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*Research article*

# On Janowski type $p$-harmonic functions associated with generalized Sălăgean operator

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**Abstract:** In this paper, some classes of Janowski type $p$-harmonic functions associated with the generalized Sălăgean operator are introduced. Further, coefficient conditions, distortion estimates and the other properties of the classes are obtained. On the one hand, the results presented here generalize the results of Yaşar and Yalçin [8]. On the other hand, we obtain some new results on sufficient convolution condition of the classes.

**Keywords:** $p$-harmonic function, Janowski function, Sălăgean operator, subordination, extreme point, convolution
**Mathematics Subject Classification:** 30C45, 30C50, 30C80

## 1. Introduction

The complex-valued function $F = u+iv$ is called $p$-harmonic function if $F$ is $2p$ ($p \geq 1, p \in \mathbb{N}$) times continuously differentiable in $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfies the equation $\Delta^p F = \Delta(\Delta^{p-1} F) = 0$, where $\Delta := \Delta^1$ represents the complex Laplacian operator:

$$\Delta = \frac{4\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It is obvious that $F$ is harmonic and bi-harmonic for $p = 1$ and $p = 2$ respectively (see [1–6]).

The function $F$ is $p$-harmonic if and only if $F$ has the following form

$$F(z) = \sum_{\mu=1}^{p} |z|^{2(\mu-1)} f_{p-\mu+1}(z), \tag{1.1}$$

where $f_{p-\mu+1}(z)$ is harmonic (or $\triangle f_{p-\mu+1} = 0$ ) (see [7]) and satisfies

$$f_{p-\mu+1} = h_{p-\mu+1} + \bar{g}_{p-\mu+1}, \tag{1.2}$$

$$h_{p-\mu+1}(z) = \sum_{k=1}^{\infty} a_{k,p-\mu+1} z^k \ \ (1 \le \mu \le p, a_{1,p} = 1) \tag{1.3}$$

and

$$g_{p-\mu+1}(z) = \sum_{k=1}^{\infty} b_{k,p-\mu+1} z^k \ \ (1 \le \mu \le p, |b_{1,p}| < 1). \tag{1.4}$$

We denote by $J_F$ the Jacobian of $F$, that is

$$J_F = |F_z|^2 - |F_{\bar{z}}|^2.$$

Then the function $F$ is sense-preserving and locally univalent if $J_F > 0$ (see [1,2]).

We denote by $SH_p$ the class of sense-preserving and univalent $p$-harmonic functions $F$ of the form (1.1). The class $SH_p$ has recently raised the interest of many researchers (see for instance [8–14]).

Furthermore, we let $TH_p$ be the subclass of $SH_p$ consisting of the functions $F$ as in (1.1), where $f_{p-\mu+1}$ has the form (1.2) and

$$h_p(z) = z - \sum_{k=2}^{\infty} |a_{k,p}| z^k, \tag{1.5}$$

$$h_{p-\mu+1}(z) = -\sum_{k=1}^{\infty} |a_{k,p-\mu+1}| z^k \ (2 \le \mu \le p), \tag{1.6}$$

$$g_{p-\mu+1}(z) = -\sum_{k=1}^{\infty} |b_{k,p-\mu+1}| z^k \ (1 \le \mu \le p). \tag{1.7}$$

Let

$$F(z) = \sum_{\mu=1}^{p} |z|^{2(\mu-1)} (\sum_{k=1}^{\infty} a_{k,p-\mu+1} z^k + \sum_{k=1}^{\infty} \bar{b}_{k,p-\mu+1} \bar{z}^k) \in SH_p$$

and

$$G(z) = \sum_{\mu=1}^{p} |z|^{2(\mu-1)} (\sum_{k=1}^{\infty} A_{k,p-\mu+1} z^k + \sum_{k=1}^{\infty} \bar{B}_{k,p-\mu+1} \bar{z}^k) \in SH_p,$$

where $a_{1,p} = 1$ and $A_{1,p} = 1$. In the following, we define the convolution of $F$ and $G$ by

$$(F * G)(z) = \sum_{\mu=1}^{p} |z|^{2(\mu-1)} (\sum_{k=1}^{\infty} a_{k,p-\mu+1} A_{k,p-\mu+1} z^k + \sum_{k=1}^{\infty} \bar{b}_{k,p-\mu+1} \bar{B}_{k,p-\mu+1} \bar{z}^k). \tag{1.8}$$

For $F \in SH_p$, Yaşar and Yalçin [8] introduced the generalized Sălăgean operator $D_\lambda^n$ as follows,

$$\begin{aligned} &D_\lambda^0 F(z) = F(z), \\ &D_\lambda^1 F(z) = (1-\lambda) D_\lambda^0 F(z) + \lambda[z(D_\lambda^0 F(z))_z + \bar{z}(D_\lambda^0 F(z))_{\bar{z}}], \\ &D_\lambda^n F(z) = D_\lambda^1 (D_\lambda^{n-1} F(z)), \end{aligned} \tag{1.9}$$

where $\lambda \ge 0$ and $n \in \mathbb{N} = \{1, 2, \cdots\}$.

From (1.1) and (1.9), we have

$$D_\lambda^n F(z) = \sum_{\mu=1}^{p} |z|^{2(\mu-1)} \sum_{k=1}^{\infty} [1+(k-1)\lambda+2(\mu-1)\lambda]^n a_{k,p-\mu+1} z^k,$$
$$+ \sum_{\mu=1}^{p} |z|^{2(\mu-1)} \sum_{k=1}^{\infty} [1+(k-1)\lambda+2(\mu-1)\lambda]^n \bar{b}_{k,p-\mu+1} \bar{z}^k, \tag{1.10}$$

where $a_{1,p} = 1$ and $|b_{1,p}| < 1$.

In particular, for $p = 1$, we obtain univalent harmonic functions with the generalized Sălăgean operator defined by Li and Liu [15]. Let $p = 1$ and $g(z) = 0$, the generalized Sălăgean operator of univalent functions are obtained by Al-Oboudi [16]. Also let $p = 1$, $\lambda = 0$ and $g(z) = 0$, then we get the classical Sălăgean operator [17].

According to (1.10) and the above definition of convolution, we obtain

$$D_\lambda^n F(z) = F(z) * \underbrace{\sum_{\mu=1}^{p} |z|^{2(\mu-1)}(\varphi_\mu(z)+\varphi_\mu(\bar{z})) * \ldots * \sum_{\mu=1}^{p} |z|^{2(\mu-1)}(\varphi_\mu(z)+\varphi_\mu(\bar{z}))}_{n\ times},$$

where

$$\varphi_\mu(z) = \frac{[1+2(\mu-1)\lambda]z - [1+(2(\mu-1)-1)\lambda]z^2}{(1-z)^2}. \tag{1.11}$$

Using the operator $D_\lambda^n$, Yaşar and Yalçin [8] introduced and studied the subclass $SH_p(n,\lambda,\beta)$ of $SH_p$ satisfying the condition

$$\mathrm{Re}\frac{D_\lambda^{n+1}F(z)}{D_\lambda^n F(z)} > \beta \quad (\lambda \geq 0, 0 \leq \beta < 1).$$

An analytic function $s : \mathcal{U} \to \mathbb{C}$ is subordinate to an analytic function $t : \mathcal{U} \to \mathbb{C}$, if there is a function $\nu$ satisfying $\nu(0) = 0$ and $|\nu(z)| < 1$ $(z \in \mathcal{U})$, such that $s(z) = t(\nu(z))(z \in \mathcal{U})$, we note that $s(z) \prec t(z)$. In particular, if $t$ is univalent in $\mathcal{U}$, then the following conclusion is true

$$s(z) \prec t(z) \Longleftrightarrow s(0) = t(0) \text{ and } s(\mathcal{U}) \subset t(\mathcal{U}).$$

Inspired by Janowski [18], we define the subclass of $SH_p$ as below.

**Definition 1.** *Suppose $\lambda \geq 0, -1 \leq B < 0 < A \leq 1, p \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The function $F$ is in $HL_p^n(\lambda, A, B)$ if it satisfies*

$$\frac{D_\lambda^{n+1}F(z)}{D_\lambda^n F(z)} \prec \frac{1+Az}{1+Bz}, \tag{1.12}$$

*where $D_\lambda^n F(z)$ is given by (1.10).*

For $A = 1-2\beta$ $(0 \leq \beta < 1)$, $B = -1$, the class $HL_p^n(\lambda, A, B)$ reduces to the class $SH_p(n,\lambda,\beta)$.

In particular, let

$$\widetilde{HL}_p^n(\lambda, A, B) = HL_p^n(\lambda, A, B) \cap TH_p. \tag{1.13}$$

In this paper, convolution properties, coefficient conditions, distortion estimates, extreme functions and convex combination of the class $\widetilde{HL}_p^n(\lambda, A, B)$ are obtained. On the one hand, the results presented here generalize the results of Yaşar and Yalçin [8]. On the other hand, we obtain some new results on sufficient convolution condition of the class.

## 2. Basic properties

First of all, we provide the necessary and sufficient convolution conditions.

**Theorem 1.** *The function $F \in HL_p^n(\lambda, A, B)$ iff*

$$D_\lambda^n F(z) * \sum_{\mu=1}^{p} |z|^{2(\mu-1)}(\Phi_\mu(z) + \Phi_\mu(\bar{z})) \neq 0 \ \ (z \in \mathcal{U}),$$

*where*

$$\Phi_\mu(z) = (1 + B\chi)\varphi_\mu(z) - (1 + A\chi)\frac{z}{1-z} \quad (\chi \in \mathbb{C}, |\chi| = 1)$$

*and $\varphi_\mu(z)$ given by (1.11).*

*Proof.* Let $F \in HL_p^n(\lambda, A, B)$. According to Definition 1 and the subordination relationship, there exists an analytic function $\omega$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$\frac{D_\lambda^{n+1}F(z)}{D_\lambda^n F(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad (z \in \mathcal{U}),$$

which is equivalent for $\chi \in \mathbb{C}$ with $|\chi| = 1$

$$\frac{D_\lambda^{n+1}F(z)}{D_\lambda^n F(z)} \neq \frac{1 + A\chi}{1 + B\chi}. \tag{2.1}$$

Now for

$$D_\lambda^n F(z) = D_\lambda^n F(z) * \sum_{\mu=1}^{p} |z|^{2(\mu-1)} \left(\frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}\right)$$

and

$$D_\lambda^{n+1} F(z) = D_\lambda^n F(z) * \sum_{\mu=1}^{p} |z|^{2(\mu-1)} \left(\varphi_\mu(z) + \varphi_\mu(\bar{z})\right),$$

where $\varphi_\mu(z)$ is defined by (1.11).

The inequality (2.1) yields

$$\begin{aligned}
&(1 + B\chi)(D_\lambda^{n+1}F(z)) - (1 + A\chi)(D_\lambda^n F(z)) \\
&= D_\lambda^n F(z) * \sum_{\mu=1}^{p} |z|^{2(\mu-1)}[(1 + B\chi)\varphi_\mu(z) - \frac{(1 + A\chi)z}{1-z} + (1 + B\chi)\varphi_\mu(\bar{z}) - \frac{(1 + A\chi)\bar{z}}{1-\bar{z}}] \\
&\neq 0,
\end{aligned}$$

which is the required necessary condition.

The sufficiency of Theorem 1 is proved as follows.

Let

$$M(z) = \frac{D_\lambda^{n+1}F(z)}{D_\lambda^n F(z)} \quad \text{and} \quad N(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U}).$$

It is clear that $M(z)$ is harmonic in $\mathcal{U}$ and $N(z)$ is univalent in $\mathcal{U}$. From (2.1), it is easy to see that $M(\mathcal{U}) \cap N(\partial\mathcal{U}) = \emptyset$, that is, $M(\mathcal{U}) \subset \mathbb{C}\backslash N(\partial\mathcal{U})$. By $M(0) = N(0) = 1$, we have $M(\mathcal{U}) \subset N(\mathcal{U})$. According to the subordination relationship, we obtain

$$M(z) \prec N(z),$$

that is,

$$\frac{D_\lambda^{n+1}F(z)}{D_\lambda^n F(z)} \prec \frac{1+Az}{1+Bz}, \quad (z \in \mathcal{U}).$$

Therefore, we complete the proof of Theorem 1. □

**Theorem 2.** *Let $\lambda \geq 1$ and $a_{1,p} = 1$. For the class of $HL_p^n(\lambda, A, B)$, the sufficient condition on the coefficients of a function $F$ of the class to be sense preserving and univalent in $\mathcal{U}$ is*

$$\sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\phi_{k,\mu}^n(\lambda, A, B)\left(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|\right) \leq 2(A-B), \tag{2.2}$$

*where*

$$\phi_{k,\mu}^n(\lambda, A, B) = \left[(A-B) + (1-B)\left((k-1) + 2(\mu-1)\right)\lambda\right]\left[1 + (k-1)\lambda + 2(\mu-1)\lambda\right]^n. \tag{2.3}$$

*Proof.* In order to prove $F$ is sense preserving. We only need to show

$$J_F = |F_z|^2 - |F_{\bar{z}}|^2 > 0.$$

For $z \neq 0$, we have

$$\begin{aligned}
J_F(z) = {} & (|F_z| + |F_{\bar{z}}|)\Bigg[\Bigg|1 + \sum_{k=2}^{\infty} k a_{k,p} z^{k-1} + \sum_{\mu=2}^{p}\frac{|z|^{2(\mu-1)}}{z}\sum_{k=1}^{\infty}[(k+\mu-1)a_{k,p-\mu+1}z^k + (\mu-1)\bar{b}_{k,p-\mu+1}\bar{z}^k]\Bigg| \\
& - \Bigg|\sum_{k=1}^{\infty} k\bar{b}_{k,p}\bar{z}^{k-1} + \sum_{\mu=2}^{p}\frac{|z|^{2(\mu-1)}}{\bar{z}}\sum_{k=1}^{\infty}[(\mu-1)a_{k,p-k+1}z^k + (k+\mu-1)\bar{b}_{k,p-\mu+1}\bar{z}^k]\Bigg|\Bigg] \\
\geq {} & (|F_z| + |F_{\bar{z}}|)\Bigg[2 - \sum_{\mu=1}^{p}\sum_{k=1}^{\infty}(2(\mu-1)+k)(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|)\Bigg] \\
> {} & (|F_z| + |F_{\bar{z}}|)\Bigg[2 - \sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\frac{\phi_{k,\mu}^n(\lambda, A, B)}{(A-B)}(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|)\Bigg] \\
\geq {} & 0.
\end{aligned}$$

It is easy to show that $J_F(0) > 0$. Thus, $F$ is sense preserving.

For $z_1, z_2 \in \mathcal{U}$ and $z_1 \neq z_2$, according to the condition (2.2), we get

$$|F(z_1) - F(z_2)| = \left|\sum_{\mu=1}^{p}(|z_1|^{2(\mu-1)}f_{p-\mu+1}(z_1) - |z_2|^{2(\mu-1)}f_{p-\mu+1}(z_2))\right|$$

$$\geq |z_1 - z_2|\Big\{1 - \Big|\sum_{k=2}^{\infty} a_{k,p}\frac{z_1^k - z_2^k}{z_1 - z_2} + \sum_{k=1}^{\infty} \bar{b}_{k,p}\frac{\bar{z}_1^k - \bar{z}_2^k}{z_1 - z_2}\Big|$$

$$- \Big|\sum_{\mu=2}^{p}\Big(\sum_{k=1}^{\infty} a_{k,p-\mu+1}\frac{|z_1|^{2(\mu-1)}z_1^k - |z_2|^{2(\mu-1)}z_2^k}{z_1 - z_2} + \sum_{k=1}^{\infty} \bar{b}_{k,p-\mu+1}\frac{|z_1|^{2(\mu-1)}\bar{z}_1^k - |z_2|^{2(\mu-1)}\bar{z}_2^k}{z_1 - z_2}\Big)\Big|\Big\}$$

$$\geq |z_1 - z_2|\Big[1 - |b_{1,p}| - \sum_{k=2}^{\infty} k(|a_{k,p}| + |b_{k,p}|) - \sum_{\mu=2}^{p}\sum_{k=1}^{\infty}(2(\mu-1) + k)(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|)\Big]$$

$$> |z_1 - z_2|\Big[2 - \sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\frac{\phi_{k,\mu}^n(\lambda, A, B)}{(A - B)}(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|)\Big]$$

$$\geq 0.$$

Therefore, $F$ is univalent in $\mathcal{U}$.

According to Definition 1 and the subordination relationship, we have $F \in HL_p^n(\lambda, A, B)$ iff

$$\left|\frac{D_\lambda^{n+1}F(z) - D_\lambda^n F(z)}{AD_\lambda^n F(z) - BD_\lambda^{n+1}F(z)}\right| < 1 \ (z \in \mathcal{U}),$$

that is,

$$|AD_\lambda^n F(z) - BD_\lambda^{n+1}F(z)| - |D_\lambda^{n+1}F(z) - D_\lambda^n F(z)| > 0 \ (z \in \mathcal{U}). \tag{2.4}$$

Thus, from (1.10) and (2.4) we get

$$|AD_\lambda^n F(z) - BD_\lambda^{n+1}F(z)| - |D_\lambda^{n+1}F(z) - D_\lambda^n F(z)|$$

$$\geq (A - B)|z| - \sum_{k=2}^{\infty}(1 + (k-1)\lambda)^n[(A - B) + (1 - B)(k-1)\lambda]|a_{k,p}||z|^k$$

$$- \sum_{k=1}^{\infty}(1 + (k-1)\lambda)^n[(A - B) + (1 - B)(k-1)\lambda]|\bar{b}_{k,p}||z|^k$$

$$- \sum_{\mu=2}^{p}|z|^{2(\mu-1)}\sum_{k=1}^{\infty}\phi_{k,\mu}^n(\lambda, A, B)[|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|]|z|^k$$

$$> |z|\Big[2(A - B) - \sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\phi_{k,\mu}^n(\lambda, A, B)(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}||z|^{k-1})\Big].$$

Consequently, we infer that the sufficient condition (2.2) for $F \in HL_p^n(\lambda, A, B)$ holds true. □

**Theorem 3.** *The coefficient condition (2.2) characterizes the elements of* $\widetilde{HL}_p^n(\lambda; A, B)$.

*Proof.* By Theorem 2, the sufficient part is true. For the necessary part, let $F \in \widetilde{HL}_p^n(\lambda; A, B)$. By (1.12) and the relationship of subordination, we get

$$\left|\frac{D_\lambda^{n+1}F(z) - D_\lambda^n F(z)}{AD_\lambda^n F(z) - BD_\lambda^{n+1}F(z)}\right| < 1 \ \ (z \in \mathcal{U}), \tag{2.5}$$

that is,

$$\left|\frac{\sum\limits_{\mu=1}^{p}|z|^{2(\mu-1)}\sum\limits_{k=1}^{\infty}N_{k,\mu}(|a_{k,p-\mu+1}|z^k+|b_{k,p-\mu+1}|\bar{z}^k)}{2(A-B)z-\sum\limits_{k=1}^{\infty}M_{k,1}|a_{k,p}|z^k-\sum\limits_{k=1}^{\infty}M_{k,1}|b_{k,p}|\bar{z}^k-\sum\limits_{\mu=2}^{p}|z|^{2(\mu-1)}\sum\limits_{k=1}^{\infty}M_{k,\mu}(|a_{k,p-\mu+1}|z^k+|b_{k,p-\mu+1}|\bar{z}^k)}\right|<1, \tag{2.6}$$

where

$$M_{k,\mu}=(1+(k-1)\lambda+2(\mu-1)\lambda)^n\left[(A-B)-B\left((k-1)\lambda+2(\mu-1)\lambda\right)\right]$$

and

$$N_{k,\mu}=(1+(k-1)\lambda+2(\mu-1)\lambda)^n\left[(k-1)\lambda+2(\mu-1)\lambda\right].$$

From (2.6), we have

$$\operatorname{Re}\left\{\frac{\sum\limits_{\mu=1}^{p}|z|^{2(\mu-1)}\sum\limits_{k=1}^{\infty}N_{k,\mu}(|a_{k,p-\mu+1}|z^k+|b_{k,p-\mu+1}|\bar{z}^k)}{(A-B)z-\sum\limits_{k=2}^{\infty}M_{k,1}|a_{k,p}|z^k-\sum\limits_{k=1}^{\infty}M_{k,1}|b_{k,p}|\bar{z}^k-\sum\limits_{\mu=2}^{p}|z|^{2(\mu-1)}\sum\limits_{k=1}^{\infty}M_{k,\mu}(|a_{k,p-\mu+1}|z^k+|b_{k,p-\mu+1}|\bar{z}^k)}\right\}<1, \tag{2.7}$$

which is equivalent to

$$\operatorname{Re}\left\{\frac{\sum\limits_{\mu=1}^{p}|z|^{2(\mu-1)}\sum\limits_{k=1}^{\infty}N_{k,\mu}(|a_{k,p-\mu+1}|z^k+|b_{k,p-\mu+1}|\bar{z}^k)}{2(A-B)z-\sum\limits_{\mu=1}^{p}|z|^{2(\mu-1)}\sum\limits_{k=1}^{\infty}M_{k,\mu}(|a_{k,p-\mu+1}|z^k+|b_{k,p-\mu+1}|\bar{z}^k)}\right\}<1. \tag{2.8}$$

Let $z=r$ $(0\le r<1)$, from (2.8), we have

$$\sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\phi_{k,\mu}^{n}(\lambda,A,B)\left(|a_{k,p-\mu+1}|+|b_{k,p-\mu+1}|\right)r^{k-1}<2(A-B), \tag{2.9}$$

where $\phi_{k,\mu}^{n}(\lambda,A,B)$ is given by (2.3).

Setting $r\to 1^-$ in (2.9), we will get (2.2). Thus, the proof is completed. □

**Theorem 4.** *Let* $|z|=r<1$. *If* $F\in\widetilde{HL}_p^n(\lambda;A,B)$, *then*

$$|F(z)|\le\left(\sum_{\mu=1}^{p}(|a_{1,p-\mu+1}|+|b_{1,p-\mu+1}|)\right)r+\left(\frac{2(A-B)}{\phi_{2,1}^{n}(\lambda,A,B)}-\sum_{\mu=1}^{p}\frac{\phi_{1,\mu}^{n}(\lambda,A,B)}{\phi_{2,1}^{n}(\lambda,A,B)}(|a_{1,p-\mu+1}|+|b_{1,p-\mu+1}|)\right)r^2$$

*and*

$$|F(z)|\ge\left(2-\sum_{\mu=1}^{p}(|a_{1,p-\mu+1}|+|b_{1,p-\mu+1}|)\right)r-\left(\frac{2(A-B)}{\phi_{2,1}^{n}(\lambda,A,B)}-\sum_{\mu=1}^{p}\frac{\phi_{1,\mu}^{n}(\lambda,A,B)}{\phi_{2,1}^{n}(\lambda,A,B)}(|a_{1,p-\mu+1}|+|b_{1,p-\mu+1}|)\right)r^2,$$

*where* $\phi_{k,\mu}^{n}(\lambda,A,B)$ *is given by (2.3).*

*Proof.* For $F \in \widetilde{HL}_p^n(\lambda; A, B)$, we obtain

$$|F(z)| \leq \left(\sum_{\mu=1}^{p}(|a_{1,p-\mu+1}| + |b_{1,p-\mu+1}|)\right)r + \left(\sum_{\mu=1}^{p}\sum_{k=2}^{\infty}(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|)\right)r^2.$$

Using the fact that $\phi_{k,\mu}^n(\lambda, A, B)$ is an increasing function with respect to $k$ and $\mu$ satisfying $\phi_{k,\mu}^n(\lambda, A, B) \geq \phi_{2,1}^n(\lambda, A, B)$, we have

$$|F(z)| \leq \left(\sum_{\mu=1}^{p}(|a_{1,p-\mu+1}| + |b_{1,p-\mu+1}|)\right)r + \left(\frac{2(A-B)}{\phi_{2,1}^n(\lambda, A, B)}\sum_{\mu=1}^{p}\sum_{k=2}^{\infty}\frac{\phi_{k,\mu}^n(\lambda, A, B)}{2(A-B)}(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|)\right)r^2.$$

Applying Theorem 3, we have

$$|F(z)| \leq \left(\sum_{\mu=1}^{p}(|a_{1,p-\mu+1}| + |b_{1,p-\mu+1}|)\right)r + \frac{2(A-B)}{\phi_{2,1}^n(\lambda, A, B)}\left(1 - \sum_{\mu=1}^{p}\frac{\phi_{1,\mu}^n(\lambda, A, B)}{2(A-B)}(|a_{1,p-\mu+1}| + |b_{1,p-\mu+1}|)\right)r^2.$$

Using the same methods above, we get

$$\begin{aligned}|F(z)| &\geq \left(1 - |b_{1,p}| - \sum_{\mu=2}^{p}(|a_{1,p-\mu+1}| + |b_{1,p-\mu+1}|)\right)r - \left(\sum_{\mu=1}^{p}\sum_{k=2}^{\infty}(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|)\right)r^2 \\ &\geq \left(2 - \sum_{\mu=1}^{p}(|a_{1,p-\mu+1}| + |b_{1,p-\mu+1}|)\right)r - \frac{2(A-B)}{\phi_{2,1}^n(\lambda, A, B)}\left(1 - \sum_{\mu=1}^{p}\frac{\phi_{1,\mu}^n(\lambda, A, B)}{2(A-B)}(|a_{1,p-\mu+1}| + |b_{1,p-\mu+1}|)\right)r^2.\end{aligned}$$

Hence the proof is completed. □

**Corollary 1.** *Let $F$ be given by (1.1). If $F \in \widetilde{HL}_p^n(\lambda; A, B)$, then*

$$\{\omega : |\omega| < \rho\} \subset F(\mathcal{U}),$$

*where*

$$\rho = 2\left(1 - \frac{A-B}{\phi_{2,1}^n(\lambda, A, B)}\right) + \sum_{\mu=1}^{p}\left(\frac{\phi_{1,\mu}^n(\lambda, A, B)}{\phi_{2,1}^n(\lambda, A, B)} - 1\right)(|a_{1,p-\mu+1}| + |b_{1,p-\mu+1}|)$$

*and $\phi_{k,\mu}^n(\lambda, A, B)$ is given by (2.3).*

**Theorem 5.** *If the function $F$ is given by (1.1) and $\phi_{k,\mu}^n(\lambda, A, B)$ is given by (2.3), then $F$ lies in $\widetilde{HL}_p^n(\lambda; A, B)$ if and only if*

$$F(z) = \sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\left(X_{k,p-\mu+1}h_{k,p-\mu+1}(z) + Y_{k,p-\mu+1}g_{k,p-\mu+1}(z)\right),$$

*where*

$$h_{1,p}(z) = z,\ h_{k,p}(z) = z - \frac{2(A-B)}{\phi_{k,1}^n(\lambda, A, B)}z^k\ (k \geq 2),$$

$$
\begin{aligned}
g_{k,p}(z) &= z - \frac{2(A-B)}{\phi^n_{k,1}(\lambda,A,B)}\bar{z}^k \ (k \geq 1),\\
h_{k,p-\mu+1}(z) &= z - |z|^{2(\mu-1)}\frac{2(A-B)}{\phi^n_{k,\mu}(\lambda,A,B)}z^k \ (2 \leq \mu \leq p; k \geq 1),\\
g_{k,p-\mu+1}(z) &= z - |z|^{2(\mu-1)}\frac{2(A-B)}{\phi^n_{k,\mu}(\lambda,A,B)}\bar{z}^k \ (2 \leq \mu \leq p; k \geq 1)
\end{aligned}
$$

*and*

$$
\sum_{\mu=1}^{p}\sum_{k=1}^{\infty}(X_{k,p-\mu+1}+Y_{k,p-\mu+1}) = 1,\ (X_{k,p-\mu+1} \geq 0, Y_{k,p-\mu+1} \geq 0).
$$

*In particular, for $k \geq 1$ and $1 \leq \mu \leq p$, $\{h_{k,p-\mu+1}(z)\}$ and $\{g_{k,p-\mu+1}(z)\}$ are the extreme functions of the class $\widetilde{HL}^n_p(\lambda; A, B)$.*

*Proof.* Since

$$
\begin{aligned}
F(z) =& \sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\Big(X_{k,p-\mu+1}h_{k,p-\mu+1}(z) + Y_{k,p-\mu+1}g_{k,p-\mu+1}(z)\Big)\\
=& z - \sum_{k=2}^{\infty}\frac{2(A-B)}{\phi^n_{k,1}(\lambda,A,B)}X_{k,p}z^k - \sum_{k=1}^{\infty}\frac{2(A-B)}{\phi^n_{k,1}(\lambda,A,B)}Y_{k,p}\bar{z}^k\\
&- \sum_{\mu=2}^{p}|z|^{2(\mu-1)}\sum_{k=1}^{\infty}\frac{2(A-B)}{\phi^n_{k,\mu}(\lambda,A,B)}\left[X_{k,p-\mu+1}z^k + Y_{k,p-\mu+1}\bar{z}^k\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=2}^{\infty}\frac{\phi^n_{k,1}(\lambda,A,B)}{2(A-B)}\cdot\frac{2(A-B)}{\phi^n_{k,1}(\lambda,A,B)}X_{k,p} + \sum_{k=1}^{\infty}\frac{\phi^n_{k,1}(\lambda,A,B)}{2(A-B)}\cdot\frac{2(A-B)}{\phi^n_{k,1}(\lambda,A,B)}Y_{k,p}\\
&+ \sum_{\mu=2}^{p}\sum_{k=1}^{\infty}\frac{\phi^n_{k,\mu}(\lambda,A,B)}{2(A-B)}\cdot\frac{2(A-B)}{\phi^n_{k,\mu}(\lambda,A,B)}[X_{k,p-\mu+1} + X_{k,p-\mu+1}]\\
&= \sum_{\mu=2}^{p}\sum_{k=1}^{\infty}(X_{k,p-\mu+1}+Y_{k,p-\mu+1}) + \sum_{k=2}^{\infty}(X_{k,p}+Y_{k,p}) + Y_{1,p}\\
&\leq 1 - X_{1,p} \leq 1.
\end{aligned}
$$

Using Theorem 3, we obtain that $F \in \widetilde{HL}^n_p(\lambda; A, B)$.

For $F \in \widetilde{HL}^n_p(\lambda; A, B)$, let

$$
\begin{aligned}
X_{k,p} &= \frac{\phi^n_{k,1}(\lambda,A,B)}{2(A-B)}|a_{k,p}| \ (k \geq 2),\\
Y_{k,p} &= \frac{\phi^n_{k,1}(\lambda,A,B)}{2(A-B)}|b_{k,p}| \ (k \geq 1),\\
X_{k,p-\mu+1} &= \frac{\phi^n_{k,\mu}(\lambda,A,B)}{2(A-B)}|a_{k,p-\mu+1}| \ (2 \leq \mu \leq p, k \geq 1),
\end{aligned}
$$

$$Y_{k,p-\mu+1} = \frac{\phi^n_{k,\mu}(\lambda, A, B)}{2(A-B)}|b_{k,p-\mu+1}| \; (2 \le \mu \le p, k \ge 1)$$

and

$$X_{1,p} = 1 - \sum_{\mu=2}^{p}\sum_{k=1}^{\infty}(X_{k,p-\mu+1} + Y_{k,p-\mu+1}) - \sum_{k=2}^{\infty}(X_{k,p} + Y_{k,p}) - Y_{1,p},$$

where $X_{1,p} \ge 0$. According to Theorem 3, we have

$$F(z) = \sum_{\mu=1}^{p}\sum_{k=1}^{\infty}(X_{k,p-\mu+1}h_{k,p-\mu+1}(z) + Y_{k,p-\mu+1}g_{k,p-\mu+1}(z)).$$

Thus, we complete the proof. □

**Theorem 6.** *The class $\widetilde{HL}^n_p(\lambda; A, B)$ is convex.*

*Proof.* Suppose $F_i(z) \in \widetilde{HL}^n_p(\lambda; A, B)$, where

$$F_i(z) = z - \sum_{k=2}^{\infty}|a_{i_k,p}|z^k - \sum_{k=1}^{\infty}|b_{i_k,p}|\bar{z}^k - \sum_{\mu=2}^{p}|z|^{2(\mu-1)}\sum_{k=1}^{\infty}\left(|a_{i_k,p-\mu+1}|z^k + |\bar{b}_{i_k,p-\mu+1}|\bar{z}^k\right) \quad (i = 1, 2, \cdots).$$

Applying Theorem 3, we obtain

$$\sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\phi^n_{k,\mu}(\lambda, A, B)(|a_{i_k,p-\mu+1}| + |b_{i_k,p-\mu+1}|) \le 2(A-B), \tag{2.10}$$

where $\phi^n_{k,\mu}(\lambda, A, B)$ is given by (2.3).

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1$, we can write the convex combination of $F_i$ as follows

$$\sum_{i=1}^{\infty} t_iF_i = z - \sum_{k=2}^{\infty}\Big(\sum_{i=1}^{\infty} t_i[|a_{ik,p}|z^k + |b_{ik,p}|\bar{z}^k]\Big) - \sum_{\mu=2}^{p}|z|^{2(\mu-1)}\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{\infty} t_i[|a_{ik,p-\mu+1}|z^k + |b_{ik,p-\mu+1}|\bar{z}^k]\Big).$$

From (2.10), we obtain

$$\begin{aligned}
&\sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\frac{\phi^n_{k,\mu}(\lambda, A, B)}{2(A-B)} \cdot \left(\sum_{i=1}^{\infty} t_i[|a_{ik,p-\mu+1} + |b_{ik,p-\mu+1}|]\right)\\
&= \sum_{i=1}^{\infty} t_i\left[\sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\frac{\phi^n_{k,\mu}(\lambda, A, B)}{2(A-B)} \cdot \left(|a_{ik,p-\mu+1}| + |b_{ik,p-\mu+1}|\right)\right]\\
&\le \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}$$

Using Theorem 3, we get $\sum_{i=1}^{\infty} t_iF_i \in \widetilde{HL}^n_p(\lambda; A, B)$. Therefore, the proof is completed. □

**Remark 1.** *In particular, let $A = 1 - 2\beta$ $(0 \le \beta < 1)$ and $B = -1$. Then Theorem 1, Theorem 2, Theorem 4, Theorem 3 and Theorem 6 in [8] are particular cases of Theorem 2, Theorem 4, Theorem 5 and Theorem 6.*

## 3. Convolution

First of all, we provide a new theorem for convolution of the class $\widetilde{HL}_p^n(\lambda; A, B)$.

**Theorem 7.** *Let $\lambda \geq 1$, $p \in \{1, 2, \ldots\}$, $A_{1,p-\mu+1} = a_{1,p-\mu+1} = 0 (2 \leq \mu \leq p)$ and $B_{1,p-\mu+1} = b_{1,p-\mu+1} = 0 (1 \leq \mu \leq p)$. If $F$, $G \in \widetilde{HL}_p^n(\lambda; A, B)$, then $F * G$ belongs to the class $\widetilde{HL}_p^n(\lambda; A, B)$, where $\phi_{k,\mu}^n(\lambda, A, B)$ is given by (2.3) and*

$$\phi_{2,1}^n(\lambda, A, B) \geq 2p(A - B). \tag{3.1}$$

*Proof.* Let $F, G \in \widetilde{HL}_p^n(\lambda; A, B)$. Then the convolution $F * G$ is in $\widetilde{HL}_p^n(\lambda; A, B)$ if

$$\sum_{\mu=1}^{p} \sum_{k=2}^{\infty} \frac{\phi_{k,\mu}^n(\lambda, A, B)}{2(A - B)} (|A_{k,p-\mu+1}||a_{k,p-\mu+1}| + |B_{k,p-\mu+1}||b_{k,p-\mu+1}|) \leq 1. \tag{3.2}$$

For $F, G \in \widetilde{HL}_p^n(\lambda; A, B)$, we have

$$\sum_{\mu=1}^{p} \sum_{k=2}^{\infty} \frac{\phi_{k,\mu}^n(\lambda, A, B)}{2(A - B)} (|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|) \leq 1 \tag{3.3}$$

and

$$\sum_{\mu=1}^{p} \sum_{k=2}^{\infty} \frac{\phi_{k,\mu}^n(\lambda, A, B)}{2(A - B)} (|A_{k,p-\mu+1}| + |B_{k,p-\mu+1}|) \leq 1. \tag{3.4}$$

From (3.3) and (3.4), we get

$$\sum_{k=2}^{\infty} \frac{\phi_{k,\mu}^n(\lambda, A, B)}{2(A - B)} (|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|) \leq 1 \tag{3.5}$$

and

$$\sum_{k=2}^{\infty} \frac{\phi_{k,\mu}^n(\lambda, A, B)}{2(A - B)} (|A_{k,p-\mu+1}| + |B_{k,p-\mu+1}|) \leq 1. \tag{3.6}$$

Applying Cauchy-Schwarz inequality to (3.5) and (3.6), we obtain

$$\sum_{k=2}^{\infty} \frac{\phi_{k,\mu}^n(\lambda, A, B)}{2(A - B)} \sqrt{(|A_{k,p-\mu+1}| + |B_{k,p-\mu+1}|)(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|)} \leq 1. \tag{3.7}$$

Due to

$$|A_{k,p-\mu+1}||a_{k,p-\mu+1}| + |B_{k,p-\mu+1}||b_{k,p-\mu+1}| \leq (|A_{k,p-\mu+1}| + |B_{k,p-\mu+1}|)(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|), \tag{3.8}$$

from (3.7) and (3.8), we have

$$\sum_{k=2}^{\infty} \frac{\phi_{k,\mu}^n(\lambda, A, B)}{2(A - B)} \sqrt{(|A_{k,p-\mu+1}||a_{k,p-\mu+1}| + |B_{k,p-\mu+1}||b_{k,p-\mu+1}|)} \leq 1.$$

From the above inequality, it is easy to see

$$\sum_{\mu=1}^{p}\sum_{k=2}^{\infty}\frac{\phi_{k,\mu}^{n}(\lambda,A,B)}{2(A-B)}\sqrt{(|A_{k,p-\mu+1}||a_{k,p-\mu+1}|+|B_{k,p-\mu+1}||b_{k,p-\mu+1}|)}\leq p \tag{3.9}$$

and

$$\sqrt{(|A_{k,p-\mu+1}||a_{k,p-\mu+1}|+|B_{k,p-\mu+1}||b_{k,p-\mu+1}|)}\leq\frac{2(A-B)}{\phi_{k,\mu}^{n}(\lambda,A,B)}. \tag{3.10}$$

In order to obtain (3.2), we only need to show

$$(|A_{k,p-\mu+1}||a_{k,p-\mu+1}|+|B_{k,p-\mu+1}||b_{k,p-\mu+1}|)\leq\frac{1}{p}\sqrt{(|A_{k,p-\mu+1}||a_{k,p-\mu+1}|+|B_{k,p-\mu+1}||b_{k,p-\mu+1}|)},$$

that is,

$$\sqrt{(|A_{k,p-\mu+1}||a_{k,p-\mu+1}|+|B_{k,p-\mu+1}||b_{k,p-\mu+1}|)}\leq\frac{1}{p}. \tag{3.11}$$

By (3.10) and (3.11), (3.2) holds true if

$$\frac{2(A-B)}{\phi_{k,\mu}^{n}(\lambda,A,B)}\leq\frac{1}{p}.$$

For $\mu\geq 1$ and $k\geq 2$, we can get

$$\min_{\mu,k}\{\phi_{k,\mu}^{n}(\lambda,A,B)\}=\phi_{2,1}^{n}(\lambda,A,B).$$

Thus, (3.2) holds true if

$$\frac{2(A-B)}{\phi_{2,1}^{n}(\lambda,A,B)}\leq\frac{1}{p}.$$

So we get the condition (3.1) of Theorem 7 and complete the proof of Theorem 7.

Finally, we discuss the convolution properties of the class $\widetilde{HL}_{p}^{n}(\lambda;A,B)$. □

**Lemma 1.** *(see [19]) Let $-1\leq B_1\leq B_2<A_2\leq A_1\leq 1$. Then*

$$\frac{1+A_2z}{1+B_2z}\prec\frac{1+A_1z}{1+B_1z}\quad(z\in\mathcal{U}).$$

**Remark 2.** *Obviously, from Lemma 1, we get (see [8])*

$$HL_{p}^{n}(\lambda,A,B)\subseteq HL_{p}^{n}(\lambda;1-2\beta,-1)=SH_{p}(n,\lambda,\beta).$$

**Lemma 2.** *Let $\lambda_2\geq\lambda_1\geq 1,\ -1\leq B_1\leq B_2<A_2\leq A_1\leq 1,\ p\in\{1,2,\ldots\}$. Then*

$$\widetilde{HL}_{p}^{n}(\lambda_2;A_2,B_2)\subseteq\widetilde{HL}_{p}^{n}(\lambda_1;A_1,B_1).$$

*Proof.* Let $F \in \widetilde{HL}_p^n(\lambda_2; A_2, B_2)$, then $\widetilde{HL}_p^n(\lambda_2; A_2, B_2) \subseteq \widetilde{HL}_p^n(\lambda_1; A_1, B_1)$ will be proved if we can show

$$\sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\frac{\phi_{k,\mu}^n(\lambda_1, A_1, B_1)}{2(A_1 - B_1)}\left(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|\right) \le \sum_{\mu=1}^{p}\sum_{k=1}^{\infty}\frac{\phi_{k,\mu}^n(\lambda_2, A_2, B_2)}{2(A_2 - B_2)}\left(|a_{k,p-\mu+1}| + |b_{k,p-\mu+1}|\right), \quad (3.12)$$

or equivalently

$$\frac{\phi_{k,\mu}^n(\lambda_1, A_1, B_1)}{A_1 - B_1} \le \frac{\phi_{k,\mu}^n(\lambda_2, A_2, B_2)}{A_2 - B_2}. \quad (3.13)$$

Since $F \in \widetilde{HL}_p^n(\lambda_2; A_2, B_2)$, $\lambda \ge 1$, $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, $p \in \{1, 2, \ldots\}$, from (1.12) and Lemma 1, we have

$$\frac{D_{\lambda_2}^{n+1}F(z)}{D_{\lambda_2}^{n}F(z)} \prec \frac{1 + A_2 z}{1 + B_2 z} \prec \frac{1 + A_1 z}{1 + B_1 z} \quad (z \in \mathcal{U}),$$

or equivalently

$$\widetilde{HL}_p^n(\lambda_2; A_2, B_2) \subseteq \widetilde{HL}_p^n(\lambda_2; A_1, B_1). \quad (3.14)$$

Using Theorem 3 and (3.14), we get

$$\frac{\phi_{k,\mu}^n(\lambda_2, A_1, B_1)}{A_1 - B_1} \le \frac{\phi_{k,\mu}^n(\lambda_2, A_2, B_2)}{A_2 - B_2}. \quad (3.15)$$

Because $\phi_{k,\mu}^n(\lambda, A, B)$ is an increasing function of $\lambda$, so from (3.15), we obtain

$$\frac{\phi_{k,\mu}^n(\lambda_1, A_1, B_1)}{A_1 - B_1} \le \frac{\phi_{k,\mu}^n(\lambda_2, A_1, B_1)}{A_1 - B_1} \le \frac{\phi_{k,\mu}^n(\lambda_2, A_2, B_2)}{A_2 - B_2}$$

and so (3.13) is established. Also, using (3.12) and Theorem 3, we have $F \in \widetilde{HL}_p^n(\lambda_1; A_1, B_1)$. The proof is completed. □

**Theorem 8.** *Let* $\lambda_2 \ge \lambda_1 \ge 1$, $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, $p \in \{1, 2, \ldots\}$, $A_{1,p-\mu+1} = a_{1,p-\mu+1} = 0$ $(2 \le \mu \le p)$ *and* $B_{1,p-\mu+1} = b_{1,p-\mu+1} = 0$ $(1 \le \mu \le p)$. *If* $F \in \widetilde{HL}_p^n(\lambda_2; A_2, B_2)$ *and* $G \in \widetilde{HL}_p^n(\lambda_1; A_1, B_1)$, *then the convolution of F and G is in the class* $\widetilde{HL}_p^n(\lambda_2; A_2, B_2)$ *and*

$$\widetilde{HL}_p^n(\lambda_2; A_2, B_2) \subseteq \widetilde{HL}_p^n(\lambda_1; A_1, B_1).$$

*Proof.* Let $F \in \widetilde{HL}_p^n(\lambda_2; A_2, B_2)$ and $G \in \widetilde{HL}_p^n(\lambda_1; A_1, B_1)$, $k \ge 1$, $1 \le \mu \le p$. Then from Theorem 3, for $k \ge 2$, we get

$$|a_{k,p-\mu+1}| \le \frac{2(A_2 - B_2)}{\phi_{k,\mu}^n(\lambda_2, A_2, B_2)} \le \frac{2(A_2 - B_2)}{(A_2 - B_2) + (1 - B_2)(k-1)\lambda_2} \le 1 \text{ and } |b_{k,p-\mu+1}| \le 1$$

and

$$|A_{k,p-\mu+1}| \le \frac{2(A_1 - B_1)}{\phi_{k,\mu}^n(\lambda_1, A_1, B_1)} \le \frac{2(A_1 - B_1)}{(A_1 - B_1) + (1 - B_1)(k-1)\lambda_1} \le 1 \text{ and } |B_{k,p-\mu+1}| \le 1.$$

And so, we conclude that

$$\sum_{\mu=1}^{p}\sum_{k=2}^{\infty}\frac{\phi_{k,\mu}^n(\lambda_2, A_2, B_2)}{2(A_2 - B_2)}\Big(|a_{k,p-\mu+1}||A_{k,p-\mu+1}| + |b_{k,p-\mu+1}||B_{k,p-\mu+1}|\Big)$$

$$\leq \sum_{\mu=1}^{p}\sum_{k=2}^{\infty}\frac{\phi_{k,\mu}^{n}(\lambda_2,A_2,B_2)}{2(A_2-B_2)}\Big(|a_{k,p-\mu+1}|+|b_{k,p-\mu+1}|\Big)\leq 1.$$

Applying Theorem 3 and Lemma 2, we obtain $(F * G)(z) \in \widetilde{HL}_p^n(\lambda_2; A_2, B_2) \subseteq \widetilde{HL}_p^n(\lambda_1; A_1, B_1)$ and so we complete the proof of Theorem 8. □

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**Conflict of interest**

The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

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