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Research article

Note on an integral by Fritz Oberhettinger

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Abstract: In 1974 Fritz Oberhettinger published the Tables of Mellin Transforms. We derive the correct result for one of these integrals along with some interesting special cases.

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1. Introduction

In connection with the Mellin transform, the authors have the opportunity to evaluate integrals of the form

$$\int_0^\infty \left((\log(a) - ix)^k + (\log(a) + ix)^k\right) \log\left(\frac{\sin(\alpha) + \cosh(x)}{\cosh(x) - \sin(\alpha)}\right) dx \tag{1.1}$$

In our case the constants in the formula are general complex numbers subject to the restrictions given below. In Table 1.6 of [1] a formula is given which was derived by Oberhettinger. An attempt to use it led to finding that it was incorrect. The derivations follow the method used by us in [6]. The generalized Cauchy's integral formula is given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i} \int_C \frac{e^{wy}}{w^{k+1}} dw. \tag{1.2}$$

This method involves using a form of Eq (1.2) then multiplys both sides by a function, then takes a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Eq (1.2) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2. Definite integral of the contour integral

We use the method in [6]. Here the contour is in the upper left quadrant with $\Re(w) < 0$ and going round the origin with zero radius. Using a generalization of Cauchy's integral formula we first replace

y by $\log(a) + ix$ for the first equation and then y by $\log(a) - ix$ to get the second equation. Then we add these two equations, followed by multiplying both sides by $\log\left(\frac{\sin(\alpha)+\cosh(x)}{\cosh(x)-\sin(\alpha)}\right)$ and taking the definite integral over $x \in [0, \infty)$ to get

$$\begin{aligned}
&\int_0^\infty \frac{\left((\log(a) - ix)^k + (\log(a) + ix)^k\right)\log\left(\frac{\sin(\alpha)+\cosh(x)}{\cosh(x)-\sin(\alpha)}\right)}{2k!}dx \\
&= \frac{1}{2\pi i}\int_0^\infty \int_C a^w w^{-k-1}\cos(wx)\log\left(\frac{\sin(\alpha) + \cosh(x)}{\cosh(x) - \sin(\alpha)}\right)dwdx \\
&= \frac{1}{2\pi i}\int_C \int_0^\infty a^w w^{-k-1}\cos(wx)\log\left(\frac{\sin(\alpha) + \cosh(x)}{\cosh(x) - \sin(\alpha)}\right)dxdw \\
&= \frac{1}{2\pi i}\int_C \pi a^w w^{-k-2}\operatorname{sech}\left(\frac{\pi w}{2}\right)\sinh(\alpha w)dw
\end{aligned} \tag{2.1}$$

from Eq (1.9.50) in [4] where the logarithmic function is defined in Eq (4.1.2) in [2]. The integral is valid for a, k and α complex and $|\Re(\alpha)| < \pi/2$. We note that the formula in [4] is an even function of w so replacing w by $-w$ has no effect.

3. Infinite sum of the contour integral

In this section we will again use the generalized Cauchy's integral formula to derive equivalent contour integrals. First we replace y by $y + \alpha$ for the first equation and y by $y - \alpha$ for second then subtract these two equations to get

$$\frac{(y + \alpha)^k - (y - \alpha)^k}{k!} = \frac{1}{2\pi i}\int_C 2w^{-k-1}e^{wy}\sinh(\alpha w)dw \tag{3.1}$$

Next we replace y by $\log(a) + \frac{1}{2}\pi(2y + 1)$ and multiply both sides by $\pi(-1)^y$ to get

$$\begin{aligned}
&\frac{\left(\alpha + \log(a) + \frac{1}{2}\pi(2y + 1)\right)^k - \left(-\alpha + \log(a) + \frac{1}{2}\pi(2y + 1)\right)^k}{k!} \\
&= \frac{1}{2\pi i}\int_C 2w^{-k-1}\sinh(\alpha w)e^{w\left(\log(a)+\frac{1}{2}\pi(2y+1)\right)}dw
\end{aligned} \tag{3.2}$$

Next we simplify the left-hand side and take the infinite sum over $y \in [0, \infty)$ to get

$$\begin{aligned}
&\sum_{y=0}^\infty \frac{\pi^{k+1}(-1)^y\left(\left(\frac{\alpha+\log(a)}{\pi} + y + \frac{1}{2}\right)^k - \left(-\frac{\alpha}{\pi} + \frac{\log(a)}{\pi} + y + \frac{1}{2}\right)^k\right)}{k!} \\
&= \frac{1}{2\pi i}\sum_{y=0}^\infty \int_C 2\pi(-1)^y w^{-k-1}\sinh(\alpha w)e^{w\left(\log(a)+\frac{1}{2}\pi(2y+1)\right)}dw \\
&= \frac{1}{2\pi i}\int_C \sum_{y=0}^\infty 2\pi(-1)^y w^{-k-1}\sinh(\alpha w)e^{w\left(\log(a)+\frac{1}{2}\pi(2y+1)\right)}dw
\end{aligned} \tag{3.3}$$

We simplify the left-hand side in terms of the Hurwitz zeta function to get

$$\begin{aligned}
&\frac{2^{k+1}\pi^{k+2}}{(k+1)!}\left(-\zeta\left(-k-1,\frac{\log(a)-\alpha}{2\pi}+\frac{1}{4}\right)+\zeta\left(-k-1,\frac{\log(a)-\alpha}{2\pi}+\frac{3}{4}\right)\right.\\
&\left.\quad+\zeta\left(-k-1,\frac{\alpha+\log(a)}{2\pi}+\frac{1}{4}\right)-\zeta\left(-k-1,\frac{\alpha+\log(a)}{2\pi}+\frac{3}{4}\right)\right)\\
&\quad=\frac{1}{2\pi i}\int_C \pi a^w w^{-k-2}\operatorname{sech}\left(\frac{\pi w}{2}\right)\sinh(\alpha w)dw
\end{aligned} \tag{3.4}$$

from Eq (1.232.2) in [5] and $\Re(w) < 0$ where $\zeta(z,q)$ is the Hurwitz zeta function. The Hurwitz zeta function has a series representation given by

$$\zeta(z,q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^z} \tag{3.5}$$

where $\Re(z) > 1, q \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$\zeta(z,q) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}e^{-qt}}{1-e^{-t}} dt \tag{3.6}$$

where $z \neq 1$, or $0 < q < 1$.

4. Definite integral in terms of the Hurwitz zeta function

Since the right-hand sides of Eqs (2.1) and (3.4) are equal we can equate the left-hand sides and simplifying the factorials we get

$$\begin{aligned}
&\int_0^{\infty}\left((\log(a)-ix)^k+(\log(a)+ix)^k\right)\log\left(\frac{\sin(\alpha)+\cosh(x)}{\cosh(x)-\sin(\alpha)}\right)dx\\
&=\frac{(2\pi)^{k+2}}{k+1}\left(-\zeta\left(-k-1,\frac{\log(a)-\alpha}{2\pi}+\frac{1}{4}\right)+\zeta\left(-k-1,\frac{\log(a)-\alpha}{2\pi}+\frac{3}{4}\right)\right.\\
&\left.\qquad+\zeta\left(-k-1,\frac{\alpha+\log(a)}{2\pi}+\frac{1}{4}\right)-\zeta\left(-k-1,\frac{\alpha+\log(a)}{2\pi}+\frac{3}{4}\right)\right)
\end{aligned} \tag{4.1}$$

Note the left-hand side of Eq (4.1) converges for all finite k.

5. Derivation of entry 1.6.6.22 in [1]

Using Eq (4.1), setting $a = 1$, replacing k by $k-1$ and simplifying the left-hand side to get

$$\begin{aligned}
\int_0^{\infty} x^{k-1}\log\left(\frac{\sin(\alpha)+\cosh(x)}{\cosh(x)-\sin(\alpha)}\right)dx = -\frac{2^k\pi^{k+1}\csc\left(\frac{\pi k}{2}\right)}{k}&\left(\zeta\left(-k,\frac{1}{4}-\frac{\alpha}{2\pi}\right)-\zeta\left(-k,\frac{3}{4}-\frac{\alpha}{2\pi}\right)\right.\\
&\left.-\zeta\left(-k,\frac{\alpha}{2\pi}+\frac{1}{4}\right)+\zeta\left(-k,\frac{\alpha}{2\pi}+\frac{3}{4}\right)\right)
\end{aligned} \tag{5.1}$$

Next we replace k by z, x with $\frac{\pi x}{2b}$ and α by $\frac{\pi a}{2b}$ simplify to get

$$\int_0^\infty x^{z-1}\log\left(\frac{\sin\left(\frac{\pi a}{2b}\right)+\cosh\left(\frac{\pi x}{2b}\right)}{\cosh\left(\frac{\pi x}{2b}\right)-\sin\left(\frac{\pi a}{2b}\right)}\right)dx = -\frac{\pi 4^z b^z\csc\left(\frac{\pi z}{2}\right)}{z}\left(\zeta\left(-z,\frac{1}{4}-\frac{a}{4b}\right)-\zeta\left(-z,\frac{3}{4}-\frac{a}{4b}\right)\right.$$
$$\left.-\zeta\left(-z,\frac{a}{4b}+\frac{1}{4}\right)+\zeta\left(-z,\frac{a}{4b}+\frac{3}{4}\right)\right) \tag{5.2}$$

where $|\Re(b)| > |\Re(a)|$ and $\Re(z) > 0$. The result in [1] is in error.

6. Derivation of a special case in terms of the logarithm of the gamma function

Using Eq (4.1) replacing a by e^a and applying L'Hopital's rule as $k \to -1$ to the right-hand side, simplifying the left-hand side we get

$$\int_0^\infty \log\left(\frac{\sin(\alpha)+\cosh(x)}{\cosh(x)-\sin(\alpha)}\right)\frac{dx}{a^2+x^2} = \frac{\pi}{a}\log\left(\frac{\Gamma\left(\frac{a-\alpha}{2\pi}+\frac{1}{4}\right)\Gamma\left(\frac{a+\alpha}{2\pi}+\frac{3}{4}\right)}{\Gamma\left(\frac{a-\alpha}{2\pi}+\frac{3}{4}\right)\Gamma\left(\frac{a+\alpha}{2\pi}+\frac{1}{4}\right)}\right) \tag{6.1}$$

from Eq (1.10.10) in [3].

7. Derivation of a special case in terms of the Digamma function

Using equation (4.1) replacing a by e^a setting $k = -2$ and simplifying the left-hand side we get

$$\int_0^\infty \log\left(\frac{\sin(\alpha)+\cosh(x)}{\cosh(x)-\sin(\alpha)}\right)\frac{\left(a^2-x^2\right)}{(a^2+x^2)^2}dx = \frac{1}{2}\left(-\psi^{(0)}\left(\frac{a-\alpha}{2\pi}+\frac{1}{4}\right)+\psi^{(0)}\left(\frac{a-\alpha}{2\pi}+\frac{3}{4}\right)\right.$$
$$\left.+\psi^{(0)}\left(\frac{a+\alpha}{2\pi}+\frac{1}{4}\right)-\psi^{(0)}\left(\frac{a+\alpha}{2\pi}+\frac{3}{4}\right)\right) \tag{7.1}$$

from Eq (8.363.8) in [5].

8. Derivation of a special case in terms of the derivative of the Hurwitz zeta function

Using Eq (4.1) taking the first partial derivative with respect to k then setting $k = 0$ simplifying the left-hand side we get

$$\int_0^\infty \log\left(a^2+x^2\right)\log\left(\frac{\sin(\alpha)+\cosh(x)}{\cosh(x)-\sin(\alpha)}\right)dx = 2\pi\left(2\pi\left(\zeta'\left(-1,\frac{a-\alpha}{2\pi}+\frac{1}{4}\right)-\zeta'\left(-1,\frac{a-\alpha}{2\pi}+\frac{3}{4}\right)\right.\right.$$
$$\left.-\zeta'\left(-1,\frac{a+\alpha}{2\pi}+\frac{1}{4}\right)+\zeta'\left(-1,\frac{a+\alpha}{2\pi}+\frac{3}{4}\right)\right) \tag{8.1}$$
$$\left.+\alpha\log\left(\frac{2\pi}{e}\right)\right)$$

9. Discussion

The Table of Oberhettinger contain integrals of the Mellin transform. This transform is also called the two sided Laplace transform. The use of the Mellin transform in various problems in mathematical analysis is well established. Particularly widespread and effective is its application to problems arising in analytic number theory. We have been able to derive the correct version of one of the integrals which will assist readers to have the correct version while using this particular integral.

10. Conclusion

In this paper, we have presented a novel method for deriving some interesting definite integrals by Oberhettinger using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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Conflict of interest

The authors confirm there are no conflicts of interest.

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