sup_{\eta \in D^{a,s}} \|F\|_{R^*}^*\right).$$

For given function F, associate a hamiltonian vector field denoted as $X_F = \{F_J, -F_\vartheta, \mathrm{i}F_{\bar{q}}, -\mathrm{i}F_q\}$, we define the weighted norm for X_F by

$$|X_F|_{a,s,D^{a,s}\times R^*}^* = \|F_J\|_{D^{a,s}\times R^*}^* + \frac{1}{\Gamma_{D^{a,s}}^2}\|F_\vartheta\|_{D^{a,s}\times R^*}^* \\ + \frac{1}{\Gamma_{D^{a,s}}}\|F_{\bar{z}}\|_{a,s,D^{a,s}\times R^*}^* + \frac{1}{\Gamma_{D^{a,s}}}\|F_z\|_{a,s,D^{a,s}\times R^*}^*.$$

Assume $w=(q,\bar{q})\in l^{a,s}\times l^{a,s}$ is a doubly infinite complex sequence, and $A(\eta;\omega)$ be an operator from $l^{a,s}\times l^{a,s}$ to $l^{a,s}\times l^{a,s}$ for $(\eta;\omega)\in D^{a,s}\times R^{*}$, then we denote

$$\|w\|_{a,s}=\|q\|_{a,s}+\|\bar{q}\|_{a,s},$$

$$\|A(\eta;\omega)\|^{\diamond}_{a,s,D^{a,s}\times R^{*}}=\sup_{(\eta;\omega)\in D^{a,s}\times R^{*}}\sup_{w\neq 0}\frac{\|A(\eta;\omega)w\|_{a,s}}{\|w\|_{a,s}},$$

$$\|A(\eta;\omega)\|^{\star}_{a,s,D^{a,s}\times R^{*}}=\sum_{0\le\hat{j}\le 4}\|\partial^{\hat{j}}_{\omega}A\|^{\diamond}_{a,s,D^{a,s}\times R^{*}}.$$

Assume $B(\eta;\omega)$ be an operator from $D^{a,s}$ to $D^{a,s}$ for $(\eta;\omega)\in D^{a,s}\times R^{*}$, then we denote

$$|B(\eta;\omega)|^{\diamond}_{a,s,D^{a,s}\times R^{*}}=\sup_{(\eta;\omega)\in D^{a,s}\times R^{*}}\sup_{\tilde{w}\neq 0}\frac{|B(\eta;\omega)\tilde{w}|_{a,s}}{|\tilde{w}|_{a,s}},$$

$$|B(\eta;\omega)|^{\star}_{a,s,D^{a,s}\times R^{*}}=\sum_{0\le\hat{j}\le 4}|\partial^{\hat{j}}_{\omega}B|^{\diamond}_{a,s,D^{a,s}\times R^{*}}.$$

Reducibility of the autonomous Hamiltonian equation corresponding to the Hamiltonian (2.8) will be proved by an KAM iteration which involves an infinite sequence of change of variables. By utilizing the measure estimation of infinitely many small divisors, we will prove that the composition of these infinite many change of variables converges to a symplectic change of coordinates, which can reduce the Hamiltonian equation corresponding to the Hamiltonian (2.8) to constant coefficients.

At the $\nu-$step of the iteration, we consider Hamiltonian function of the form

$$H_{\nu}=H^{*}_{\nu}+P_{\nu}\tag{3.1}$$

where

$$H^{*}_{\nu}:=<\omega,J>+\sum_{j\in\mathbb{Z}^{2}_{odd}}\lambda_{j,\nu}q_{j}\bar{q}_{j},$$

$$P_{\nu}:=\varepsilon_{\nu}\sum_{j\in\mathbb{Z}^{2}_{odd}}[\eta_{j,\nu,2,0}(\vartheta,\omega)q_{j}q_{-j}+\eta_{j,\nu,1,1}(\vartheta,\omega)q_{j}\bar{q}_{j}+\eta_{j,\nu,0,2}(\vartheta,\omega)\bar{q}_{j}\bar{q}_{-j}]$$

where $\eta_{j,\nu,2,0}=\eta_{-j,\nu,2,0}$, $\eta_{j,\nu,0,2}=\eta_{-j,\nu,0,2}$, $\eta_{j,\nu,n_1,n_2}(\vartheta,\omega)=\sum_{k\in\mathbb{Z}^m}\eta_{j,\nu,k,n_1,n_2}(\omega)e^{\mathrm{i}<k,\vartheta>}$ when $n_1,n_2\in\mathbb{N},n_1+n_2=2$,

$$\eta_{j,\nu,n_1,n_2}=\lambda_j^{-1}\eta^{*}_{j,\nu,n_1,n_2},\quad \|\eta^{*}_{j,\nu,n_1,n_2}\|^{*}_{\Theta(\sigma_\nu)\times R_\nu}\le C,\quad n_1,n_2\in\mathbb{N},\quad n_1+n_2=2,\tag{3.2}$$

and

$$\lambda_{j,0}=\lambda_j,\quad \lambda_{j,\nu}=\lambda_j+\sum_{\hat{s}=0}^{\nu-1}\mu_{j,\nu,\hat{s}},\tag{3.3}$$

with

$$\mu_{j,\nu,0}=\frac{\varepsilon}{2\lambda_j}[\phi],\quad \mu_{j,\nu,\hat{s}}=\lambda_j^{-1}\varepsilon_{\hat{s}}\mu^{*}_{j,\nu,\hat{s}},\quad \|\mu^{*}_{j,\nu,\hat{s}}\|^{*}_{R_\nu}\le C,\quad \hat{s}=1,2,\dots,\nu.\tag{3.4}$$

We're going to construct a symplectic transformation

$$T_\nu : D^{a,s}_{\nu+1} \times R_{\nu+1} \longmapsto D^{a,s}_{\nu} \times R_\nu$$

and

$$H_{\nu+1} = H_\nu \circ T_\nu = H^*_{\nu+1} + P_{\nu+1} \tag{3.5}$$

satisfies all the above iterative assumptions (3.1)–(3.4) marked $\nu + 1$ on $D^{a,s}_{\nu+1} \times R_\nu$.

We assume that there is a constant C_* and a closed set R_ν satisfies

$$\text{meas}R_\nu \geq \varrho^m \left(1 - \frac{\gamma}{3} - \frac{\gamma \sum_{\hat{i}=0}^{\nu}(\delta(\hat{i}) + \hat{i})^{-2}}{3\sum_{\hat{i}=0}^{+\infty}(\delta(\hat{i}) + \hat{i})^{-2}}\right) \tag{3.6}$$

and for arbitrary $k \in \mathbb{Z}^m$, $j \in \mathbb{Z}^2_{odd}$, $\omega \in R_\nu$,

$$| < k, \omega > \pm(\lambda_{j,\nu} + \lambda_{-j,\nu})| \geq \frac{\varrho}{C_*(\delta(\nu) + \nu^2)(|k| + \delta(|k|))^{m+1}}, \tag{3.7}$$

where $\delta(x) = 1$ as $x = 0$ and $\delta(x) = 0$ as $x \neq 0$. We put its proof in the Lemma 4.1 below.

Next we will construct a parameter set $R_{\nu+1} \subset R_\nu$ and a symplectic coordinate transformation T_ν so that the transformed Hamiltonian $H_{\nu+1} = H^*_{\nu+1} + P_{\nu+1}$ satisfies the above iteration assumptions with new parameters $\varepsilon_{\nu+1}, \sigma_{\nu+1}, \Gamma_{\nu+1}$ and with $\omega \in R_{\nu+1}$.

3.1. Solving the homological equations

Let X_{Ψ_ν} be the Hamiltonian vector field for a Hamiltonian Ψ_ν :

$$\Psi_\nu = \varepsilon_\nu \Upsilon_\nu = \varepsilon_\nu \sum_{j \in \mathbb{Z}^2_{odd}} [\varpi_{j,\nu,2,0}(\vartheta;\omega) q_j q_{-j} + \varpi_{j,\nu,1,1}(\vartheta;\omega) q_j \bar{q}_j + \varpi_{j,\nu,0,2}(\vartheta;\omega) \bar{q}_j \bar{q}_{-j}]$$

where

$$\varpi_{j,\nu,2,0}(\vartheta;\omega) = \varpi_{-j,\nu,2,0}(\vartheta;\omega), \quad \varpi_{j,\nu,0,2}(\vartheta;\omega) = \varpi_{-j,\nu,0,2}(\vartheta;\omega),$$

$$\varpi_{j,\nu,n_1,n_2}(\vartheta;\omega) = \sum_{k\in\mathbb{Z}^m} \varpi_{j,\nu,k,n_1,n_2}(\omega) e^{\mathrm{i}<k,\vartheta>}, \quad n_1, n_2 \in \mathbb{N}, n_1 + n_2 = 2 \tag{3.8}$$

and $[\varpi_{j,\nu,1,1}] = 0$. Let $X^t_{\Psi_\nu}$ be its time-t map.

Let $T_\nu = X^1_{\Psi_\nu}$ where $X^1_{\Psi_\nu}$ denote the time-one map of the Hamiltonian vector field X_{Ψ_ν}, then the system (3.1)(ν) is transformed into the form (3.1)($\nu + 1$) and satisfies (3.2)($\nu + 1$), (3.3)($\nu + 1$) and (3.4)($\nu + 1$). More precisely, the new Hamiltonian $H_{\nu+1}$ can be written as follows by second order Taylor formula

$$\begin{aligned} H_{\nu+1} : &= H_\nu \circ X^1_{\Psi_\nu} \\ &= H^*_\nu + P_\nu + \{H^*_\nu, \Psi_\nu\} \\ &+ \varepsilon_\nu \int_0^1 (1-t)\{\{H^*_\nu, \Psi_\nu\}, \Upsilon_\nu\} \circ X^t_{\Psi_\nu} dt + \varepsilon_\nu \int_0^1 \{P_\nu, \Upsilon_\nu\} \circ X^t_{\Psi_\nu} dt. \end{aligned} \tag{3.9}$$

The Hamiltonian Ψ_ν is satisfies the homological equation

$$P_\nu + \{H_\nu^*, \Psi_\nu\} = \varepsilon_\nu \sum_{j\in\mathbb{Z}^2_{odd}} [\eta_{j,\nu,1,1}] q_j \bar{q}_j,$$

which is equivalent to

$$\begin{cases} -<\omega, \partial_\vartheta \varpi_{j,\nu,1,1}(\vartheta;\omega)> + \eta_{j,\nu,1,1}(\vartheta;\omega) = [\eta_{j,\nu,1,1}], \\ \mathrm{i}(\lambda_{j,\nu} + \lambda_{-j,\nu})\varpi_{j,\nu,0,2}(\vartheta;\omega) - <\omega, \partial_\vartheta \varpi_{j,\nu,0,2}(\vartheta;\omega)> + \eta_{j,\nu,0,2}(\vartheta;\omega) = 0, \\ -\mathrm{i}(\lambda_{j,\nu} + \lambda_{-j,\nu})\varpi_{j,\nu,2,0}(\vartheta;\omega) - <\omega, \partial_\vartheta \varpi_{j,\nu,2,0}(\vartheta;\omega)> + \eta_{j,\nu,2,0}(\vartheta;\omega) = 0. \end{cases} \tag{3.10}$$

Let's inserting (3.8) into (3.10)

$$\begin{cases} \mathrm{i} < k, \omega > \varpi_{j,\nu,k,1,1}(\omega) = \eta_{j,\nu,k,1,1}(\omega), \quad k \neq 0, \\ \mathrm{i}(< k, \omega > + \lambda_{j,\nu} + \lambda_{-j,\nu})\varpi_{j,\nu,k,2,0}(\omega) = \eta_{j,\nu,k,2,0}(\omega), \\ \mathrm{i}(< k, \omega > - \lambda_{j,\nu} - \lambda_{-j,\nu})\varpi_{j,\nu,k,0,2}(\omega) = \eta_{j,\nu,k,0,2}(\omega). \end{cases}$$

Thus

$$\begin{cases} \varpi_{j,\nu,1,1}(\vartheta;\omega) = \sum_{0\neq k\in\mathbb{Z}^m} \dfrac{\eta_{j,\nu,k,1,1}(\omega)}{\mathrm{i} < k, \omega >} e^{\mathrm{i}<k,\vartheta>}, \\ \varpi_{j,\nu,2,0}(\vartheta;\omega) = \sum_{k\in\mathbb{Z}^m} \dfrac{\eta_{j,\nu,k,2,0}(\omega)}{\mathrm{i}(< k, \omega > + \lambda_{j,\nu} + \lambda_{-j,\nu})} e^{\mathrm{i}<k,\vartheta>}, \\ \varpi_{j,\nu,0,2}(\vartheta;\omega) = \sum_{k\in\mathbb{Z}^m} \dfrac{\eta_{j,\nu,k,0,2}(\omega)}{\mathrm{i}(< k, \omega > - \lambda_{j,\nu} - \lambda_{-j,\nu})} e^{\mathrm{i}<k,\vartheta>}. \end{cases} \tag{3.11}$$

3.2. Estimation on the coordinate transformation

Now we're going to estimate Ψ_ν and $X^1_{\Psi_\nu}$. By Cauchy's estimate and (3.2)(ν)

$$|\eta_{j,\nu,k,n_1,n_2}| \leq \|\eta_{j,\nu,n_1,n_2}\|^*_{\Theta(\sigma_\nu)\times R_\nu} e^{-|k|\sigma_\nu} \leq C\lambda_j^{-1} e^{-|k|\sigma_\nu}, \quad n_1, n_2 \in \mathbb{N}, n_1 + n_2 = 2 \tag{3.12}$$

and

$$|\partial_\omega^{\hat{i}} \eta_{j,\nu,k,n_1,n_2}| \leq \|\eta_{j,\nu,n_1,n_2}\|^*_{\Theta(\sigma_\nu)\times R_\nu} e^{-|k|\sigma_\nu} \leq C\lambda_j^{-1} e^{-|k|\sigma_\nu}, \quad \hat{i} = 1, 2, 3, 4 \tag{3.13}$$

can be obtained. By $\omega \in R_\nu$ and (3.7)(ν),

$$\sup_{(\vartheta;\omega)\in\Theta(\sigma_{\nu+1})\times R_\nu} |\varpi_{j,\nu,1,1}| \leq CC_* \lambda_j^{-1} \varrho^{-1} \sum_{0\neq k\in\mathbb{Z}^m} |k|^{m+1} e^{-\sigma_\nu |k|} e^{\sigma_{\nu+1}|k|}$$

and

$$\sup_{(\vartheta;\omega)\in\Theta(\sigma_{\nu+1})\times R_\nu} |\varpi_{j,\nu,n_1,n_2}| \leq CC_* \lambda_j^{-1} \varrho^{-1} (\delta(\nu) + \nu^2)(1 + \sum_{0\neq k\in\mathbb{Z}^m} |k|^{m+1} e^{-\sigma_\nu |k|} e^{\sigma_{\nu+1}|k|})$$

for $n_1 = 0, n_2 = 2$ or $n_1 = 2, n_2 = 0$. According to Lemma 3.3 in [26], for $(\vartheta;\omega) \in \Theta(\sigma_{\nu+1}) \times R_\nu$,

$$|\varpi_{j,\nu,1,1}|, |\varpi_{j,\nu,2,0}|, |\varpi_{j,\nu,0,2}| \leq CC_* \lambda_j^{-1} \varrho^{-1} (\nu + 1)^{4m+4} \leq C\lambda_j^{-1} (\nu + 1)^{12m+28}, \tag{3.14}$$

where $C := CC_* \varrho^{-1}$. Moreover, in view of (3.3)(ν) and (3.4)(ν),

$$\left|\partial_\omega^{\hat{i}} \lambda_{j,\nu}\right| \leq C\varepsilon\lambda_j^{-1}, \quad \hat{i} = 1, 2, 3, 4. \tag{3.15}$$

Similarly

$$\left|\partial_{\omega}^{\hat{i}}\varpi_{j,\nu,n_1,n_2}\right| \leq C\lambda_j^{-1}(\nu+1)^{12m+28}, \quad \hat{i}=1,2,3,4, \quad n_1,n_2\in\mathbb{N}, n_1+n_2=2. \tag{3.16}$$

By (3.14) and (3.16), we have

$$\|\varpi_{j,\nu,n_1,n_2}\|^{*}_{\Theta(\sigma_{\nu+1})\times R_\nu} \leq C\lambda_j^{-1}(\nu+1)^{12m+28}. \tag{3.17}$$

Similar to the above discussion, the following estimates can be obtained

$$\|\partial_\vartheta\varpi_{j,\nu,n_1,n_2}\|^{*}_{\Theta(\sigma_{\nu+1})\times R_\nu} \leq C\lambda_j^{-1}(\nu+1)^{12m+30}, \tag{3.18}$$

$$\|\partial_{\vartheta\vartheta}\varpi_{j,\nu,n_1,n_2}\|^{*}_{\Theta(\sigma_{\nu+1})\times R_\nu} \leq C\lambda_j^{-1}(\nu+1)^{12m+32}. \tag{3.19}$$

Now let's estimate the flow $X^t_{\Psi_\nu}$, denote

$$M_{j,\nu}(\vartheta;\omega)=\begin{pmatrix} \varpi_{j,\nu,2,0}+\varpi_{-j,\nu,2,0} & \varpi_{-j,\nu,1,1} \\ \varpi_{j,\nu,1,1} & \varpi_{j,\nu,0,2}+\varpi_{-j,\nu,0,2}\end{pmatrix}, \ \mathcal{J}_2=\mathrm{i}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}.$$

By (3.17)–(3.19),

$$\|M_{j,\nu}\|^{*}_{\Theta(\sigma_{\nu+1})\times R_\nu} \leq C\lambda_j^{-1}(\nu+1)^{12m+28},$$
$$\|\partial_\vartheta M_{j,\nu}\|^{*}_{\Theta(\sigma_{\nu+1})\times R_\nu} \leq C\lambda_j^{-1}(\nu+1)^{12m+30},$$
$$\|\partial_{\vartheta\vartheta} M_{j,\nu}\|^{*}_{\Theta(\sigma_{\nu+1})\times R_\nu} \leq C\lambda_j^{-1}(\nu+1)^{12m+32}.$$

The vector field X_{Ψ_ν} is as follows

$$\begin{cases} \dot{\vartheta}=0 \\ \dfrac{d}{dt}\begin{pmatrix} q_j \\ \bar{q}_{-j}\end{pmatrix}=\varepsilon_\nu\mathcal{J}_2 M_{j,\nu}(\vartheta;\omega)\cdot\begin{pmatrix} q_j \\ \bar{q}_{-j}\end{pmatrix}, \quad j\in\mathbb{Z}^2_{odd} \\ \dot{J}=\varepsilon_\nu\sum\limits_{j\in\mathbb{Z}^2_{odd}}\left[\partial_\vartheta\varpi_{j,\nu,2,0}(\vartheta;\omega)q_jq_{-j}+\partial_\vartheta\varpi_{j,\nu,1,1}(\vartheta;\omega)q_j\bar{q}_j+\partial_\vartheta\varpi_{j,\nu,0,2}(\vartheta;\omega)\bar{q}_j\bar{q}_{-j}\right].\end{cases}$$

The integral from 0 to t of the above equation, we have $X^t_{\Psi_\nu}$:

$$\begin{cases} \vartheta=\vartheta^C \\ w(t)=exp\left(\varepsilon_\nu\mathcal{J}M_\nu(\vartheta^C;\omega)t\right)\cdot w(0) \\ J(t)=J(0)+\displaystyle\int_0^t\varepsilon_\nu\sum_{j\in\mathbb{Z}^2_{odd}}\partial_\vartheta\varpi_{j,\nu,2,0}(\vartheta^C;\omega)q_j(t)q_{-j}(t)dt \\ \qquad +\displaystyle\int_0^t\varepsilon_\nu\sum_{j\in\mathbb{Z}^2_{odd}}\left[\partial_\vartheta\varpi_{j,\nu,1,1}(\vartheta^C;\omega)q_j(t)\bar{q}_j(t)+\partial_\vartheta\varpi_{j,\nu,0,2}(\vartheta^C;\omega)\bar{q}_j(t)\bar{q}_{-j}(t)\right]dt.\end{cases} \tag{3.20}$$

where $(\vartheta^C, J(0), w(0))$ is the initial value,

$$\mathcal{J}=\mathrm{i}\begin{pmatrix} 0 & \widetilde{E}_{\infty\times\infty} \\ -\widetilde{E}_{\infty\times\infty} & 0\end{pmatrix},$$

and $M_\nu(\vartheta;\omega)$ are the corresponding matrices. According to $\varepsilon_\nu = \varepsilon^{(1+\rho)^\nu}$, then

$$|\varepsilon_\nu^{1-\rho}(\nu+1)^{12m+32}(C_*\varrho^{-1})^{5\nu}| \leq C, \quad \nu = 0, 1, \ldots \tag{3.21}$$

as $\varepsilon < 1$, where C is an absolute constant. In view of (3.17), for $\vartheta \in \Theta(\sigma_{\nu+1})$,

$$\varepsilon_\nu \mathcal{J}_2 M_{j,\nu}(\vartheta;\omega) = \lambda_j^{-1}\varepsilon_\nu(\nu+1)^{12m+28}M_{j,\nu}^{*1}(\vartheta;\omega) = \lambda_j^{-1}\varepsilon_\nu^\rho M_{j,\nu}^*(\vartheta;\omega), \|M_{j,\nu}^*(\vartheta;\omega)\|_{\Theta(\sigma_{\nu+1})\times R_\nu}^* \leq C,$$

then

$$\|\varepsilon_\nu \mathcal{J} M_\nu(\vartheta;\omega)\|_{a,s,\Theta(\sigma_{\nu+1})\times R_\nu}^\star \leq C\varepsilon_\nu^\rho. \tag{3.22}$$

In view of (3.18),

$$\partial_\vartheta\left(\varepsilon_\nu \mathcal{J}_2 M_{j,\nu}(\vartheta;\omega)\cdot\begin{pmatrix} q_j \\ \bar{q}_{-j}\end{pmatrix}\right) = \varepsilon_\nu^\rho \cdot \partial_\vartheta\left(M_{j,\nu}^*(\vartheta;\omega)\cdot\begin{pmatrix} q_j \\ \bar{q}_{-j}\end{pmatrix}\right)$$

where

$$\left\|\partial_\vartheta\left(M_{j,\nu}^*(\vartheta;\omega)\cdot\begin{pmatrix} q_j \\ \bar{q}_{-j}\end{pmatrix}\right)\right\|_{\Theta(\sigma_{\nu+1})\times R_\nu}^* \leq C(|q_j| + |\bar{q}_{-j}|)$$

then

$$\|\partial_\vartheta\left(\varepsilon_\nu \mathcal{J} M_\nu(\vartheta;\omega)\cdot w\right)\|_{D_{\nu+1}^{a,s}\times R_\nu}^* \leq C\varepsilon_\nu^\rho \Gamma_{\nu+1}. \tag{3.23}$$

By (3.22) and (3.23),

$$exp\left(\varepsilon_\nu \mathcal{J} M_\nu(\vartheta;\omega)t\right) = Id + g_\nu^\infty(\vartheta;\omega,t) \tag{3.24}$$

and for $t \in [0,1]$,

$$\|g_\nu^\infty(\vartheta;\omega,t)\|_{a,s,\Theta(\sigma_{\nu+1})\times R_\nu}^\star \leq C\varepsilon_\nu^\rho, \quad \|\partial_\vartheta\left(g_\nu^\infty(\vartheta;\omega,t)\cdot w\right)\|_{D_{\nu+1}^{a,s}\times R_\nu}^* \leq C\varepsilon_\nu^\rho\Gamma_{\nu+1}. \tag{3.25}$$

Let's define $J(t)$ in (3.20) as

$$J(t) = J + g_{J,\nu}(\vartheta, w;\omega,t). \tag{3.26}$$

By (3.18), (3.25) and (3.21),

$$\|g_{J,\nu}(\vartheta,w;\omega,t)\|_{D_{\nu+1}^{a,s}\times R_\nu}^* \leq C\varepsilon_\nu^\rho\Gamma_\nu^2, \quad t \in [0,1], \tag{3.27}$$

and for any $w' \in l^{a,s}\times l^{a,s}$,

$$\|\partial_w\left(g_{J,\nu}(\vartheta,w;\omega,t)\right)\cdot w'\|_{D_{\nu+1}^{a,s}\times R_\nu}^* \leq C\varepsilon_\nu^\rho\Gamma_\nu\cdot\|w'\|_{a,s}, \quad t\in[0,1]. \tag{3.28}$$

By (3.19), (3.25) and (3.21),

$$\|\partial_\vartheta\left(g_{J,\nu}(\vartheta,w;\omega,t)\right)\|_{D_{\nu+1}^{a,s}\times R_\nu}^* \leq C\varepsilon_\nu^\rho\Gamma_\nu^2, \quad t\in[0,1]. \tag{3.29}$$

Denote

$$X_{\Psi_\nu}^t = \Pi_{\mathcal{Z}} + g_\nu(\omega,t) : D_{\nu+1}^{a,s}\times R_{\nu+1} \mapsto D_\nu^{a,s} \tag{3.30}$$

from (3.20), (3.24) and (3.26),

$$\begin{cases} \Pi_\vartheta \circ X^t_{\Psi_\nu}(\vartheta, J, w) = \vartheta : \quad D^{a,s}_{\nu+1} \times R_{\nu+1} \mapsto \Theta(\sigma_\nu), \\ \Pi_w \circ X^t_{\Psi_\nu}(\vartheta, J, w) = \left(Id + g^\infty_\nu(\vartheta; \omega, t)\right) \cdot w : \quad D^{a,s}_{\nu+1} \times R_{\nu+1} \mapsto l^{a,s} \times l^{a,s} \\ \Pi_J \circ X^t_{\mathcal{F}_\nu}(\vartheta, J, w) = J + g_{J,\nu}(\vartheta, w; \omega, t) : \quad D^{a,s}_{\nu+1} \times R_{\nu+1} \mapsto \mathbb{C}^m \end{cases} \tag{3.31}$$

where $\Pi_{\mathcal{Z}}, \Pi_\omega$ denote the projectors

$$\Pi_{\mathcal{Z}} : \mathcal{Z}^{a,s} \times R_0 \longmapsto \mathcal{Z}^{a,s}, \quad \Pi_\omega : \mathcal{Z}^{a,s} \times R_0 \longmapsto R_0,$$

and $\Pi_\vartheta, \Pi_J, \Pi_w$ denote the projectors of $\mathcal{Z}^{a,s} = \mathbb{C}^m/2\pi\mathbb{Z}^m \times \mathbb{C}^m \times l^{a,s} \times l^{a,s}$ on the first, second and third factor respectively. According to the first equation of (3.25),(3.27) and (3.31),

$$|X^t_{\Psi_\nu} - \Pi_{\mathcal{Z}}|^*_{a,s,D^{a,s}_{\nu+1} \times R_{\nu+1}} \leq C\varepsilon^\rho_\nu. \tag{3.32}$$

By (3.31), we have

$$DX^t_{\Psi_\nu} = \begin{pmatrix} Id_{m\times m} & 0 & 0 \\ \partial_\vartheta(g^\infty_\nu(\vartheta; \omega, t)w) & Id_{\infty\times\infty} + g^\infty_\nu(\vartheta; \omega, t) & 0 \\ \partial_\vartheta(g_{J,\nu}(\vartheta, w; \omega, t)) & \partial_w(g_{J,\nu}(\vartheta, w; \omega, t)) & Id_{m\times m} \end{pmatrix}$$

where D is the differentiation operator with respect to (ϑ, w, J). In view of (3.25), (3.28) and (3.29) , for $\tilde{w} = (\vartheta', w', J'), (\vartheta, w, J) \in D^{a,s}_{\nu+1}$,

$$|\left(DX^t_{\Psi_\nu} - Id\right)\tilde{w}|_{a,s} \leq C\varepsilon^\rho_\nu |\tilde{w}|_{a,s}.$$

Thus

$$|DX^t_{\Psi_\nu} - Id|^\diamond_{a,s,D^{a,s}_{\nu+1} \times R_{\nu+1}} < C\varepsilon^\rho_\nu.$$

Similarly

$$|\partial^{\hat{i}}_\omega (DX^t_{\Psi_\nu} - Id)|^\diamond_{a,s,D^{a,s}_{\nu+1} \times R_{\nu+1}} < C\varepsilon^\rho_\nu, \quad \hat{i} = 1, 2, 3, 4$$

and

$$|DX^t_{\Psi_\nu} - Id|^\star_{a,s,D^{a,s}_{\nu+1} \times R_{\nu+1}} < C\varepsilon^\rho_\nu. \tag{3.33}$$

3.3. Estimation for the new normal form and the new smaller terms

Let

$$\lambda_{j,\nu+1} = \lambda_{j,\nu} + \varepsilon_\nu [\eta_{j,\nu,1,1}],$$

then by (3.2)(ν), it is obvious that $\lambda_{j,\nu+1}$ satisfies the conditions (3.3)($\nu + 1$) and (3.4)($\nu + 1$).

Now let's estimate the smaller terms of (3.9). Notice that those terms are polynomials of $q_j q_{-j}$, $q_j \bar{q}_j$ and $\bar{q}_j \bar{q}_{-j}$. So we can write it

$$\begin{aligned} &\varepsilon_\nu \int_0^1 (1-t) \left\{ \{H^2_\nu, \Psi_\nu\}, \Upsilon_\nu \right\} \circ X^t_{\Psi_\nu} dt + \varepsilon_\nu \int_0^1 \{P_\nu, \Upsilon_\nu\} \circ X^t_{\Psi_\nu} dt \\ &= \varepsilon^2_\nu \sum_{j \in \mathbb{Z}^2_{odd}} [\tilde{\eta}_{j,\nu+1,2,0}(\vartheta; \omega) q_j q_{-j} + \tilde{\eta}_{j,\nu+1,1,1}(\vartheta; \omega) q_j \bar{q}_j + \tilde{\eta}_{j,\nu+1,0,2}(\vartheta; \omega) \bar{q}_j \bar{q}_{-j}], \end{aligned}$$

where from

$$\{H_\nu^*, \Psi_\nu\} = \varepsilon_\nu \sum_{j\in\mathbb{Z}^2_{odd}} [\eta_{j,\nu,1,1}] q_j\bar{q}_j - P_\nu,$$

we know that $\tilde{\eta}_{j,\nu+1,n_1,n_2}(\vartheta;\omega)$ is a linear combination of the product of ϖ_{j,ν,n_1,n_2} and η_{j,ν,m_1,m_2}. By (3.17) and (3.2)(ν),

$$\varpi_{j,\nu,n_1,n_2}(\vartheta;\omega) = \lambda_j^{-1}(\nu+1)^{12m+28}\varpi^*_{j,\nu,n_1,n_2}(\vartheta;\omega), \quad \|\varpi^*_{j,\nu,n_1,n_2}\|^*_{\Theta(\sigma_{\nu+1})\times R_\nu} \le C$$

and

$$\eta_{j,\nu,n_1,n_2}(\vartheta;\omega) = \lambda_j^{-1}\eta^*_{j,\nu,n_1,n_2}(\vartheta;\omega), \quad \|\eta^*_{j,\nu,n_1,n_2}(\vartheta;\omega)\|^*_{\Theta(\sigma_{\nu+1})\times R_\nu} \le C$$

respectively. Thereby , we have

$$\tilde{\eta}_{j,\nu+1,n_1,n_2}(\vartheta;\omega) = \lambda_j^{-1}(\nu+1)^{12m+28}\tilde{\eta}^*_{j,\nu+1,n_1,n_2}(\vartheta;\omega), \quad \|\tilde{\eta}^*_{j,\nu+1,n_1,n_2}\|^*_{\Theta(\sigma_{\nu+1})\times R_\nu} \le C.$$

According to $\varepsilon_\nu^{1-\rho}(\nu+1)^{12m+28} \le 1$ as $\varepsilon < 1$, then

$$\eta_{j,\nu+1,n_1,n_2} := \varepsilon_\nu^{1-\rho}\tilde{\eta}_{j,\nu+1,n_1,n_2} = \lambda_j^{-1}\eta^*_{j,\nu+1,n_1,n_2}, \quad \|\eta^*_{j,\nu+1,n_1,n_2}\|^*_{\Theta(\sigma_{\nu+1})\times R_\nu} \le C.$$

From $\varepsilon_\nu^{2-(1-\rho)} = \varepsilon_{\nu+1}$, we have (3.1)($\nu+1$) is defined in $D^{a,s}_{\nu+1}$ and $\lambda_{j,\nu+1}$ satisfies (3.3)($\nu+1$), (3.4)($\nu+1$) and $\eta_{j,\nu+1,n_1,n_2}$ satisfies (3.2)($\nu+1$).

3.4. Convergence and reducibility theorem

The reducibility of the linear Hamiltonian systems can be summarized as follows.

Theorem 3.1. Given $\sigma_0 > 0$, $0 < \gamma < 1, 0 < \rho < 1$. Then there is a $\varepsilon^*(\gamma) > 0$ such that for any $0 < \varepsilon < \varepsilon^*(\gamma)$, there exists a set $\underline{R} \subset [\varrho, 2\varrho]^m, \varrho > 0$ with meas$\underline{R} \ge (1 - \frac{2\gamma}{3})\varrho^{m}$ and a symplectic transformation Σ^0_∞ defined on $D^a_\infty \times \underline{R}$ changes the Hamiltonian (2.8) into

$$\bar{H} \circ \Sigma^0_\infty = <\omega, J> + \sum_{j\in\mathbb{Z}^2_{odd}} \mu_j|q_j|^2,$$

where

$$\mu_j = \lambda_j + \frac{\varepsilon}{2\lambda_j}[\phi] + \frac{1}{\lambda_j}\varepsilon^{(1+\rho)}\mu_j^*, \quad \|\mu_j^*\|^*_{\underline{R}} \le C, \quad j \in \mathbb{Z}^2_{odd}.$$

Moreover, there exists a constant $C > 0$ such that

$$|\Sigma^0_\infty - id|^*_{a,s,D^{a,s}_\infty\times\underline{R}} \le C\varepsilon^\rho,$$

where *id* is identity mapping.

Proof. Let $\eta_{j,0,2,0} = \eta_{j,0,0,2} = \frac{1}{4\lambda_j}\varphi(\vartheta)$, $\eta_{j,0,1,1} = \frac{1}{2\lambda_j}\varphi(\vartheta)$, we have that $H_0 = \bar{H}$ and $\eta_{j,0,n_1,n_2} = \lambda_j^{-1}\eta^*_{j,0,n_1,n_2}, \|\eta^*_{j,0,n_1,n_2}\|^*_{\Theta(\sigma_0)\times R_0} \le C, n_1, n_2 \in \mathbb{N}, n_1 + n_2 = 2$ where C is an absolute constant. i.e., the assumptions (3.1),(3.2),(3.3),(3.4) of the iteration are satisfied when $\nu = 0$.

We obtain the following sequences:

$$R_\infty \subset \cdots \subset R_\nu \subset \cdots \subset R_1 \subset R_0 \subset [\varrho, 2\varrho]^m,$$

$$D_0^{a,s} \supset D_1^{a,s} \supset \cdots \supset D_\nu^{a,s} \supset \cdots \supset D_\infty^{a,s}.$$

From (3.30), (3.32) and (3.33), denote

$$T_\nu = X^1_{\mathcal{F}_\nu} = \Pi_{\mathcal{Z}} + g_\nu(\omega, 1) : D_{\nu+1}^{a,s} \times R_{\nu+1} \longmapsto D_\nu^{a,s} \tag{3.34}$$

then

$$|T_\nu - \Pi_{\mathcal{Z}}|^*_{a,s,D_{\nu+1}^{a,s}\times R_{\nu+1}} \leq C\varepsilon_\nu^\rho, \quad |DT_\nu - Id|^\star_{a,s,D_{\nu+1}^{a,s}\times R_{\nu+1}} \leq C\varepsilon_\nu^\rho. \tag{3.35}$$

Similar to [27], it can be seen that the limiting transformation $T_0 \circ T_1 \circ \cdots$ converges to a symplectic coordinate transformation Σ_∞^0. And there exists an absolute constant $C > 0$ independent of j such that

$$|\Sigma_\infty^0 - id|^*_{a,s,D_\infty^{a,s}\times \underline{R}} \leq C\varepsilon^\rho, \tag{3.36}$$

with *id* is identity mapping.

In view of the Hamiltonian (2.8) satisfies the conditions (3.1) − (3.4), (3.6), (3.7) with $\nu = 0$, the above iterative procedure can run repeatedly. Thus the transformation Σ_∞^0 changes the Hamiltonian (2.8) to

$$\bar{H} \circ \Sigma_\infty^0 =< \omega, J > + \sum_{j\in\mathbb{Z}^2_{odd}} \mu_j |q_j|^2, \tag{3.37}$$

with

$$\mu_j = \lambda_j + \frac{\varepsilon}{2\lambda_j}[\phi] + \frac{1}{\lambda_j}\varepsilon^{(1+\rho)}\mu_j^*, \quad \|\mu_j^*\|^*_{\underline{R}} \leq C, \quad j \in \mathbb{Z}^2_{odd}. \tag{3.38}$$

□

We present the following lemma which has been used in the above iterative procedure. The proof is similar to Lemma 3.1 in [15].

Lemma 3.1. For any given $k \in \mathbb{Z}^m$, $j \in \mathbb{Z}^2_{odd}$, $\hat{l} \in \mathbb{N}$, denote

$$\mathcal{I}_k^1 = \left\{\omega \in [\varrho, 2\varrho]^m : | < k, \omega > | \leq \frac{\varrho}{C_*|k|^{m+1}}\right\}, \quad k \neq 0,$$

$$\mathcal{I}_{k,j,\hat{l}}^{2,+} = \left\{\omega \in [\varrho, 2\varrho]^m : \left| < k, \omega > + \lambda_{j,\hat{l}} + \lambda_{-j,\hat{l}}\right| < \frac{\varrho}{C_*(\delta(\hat{l}) + \hat{l}^2)(|k| + \delta(|k|))^{m+1}}\right\},$$

$$\mathcal{I}_{k,j,\hat{l}}^{2,-} = \left\{\omega \in [\varrho, 2\varrho]^m : \left| < k, \omega > - \lambda_{j,\hat{l}} - \lambda_{-j,\hat{l}}\right| < \frac{\varrho}{C_*(\delta(\hat{l}) + \hat{l}^2)(|k| + \delta(|k|))^{m+1}}\right\},$$

$$R^1 = \bigcup_{0\neq k\in\mathbb{Z}^m} \mathcal{I}_k^1, \quad R^2_{\hat{l}} = \bigcup_{j\in\mathbb{Z}^2_{odd}} \bigcup_{k\in\mathbb{Z}^m} \left(\mathcal{I}_{k,j,\hat{l}}^{2,+} \bigcup \mathcal{I}_{k,j,\hat{l}}^{2,-}\right)$$

where $\delta(x) = 1$ as $x = 0$ and $\delta(x) = 0$ as $x \neq 0$. Then the sets $R^1, R^2_{\hat{l}}$ is measurable and

$$\text{meas}R^1 \leq \frac{1}{3}\gamma\varrho^m, \quad \text{meas}R^2_{\hat{l}} \leq \frac{\gamma(\delta(\hat{l}) + \hat{l})^{-2}}{3\sum_{\hat{i}=0}^{+\infty}(\delta(\hat{i}) + \hat{i})^{-2}}\varrho^m \tag{3.39}$$

if $C_* \gg 1$ large enough.

Let

$$R_{00} = [\varrho, 2\varrho]^m \backslash R^1, \quad R_0 = R_{00} \setminus R_0^2, \quad R_{\hat{l}+1} = R_{\hat{l}} \setminus R_{\hat{l}+1}^2, \quad \hat{l} = 0, 1, \cdots. \tag{3.40}$$

Then we have (3.6) and (3.7). Denote

$$\underline{R} = \bigcap_{\hat{l}=1}^{\infty} R_{\hat{l}} \tag{3.41}$$

then by (3.6),

$$\text{meas}\underline{R} > (1 - \frac{2\gamma}{3})\varrho^m. \tag{3.42}$$

3.5. The Hamiltonian after the iterative procedure

In view of the symplectic coordinate transformation Σ_∞^0 is linear, and (3.36), then

$$q_j \circ \Sigma_\infty^0 = q_j + \lambda_j^{-1} \varepsilon^\rho \tilde{g}_{j,1,\infty}^*(\vartheta; \omega) q_j + \lambda_j^{-1} \varepsilon^\rho \tilde{g}_{j,2,\infty}^*(\vartheta; \omega) \bar{q}_{-j}$$

where

$$\|\tilde{g}_{j,\hat{l},\infty}^*(\vartheta; \omega)\|_{\Theta(\sigma_0/2)\times\underline{R}}^* \leq C, \quad \hat{l} = 1, 2.$$

Thus from (3.37), the Hamiltonian (2.8) is transformed into by Σ_∞^0

$$H_{00} := \bar{H} \circ \Sigma_\infty^0 =< \omega, J > + \sum_{j \in \mathbb{Z}_{odd}^2} \mu_j q_j \bar{q}_j, \tag{3.43}$$

and the Hamiltonian (2.9) is transformed into

$$\begin{aligned}
\tilde{G}^4 = G^4 \circ \Sigma_\infty^0 = \frac{3}{32\pi^2} & \sum_{\substack{i-j+d-l=0 \\ i,j,d,l \in \mathbb{Z}_{odd}^2}} \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_d \lambda_l}} \tilde{G}_{ijdl}^{2,2}(\vartheta; \omega) q_i \bar{q}_j q_d \bar{q}_l \\
& + \frac{1}{64\pi^2} \sum_{\substack{i+j+d+l=0 \\ i,j,d,l \in \mathbb{Z}_{odd}^2}} \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_d \lambda_l}} (\tilde{G}_{ijdl}^{4,0}(\vartheta; \omega) q_i q_j q_d q_l + \tilde{G}_{ijdl}^{0,4}(\vartheta; \omega) \bar{q}_i \bar{q}_j \bar{q}_d \bar{q}_l) \\
& + \frac{1}{16\pi^2} \sum_{\substack{i+j+d-l=0 \\ i,j,d,l \in \mathbb{Z}_{odd}^2}} \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_d \lambda_l}} (\tilde{G}_{ijdl}^{3,1}(\vartheta; \omega) q_i q_j q_d \bar{q}_l + \tilde{G}_{ijdl}^{1,3}(\vartheta; \omega) \bar{q}_i \bar{q}_j \bar{q}_d q_l)
\end{aligned} \tag{3.44}$$

where

$$\tilde{G}_{ijdl}^{n_1,n_2}(\vartheta; \omega) = G_{ijdl}^{n_1,n_2}(\vartheta) \left(1 + \frac{\varepsilon^\rho G_{ijdl}^{n_1,n_2,*}(\vartheta; \omega)}{\min(|i|^2, |j|^2, |d|^2, |l|^2)}\right), \quad \|G_{ijdl}^{n_1,n_2,*}(\vartheta; \omega)\|_{\Theta(\sigma_0/2)\times\underline{R}}^* \leq C, \tag{3.45}$$

with $n_1, n_2 \in \mathbb{N}, n_1 + n_2 = 4, n_1, n_2 = 0, 1, 2, 3, 4$.

This means that the transformation Σ_∞^0 changes the Hamiltonian (2.7) into

$$H = H_{00} + \varepsilon \tilde{G}^4. \tag{3.46}$$

The following Lemma gives a regularity result, the proof is similar to [13] and is omitted.

Lemma 3.2. For $a \geq 0$ and $s > 0$, the gradients $\tilde{G}^4_q, \tilde{G}^4_{\bar{q}}$ are real analytic as maps from some neighborhood of origin in $l^{a,s} \times l^{a,s}$ into $l^{a,s}$ with $\|\tilde{G}^4_q\|_{a,s} = O(\|q\|^3_{a,s})$, $\|\tilde{G}^4_{\bar{q}}\|_{a,s} = O(\|q\|^3_{a,s})$.

4. Partial Birkhoff normal form

As in [13], Let S is an admissible set. We define $\mathbb{Z}^2_* = \mathbb{Z}^2_{odd} \setminus S$. For simplicity, we define the following three sets:

$$S_1 = \left\{ (i,j,d,l) \in (\mathbb{Z}^2_{odd})^4 : \begin{array}{l} i - j + d - l = 0, \\ |i|^2 - |j|^2 + |d|^2 - |l|^2 \neq 0, \\ \#(S \cap \{i,j,d,l\}) \geq 2 \end{array} \right\} \tag{4.1}$$

and

$$S_2 = \left\{ (i,j,d,l) \in (\mathbb{Z}^2_{odd})^4 : \begin{array}{l} i + j + d + l = 0, \\ |i|^2 + |j|^2 + |d|^2 + |l|^2 \neq 0, \\ \#(S \cap \{i,j,d,l\}) \geq 2 \end{array} \right\} \tag{4.2}$$

$$S_3 = \left\{ (i,j,d,l) \in (\mathbb{Z}^2_{odd})^4 : \begin{array}{l} i + j + d - l = 0, \\ |i|^2 + |j|^2 + |d|^2 - |l|^2 \neq 0, \\ \#(S \cap \{i,j,d,l\}) \geq 2. \end{array} \right\}. \tag{4.3}$$

Obviously, the set

$$\left\{ (i,j,d,l) \in (\mathbb{Z}^2_{odd})^4 : \begin{array}{l} i + j + d + l = 0, \\ |i|^2 + |j|^2 + |d|^2 + |l|^2 = 0, \end{array} \right\}$$

is empty. Similar to [13], the set

$$\left\{ (i,j,d,l) \in (\mathbb{Z}^2_{odd})^4 : \begin{array}{l} i + j + d - l = 0, \\ |i|^2 + |j|^2 + |d|^2 - |l|^2 = 0, \end{array} \right\}$$

is empty.

For Proposition 4.1, we give the following lemma that will be proved in the "Appendix".

Lemma 4.1. Given $\varrho > 0, 0 < \gamma < 1$, and C_* large enough, ε small enough, then there is a subset $\overline{R} \subset [\varrho, 2\varrho]^m$ with

$$\text{meas}\overline{R} > (1 - \frac{\gamma}{3})\varrho^m \tag{4.4}$$

so that the following statements hold:

(i) If $(i,j,d,l) \in S_1$ or $i - j + d - l = 0, |i|^2 - |j|^2 + |d|^2 - |l|^2 = 0, \#(S \cap \{i,j,d,l\}) = 2$ and $k \neq 0$, then for any $\omega \in \overline{R}$,

$$\left| \mu_i - \mu_j + \mu_d - \mu_l + < k, \omega > \right| \geq \frac{\varrho}{C_*(|k| + \delta(|k|))^{m+1}}, \qquad \forall k \in \mathbb{Z}^m; \tag{4.5}$$

(ii) If $(i, j, d, l) \in S_2$, then for any $\omega \in \overline{R}$,

$$\left|\mu_i + \mu_j + \mu_d + \mu_l + < k, \omega >\right| \geq \frac{\varrho}{C_*(|k| + \delta(|k|))^{m+1}}, \qquad \forall k \in \mathbb{Z}^m; \tag{4.6}$$

(iii) If $(i, j, d, l) \in S_3$, then for any $\omega \in \overline{R}$,

$$\left|\mu_i + \mu_j + \mu_d - \mu_l + < k, \omega >\right| \geq \frac{\varrho}{C_*(|k| + \delta(|k|))^{m+1}}, \qquad \forall k \in \mathbb{Z}^m; \tag{4.7}$$

where $\delta(x) = 1$ as $x = 0$ and $\delta(x) = 0$ as $x \neq 0$.

Let

$$R = \underline{R} \cap \overline{R},$$

then

$$\text{meas} R \geq (1 - \gamma)\varrho^m.$$

Next we transform the Hamiltonian (3.46) into some partial Birkhoff form of order four.

Proposition 4.1. For each admissible set S there exists a symplectic change of coordinates X_F^1 that changes the hamiltonian $H = H_{00} + \varepsilon\tilde{G}^4$ with nonlinearity (3.44) into

$$H \circ X_F^1 = N + \mathcal{A} + \mathcal{B} + \bar{\mathcal{B}} + P, \tag{4.8}$$

with

$$\begin{aligned} N = {} & \varepsilon^{-4} < \omega, J > + \varepsilon^{-4} \sum_{j \in S} \mu_j I_j + \varepsilon^{-4} \sum_{j \in \mathbb{Z}_*^2} \mu_j |z_j|^2 + \frac{3}{16\pi^2} \sum_{i \in S} \frac{1}{\lambda_i^2} [\tilde{G}_{iiii}^{2,2}] \tilde{\xi}_i I_i \\ & + \frac{3}{8\pi^2} \sum_{i,j \in S, i \neq j} \frac{1}{\lambda_i \lambda_j} [\tilde{G}_{iijj}^{2,2}] \tilde{\xi}_i I_j + \frac{3}{8\pi^2} \sum_{i \in S, j \in \mathbb{Z}_*^2} \frac{1}{\lambda_i \lambda_j} [\tilde{G}_{iijj}^{2,2}] \tilde{\xi}_i |z_j|^2 \end{aligned} \tag{4.9}$$

$$\mathcal{A} = \frac{3}{8\pi^2} \sum_{d \in \mathcal{L}_1} \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_d \lambda_l}} [\tilde{G}_{ijdl}^{2,2}] \sqrt{\tilde{\xi}_i \tilde{\xi}_j} e^{\mathrm{i}(\theta_i - \theta_j)} z_d \bar{z}_l \tag{4.10}$$

$$\mathcal{B} = \frac{3}{8\pi^2} \sum_{d \in \mathcal{L}_2} \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_d \lambda_l}} [\tilde{G}_{dilj}^{2,2}] \sqrt{\tilde{\xi}_i \tilde{\xi}_j} e^{-\mathrm{i}(\theta_i + \theta_j)} z_d z_l \tag{4.11}$$

$$\bar{\mathcal{B}} = \frac{3}{8\pi^2} \sum_{d \in \mathcal{L}_2} \frac{1}{\sqrt{\lambda_i \lambda_j \lambda_d \lambda_l}} [\tilde{G}_{idjl}^{2,2}] \sqrt{\tilde{\xi}_i \tilde{\xi}_j} e^{\mathrm{i}(\theta_i + \theta_j)} \bar{z}_d \bar{z}_l. \tag{4.12}$$

$$\begin{aligned} P = {} & O(\varepsilon^2 |I|^2 + \varepsilon^2 |I| \|z\|_{a,s}^2 + \varepsilon |\tilde{\xi}|^{\frac{1}{2}} \|z\|_{a,s}^3 + \varepsilon^2 \|z\|_{a,s}^4 + \varepsilon^2 |\tilde{\xi}|^3 \\ & + \varepsilon^3 |\tilde{\xi}|^{\frac{5}{2}} \|z\|_{a,s} + \varepsilon^4 |\tilde{\xi}|^2 \|z\|_{a,s}^2 + \varepsilon^5 |\tilde{\xi}|^{\frac{3}{2}} \|z\|_{a,s}^3). \end{aligned} \tag{4.13}$$

Proof. Denote

$$\tilde{G}^{n_1,n_2}_{ijdl}(\vartheta,\omega)=\sum_{k\in\mathbb{Z}^m}G^{n_1,n_2}_{ijdl,k}(\omega)e^{\mathrm{i}<k,\vartheta>},\quad n_1,n_2=0,1,2,3,4,\quad n_1+n_2=4. \tag{4.14}$$

We find a Hamiltonian

$$\begin{aligned}
F=&\frac{3}{32\pi^2}\sum_{i\in S}\sum_{k\neq 0}\frac{1}{\lambda_i^2}\cdot\frac{G^{2,2}_{iiii,k}}{\mathrm{i}<k,\omega>}e^{\mathrm{i}<k,\vartheta>}|q_i|^4\\
&+\frac{3}{8\pi^2}\sum_{i,j\in S,i\neq j}\sum_{k\neq 0}\frac{1}{\lambda_i\lambda_j}\cdot\frac{G^{2,2}_{iijj,k}}{\mathrm{i}<k,\omega>}e^{\mathrm{i}<k,\vartheta>}|q_i|^2|q_j|^2\\
&+\frac{3}{8\pi^2}\sum_{i\in S,j\in\mathbb{Z}^2_*}\sum_{k\neq 0}\frac{1}{\lambda_i\lambda_j}\cdot\frac{G^{2,2}_{iijj,k}}{\mathrm{i}<k,\omega>}e^{\mathrm{i}<k,\vartheta>}|q_i|^2|q_j|^2\\
&+\frac{3}{8\pi^2}\sum_{d\in\mathcal{L}_1}\sum_{k\neq 0}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}\cdot\frac{G^{2,2}_{ijdl,k}}{\mathrm{i}(\mu_i-\mu_j+\mu_d-\mu_l+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_jq_d\bar{q}_l\\
&+\frac{3}{8\pi^2}\sum_{d\in\mathcal{L}_2}\sum_{k\neq 0}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}\cdot\frac{G^{2,2}_{dilj,k}}{\mathrm{i}(\mu_d+\mu_l-\mu_i-\mu_j+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}\bar{q}_i\bar{q}_jq_dq_l\\
&+\frac{3}{8\pi^2}\sum_{d\in\mathcal{L}_2}\sum_{k\neq 0}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}\cdot\frac{G^{2,2}_{idjl,k}}{\mathrm{i}(\mu_i-\mu_d+\mu_j-\mu_l+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}q_iq_j\bar{q}_d\bar{q}_l\\
&+\frac{3}{8\pi^2}\sum_{(i,j,d,l)\in S_1}\sum_{k\in\mathbb{Z}^m}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}\cdot\frac{G^{2,2}_{ijdl,k}}{\mathrm{i}(\mu_i-\mu_j+\mu_d-\mu_l+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}q_i\bar{q}_jq_d\bar{q}_l\\
&+\frac{1}{64\pi^2}\sum_{(i,j,d,l)\in S_2}\sum_{k\in\mathbb{Z}^m}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}\cdot\frac{G^{4,0}_{ijdl,k}}{\mathrm{i}(\mu_i+\mu_j+\mu_d+\mu_l+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}q_iq_jq_dq_l\\
&+\frac{1}{64\pi^2}\sum_{(i,j,d,l)\in S_2}\sum_{k\in\mathbb{Z}^m}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}\cdot\frac{G^{0,4}_{ijdl,k}}{\mathrm{i}(-\mu_i-\mu_j-\mu_d-\mu_l+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}\bar{q}_i\bar{q}_j\bar{q}_d\bar{q}_l\\
&+\frac{1}{16\pi^2}\sum_{(i,j,d,l)\in S_3}\sum_{k\in\mathbb{Z}^m}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}\cdot\frac{G^{3,1}_{ijdl,k}}{\mathrm{i}(\mu_i+\mu_j+\mu_d-\mu_l+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}q_iq_jq_d\bar{q}_l\\
&+\frac{1}{16\pi^2}\sum_{(i,j,d,l)\in S_3}\sum_{k\in\mathbb{Z}^m}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}\cdot\frac{G^{1,3}_{ijdl,k}}{\mathrm{i}(-\mu_i-\mu_j-\mu_d+\mu_l+<k,\omega>)}e^{\mathrm{i}<k,\vartheta>}\bar{q}_i\bar{q}_j\bar{q}_dq_l.
\end{aligned} \tag{4.15}$$

Let X^1_F be the time-1 map of the Hamiltonian vector field of εF and denote variables as follows

$$q_j=\begin{cases}q_j, & j\in S,\\ z_j, & j\in\mathbb{Z}^2_*,\end{cases}$$

then it satisfies

$$\begin{aligned}
\widehat{H}&=H\circ X^1_F=H_{00}+\varepsilon\tilde{G}^4+\varepsilon\{H_{00},F\}+\varepsilon^2\{\tilde{G}^4,F\}+\varepsilon^2\int_0^1(1-t)\{\{H,F\},F\}\circ X^t_Fdt\\
&=<\omega,J>+\sum_{j\in S}\mu_j|q_j|^2+\sum_{j\in\mathbb{Z}^2_*}\mu_j|z_j|^2
\end{aligned}$$

$$+\frac{3\varepsilon}{32\pi^2}\sum_{i\in S}\frac{1}{\lambda_i^2}[\tilde{G}^{2,2}_{iiii}]|q_i|^4+\frac{3\varepsilon}{8\pi^2}\sum_{i,j\in S,i\neq j}\frac{1}{\lambda_i\lambda_j}[\tilde{G}^{2,2}_{iijj}]|q_i|^2|q_j|^2$$
$$+\frac{3\varepsilon}{8\pi^2}\sum_{i\in S,j\in\mathbb{Z}^2_*}\frac{1}{\lambda_i\lambda_j}[\tilde{G}^{2,2}_{iijj}]|q_i|^2|q_j|^2+\frac{3\varepsilon}{8\pi^2}\sum_{d\in\mathcal{L}_1}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}[\tilde{G}^{2,2}_{ijdl}]q_i\bar{q}_jq_d\bar{q}_l$$
$$+\frac{3\varepsilon}{8\pi^2}\sum_{d\in\mathcal{L}_2}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}[\tilde{G}^{2,2}_{dilj}]\bar{q}_i\bar{q}_jq_dq_l+\frac{3\varepsilon}{8\pi^2}\sum_{d\in\mathcal{L}_2}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}[\tilde{G}^{2,2}_{idjl}]q_iq_j\bar{q}_d\bar{q}_l$$
$$+O(\varepsilon|q|\|z\|^3_{a,s}+\varepsilon\|z\|^4_{a,s}+\varepsilon^2|q|^6+\varepsilon^2|q|^5\|z\|_{a,s}+\varepsilon^2|q|^4\|z\|^2_{a,s}+\varepsilon^2|q|^3\|z\|^3_{a,s}).$$

Now we introduce the parameter vector $\tilde{\xi}=(\tilde{\xi}_j)_{j\in S}$ and the action-angle variable by setting

$$q_j=\sqrt{I_j+\tilde{\xi}_j}e^{\mathrm{i}\theta_j},\qquad \bar{q}_j=\sqrt{I_j+\tilde{\xi}_j}e^{-\mathrm{i}\theta_j},\quad j\in S. \tag{4.16}$$

From the symplectic transformation (4.16), the Hamiltonian $\widehat{H}$ is changed into

$$\begin{aligned}\widehat{H} &= <\omega,J>+\sum_{j\in S}\mu_jI_j+\sum_{j\in\mathbb{Z}^2_*}\mu_j|z_j|^2+\frac{3\varepsilon}{16\pi^2}\sum_{i\in S}\frac{1}{\lambda_i^2}[\tilde{G}^{2,2}_{iiii}]\tilde{\xi}_iI_i\\
&\quad+\frac{3\varepsilon}{8\pi^2}\sum_{i,j\in S,i\neq j}\frac{1}{\lambda_i\lambda_j}[\tilde{G}^{2,2}_{iijj}]\tilde{\xi}_iI_j+\frac{3\varepsilon}{8\pi^2}\sum_{i\in S,j\in\mathbb{Z}^2_*}\frac{1}{\lambda_i\lambda_j}[\tilde{G}^{2,2}_{iijj}]\tilde{\xi}_i|z_j|^2\\
&\quad+\frac{3\varepsilon}{8\pi^2}\sum_{d\in\mathcal{L}_1}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}[\tilde{G}^{2,2}_{ijdl}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}e^{\mathrm{i}(\theta_i-\theta_j)}z_d\bar{z}_l\\
&\quad+\frac{3\varepsilon}{8\pi^2}\sum_{d\in\mathcal{L}_2}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}[\tilde{G}^{2,2}_{dilj}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}e^{-\mathrm{i}(\theta_i+\theta_j)}z_dz_l\\
&\quad+\frac{3\varepsilon}{8\pi^2}\sum_{d\in\mathcal{L}_2}\frac{1}{\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}}[\tilde{G}^{2,2}_{idjl}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}e^{\mathrm{i}(\theta_i+\theta_j)}\bar{z}_d\bar{z}_l\\
&\quad+O(\varepsilon|I|^2+\varepsilon|I|\|z\|^2_{a,s}+\varepsilon|\tilde{\xi}|^{\frac{1}{2}}\|z\|^3_{a,s}+\varepsilon\|z\|^4_{a,s}+\varepsilon^2|\tilde{\xi}|^3\\
&\qquad+\varepsilon^2|\tilde{\xi}|^{\frac{5}{2}}\|z\|_{a,s}+\varepsilon^2|\tilde{\xi}|^2\|z\|^2_{a,s}+\varepsilon^2|\tilde{\xi}|^{\frac{3}{2}}\|z\|^3_{a,s})\end{aligned}$$

Through scaling variables

$$\tilde{\xi}\to\varepsilon^3\tilde{\xi},\quad J\to\varepsilon^5J,\quad I\to\varepsilon^5I,\quad \vartheta\to\varepsilon^4\vartheta,\quad \theta\to\theta,\quad z\to\varepsilon^{\frac{5}{2}}z,\quad \bar{z}\to\varepsilon^{\frac{5}{2}}\bar{z},$$

and scaling time $t\to\varepsilon^9t$, the rescaled Hamiltonian can be obtained

$$H=\varepsilon^{-9}\widehat{H}(\varepsilon^3\tilde{\xi};\varepsilon^9J,\varepsilon^5I,\vartheta,\theta,\varepsilon^{\frac{5}{2}}z,\varepsilon^{\frac{5}{2}}\bar{z}).$$

Then H satisfies the equation (4.8)–(4.13). □

Now let's give the estimates of the perturbation P. For this purpose, we need to introduce the notations which are taken from [13]. Let $l^{a,s}$ is now the Hilbert space of all complex sequence $w=(\ldots,w_j,\ldots)_{j\in\mathbb{Z}^2_*}$ with

$$\|w\|_{a,s}=\sum_{j\in\mathbb{Z}^2_*}|w_j|e^{a|j|}\cdot|j|^s<\infty,\quad a>0,s>0.$$

Let $x = \vartheta \oplus \theta$ with $\theta = (\theta_j)_{j\in S}$, $y = J \oplus I$, $z = (z_j)_{j\in\mathbb{Z}^2_*}$ and $\zeta = \omega \oplus (\tilde{\xi}_j)_{j\in S}$, and let's introduce the phase space

$$\mathcal{P}^{a,s} = \widehat{\mathbb{T}}^{m+n} \times \mathbb{C}^{m+n} \times l^{a,s} \times l^{a,s} \ni (x, y, z, \bar{z})$$

where $\widehat{\mathbb{T}}^{m+n}$ is the complexiation of the usual $(m + n)$-torus $\mathbb{T}^{m+n}$. Let

$$D_{a,s}(s', r) := \{(x, y, z, \bar{z}) \in \mathcal{P}^{a,s} : |\mathrm{Im}x| < s', |y| < r^2, \|z\|_{a,s} + \|\bar{z}\|_{a,s} < r\},$$

and

$$|W|_r = |x| + \frac{1}{r^2}|y| + \frac{1}{r}\|z\|_{a,s} + \frac{1}{r}\|\bar{z}\|_{a,s}$$

for $W = (x, y, z, \bar{z}) \in \mathcal{P}^{a,s}$. Set $\alpha \equiv (\ldots, \alpha_j, \ldots)_{j\in\mathbb{Z}^2_*}$, $\beta \equiv (\ldots, \beta_j, \ldots)_{j\in\mathbb{Z}^2_*}$, α_j and $\beta_j \in \mathbb{N}$ with finitely many nonzero components of positive integers. The product $z^\alpha \bar{z}^\beta$ denotes $\prod_j z_j^{\alpha_j} \bar{z}_j^{\beta_j}$. Let

$$P(x, y, z, \bar{z}) = \sum_{\alpha,\beta} P_{\alpha\beta}(x, y) z^\alpha \bar{z}^\beta,$$

where $P_{\alpha\beta} = \sum_{k,b} P_{kb\alpha\beta} y^b e^{\mathrm{i}<k,x>}$ are C^4_W functions in parameter ζ in the sense of Whitney. Let

$$\|P\|_{D_{a,s}(s',r),\underline{\Sigma}} \equiv \sup_{\|z\|_{a,s}<r, \|\bar{z}\|_{a,s}<r} \sum_{\alpha,\beta} \|P_{\alpha\beta}\| |z^\alpha| |\bar{z}^\beta|,$$

where, if $P_{\alpha,\beta} = \sum_{k\in\mathbb{Z}^{m+n}, b\in\mathbb{N}^{m+n}} P_{kb\alpha\beta}(\zeta) y^b e^{\mathrm{i}<k,x>}$, $P_{\alpha\beta}$ is short for

$$\|P_{\alpha\beta}\| \equiv \sum_{k,b} |P_{kb\alpha\beta}|_{\underline{\Sigma}} r^{2|b|} e^{|k|s'}, \quad |P_{kb\alpha\beta}|_{\underline{\Sigma}} \equiv \sup_{\zeta\in\underline{\Sigma}} \sum_{0\le s\le 4} |\partial^s_\zeta P_{kb\alpha\beta}|$$

the derivatives with respect to ζ are in the sense of Whitney. Denote by X_P the vector field corresponding the Hamiltonian P with respect to the symplectic structure $dx \wedge dy + \mathrm{i}dz \wedge d\bar{z}$, namely,

$$X_P = (\partial_y P, -\partial_x P, \mathrm{i}\nabla_{\bar{z}} P, -\mathrm{i}\nabla_z P).$$

Its weighted norm is defined by

$$\begin{aligned} \|X_P\|_{D_{a,s}(s',r),\underline{\Sigma}} \equiv{}& \|P_y\|_{D_{a,s}(s',r),\underline{\Sigma}} + \frac{1}{r^2}\|P_x\|_{D_{a,s}(s',r),\underline{\Sigma}} \\ &+ \frac{1}{r}\Big(\sum_{j\in\mathbb{Z}^2_*} \|P_{z_j}\|_{D_{a,s}(s',r),\underline{\Sigma}} e^{|j|a} + \sum_{j\in\mathbb{Z}^2_*} \|P_{\bar{z}_j}\|_{D_{a,s}(s',r),\underline{\Sigma}} e^{|j|a}\Big). \end{aligned}$$

The following Lemma can be obtained and the proof is similar to Lemma 3.2 in [27].

Lemma 4.2. For given $s', r > 0$, the perturbation $P(x, y, z, \bar{z}; \zeta)$ is real analytic for $(x, y, z, \bar{z}) \in D_{a,s}(s', r)$ and Lipschitz in the parameters $\zeta \in \underline{\Sigma}$, and for any $\zeta \in \underline{\Sigma}$, its gradients with respect to $z, \bar{z}$ satisfy

$$\partial_z P, \quad \partial_{\bar{z}} P \in \mathcal{A}(l^{a,s}, l^{a,s}),$$

and

$$\|X_P\|_{D_{a,s+1}(s',r),\underline{\Sigma}} \le C\varepsilon,$$

where $s' = \sigma_0/3$ and $r = \sqrt{\varepsilon}$.

5. An infinite-dimensional KAM theorem for partial differential equations

In order to prove our main result (Theorem 1.1), we need to state a KAM theorem which was proved by Geng-Zhou [13]. Here we recite the theorem from [13].

Let us consider the perturbations of a family of Hamiltonian

$$H_{00} = N + \mathcal{A} + \mathcal{B} + \bar{\mathcal{B}},$$

where

$$\begin{aligned} N &= \sum_{j\in S} \widehat{\omega}_j(\xi) y_j + \sum_{j\in\mathbb{Z}_*^2} \widehat{\Omega}_j(\xi) z_j \bar{z}_j \\ \mathcal{A} &= \sum_{d\in\mathcal{L}_1} a_d(\xi) e^{\mathrm{i}(x_i - x_j)} z_d \bar{z}_l \\ \mathcal{B} &= \sum_{d\in\mathcal{L}_2} a_d(\xi) e^{-\mathrm{i}(x_i + x_j)} z_d z_l \\ \bar{\mathcal{B}} &= \sum_{d\in\mathcal{L}_2} \bar{a}_d(\xi) e^{\mathrm{i}(x_i + x_j)} \bar{z}_d \bar{z}_l. \end{aligned}$$

in n-dimensional angle-action coordinates (x, y) and infinite-dimensional coordinates $(z, \bar{z})$ with symplectic structure

$$\sum_{j\in S} dx_j \wedge dy_j + \mathrm{i} \sum_{j\in\mathbb{Z}_*^2} dz_j \wedge d\bar{z}_j.$$

The tangent frequencies $\widehat{\omega} = (\widehat{\omega}_j)_{j\in S}$ and normal ones $\widehat{\Omega} = (\widehat{\Omega}_j)_{j\in\mathbb{Z}_*^2}$ depend on n parameters

$$\xi \in \Pi \subset \mathbb{R}^n,$$

with Π a closed bounded set of positive Lebesgue measure.

For each ξ there is an invariant n-torus $\mathcal{T}_0^n = \mathbb{T}^n \times \{0, 0, 0\}$ with frequencies $\widehat{\omega}(\xi)$. The aim is to prove the persistence of a large portion of this family of rotational torus under small perturbations $H = H_{00} + P$ of H_{00}. To this end the following assumptions are made.

Assumption A1. (Non-degeneracy): The map $\xi \mapsto \widehat{\omega}(\xi)$ is a C_W^4 diffeomorphism between Π and its image.

Assumption A2. (Asymptotics of normal frequencies):

$$\widehat{\Omega}_j = \varepsilon^{-\varsigma} |j|^2 + \widetilde{\Omega}_j, \quad \varsigma > 0$$

where $\widetilde{\Omega}_j$ is a C_W^4 functions of ξ and $\widetilde{\Omega}_j = O(|j|^{-\iota}), \iota > 0$.

Assumption A3. (Melnikov conditions): Let $B_d = \widehat{\Omega}_d$ for $d \in \mathbb{Z}_*^2 \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$, and let

$$B_d = \begin{pmatrix} \widehat{\Omega}_d + \widehat{\omega}_i & a_d \\ a_l & \widehat{\Omega}_l + \widehat{\omega}_j \end{pmatrix}, \quad d \in \mathcal{L}_1$$

$$B_d = \begin{pmatrix} \widehat{\Omega}_d - \widehat{\omega}_i & a_d \\ \bar{a}_l & \widehat{\Omega}_l - \widehat{\omega}_j \end{pmatrix}, \quad d \in \mathcal{L}_2$$

there exist $\gamma', \tau > 0$ (here I_2 is 2×2 identity matrix) such that

$$|<k,\widehat{\omega}>| \geq \frac{\gamma'}{|k|^\tau}, \quad k \neq 0,$$

$$|\det(<k,\widehat{\omega}> I + B_d)| \geq \frac{\gamma'}{|k|^\tau},$$

$$|\det(<k,\widehat{\omega}> I \pm B_d \otimes I_2 \pm I_2 \otimes B_{d'})| \geq \frac{\gamma'}{|k|^\tau}, \quad k \neq 0,$$

where I means the identity matrix.

Assumption A4. (Regularity): $\mathcal{A}+\mathcal{B}+\bar{\mathcal{B}}+P$ is real analytic in $x, y, z, \bar{z}$ and Whitney smooth in ξ; and we have

$$\|X_{\mathcal{A}}\|_{D_{a,s}(s',r),\Pi} + \|X_{\mathcal{B}}\|_{D_{a,s}(s',r),\Pi} + \|X_{\bar{\mathcal{B}}}\|_{D_{a,s}(s',r),\Pi} < 1, \quad \|X_P\|_{D_{a,s}(s',r),\Pi} < \varepsilon.$$

Assumption A5. (Zero-momentum condition): The normal form part $\mathcal{A}+\mathcal{B}+\bar{\mathcal{B}}+P$ satisfies the following condition

$$\mathcal{A}+\mathcal{B}+\bar{\mathcal{B}}+P = \sum_{k\in\mathbb{Z}^n, b\in\mathbb{N}^n, \alpha,\beta} (\mathcal{A}+\mathcal{B}+\bar{\mathcal{B}}+P)_{kb\alpha\beta}(\xi) y^b e^{\mathrm{i}<k,x>} z^\alpha \bar{z}^\beta$$

and we have

$$(\mathcal{A}+\mathcal{B}+\bar{\mathcal{B}}+P)_{kb\alpha\beta} \neq 0 \Rightarrow \sum_{\hat{s}=1}^{n} k_{\hat{s}} i_{\hat{s}} + \sum_{d\in\mathbb{Z}^2_*} (\alpha_d - \beta_d) d = 0.$$

Now we state the basic KAM theorem which is attributed to Geng-Zhou [13], and as a corollary, we get Theorem 1.1.

Theorem 5.1.([13] Theorem 2) Assume that the Hamiltonian $H = N + \mathcal{A}+\mathcal{B}+\bar{\mathcal{B}}+P$ satisfies (**A1**) − (**A5**). Let $\gamma' > 0$ be sufficiently small, then there exists $\varepsilon > 0$ and $a, s > 0$ such that if $\|X_P\|_{D_{a,s}(s',r),\Pi} < \varepsilon$, the following holds: there exists a Cantor subset $\Pi_{\gamma'} \subset \Pi$ with $\mathrm{meas}(\Pi \setminus \Pi_{\gamma'}) = O(\gamma'^{\varsigma})$ (ς is a positive constant) and two maps which are analytic in x and C^4_W in ξ,

$$\Psi : \mathbb{T}^n \times \Pi_{\gamma'} \to D_{a,s}(s', r), \quad \tilde{\omega} : \Pi_{\gamma'} \to \mathbb{R}^n,$$

where Ψ is $\frac{\varepsilon}{(\gamma')^{16}}$-close to the trivial embedding $\Psi_0 : \mathbb{T}^n \times \Pi \to \mathbb{T}^n \times \{0,0,0\}$ and $\tilde{\omega}$ is ε-close to the unperturbed frequency $\widehat{\omega}$, such that for any $\xi \in \Pi_{\gamma'}$ and $x \in \mathbb{T}^n$, the curve $t \to \Psi(x + \tilde{\omega}(\xi)t, \xi)$ is a quasi-periodic solution of the Hamiltonian equations governed by $H = N + \mathcal{A}+\mathcal{B}+\bar{\mathcal{B}}+P$.

In order to apply the above theorem to our problem, we need to introduce a new parameter $\bar{\omega}$ below. Given $\omega_- \in R$, for $\omega \in \bar{\bar{R}} := \{\omega \in R \mid |\omega - \omega_-| \leq \varepsilon\}$, we introduce new parameter $\bar{\omega}$ by

$$\omega = \omega_- + \varepsilon\bar{\omega}, \quad \bar{\omega} \in [0,1]^m. \tag{5.1}$$

Then the Hamiltonian (4.8) is changed into

$$H = <\widehat{\omega}(\xi), \hat{y}> + <\widehat{\Omega}(\xi), \hat{z}> + \mathcal{A}+\mathcal{B}+\bar{\mathcal{B}}+P \tag{5.2}$$

where $\widehat{\omega}(\xi) = (\varepsilon^{-4}\omega) \oplus \breve{\omega}, \xi = \bar{\omega} \oplus \tilde{\xi}, \hat{z} = (|z_j|^2)_{j\in\mathbb{Z}_*^2}, \hat{x} = \vartheta \oplus \theta, \hat{y} = J \oplus I$ with

$$\breve{\omega}_i = \varepsilon^{-4}\mu_i + \frac{3}{16\pi^2}\frac{1}{\lambda_i^2}[\tilde{G}^{2,2}_{iiii}]\tilde{\xi}_i + \frac{3}{8\pi^2}\sum_{j\in S}\frac{1}{\lambda_i\lambda_j}[\tilde{G}^{2,2}_{iijj}]\tilde{\xi}_j, \quad i \in S, \tag{5.3}$$

$$\widehat{\Omega}_d = \varepsilon^{-4}\mu_d + \frac{3}{8\pi^2}\sum_{j\in S}\frac{1}{\lambda_j\lambda_d}[\tilde{G}^{2,2}_{jjdd}]\tilde{\xi}_j, \quad d \in \mathbb{Z}_*^2. \tag{5.4}$$

Denote $\breve{\omega}(\xi) = \varepsilon^{-4}\tilde{\alpha} + A\tilde{\xi}$, $\widehat{\Omega}(\xi) = \varepsilon^{-4}\tilde{\beta} + B\tilde{\xi}$, where

$$\begin{aligned} \tilde{\alpha} &= (\ldots, \mu_i, \ldots)_{i\in S}, \quad \tilde{\beta} = (\ldots, \mu_d, \ldots)_{d\in\mathbb{Z}_*^2}, \\ A &= (\tilde{G}_{ij})_{i,j\in S}, \quad B = (\tilde{G}_{ij})_{i\in\mathbb{Z}_*^2, j\in S}, \end{aligned} \tag{5.5}$$

with

$$\tilde{G}_{ij} = \frac{3\cdot(2-\delta_{ij})}{16\pi^2\lambda_i\lambda_j}[\tilde{G}^{2,2}_{iijj}], \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{5.6}$$

Lemma 5.1. Let $\Pi = [0,1]^{m+n}$, for any $\varepsilon > 0$ sufficiently small, $r = \sqrt{\varepsilon}$, then we have

$$\|X_P\|_{D_{a,s+1}(s',r)\times\Pi} \leq C\varepsilon.$$

The proof of the above lemma is the same as one of Lemma 4.2.

6. Proof of main theorem

In this section, we prove that the Hamiltonian (5.2) satisfies the assumptions (**A1**) – (**A5**). In view of (5.5),(5.6),(2.10) and (3.45),

$$\lim_{\varepsilon\to 0} A = \frac{3[\phi]}{16\pi^2}\begin{pmatrix} \frac{1}{\lambda_1^2} & \frac{2}{\lambda_1\lambda_2} & \cdots & \frac{2}{\lambda_1\lambda_n} \\ \frac{2}{\lambda_2\lambda_1} & \frac{1}{\lambda_2^2} & \cdots & \frac{2}{\lambda_2\lambda_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{2}{\lambda_n\lambda_1} & \frac{2}{\lambda_n\lambda_2} & \cdots & \frac{1}{\lambda_n^2} \end{pmatrix}_{n\times n} := \widetilde{A} =: [\phi]\widehat{A},$$

Verifying (A1) : From (5.3),

$$\frac{\partial\widehat{\omega}}{\partial\xi} = \begin{pmatrix} \varepsilon^{-3}I_m & 0 \\ \varepsilon^{-3}\cdot\frac{\partial\tilde{\alpha}}{\partial\omega} + \varepsilon\cdot\frac{\partial(A\tilde{\xi})}{\partial\omega} & A \end{pmatrix}, \quad \text{for} \quad \xi \in \Pi,$$

where I_m denotes the unit $m\times m$-matrix. It is obvious that $\det\widetilde{A} \neq 0$. So $\det A \neq 0$ can be obtained by assuming $0 < \varepsilon \ll 1$. Thus assumption (A1) is verified.

Verifying (A2) : Take $\varsigma = 4, \iota = 4$, the proof is obvious.

Verifying (A3) : For (5.2), B_d is defined as follows,

$$B_d = \widehat{\Omega}_d \quad d \in \mathbb{Z}^2_* \setminus (\mathcal{L}_1 \cup \mathcal{L}_2),$$

and

$$B_d = \begin{pmatrix} \widehat{\Omega}_d + \breve{\omega}_i & \dfrac{3[\tilde{G}^{2,2}_{ijdl}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}}{8\pi^2\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}} \\ \dfrac{3[\tilde{G}^{2,2}_{ijdl}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}}{8\pi^2\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}} & \widehat{\Omega}_l + \breve{\omega}_j \end{pmatrix}, \quad d \in \mathcal{L}_1$$

$$B_d = \begin{pmatrix} \widehat{\Omega}_d - \breve{\omega}_i & \dfrac{3[\tilde{G}^{2,2}_{dilj}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}}{8\pi^2\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}} \\ \dfrac{3[\tilde{G}^{2,2}_{idjl}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}}{8\pi^2\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}} & \widehat{\Omega}_l - \breve{\omega}_j \end{pmatrix}, \quad d \in \mathcal{L}_2$$

where (i, j, l) is uniquely determined by d. In the following, we only prove (A3) for $\det[< k, \widehat{\omega}(\xi) > I \pm B_d \otimes I_2 \pm I_2 \otimes B_{d'}]$ which is the most complicated case. For $k \in \mathbb{Z}^{m+n}, b \in \mathbb{N}^{m+n}$, denote

$$k = (k_1, k_2), \quad b = (b_1, b_2), \quad k_1 \in \mathbb{Z}^m, k_2 \in \mathbb{Z}^n, \quad b_1 \in \mathbb{N}^m, b_2 \in \mathbb{N}^n.$$

Let

$$\begin{aligned} \mathcal{Z}(\xi) &= < k, \widehat{\omega}(\xi) > I \pm B_d \otimes I_2 \pm I_2 \otimes B_{d'} \\ &= (\varepsilon^{-4} < k_1, \omega > + \varepsilon^{-4} < k_2, \tilde{\alpha} > + < k_2, A\tilde{\xi} >) I \pm B_d \otimes I_2 \pm I_2 \otimes B_{d'}. \end{aligned}$$

We need to prove that $|\mathcal{Z}(\xi)| \geq \dfrac{\gamma'}{|k|^\tau}, (k \neq 0)$. For this purpose, we need to divide into the following two cases.

Case 1. When $k_1 \neq 0$, notice that

$$\frac{\partial\Big((\varepsilon^{-4} < k_2, \tilde{\alpha} > + < k_2, A\tilde{\xi} >) I \pm B_d \otimes I_2 \pm I_2 \otimes B_{d'}\Big)}{\partial \bar{\omega}} = \varepsilon^{-3} \cdot O(\varepsilon^{1+\rho}),$$

and from

$$\frac{\partial < k_1, \varepsilon^{-4}\omega >}{\partial \bar{\omega}} + \varepsilon^{-3} \cdot O(\varepsilon^{1+\rho}) = \varepsilon^{-3}(k_1 + O(\varepsilon^{1+\rho})) \neq 0, \quad 0 < \varepsilon \ll 1$$

then all the eigenvalues of $\mathcal{Z}(\xi)$ are not identically zero.

Case 2. When $k_1 = 0$, then

$$\begin{aligned} \mathcal{Z}(\xi) &= (\varepsilon^{-4} < k_1, \omega > + \varepsilon^{-4} < k_2, \tilde{\alpha} > + < k_2, A\tilde{\xi} >) I \pm B_d \otimes I_2 \pm I_2 \otimes B_{d'} \\ &= (\varepsilon^{-4} < k_2, \tilde{\alpha} > + < k_2, A\tilde{\xi} >) I \pm B_d \otimes I_2 \pm I_2 \otimes B_{d'}, \end{aligned}$$

We assert that all the eigenvalues of $\mathcal{Z}(\xi)$ are not identically zero. Here we're just proving it for $d, d' \in \mathcal{L}_1$, and everything else is similar. Let

$$B_d = \varepsilon^{-4} B_d^1 + B_d^2, \quad \forall d \in \mathcal{L}_1$$

where

$$B_d^1 = \begin{pmatrix} \mu_d + \mu_i & 0 \\ 0 & \mu_l + \mu_j \end{pmatrix},$$

$$B_d^2 = \begin{pmatrix} -\frac{3[\tilde{G}^{2,2}_{iiii}]\tilde{\xi}_i}{16\pi^2\lambda_i^2} + \frac{3\sum\limits_{\kappa\in S}(\frac{[\tilde{G}^{2,2}_{\kappa\kappa ii}]}{\lambda_i\lambda_\kappa} + \frac{[\tilde{G}^{2,2}_{\kappa\kappa dd}]}{\lambda_\kappa\lambda_d})\tilde{\xi}_\kappa}{8\pi^2} & \frac{3[\tilde{G}^{2,2}_{ijdl}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}}{8\pi^2\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}} \\ \frac{3[\tilde{G}^{2,2}_{ijdl}]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}}{8\pi^2\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}} & -\frac{3[\tilde{G}^{2,2}_{jjjj}]\tilde{\xi}_j}{16\pi^2\lambda_j^2} + \frac{3\sum\limits_{\kappa\in S}(\frac{[\tilde{G}^{2,2}_{\kappa\kappa jj}]}{\lambda_\kappa\lambda_j} + \frac{[\tilde{G}^{2,2}_{\kappa\kappa ll}]}{\lambda_\kappa\lambda_l})\tilde{\xi}_\kappa}{8\pi^2} \end{pmatrix}.$$

Then

$$\mathcal{Z}(\xi) = \varepsilon^{-4}(< k_2, \tilde{\alpha} > I \pm B_d^1 \otimes I_2 \pm I_2 \otimes B_{d'}^1) + (< k_2, A\tilde{\xi} > I \pm B_d^2 \otimes I_2 \pm I_2 \otimes B_{d'}^2).$$

In view of $|i|^2 + |d|^2 = |j|^2 + |l|^2$ and (2.10),(3.45),

$$\lim_{\varepsilon\to 0} B_d^1 = \begin{pmatrix} |i|^2 + |d|^2 & 0 \\ 0 & |i|^2 + |d|^2 \end{pmatrix} := \widehat{B_d^1},$$

$$\begin{aligned}\lim_{\varepsilon\to 0} B_d^2 &= \begin{pmatrix} -\frac{3[\phi]\tilde{\xi}_i}{16\pi^2\lambda_i^2} + \frac{3[\phi]\sum\limits_{\kappa\in S}(\frac{1}{\lambda_\kappa\lambda_i} + \frac{1}{\lambda_\kappa\lambda_d})\tilde{\xi}_\kappa}{8\pi^2} & \frac{3[\phi]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}}{8\pi^2\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}} \\ \frac{3[\phi]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}}{8\pi^2\sqrt{\lambda_i\lambda_j\lambda_d\lambda_l}} & -\frac{3[\phi]\tilde{\xi}_j}{16\pi^2\lambda_j^2} + \frac{3[\phi]\sum\limits_{\kappa\in S}(\frac{1}{\lambda_\kappa\lambda_j} + \frac{1}{\lambda_\kappa\lambda_l})\tilde{\xi}_\kappa}{8\pi^2} \end{pmatrix} \\ &:= \widetilde{B_d^2} := [\phi]\widehat{B_d^2},\end{aligned}$$

Thus,

$$\begin{aligned}&\lim_{\varepsilon\to 0} \mathcal{Z}(\xi) \\ &= \varepsilon^{-4}(< k_2, \hat{\alpha} > I \pm \widehat{B_d^1} \otimes I_2 \pm I_2 \otimes \widehat{B_{d'}^1}) + [\phi](< k_2, \widehat{A}\tilde{\xi} > I \pm \widehat{B_d^2} \otimes I_2 \pm I_2 \otimes \widehat{B_{d'}^2}) \\ &= \varepsilon^{-4}\left(< k_2, \hat{\alpha} > \pm(|i|^2 + |d|^2) \pm (|i'|^2 + |d'|^2)\right) I \\ &\quad + [\phi] < \widehat{A}k_2 \pm (\tfrac{1}{\lambda_i} + \tfrac{1}{\lambda_d})\hat{\beta} \pm (\tfrac{1}{\lambda_j} + \tfrac{1}{\lambda_l})\hat{\beta}, \tilde{\xi} > I \\ &\quad \pm \begin{pmatrix} \frac{-3[\phi]\tilde{\xi}_i}{16\pi^2\lambda_i^2} & \frac{3[\phi]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}}{8\pi^2\lambda_i\lambda_j} \\ \frac{3[\phi]\sqrt{\tilde{\xi}_i\tilde{\xi}_j}}{8\pi^2\lambda_i\lambda_j} & -\frac{3[\phi]\tilde{\xi}_j}{16\pi^2\lambda_j^2} \end{pmatrix} \otimes I_2 \pm I_2 \otimes \begin{pmatrix} -\frac{3[\phi]\tilde{\xi}_{i'}}{16\pi^2\lambda_{i'}^2} & \frac{3[\phi]\sqrt{\tilde{\xi}_{i'}\tilde{\xi}_{j'}}}{8\pi^2\lambda_{i'}\lambda_{j'}} \\ \frac{3[\phi]\sqrt{\tilde{\xi}_{i'}\tilde{\xi}_{j'}}}{8\pi^2\lambda_{i'}\lambda_{j'}} & -\frac{3[\phi]\tilde{\xi}_{j'}}{16\pi^2\lambda_{j'}^2} \end{pmatrix} := \widehat{\mathcal{Z}}(\xi)\end{aligned}$$

with $\hat{\alpha} = (\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n})$, $\hat{\beta} = (\frac{3}{8\pi^2\lambda_{i_1}}, \frac{3}{8\pi^2\lambda_{i_2}}, \ldots, \frac{3}{8\pi^2\lambda_{i_n}})$ and $\tilde{\xi} = (\tilde{\xi}_{i_1}, \tilde{\xi}_{i_2}, \ldots, \tilde{\xi}_{i_n})$. The eigenvalues of $\widehat{\mathcal{Z}}(\xi)$ are

$$\varepsilon^{-4}(< k_2, \hat{\alpha} > \pm(|i|^2 + |d|^2) \pm (|i'|^2 + |d'|^2)) + [\phi] < \widehat{A}k_2 \pm (\frac{1}{\lambda_i} + \frac{1}{\lambda_d})\hat{\beta} \pm (\frac{1}{\lambda_j} + \frac{1}{\lambda_l})\hat{\beta}, \tilde{\xi} >$$
$$\pm\frac{3[\phi]}{32\pi^2}[(- \frac{\tilde{\xi}_i}{\lambda_i^2} - \frac{\tilde{\xi}_j}{\lambda_j^2} \pm \sqrt{\frac{\tilde{\xi}_i^2}{\lambda_i^4} + 14\frac{\tilde{\xi}_i\tilde{\xi}_j}{\lambda_i^2\lambda_j^2} + \frac{\tilde{\xi}_j^2}{\lambda_j^4}}) \pm (- \frac{\tilde{\xi}_{i'}}{\lambda_{i'}^2} - \frac{\tilde{\xi}_{j'}}{\lambda_{j'}^2} \pm \sqrt{\frac{\tilde{\xi}_{i'}^2}{\lambda_{i'}^4} + 14\frac{\tilde{\xi}_{i'}\tilde{\xi}_{j'}}{\lambda_{i'}^2\lambda_{j'}^2} + \frac{\tilde{\xi}_{j'}^2}{\lambda_{j'}^4}})].$$

Similar to [10], we know that all the eigenvalues are not identically zero. Thus all the eigenvalues of $\mathcal{Z}(\xi)$ are not identically zero as $0 < \varepsilon \ll 1$. Moreover, they are similar to $d \in \mathcal{L}_1, d' \in \mathcal{L}_2$ or $d \in \mathcal{L}_2, d' \in \mathcal{L}_2$, and omit them here.

Hence all eigenvalues of $\mathcal{Z}(\xi)$ are not identically zero for $k \neq 0$. According to Lemma 3.1 in [10], $\det(\mathcal{Z}(\xi))$ is polynomial function in ξ of order at most four. Thus

$$\left|\partial_\xi^4(\det(\mathcal{Z}(\xi)))\right| \geq \frac{1}{2}|k| \neq 0.$$

By excluding some parameter set with measure $O(\sqrt[4]{\gamma'})$, we get

$$|\det(\mathcal{Z}(\xi))| \geq \frac{\gamma'}{|k|^\tau}, \quad k \neq 0.$$

(A3) is verified.

Verifying (A4) : Assumption (A4) can be verified easily fulfilled by Lemma 5.1.

Verifying (A5) : The proof is similar to [27].

By applying Theorem 5.1([13] Theorem 2), we get Theorem 1.1.

7. Appendix

Proof of Lemma 4.1. Case 1. Similar to Lemma 3.1 in [27], there exists a set $R^{3,1}$ so that $\forall\omega \in [\varrho, 2\varrho]^m \setminus R^{3,1}$, Lemma 4.1(i) is true, and meas$R^{3,1} \leq \frac{\gamma}{9}\varrho^m$. We omit the proof.

Case 2. Assume $i + j + d + l = 0, |i|^2 + |j|^2 + |d|^2 + |l|^2 \neq 0$ and $\#(S \cap \{i, j, d, l\}) \geq 2$. First of all, we have $\left||i|^2 + |j|^2 + |d|^2 + |l|^2\right| \geq 1$. Denote $f(\varepsilon) = \mu_i + \mu_j + \mu_d + \mu_l$, then by $\mu_j = \lambda_j + \frac{\varepsilon}{2\lambda_j}[\phi] + \frac{1}{\lambda_j}\varepsilon^{(1+\rho)}\mu_j^*$ we have

$$f(\varepsilon) = |i|^2 + |j|^2 + |d|^2 + |l|^2 + \varepsilon[\phi](\frac{1}{2\lambda_i} + \frac{1}{2\lambda_j} + \frac{1}{2\lambda_d} + \frac{1}{2\lambda_l}) + \varepsilon^{(1+\rho)}(\frac{\mu_i^*}{\lambda_i} + \frac{\mu_j^*}{\lambda_j} + \frac{\mu_d^*}{\lambda_d} + \frac{\mu_l^*}{\lambda_l}).$$

Case 1.1. For $k = 0$, then

$$|f(\varepsilon)+ < k, \omega > | = |f(\varepsilon)| \geq 1 - C\varepsilon \geq \frac{\varrho}{C_*}$$

when ε small enough and C_* large enough.

Case 1.2. For $k \neq 0$, denote

$$\mathcal{I}_{ijdl,k}^{3,2} = \left\{\omega \in [\varrho, 2\varrho]^m : |f(\varepsilon)+ < k, \omega > | < \frac{\varrho}{C_*|k|^{m+1}}\right\},$$

and

$$R^{3,2}=\bigcup_{0\neq k\in\mathbb{Z}^m}\bigcup_{i,j,d,l}\mathcal{I}^{3,2}_{ijdl,k}.$$

Case 1.2.1. When $\#(S\cap\{i,j,d,l\})=4$. Denote

$$\mathcal{I}^{3,2,1}_{ijdl,k}=\left\{\omega\in[\varrho,2\varrho]^m:|f(\varepsilon)+<k,\omega>|<\frac{\varrho}{C_*|k|^m}\right\},$$

$$R^{3,2,1}=\bigcup_{0\neq k\in\mathbb{Z}^m}\bigcup_{i\in S,j\in S,d\in S,l\in S}\mathcal{I}^{3,2,1}_{ijdl,k},$$

we have

$$\text{meas}\mathcal{I}^{3,2,1}_{ijdl,k}\leq\frac{2\varrho^m}{C_*|k|^{m+1}}.$$

Let

$$|k|_\infty=\max\{|k_1|,|k_2|,\ldots,|k_m|\},$$

in view of

$$\sum_{|k|_\infty=p}1\leq 2m(2p+1)^{m-1},$$

$$|k|_\infty\leq|k|\leq m|k|_\infty,$$

we have

$$\begin{aligned}\text{meas}R^{3,2,1} &= \text{meas}\bigcup_{0\neq k\in\mathbb{Z}^m}\bigcup_{i\in S,j\in S,d\in S,l\in S}\mathcal{I}^{3,2,1}_{ijdl,k}\leq\sum_{0\neq k\in\mathbb{Z}^m}n^4\frac{2\varrho^m}{C_*|k|^{m+1}}\\ &\leq \frac{C_1''}{C_*}\varrho^m\sum_{0\neq k\in\mathbb{Z}^m}\frac{1}{|k|^{m+1}}\leq\frac{C_1'}{C_*}\varrho^m\sum_{p=1}^{\infty}(2p+1)^{m-1}p^{-(m+1)}\leq\frac{C_1}{C_*}\varrho^m\end{aligned}$$

where the constant C_1 depends on n,m. Thus

$$\text{meas}R^{3,2,1}\leq\frac{\gamma}{27}\varrho^m$$

provided C_* large enough.

Case 1.2.2. When $\#(S\cap\{i,j,d,l\})=3$. Assume $i,j,d\in S,l\in\mathbb{Z}^2_*$ without loss of generality. Then $l=-i-j-d$ is at most n^3 different values. Denote

$$\mathcal{I}^{3,2,2}_{ijdl,k}=\left\{\omega\in[\varrho,2\varrho]^m:|f(\varepsilon)+<k,\omega>|<\frac{\varrho}{C_*|k|^m}\right\},$$

$$R^{3,2,2}=\bigcup_{0\neq k\in\mathbb{Z}^m}\bigcup_{i\in S,j\in S,d\in S,l=-i-j-d}\mathcal{I}^{3,2,2}_{ijdl,k},$$

then

$$\text{meas}\mathcal{I}^{3,2,2}_{ijdl,k}\leq\frac{2\varrho^m}{C_*|k|^{m+1}}.$$

We obtain

$$\text{meas}R^{3,2,2} = \text{meas} \bigcup_{0\neq k\in\mathbb{Z}^m} \bigcup_{i\in S, j\in S, d\in S, l=-i-j-d} \mathcal{I}^{3,2,2}_{ijdl,k} \leq \sum_{0\neq k\in\mathbb{Z}^m} n^6 \frac{2\varrho^m}{C_*|k|^{m+1}} \leq \frac{C_2}{C_*}\varrho^m$$

where the constant C_2 depends on n, m. Thus

$$\text{meas}R^{3,2,2} \leq \frac{\gamma}{27}\varrho^m$$

provided C_* large enough.

Case 1.2.3. When $\#(S \cap \{i, j, d, l\}) = 2$. Assume $i, j \in S, d, l \in \mathbb{Z}^2_*$ without loss of generality. Then we have $l = -i - j - d$ and

$$\begin{aligned} f(\varepsilon) &= |i|^2 + |j|^2 + |d|^2 + |i + j + d|^2 \\ &\quad +\varepsilon[\phi](\tfrac{1}{2\lambda_i} + \tfrac{1}{2\lambda_j} + \tfrac{1}{2\lambda_d} + \tfrac{1}{2\lambda_l}) + \varepsilon^{(1+\rho)}(\tfrac{\mu^*_i}{\lambda_i} + \tfrac{\mu^*_j}{\lambda_j} + \tfrac{\mu^*_d}{\lambda_d} + \tfrac{\mu^*_l}{\lambda_l}) \\ &= g + \varepsilon[\phi](\tfrac{1}{2\lambda_i} + \tfrac{1}{2\lambda_j} + \tfrac{1}{2\lambda_d} + \tfrac{1}{2\lambda_l}) + \varepsilon^{(1+\rho)}(\tfrac{\mu^*_i}{\lambda_i} + \tfrac{\mu^*_j}{\lambda_j} + \tfrac{\mu^*_d}{\lambda_d} + \tfrac{\mu^*_l}{\lambda_l}) \end{aligned}$$

where $g = |i|^2 + |j|^2 + |d|^2 + |i + j + d|^2 \in \mathbb{Z}^+$. Denote

$$\mathcal{I}^{3,2,3}_{ijdl,k} = \left\{\omega \in [\varrho, 2\varrho]^m : |f(\varepsilon) + < k, \omega > | < \frac{\varrho}{C_*|k|^{m+1}}\right\},$$

$$R^{3,2,3} = \bigcup_{0\neq k\in\mathbb{Z}^m} \bigcup_{i\in S, j\in S, d\in\mathbb{Z}^2_*, l=-i-j-d} \mathcal{I}^{3,2,3}_{ijdl,k}.$$

For given i, j, g, denote

$$d^*_{ijg} = \left\{d \in \mathbb{Z}^2_* : g = |i|^2 + |j|^2 + |d|^2 + |i + j + d|^2\right\}$$

$$\mu^*_{ijg,1} = \sup_{d\in d^*_{ijg}} \left\{\frac{\mu^*_d}{\lambda_d} + \frac{\mu^*_{-i-j-d}}{\lambda_{-i-j-d}}\right\}, \qquad \mu^*_{ijg,2} = \inf_{d\in d^*_{ijg}} \left\{\frac{\mu^*_d}{\lambda_d} + \frac{\mu^*_{-i-j-d}}{\lambda_{-i-j-d}}\right\}$$

$$g^* = g + \varepsilon[\phi](\frac{1}{2\lambda_i} + \frac{1}{2\lambda_j} + \frac{1}{2\lambda_d} + \frac{1}{2\lambda_l})$$

$$\mathcal{I}^{3,2,3,1}_{ijg,k} = \left\{\omega \in [\varrho, 2\varrho]^m : | < k, \omega > +g^* + \varepsilon^{(1+\rho)}(\frac{\mu^*_i}{\lambda_i} + \frac{\mu^*_j}{\lambda_j} + \mu^*_{ijg,1})| < \frac{\varrho}{C_*|k|^{m+1}}\right\},$$

$$\mathcal{I}^{3,2,3,2}_{ijg,k} = \left\{\omega \in [\varrho, 2\varrho]^m : | < k, \omega > +g^* + \varepsilon^{(1+\rho)}(\frac{\mu^*_i}{\lambda_i} + \frac{\mu^*_j}{\lambda_j} + \mu^*_{ijg,2})| < \frac{\varrho}{C_*|k|^{m+1}}\right\},$$

then for $l = -i - j - d, d \in d^*_{ijg}$, from $\varepsilon^{(1+\rho)}(\frac{\mu^*_d}{\lambda_d} + \frac{\mu^*_{-i-j-d}}{\lambda_{-i-j-d}})$ is sufficiently small,

$$\mathcal{I}^{3,2,3}_{ijdl,k} \subset \mathcal{I}^{3,2,3,1}_{ijg,k} \bigcup \mathcal{I}^{3,2,3,2}_{ijg,k}.$$

Thus

$$\bigcup_{l=-i-j-d} \bigcup_{d\in d^*_{ijg}} \mathcal{I}^{3,2,3}_{ijdl,k} \subset (\mathcal{I}^{3,2,3,1}_{ijg,k} \bigcup \mathcal{I}^{3,2,3,2}_{ijg,k}).$$

We get

$$\mathrm{meas}\mathcal{I}^{3,2,3,1}_{ijg,k} \le \frac{2\varrho^m}{C_*|k|^{m+2}}, \quad \mathrm{meas}\mathcal{I}^{3,2,3,2}_{ijg,k} \le \frac{2\varrho^m}{C_*|k|^{m+2}}.$$

When $|g| > |k|\varrho + 4$, the sets $\mathcal{I}^{3,2,3,1}_{ijg,k}, \mathcal{I}^{3,2,3,2}_{ijg,k}$ are empty. So let

$$R^{3,2,3} = \bigcup_{0\neq k\in\mathbb{Z}^m} \bigcup_{i\in S, j\in S} \bigcup_{d\in\mathbb{Z}^2_*} \bigcup_{l=-i-j-d} \mathcal{I}^{3,2,3}_{ijdl,k} \subset \bigcup_{0\neq k\in\mathbb{Z}^m} \bigcup_{i\in S, j\in S} \bigcup_{g\in\mathbb{Z}} (\mathcal{I}^{3,2,3,1}_{ijg,k} \bigcup \mathcal{I}^{3,2,3,2}_{ijg,k}),$$

then

$$\begin{aligned} \mathrm{meas}R^{3,2,3} &\le \mathrm{meas} \bigcup_{0\neq k\in\mathbb{Z}^m} \bigcup_{i\in S, j\in S} \bigcup_{g\in\mathbb{Z}} (\mathcal{I}^{3,2,3,1}_{ijg,k} \bigcup \mathcal{I}^{3,2,3,2}_{ijg,k}) \\ &= \mathrm{meas} \bigcup_{0\neq k\in\mathbb{Z}^m} \bigcup_{i\in S, j\in S} \bigcup_{1\le|g|\le|k|\varrho+4} (\mathcal{I}^{2,3,1}_{ijg,k} \bigcup \mathcal{I}^{2,3,2}_{ijg,k}) \\ &\le \sum_{0\neq k\in\mathbb{Z}^m} 4n^2(|k|\varrho + 4)\frac{2\varrho^m}{C_*|k|^{m+2}} \le \frac{C_3}{C_*}\varrho^m, \end{aligned}$$

where the constant C_3 depends on n, m. Thus

$$\mathrm{meas}R^{3,2,3} \le \frac{\gamma}{27}\varrho^m$$

provided C_* large enough. Denote

$$R^{3,2} = R^{3,2,1} \cup R^{3,2,2} \cup R^{3,2,3},$$

then we have $\mathrm{meas}R^{3,2} \le \frac{\gamma}{9}\varrho^m$.

Case 3. Similar to Case 2, there exists a set $R^{3,3}$ so that $\forall\omega \in [\varrho, 2\varrho]^m \backslash R^{3,3}$, Lemma 4.1(iii) is true, and $\mathrm{meas}R^{3,3} \le \frac{\gamma}{9}\varrho^m$. We omit the proof.

Denote

$$\overline{R} = [\varrho, 2\varrho]^m \setminus \left(R^{3,1} \cup R^{3,2} \cup R^{3,3}\right),$$

then it satisfies as required and

$$\mathrm{meas}\overline{R} \ge (1 - \frac{\gamma}{3})\varrho^m.$$

□

Symbol description

$\mathbb{N}$ is the set of natural Numbers, $\mathbb{Z}$ is the set of integers, $\mathbb{Z}^n$ is an n-dimensional integer space, $\mathbb{R}$ is the set of real Numbers, $\mathbb{R}^n$ is an n-dimensional Euclidean space, $\mathbb{T}^n$ is an n-dimensional torus.

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Conflict of interest

The authors declare that they have no competing interests in this paper.

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