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Research article

Attractors for a quasilinear viscoelastic equation with nonlinear damping and memory

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Abstract: In this paper, the long time behavior of a quasilinear viscoelastic equation with nonlinear damping is considered. Under suitable assumptions, the existence of global attractors is established.

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1. Introduction

In this paper we investigate the long-time dynamics of solutions for the quasilinear viscoelastic equation with nonlinear damping and memory

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u_{tt} - \alpha\Delta u + \int_{-\infty}^{t} \mu(t-s)\Delta u(s)\mathrm{d}s + f(u) + g(u_t) = h(x), \\ u|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N (N \geqslant 1)$ with smooth boundary $\partial\Omega$, u_0 is the prescribed past history of u.

Problem (1.1) can be seen as an extension, accounting for memory effects in the material, of equations of the form

$$f(u_t)u_{tt} - \Delta u - \Delta u_{tt} = 0. \tag{1.2}$$

This equation is interesting not only from the point of view of PDE general theory, but also due to its applications in Mechanics. In the case $f(u_t)$ is a constant, Eq (1.2) has been used to model extensional vibrations of thin rods (see Love [1, Chapter 20]). In the case $f(u_t)$ is not a constant, Eq (1.2) can model materials whose density depends on the velocity u_t. We refer the reader to Fabrizio and Morro [2] for several other related models.

When $\rho = 0$ and Δu_{tt} is dropped in (1.1), the related problem has been extensively studied and several results concerning the global existence, decay of global solution and finite time blow up have been established. In this direction, we refer the readers to see Ref. [3–12] and the references therein.

Now let us recall some results concerning quasilinear viscoelastic wave equations. In [13], Cavalcanti et al. studied the following equation with Dirichlet boundary conditions

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s - \gamma\Delta u_t = 0. \tag{1.3}$$

A global existence result for $\gamma \geqslant 0$, as well as an exponential decay for $\gamma > 0$, has been established. This last result has been extended by Messaoudi and Tatar [14] to the case $\gamma = 0$.

In [15], Messaoudi and Tatar studied the following equation

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = b|u|^{p-2}u. \tag{1.4}$$

By introducing a new functional and using a potential well method, they obtained the global existence of solutions and the uniform decay of the energy if the initial data are in some stable set. In the case $b = 0$ in (1.4), Messaoudi and Tatar [16] proved the exponential decay of global solutions to (1.4), without smallness of initial data, considering only the dissipation effect given by the memory. Liu [17] proved that for certain initial data in the stable set, the solution decays exponentially, and for certain initial data in the unstable set, the solution blows up in finite time.

Replacing strong damping by weak damping in Eq (1.3), several authors have studied the energy decay rates of the related problems like

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + h(u_t) = 0. \tag{1.5}$$

When $h(u_t) = u_t$, Han and Wang [18] investigated the global existence and exponential stability of the energy for solutions for Eq (1.5). When $h(u_t) = |u_t|^m u_t (m > 0)$, the same authors [19] proved the general decay of energy for Eq (1.5). Later, Park and Park [20] established the general decay for Eq (1.5) with general nonlinear weak damping.

Now, we list some important literature on the nonlinear evolution equation with hereditary memory and variable density. Araújo et al. [21] studied the following equation

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \alpha\Delta u + \int_{-\infty}^t \mu(t-s)\Delta u(s)\mathrm{d}s - \gamma\Delta u_t + f(u) = h(x), \tag{1.6}$$

and proved the global existence, uniqueness and exponential stability of solutions and existence of the global attractor. Subsequently, Qin et al. [22,23] proved the upper semicontinuity of pullback attractors and the existence of uniform attractors by assuming $f(u) = 0$ and taking a frictional damping u_t instead of strong damping $-\Delta u_t$. However, their argument for uniqueness rely on the differentiability of the map $\sigma(x) = |x|^\rho$ at zero, which introduces the further restriction $\rho > 1$.

Lately, the authors [24] established an existence, uniqueness and continuous dependence result for the weak solutions to the semigroup generated for the system (1.6) in a three-dimensional space when $\rho \in [0,4]$ and f has polynomial growth of (at most) critical order 5. Then, based on the [24], the same authors [25] established the existence of the global attractor of optimal regularity for Eq (1.6) when $\rho \in [0,4)$. Recently, the authors [26] proved that the sole weak dissipation ($\gamma = 0$) given by the memory term is enough to ensure existence and optimal regularity of the global attractor. Leuyacc and Parejas [27] proved the upper semicontinuity of global attractors when $\rho \to 0^+$ in (1.6). Li and Jia [28] proved the existence of a global solution by means of the Galerkin method, establish the exponential stability result and the polynomial stability result when the kernel $\mu(s)$ satisfies $\mu'(s) \leqslant -k_1\mu^q(s),\ 1 \leqslant q < 3/2$.

Motivated by the works above mentioned, our aim is to present the existence of global attractors for the problem (1.1).

As in [29–31], we shall introduce a new variable η^t to the system which corresponds to the relative displacement history.

Let us define

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \quad (x,s) \in \Omega \times \mathbb{R}^+,\ t \geqslant 0.$$

Note that

$$\eta^t_t(x,s) = -\eta^t_s(x,s) + u_t(x,s).$$

Thus, the original memory term can be rewritten as

$$\int_{-\infty}^{t} \mu(t-s)\Delta u(s)\mathrm{d}s = \int_0^{\infty} \mu(s)\Delta u(t-s)\mathrm{d}s = \int_0^{\infty} \mu(s)\mathrm{d}s\Delta u - \int_0^{\infty} \mu(s)\Delta\eta^t(s)\mathrm{d}s,$$

and Eq (1.1) becomes

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \left(\alpha - \int_0^{\infty} \mu(s)\mathrm{d}s\right)\Delta u - \int_0^{\infty} \mu(s)\Delta\eta^t(s)\mathrm{d}s + f(u) + g(u_t) = h(x).$$

Assuming for simplicity that $\alpha - \int_0^\infty \mu(s)\mathrm{d}s = 1$, we have the new system

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u - \displaystyle\int_0^{\infty} \mu(s)\Delta\eta^t(s)\mathrm{d}s + f(u) + g(u_t) = h(x), \\ \eta^t_t(x,s) = -\eta^t_s(x,s) + u_t(x,s), \\ u|_{\partial\Omega} = 0, \quad \eta^t|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \eta^0(x,s) = \eta_0(x,s), \end{cases} \tag{1.7}$$

where

$$\begin{cases} u_0(x) = u_0(x,0), \quad x \in \Omega, \\ u_1(x) = \partial_t u_0(x,t)|_{t=0}, \quad x \in \Omega, \\ \eta_0(x,s) = u_0(x,0) - u_0(x,-s), \quad (x,s) \in \Omega \times \mathbb{R}^+. \end{cases}$$

2. Assumptions and the main result

We begin with precise hypotheses on problem (1.7). Assume that

$$0 < \rho < \frac{4}{N-2} \quad \text{if} \quad N \geqslant 3 \quad \text{and} \quad \rho > 0 \quad \text{if} \quad N = 1, 2, \tag{2.1}$$

Concerning the source term $f : \mathbb{R} \to \mathbb{R}$, we assume that

$$f(0) = 0, \qquad |f(u) - f(v)| \leqslant c_0(1 + |u|^p + |v|^p)|u - v|, \quad \forall\, u, v \in \mathbb{R}, \tag{2.2}$$

where $c_0 > 0$ and

$$0 < p < \frac{4}{N-2} \quad \text{if} \quad N \geqslant 3 \quad \text{and} \quad p > 0 \quad \text{if} \quad N = 1, 2. \tag{2.3}$$

In addition, we assume that

$$f(u)u \geqslant F(u) \geqslant 0, \quad \forall\, u \in \mathbb{R}, \tag{2.4}$$

where $F(u) = \int_0^u f(s)ds$.

The damping function $g \in C^1(\mathbb{R})$ is a non-decreasing function with $g(0) = 0$ and satisfies the polynomial condition

$$g'(s) \geqslant 0, \qquad |g(u) - g(v)| \leqslant c_1(1 + |u|^q + |v|^q)|u - v|, \quad \forall\, u, v \in \mathbb{R}, \tag{2.5}$$

where $c_1 > 0$ and

$$0 < q \leqslant \frac{4}{N-2} \quad \text{if} \quad N \geqslant 3 \quad \text{and} \quad q > 0 \quad \text{if} \quad N = 1, 2. \tag{2.6}$$

With respect to the memory component, we assume that

$$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu'(s) \leqslant 0, \quad 0 \leqslant \mu(s) < \infty, \tag{2.7}$$

and there exist $k_0, k_1 > 0$ such that

$$\int_0^\infty \mu(s)\mathrm{d}s = k_0, \tag{2.8}$$

and

$$\mu'(s) + k_1\mu(s) \leqslant 0, \quad \forall\, s \in \mathbb{R}^+. \tag{2.9}$$

As usual, $\|\cdot\|_p$ denotes the L^p-norms as well as $(\cdot,\cdot)$ denotes either the L^2-inner product. Let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

In order to consider the relative displacement η^t as a new variable, one introduces the weighted L^2-space

$$\mathcal{M} = L^2_\mu(\mathbb{R}^+; H_0^1(\Omega)) = \left\{\xi : \mathbb{R}^+ \to H_0^1(\Omega) \middle| \int_0^\infty \mu(s)\|\nabla\xi(s)\|_2^2\mathrm{d}s < \infty\right\},$$

which is a Hilbert space endowed with inner product and norm

$$(\xi,\zeta)_{\mathcal{M}} = \int_0^\infty \mu(s)(\nabla\xi, \nabla\zeta)\mathrm{d}s \quad \text{and} \quad \|\xi\|^2_{\mathcal{M}} = \int_0^\infty \mu(s)\|\nabla\xi\|_2^2\mathrm{d}s,$$

respectively.

Next let us introduce the phase space

$$\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times \mathcal{M},$$

endowed with the norm

$$\|z\|_{\mathcal{H}} = \|(u, v, \eta)\|^2_{\mathcal{H}} = \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|\eta\|^2_{\mathcal{M}}.$$

Then, the energy of problem (1.7) is given by

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2 + \int_\Omega (F(u) - hu)\mathrm{d}x. \tag{2.10}$$

According to the arguments [24] with slightly modified, we can obtain the following well-posedness result.

Theorem 2.1. *Assume the assumptions* (2.1)-(2.7) *hold. If initial data* $z_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$ *and* $h \in L^2(\Omega)$*, then the problem* (1.7) *admits a unique global solution*

$$z = (u, u_t, \eta^t) \in C([0,T], \mathcal{H}), \tag{2.11}$$

satisfying

$$\begin{aligned} u \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)), \qquad & u_t \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)), \\ u_{tt} \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)), \qquad & \eta^t \in L^\infty(\mathbb{R}^+; \mathcal{M}). \end{aligned}$$

Remark 1. The well-posedness of problem (1.7) given by Theorem 2.1 implies that the one-parameter family of operators $S(t) : \mathcal{H} \to \mathcal{H}$ defined by

$$S(t)z_0 = (u(t), u_t(t), \eta^t(t)) = z, \qquad t \geqslant 0, \tag{2.12}$$

where $z = (u(t), u_t(t), \eta^t(t))$ is the unique weak solution of the system (1.7), satisfies the semigroup properties

$$S(0) = I \qquad \text{and} \qquad S(t+s) = S(t) \circ S(s), \qquad t, s \geqslant 0,$$

and defines a nonlinear C_0-semigroup. Then problem (1.7) can be viewed as a nonlinear infinite dynamical system $(\mathcal{H}, S(t))$.

Now we give the following result concerning the global attractors.

Theorem 2.2. *Assume the assumptions* (2.1)-(2.7) *hold and* $h \in L^2(\Omega)$*. Then the dynamical system* $(\mathcal{H}, S(t))$ *generated by* (1.7) *has a compact global attractor* $\mathcal{A} \subset \mathcal{H}$*.*

3. Global attractors

Before presenting our results we recall some fundamentals of the theory of infinite-dimensional dynamical systems which can be founded in the book by Chueshov and Lasiecka [32, 33].

Theorem 3.1. *A dissipative dynamical system* $(X, S(t))$ *has a compact global attractor if and only if it is asymptotically smooth.*

The proof of asymptotic smoothness property can be very delicate. Here we use the following "compensated compactness"result [32, 34].

Theorem 3.2. *Let* $(X, S(t))$ *be a dynamical system on a complete metric space* X *endowed with a metric d. Suppose that for any bounded positively invariant set* $B \subset X$ *and for any* $\varepsilon > 0$*, there exists* $T = T(\varepsilon, B)$ *such that*

$$\|S(T)x - S(T)y\|_X \leqslant \varepsilon + \Phi_T(x, y), \quad \forall\, x, y \in B,$$

where $\Phi_T : B \times B \to \mathbb{R}$ satisfies

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \Phi_T(z_n, z_m) = 0, \tag{3.1}$$

for any sequence $\{z_n\}_{n\in\mathbb{N}}$ in B. Then $S(t)$ is asymptotic smooth in $\mathcal{X}$.

In the sequel we will apply the abstract results presented above to prove Theorem 2.2. Firstly, we show that the dynamical system $(\mathcal{H}, S(t))$ is dissipative. The next step is to verify the asymptotic smoothness. Then the existence of a compact global attractor is guaranteed by Theorem 3.1. In what follows, the generic positive constants will be denoted as C, while $Q(\cdot)$ will stand for a generic increasing positive function.

3.1. Existence of an absorbing set

In this section, our aim is to show that the dynamical system $(\mathcal{H}, S(t))$ is dissipative. To this aim, we first give some priori estimates used later.

Proposition 3.1. *For any initial data z_0 with $\|z_0\|_{\mathcal{H}} \leqslant R$, we have the uniform estimate*

$$\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + \|\eta^t\|_{\mathcal{M}}^2 + \|\nabla u_{tt}\|_2^2 \leqslant Q(R), \quad \forall t \geqslant 0.$$

Proof. Multiplying (1.7) by (u_t, η^t), we obtain

$$E'(t) = -(g(u_t), u_t) - (\eta_s^t, \eta^t)_{\mathcal{M}}. \tag{3.2}$$

Owing to (2.7), one can easily see that

$$\begin{aligned}(\eta_s^t, \eta^t)_{\mathcal{M}} &= \frac{1}{2}\int_\Omega \Big(\int_0^\infty \mu(s)\frac{\mathrm{d}}{\mathrm{d}s}|\nabla\eta^t(s)|^2 \mathrm{d}s\Big)\mathrm{d}x \\ &= -\frac{1}{2}\int_\Omega \Big(\int_0^\infty \mu'(s)|\nabla\eta^t(s)|^2 \mathrm{d}s\Big)\mathrm{d}x. \end{aligned}\tag{3.3}$$

Combining (3.3) and (3.2), we have

$$E'(t) = -(g(u_t), u_t) + \frac{1}{2}\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|_2^2 \mathrm{d}s. \tag{3.4}$$

Since $\mu(s)$ is decreasing, $g' \geqslant 0$ and $g(0) = 0$, we get

$$E'(t) \leqslant 0,$$

and consequently

$$E(t) \leqslant E(0).$$

Applying Young inequality yields

$$\int_\Omega hu dx \leqslant \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{\lambda_1}\|h\|_2^2.$$

It follows promptly from (2.10) that

$$E(t) \geqslant \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2 + \int_\Omega F(u)\mathrm{d}x - \frac{1}{\lambda_1}\|h\|_2^2.$$

Then making use of (2.4) we obtain

$$\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2 \leqslant E(t) + \frac{1}{\lambda_1}\|h\|_2^2 \leqslant E(0) + \frac{1}{\lambda_1}\|h\|_2^2.$$

This means that

$$\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + \|\eta^t\|_{\mathcal{M}}^2 \leqslant Q(R), \quad t \geqslant 0. \tag{3.5}$$

A multiplication of (1.7) by u_{tt} gives

$$\int_\Omega |u_t|^\rho u_{tt}^2\mathrm{d}x + \|\nabla u_{tt}\|_2^2 = -\int_\Omega \nabla u\cdot\nabla u_{tt}\mathrm{d}x - (\eta^t, u_{tt})_{\mathcal{M}} - \int_\Omega f(u)u_{tt}\mathrm{d}x$$
$$- \int_\Omega g(u_t)u_{tt}\mathrm{d}x + \int_\Omega hu_{tt}\mathrm{d}x. \tag{3.6}$$

Next, we estimate each term individually. By Hölder inequality, Poincaré inequality and Young inequality, we have

$$-\int_\Omega \nabla u\cdot\nabla u_{tt}\mathrm{d}x \leqslant \frac{1}{6}\|\nabla u_{tt}\|_2^2 + \frac{3}{2}\|\nabla u\|_2^2,$$
$$-(\eta^t, u_{tt})_{\mathcal{M}} \leqslant \frac{1}{6}\|\nabla u_{tt}\|_2^2 + \frac{3k_0}{2}\|\eta^t\|_{\mathcal{M}}^2,$$

and

$$\int_\Omega hu_{tt}dx \leqslant \frac{1}{6}\|\nabla u_{tt}\|_2^2 + \frac{3}{2\lambda_1}\|h\|_2^2.$$

Using Hölder inequality, Poincaré inequality, Young inequality, and (3.5), we have

$$\begin{aligned} -\int_\Omega f(u)u_{tt}dx &\leqslant C(\|u\|_{p+2} + \|u\|_{p+2}^{p+1})\|u_{tt}\|_{p+2} \\ &\leqslant C(\|\nabla u\|_2 + \|\nabla u\|_2^{p+1})\|\nabla u_{tt}\|_2 \\ &\leqslant \frac{1}{6}\|\nabla u_{tt}\|_2^2 + Q(R), \end{aligned}$$

and

$$\begin{aligned} -\int_\Omega g(u_t)u_{tt}\mathrm{d}x &\leqslant C(\|u_t\|_{q+2} + \|u_t\|_{q+2}^{q+1})\|u_{tt}\|_{q+2} \\ &\leqslant C(\|\nabla u_t\|_2 + \|\nabla u_t\|_{q+2}^{q+1})\|u_{tt}\|_2 \\ &\leqslant \frac{1}{6}\|\nabla u_{tt}\|_2^2 + Q(R). \end{aligned}$$

Substituting all the above inequalities into (3.6) and taking (3.5) into account, we derive that

$$\int_\Omega |u_t|^\rho u_{tt}^2\mathrm{d}x + \frac{1}{6}\|\nabla u_{tt}\|_2^2 \leqslant Q(R).$$

This means that

$$\|\nabla u_{tt}\|_2^2 \leqslant Q(R). \tag{3.7}$$

In light of (3.5) and (3.7), we obtain the Proposition 3.1. □

Now, we introduce the following two functionals

$$\Phi(t) = \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u \mathrm{d}x + \int_\Omega \nabla u_t \nabla u \mathrm{d}x, \tag{3.8}$$

and

$$\Psi(t) = \int_\Omega \left(\Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t\right)\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\mathrm{d}x. \tag{3.9}$$

Lemma 3.3. *There exists a positive constant C_1, such that*

$$\Phi'(t) \leqslant -E(t) - \frac{1}{4}\|\nabla u\|_2^2 + C_1\|\nabla u_t\|_2^2 + \frac{2}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - C_1\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|_2^2\mathrm{d}s. \tag{3.10}$$

Proof. A multiplication of the first equation of (1.7) by u gives

$$\begin{aligned}
\Phi'(t) &= \int_\Omega (|u_t|^\rho u_{tt} - \Delta u_{tt})u\mathrm{d}x + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 \\
&= -\|\nabla u\|_2^2 + \int_0^\infty \mu(s)\Big(\int_\Omega \Delta\eta^t(s)u(t)\mathrm{d}x\Big)\mathrm{d}s - \int_\Omega f(u)u\mathrm{d}x - \int_\Omega g(u_t)u\mathrm{d}x \\
&\quad + \int_\Omega hu\mathrm{d}x + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2.
\end{aligned}$$

By Hölder inequality and Cauchy inequality, we obtain

$$\begin{aligned}
\int_0^\infty \mu(s)\Big(\int_\Omega \Delta\eta^t(s)u(t)\mathrm{d}x\Big)\mathrm{d}s &\leqslant \|\nabla u(t)\|_2\int_0^\infty \mu(s)\|\nabla\eta^t\|_2 ds \\
&\leqslant \frac{1}{8}\|\nabla u\|_2^2 + 2k_0\|\nabla\eta^t\|_{\mathcal{M}}^2.
\end{aligned}$$

Using Hölder inequality, Young inequality and Sobolev inequality, taking into account (2.5) and (2.6), we arrive at

$$\begin{aligned}
\int_\Omega g(u_t)u\mathrm{d}x \leqslant c_1\int_\Omega (1+|u_t|^q)|u_t||u|\mathrm{d}x &\leqslant C(1+\|u_t\|_{q+2}^q)\|u_t\|_{q+2}\|u\|_{q+2} \\
&\leqslant C(1+\|\nabla u_t\|_2^q)\|\nabla u_t\|_2\|\nabla u\|_2 \\
&\leqslant \frac{1}{8}\|\nabla u\|_2^2 + C(1+\|\nabla u_t\|_2^{2q})\|\nabla u_t\|_2^2 \\
&\leqslant \frac{1}{8}\|\nabla u\|_2^2 + Q(R)\|\nabla u_t\|_2^2.
\end{aligned}$$

Combining the last two estimates, we end up with

$$\Phi'(t) \leqslant \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - \frac{3}{4}\|\nabla u\|_2^2 + C\|\nabla u_t\|_2^2 + 2k_0\|\eta^t\|_{\mathcal{M}}^2 - \int_\Omega f(u)u\mathrm{d}x + \int_\Omega hu\mathrm{d}x.$$

Besides, in light of (2.9), we get

$$\|\eta^t\|_{\mathcal{M}}^2 \leqslant -\frac{1}{k_1}\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|_2^2\mathrm{d}s. \tag{3.11}$$

Using (2.10), (2.4) and (3.11) yields

$$\Phi'(t) \leqslant -E(t) - \frac{1}{4}\|\nabla u\|_2^2 + \frac{2}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + C_1\|\nabla u_t\|_2^2 - C_1\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|_2^2 \mathrm{d}s.$$

This lemma is complete. □

Lemma 3.4. *For any $\delta_1 > 0$, there exists $C_2 > 0$ such that*

$$\begin{aligned}\Psi'(t) \leqslant & \delta_1 C_2\|\nabla u(t)\|_2^2 - (k_0 - \delta_1 C_2)\|\nabla u_t(t)\|_2^2 - \frac{k_0}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2} \\ & + k_0\|h\|_2^2 - C_2\int_0^\infty \mu'(s)\|\nabla\eta^t\|_2^2 \mathrm{d}s. \end{aligned} \tag{3.12}$$

Proof. Taking the time derivative of $\Psi(t)$, in light of the first equation of (1.7), we get

$$\begin{aligned}\Psi'(t) =& \int_\Omega \big(- |u_t|^\rho u_{tt} + \Delta u_{tt}\big)\Big(\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\Big)\mathrm{d}x \\ &+ \int_\Omega \Big(- \frac{|u_t|^\rho u_t}{\rho+1} + \Delta u_t\Big)\Big(\int_0^\infty \mu(s)\eta^t_t(s)\mathrm{d}s\Big)\mathrm{d}x \\ =& \int_\Omega \Big(- \Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)\mathrm{d}s + g(u_t) + f(u) - h\Big)\Big(\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\Big)\mathrm{d}x \\ &+ \int_\Omega \Big(- \frac{|u_t|^\rho u_t}{\rho+1} + \Delta u_t\Big)\Big(\int_0^\infty \mu(s)\eta^t_t(s)\mathrm{d}s\Big)\mathrm{d}x. \end{aligned}$$

Next, we will estimate the right side of the above identity. Integrating by parts with respect to x and using Young inequality, we obtain

$$\begin{aligned}-\int_\Omega \Delta u\Big(\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\Big)\mathrm{d}x =& \int_\Omega \nabla u\cdot\Big(\int_0^\infty \mu(s)\nabla\eta^t(s)\mathrm{d}s\Big)\mathrm{d}x \\ \leqslant & \delta_1\|\nabla u\|_2^2 + \frac{k_0}{4\delta_1}\|\eta^t\|_{\mathcal{M}}^2, \end{aligned}$$

and

$$\begin{aligned}-\int_\Omega \Big(\int_0^\infty \mu(s)\Delta\eta^t(s)\mathrm{d}s\Big)\Big(\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\Big)\mathrm{d}x =& \int_\Omega \sum_{j=1}^N \Big(\int_0^\infty \mu(s)\frac{\partial\eta^t(s)}{\partial x_j}\mathrm{d}s\Big)^2 dx \\ \leqslant & k_0\int_\Omega \sum_{j=1}^N \Big(\int_0^\infty \mu(s)\Big|\frac{\partial\eta^t(s)}{\partial x_j}\Big|^2 \mathrm{d}s\Big)\mathrm{d}x \\ =& k_0\|\eta^t\|_{\mathcal{M}}^2. \end{aligned}$$

Applying (2.5), Hölder inequality, Sobolev embedding inequality, Young inequality and Proposition 3.1, we obtain

$$\int_\Omega g(u_t)\Big(\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\Big)\mathrm{d}x = \int_0^\infty \mu(s)\Big(\int_\Omega g(u_t)\eta^t(s)\mathrm{d}x\Big)\mathrm{d}s$$

$$
\begin{aligned}
&\leqslant C\int_0^\infty \mu(s)(1+\|u_t(t)\|_{q+2}^q)\|u_t(t)\|_{q+2}\|\eta^t(s)\|_{q+2}\mathrm{d}s\\
&\leqslant C\int_0^\infty (1+\|\nabla u_t(t)\|_2^q)\|\nabla u_t(t)\|_2\mu(s)\|\nabla\eta^t\|_2\mathrm{d}s\\
&\leqslant \delta_1 Q(R)\|\nabla u_t\|_2^2+\frac{1}{4\delta_1}\|\eta^t\|_{\mathcal{M}}^2.
\end{aligned}
$$

Analogously, but using (2.2) instead of (2.5), we have

$$
\begin{aligned}
\int_\Omega f(u)\Big(\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\Big)\mathrm{d}x &= \int_0^\infty \mu(s)\Big(\int_\Omega f(u)\eta^t(s)\mathrm{d}x\Big)\mathrm{d}s\\
&\leqslant C\int_0^\infty \mu(s)(1+\|u(t)\|_{p+2}^p)\|u(t)\|_{p+2}\|\eta^t(s)\|_{p+2}\mathrm{d}s\\
&\leqslant C\int_0^\infty \mu(s)(1+\|\nabla u(t)\|_2^p)\|\nabla u(t)\|_2\|\nabla\eta^t\|_2\mathrm{d}s\\
&\leqslant \delta_1 Q(R)\|\nabla u\|_2^2+\frac{1}{4\delta_1}\|\eta^t\|_{\mathcal{M}}^2.
\end{aligned}
$$

Using Hölder inequality, Young's inequality and Sobolev embedding inequality, we get

$$
\begin{aligned}
-\int_\Omega h\Big(\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\Big)\mathrm{d}x &\leqslant \int_0^\infty \mu(s)\|h\|_2\|\eta^t(s)\|_2\mathrm{d}s\\
&\leqslant k_0\|h\|_2^2+C\|\eta^t(s)\|_{\mathcal{M}}^2.
\end{aligned}
$$

On the other hand, since

$$
\int_0^\infty \mu(s)\eta_t^t(s)\mathrm{d}s=-\int_0^\infty \mu(s)\eta_s^t(s)\mathrm{d}s+\int_0^\infty \mu(s)u_t(t)\mathrm{d}s=\int_0^\infty \mu'(s)\eta^t(s)\mathrm{d}s+k_0u_t(t),
$$

we find

$$
\begin{aligned}
&\int_\Omega\Big(-\frac{|u_t|^\rho u_t}{\rho+1}+\Delta u_t\Big)\Big(\int_0^\infty \mu(s)\eta_t^t(s)\mathrm{d}s\Big)\mathrm{d}x\\
&=-k_0\|\nabla u_t(t)\|_2^2-\frac{k_0}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2}+\int_0^\infty \mu'(s)\Big(\int_\Omega \Delta u_t(t)\eta^t(s)\mathrm{d}x\Big)\mathrm{d}s\\
&\quad-\frac{1}{\rho+1}\int_0^\infty \mu'(s)\Big(\int_\Omega |u_t(t)|^\rho u_t(t)\eta^t(s)\Big)\mathrm{d}s.
\end{aligned}
$$

Applying Hölder inequality, Young's inequality and Sobolev embedding inequality, we obtain

$$
\begin{aligned}
\int_0^\infty \mu'(s)\Big(\int_\Omega \Delta u_t(t)\eta^t(s)\mathrm{d}x\Big)\mathrm{d}s &\leqslant -\int_0^\infty \mu'(s)\|\nabla u_t(t)\|_2\|\nabla\eta^t(s)\|_2\mathrm{d}s\\
&\leqslant \delta_1\|\nabla u_t(t)\|_2^2-\frac{\mu(0)}{4\delta_1}\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|_2^2\mathrm{d}s,
\end{aligned}
$$

and

$$
-\frac{1}{\rho+1}\int_0^\infty \mu'(s)\Big(\int_\Omega |u_t(t)|^\rho u_t(t)\eta^t(s)\Big)\mathrm{d}s\leqslant -\frac{1}{\rho+1}\int_0^\infty \mu'(s)\|u_t\|_{\rho+2}^{\rho+1}\|\eta^t\|_{\rho+2}\mathrm{d}s
$$

$$\leqslant \frac{\delta_1\mu(0)}{\rho+1}\|u_t\|_{\rho+2}^{2(\rho+1)} - \frac{1}{4\delta_1(\rho+1)}\int_0^\infty \mu'(s)\|\eta^t\|_{\rho+2}^2 \mathrm{d}s$$
$$\leqslant \delta_1 C\|\nabla u_t\|_2^{2(\rho+1)} - C\int_0^\infty \mu'(s)\|\nabla\eta^t\|_2^2 \mathrm{d}s$$
$$\leqslant \delta_1 Q(R)\|\nabla u_t\|_2^2 - C\int_0^\infty \mu'(s)\|\nabla\eta^t\|_2^2 \mathrm{d}s.$$

Collecting all the above inequalities and (3.11), we end up with

$$\Psi'(t) \leqslant \delta_1 C_2\|\nabla u(t)\|_2^2 - (k_0 - \delta_1 C_2)\|\nabla u_t(t)\|_2^2 - \frac{k_0}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2} + C_2\|h\|_2^2 - C_2\int_0^\infty \mu'(s)\|\nabla\eta^t\|_2^2 \mathrm{d}s.$$

This lemma is complete. □

Lemma 3.5 (Absorbing set)**.** *Under the hypotheses of Theorem 2.2, the dynamical system* $(\mathcal{H}, S(t))$ *corresponding to problem* (1.7) *has a bounded absorbing set.*

Proof. Let us consider a perturbed functional

$$\mathcal{L}(t) = ME(t) + \varepsilon\Phi(t) + \Psi(t) \tag{3.13}$$

where $\Phi(t)$ and $\Psi(t)$ are the same functional defined in (3.8) and (3.9).

Since

$$\int_\Omega hu\mathrm{d}x \leqslant \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{\lambda_1}\|h\|_2^2,$$

we get

$$E(t) \geqslant \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\eta^t\|_{\mathcal{M}}^2 + \int_\Omega F(u)\mathrm{d}x - \frac{1}{\lambda_1}\|h\|_2^2.$$

Then making use of (2.4) we obtain

$$\frac{1}{4}(\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2 + \|\eta^t\|_{\mathcal{M}}^2) \leqslant E(t) + \frac{1}{\lambda_1}\|h\|_2^2. \tag{3.14}$$

Now we claim that there exist three constants $\alpha_1, \alpha_2, b > 0$ such that

$$\alpha_1 E(t) - b\|h\|_2^2 \leqslant \mathcal{L}(t) \leqslant \alpha_2 E(t) + b\|h\|_2^2. \tag{3.15}$$

To prove this, we first note that

$$|\Phi(t)| \leqslant C_0(\|\nabla u_t\|_2^2 + \|\nabla u\|_2^2) \leqslant 4C_0\Big(E(t) + \frac{1}{\lambda_1}\|h\|_2^2\Big),$$

$$|\Psi(t)| \leqslant C_0(\|\nabla u_t\|_2^2 + \|\nabla u\|_2^2) \leqslant 4C_0\Big(E(t) + \frac{1}{\lambda_1}\|h\|_2^2\Big).$$

Hence there exists a constant $b > 0$ such that

$$|\varepsilon\Phi(t) + \Psi(t)| \leqslant b\Big(E(t) + \|h\|_2^2\Big).$$

Then taking $M > b$, we get (3.15) with $\alpha_1 = M - b$ and $\alpha_2 = M - b$.

Combining (3.4), (3.13), Lemma 3.3 and 3.4, we arrive at

$$\begin{aligned}\mathcal{L}'(t) =& ME'(t) + \varepsilon\Phi'(t) + \Psi'(t)\\ \leqslant& -\varepsilon E(t) - \Big(k_0 - \delta_1 C_2 - \varepsilon C_1\Big)\|\nabla u_t\|_2^2 - \Big(\frac{\varepsilon}{4} - \delta_1 C_2\Big)\|\nabla u\|_2^2 + k_0\|h\|_2^2\\ &+ \left[\frac{M}{2} - (\varepsilon C_1 + C_2)\right]\int_0^\infty \mu'(s)\|\nabla\eta^t\|_2^2 \mathrm{d}s.\end{aligned}$$

Choosing $\varepsilon, \delta_1 > 0$ small enough and $M > 0$ sufficiently large such that

$$k_0 - \delta_1 C_2 - \varepsilon C_1 > 0, \qquad \varepsilon - 4\delta_1 C_2 > 0, \qquad M - 2(\varepsilon C_1 + C_2) > 0,$$

then we have

$$\mathcal{L}'(t) \leqslant -\varepsilon E(t) + k_0\|h\|_2^2,$$

which together with (3.15) implies that

$$\mathcal{L}'(t) \leqslant -\frac{\varepsilon}{\alpha_2}\mathcal{L}(t) + \left(\frac{b\varepsilon}{\alpha_2} + k_0\right)\|h\|_2^2.$$

Integrating the above inequality over $[0, t]$, we can derive

$$\mathcal{L}(t) \leqslant \mathcal{L}(0)e^{-\frac{\varepsilon}{\alpha_2}t} + (1 - e^{-\frac{\varepsilon}{\alpha_2}t})\Big(b + \frac{\alpha_2 k_0}{\varepsilon}\Big)\|h\|_2^2.$$

Using again (3.15) yields

$$E(t) \leqslant \frac{\alpha_2}{\alpha_1}E(0)e^{-\frac{\varepsilon}{\alpha_2}t} + \frac{1}{\alpha_1}\Big(2b + \frac{\alpha_2 k_0}{\varepsilon}\Big)\|h\|_2^2.$$

Recalling (3.14), we obtain

$$\|(u, u_t, \eta^t)\|_{\mathcal{H}}^2 \leqslant \frac{4\alpha_2}{\alpha_1}E(0)e^{-\frac{\varepsilon}{\alpha_2}t} + R_0^2,$$

where

$$R_0^2 = \frac{1}{\alpha_1}\Big(2b + \frac{\alpha_2 k_0}{\varepsilon}\Big)\|h\|_2^2 + 4b\|h\|_2^2.$$

This shows that any closed ball $\bar{B}(0, R)$ with $R > R_0$ is a bounded absorbing set of $(\mathcal{H}, S(t))$. The proof of Lemma 3.5 is now complete. □

As a straight consequence of Lemma 3.5, we have that the solutions of problem (1.7) are globally bounded provided initial data lying in bounded sets $B \subset \mathcal{H}$. Namely, let $z = (u, u_t, \eta^t)$ be a solution of (1.7) with initial data $z_0 = (u_0, u_1, \eta_0)$ in a bounded set B. Then one has

$$\|z\|_{\mathcal{H}} \leqslant C_B, \quad \forall t \geqslant 0, \tag{3.16}$$

where C_B is a constant depending on B. Lemma 3.5 also ensures the existence of bounded positively invariant sets.

3.2. A stability inequality

Lemma 3.6 (Stabilizability)**.** *Under the hypotheses of Theorem* (2.2)*, given a bounded set $B \subset \mathcal{H}$, let $z_1 = (u, u_t, \eta^t)$ and $z_2 = (v, v_t, \xi^t)$ be two weak solutions of problem* (1.7) *such that $z_1(0) = (u_0, u_1, \eta_0)$ and $z_2(0) = (v_0, v_1, \xi_0)$ are in B. Then*

$$\begin{aligned}\|z_1(t) - z_2(t)\|_{\mathcal{H}}^2 \leqslant & C_B e^{-\gamma t}\|z_1(0) - z_2(0)\|_{\mathcal{H}}^2 \\ & + C_B \int_0^t e^{-\gamma(t-\tau)}(\|w(\tau)\|_{\rho+2} + \|w(\tau)\|_{p+2} + \|w_t(\tau)\|_{\rho+2} + \|w_t(\tau)\|_{p+2})\mathrm{d}\tau,\end{aligned}$$

where $w = u - v$ and γ, C_B are positive constants depending on B.

Proof. Let us write $w = u - v$ and $\zeta^t = \eta^t - \xi^t$. Then (w, ζ^t) satisfies

$$\begin{cases} \Delta w + \Delta w_{tt} + \int_0^\infty \mu(s)\Delta\zeta^t(s)\mathrm{d}s = |u_t|^\rho u_{tt} - |v_t|^\rho v_{tt} + f(u) - f(v) + g(u_t) - g(v_t), \\ \zeta_t^t = -\zeta_s^t + w_t, \\ w(0) = u_0 - v_0, \quad w_t(0) = u_1 - v_1, \quad \zeta_0 = \eta_0 - \xi_0. \end{cases} \tag{3.17}$$

Now we consider the functional

$$E_w(t) = \frac{1}{2}\|\nabla w_t\|_2^2 + \frac{1}{2}\|\nabla w\|_2^2 + \frac{1}{2}\|\zeta^t\|_{\mathcal{M}}^2 \tag{3.18}$$

and its perturbation

$$\mathcal{G}(t) = ME_w(t) + \varepsilon\phi(t) + \psi(t), \quad \varepsilon > 0, \tag{3.19}$$

where

$$\phi(t) = -\int_\Omega \Delta w_t(t)w(t)\mathrm{d}x,$$

$$\psi(t) = \int_\Omega \Big[\Delta w_t - \frac{1}{\rho+1}(|u_t|^\rho u_t - |v_t|^\rho v_t)\Big]\Big(\int_0^\infty \mu(s)\zeta^t(s)\mathrm{d}s\Big)\mathrm{d}x.$$

We divide the remaining of the proof into five steps. Hereafter, we use C_B to denote several positive constants.

Step 1. For $M > 0$ sufficiently large, there exists ε_0 such that

$$\frac{M}{2}E_w(t) \leqslant \mathcal{G}(t) \leqslant \frac{3M}{2}E_w(t), \quad t \geqslant 0, \quad \varepsilon \in (0, \varepsilon_0]. \tag{3.20}$$

To prove this, we first observe that

$$|\phi(t)| \leqslant \frac{1}{2}\|\nabla w\|_2^2 + \frac{1}{2}\|\nabla w_t\|_2^2 \leqslant E_w(t).$$

Besides, using Hölder inequality, Sobolev inequality, Young inequality and (3.16), we can derive that

$$|\psi(t)| \leqslant \int_0^\infty \mu(s)\|\nabla w_t\|_2\|\nabla\zeta^t(s)\|_2\mathrm{d}s$$

$$
\begin{aligned}
&+ 2^{\rho}\int_0^{\infty}\mu(s)(\|u_t\|_{\rho+2}^{\rho} + \|v_t\|_{\rho+2}^{\rho})\|w_t\|_{\rho+2}\|\zeta^t\|_{\rho+2}\mathrm{d}s\\
\leqslant&\frac{k_0}{2}\|\nabla w_t\|_2^2 + \frac{1}{2}\|\zeta^t\|_{\mathcal{M}}^2 + C_B\int_0^{\infty}\mu(s)(\|\nabla u_t\|_2^{\rho} + \|\nabla v_t\|_2^{\rho})\|\nabla w_t\|_2\|\nabla\zeta^t\|_2\mathrm{d}s\\
\leqslant&\frac{k_0}{2}\|\nabla w_t\|_2^2 + \frac{1}{2}\|\zeta^t\|_{\mathcal{M}}^2 + C_B\|\nabla w_t\|_2^2 + \frac{1}{2}\|\zeta^t\|_{\mathcal{M}}^2\\
\leqslant&C_BE_w(t).
\end{aligned}
$$

Collecting the above two estimates and (3.19), we obtain

$$
(M - C_B - \varepsilon)E_w(t) \leqslant \mathcal{G}(t) \leqslant (M + C_B + \varepsilon)E_w(t).
$$

Now let us put $\varepsilon_0 = M/2 - C_B$. Then for all $\varepsilon \leqslant \epsilon_0$, the inequality (3.20) holds.
Step 2. There exists a constant $C_3 > 0$ such that

$$
E_w'(t) \leqslant \frac{1}{2}\int_0^{\infty}\mu'(s)\|\nabla\zeta^t(s)\|_2^2\mathrm{d}s + C_3(\|w_t\|_{\rho+2} + \|w_t\|_{p+2}). \tag{3.21}
$$

In fact, differentiating (3.18) with respect to t and using (3.17), we have

$$
\begin{aligned}
E_w'(t) =& \int_\Omega(|v_t|^{\rho}v_{tt} - |u_t|^{\rho}u_{tt})w_t\mathrm{d}x - \int_\Omega(f(u) - f(v))w_t\mathrm{d}x\\
&- \int_\Omega(g(u_t) - g(v_t))w_t\mathrm{d}x + \frac{1}{2}\int_0^{\infty}\mu'(s)\|\nabla\zeta^t(s)\|_2^2\mathrm{d}s.
\end{aligned} \tag{3.22}
$$

Using Hölder inequality, Sobolev inequality, estimate (3.7) and (3.16) , we have

$$
\begin{aligned}
\int_\Omega(|v_t|^{\rho}v_{tt} - |u_t|^{\rho}u_{tt})w_t\mathrm{d}x &\leqslant \int_\Omega|v_t|^{\rho}|v_{tt}||w_t|\mathrm{d}x + \int_\Omega|u_t|^{\rho}|u_{tt}||w_t|\mathrm{d}x\\
&\leqslant (\|v_t\|_{\rho+2}^{\rho}\|v_{tt}\|_{\rho+2} + \|u_t\|_{\rho+2}^{\rho}\|u_{tt}\|_{\rho+2})\|w_t\|_{\rho+2}\\
&\leqslant C_B(\|\nabla v_t\|_2^{\rho}\|\nabla v_{tt}\|_2 + \|\nabla u_t\|_2^{\rho}\|\nabla u_{tt}\|_2)\|w_t\|_{\rho+2}\\
&\leqslant C_B\|w_t\|_{\rho+2}.
\end{aligned}
$$

Combining (2.2), Hölder inequality, Sobolev inequality, and estimate (3.16) yields

$$
\begin{aligned}
-\int_\Omega(f(u) - f(v))w_t\mathrm{d}x &\leqslant C_B(1 + \|u\|_{p+2}^{p} + \|v\|_{p+2}^{p})\|w\|_{p+2}\|w_t\|_{p+2}\\
&\leqslant C_B(1 + \|\nabla u\|_2^{p} + \|\nabla v\|_2^{p})\|\nabla w\|_2\|w_t\|_{p+2}\\
&\leqslant C_B\|w_t\|_{p+2}.
\end{aligned}
$$

Since g is a non-decreasing function, this yields

$$
\int_\Omega(g(u_t) - g(v_t))w_t\mathrm{d}x \geqslant 0.
$$

Inserting last three estimates into (3.22), we arrive at

$$
E_w'(t) \leqslant \frac{1}{2}\int_0^{\infty}\mu'(s)\|\nabla\zeta^t(s)\|_2^2\mathrm{d}s + C_B(\|w_t\|_{\rho+2} + \|w_t\|_{p+2}).
$$

Step 3. There exists $C_4 > 0$ such that

$$\phi'(t) \leqslant -E_w(t) - \frac{1}{4}\|\nabla w\|_2^2 + C_4(\|w\|_{\rho+2} + \|w\|_{p+2}) + C_4\|\nabla w_t\|_2^2 - C_4\int_0^\infty \mu'(s)\|\nabla\zeta^t(s)\|_2^2 \mathrm{d}s. \tag{3.23}$$

Taking the derivative of $\phi(t)$, it follows from (3.17) that

$$\begin{aligned}\phi'(t) &= -\int_\Omega \Delta w_{tt} w \mathrm{d}x + \|\nabla w_t\|_2^2 \\ &= \int_\Omega (|v_t|^\rho v_{tt} - |u_t|^\rho u_{tt}) w \mathrm{d}x - \int_\Omega (f(u) - f(v)) w \mathrm{d}x - \int_\Omega (g(u_t) - g(v_t)) w \mathrm{d}x \\ &\quad + \int_0^\infty \mu(s)\int_\Omega \Delta\zeta^t(s) w \mathrm{d}x\mathrm{d}s - \|\nabla w\|_2^2 + \|\nabla w_t\|_2^2. \end{aligned} \tag{3.24}$$

From Hölder inequality, Sobolev embedding, estimate (3.7) and (3.16), we have

$$\begin{aligned}\int_\Omega (|v_t|^\rho v_{tt} - |u_t|^\rho u_{tt}) w \mathrm{d}x &\leqslant \int_\Omega |v_t|^\rho |v_{tt}||w| \mathrm{d}x + \int_\Omega |u_t|^\rho |u_{tt}||w| \mathrm{d}x \\ &\leqslant (\|v_t\|_{\rho+2}^\rho \|v_{tt}\|_{\rho+2} + \|u_t\|_{\rho+2}^\rho \|u_{tt}\|_{\rho+2})\|w\|_{\rho+2} \\ &\leqslant C_B(\|\nabla v_t\|_2^\rho \|\nabla v_{tt}\|_2 + \|\nabla u_t\|_2^\rho \|\nabla u_{tt}\|_2)\|w\|_{\rho+2} \\ &\leqslant C_B\|w\|_{\rho+2}.\end{aligned}$$

Applying (2.2), Hölder inequality, Sobolev embedding, and estimate (3.16), we get

$$\begin{aligned}-\int_\Omega (f(u) - f(v)) w \mathrm{d}x &\leqslant (1 + \|u\|_{p+2}^p + \|v\|_{p+2}^p)\|w\|_{p+2}\|w\|_{p+2} \\ &\leqslant C_B(1 + \|\nabla u\|_2^p + \|\nabla v\|_2^p)\|\nabla w\|_2\|w\|_{p+2} \\ &\leqslant C_B\|w\|_{p+2}.\end{aligned}$$

Similarly, in light of (2.5), Hölder inequality, Young inequality, Sobolev embedding, and estimate (3.16), we obtain

$$\begin{aligned}-\int_\Omega (g(u_t) - g(v_t)) w \mathrm{d}x &\leqslant C_B(1 + \|u_t\|_{q+2}^q + \|v\|_{q+2}^q)\|w_t\|_{q+2}\|w\|_{q+2} \\ &\leqslant C_B\|\nabla w_t\|_2\|\nabla w\|_2 \leqslant \frac{1}{8}\|\nabla w\|_2^2 + C_B\|\nabla w_t\|_2^2.\end{aligned}$$

Using Young inequality gives

$$\int_0^\infty \mu(s)\int_\Omega \Delta\zeta^t(s) w \mathrm{d}x\mathrm{d}s \leqslant \frac{1}{8}\|\nabla w\|_2^2 + 2k_0\|\zeta^t\|_{\mathcal{M}}^2.$$

Combining these six last estimates with (3.18) we end up with

$$\phi'(t) \leqslant -E_w(t) - \frac{1}{4}\|\nabla w\|_2^2 + C_B\|\nabla w_t\|_2^2 + (\frac{1}{2} + 2k_0)\|\zeta^t\|_{\mathcal{M}} + C_B(\|w\|_{\rho+2} + \|w\|_{p+2}),$$

which together with (3.11) implies that inequality (3.23) holds for some $C_4 > 0$.

Step 4. For any $\delta_2 > 0$, there exists $C_5 > 0$ such that

$$\psi'(t) \leqslant -\frac{k_0}{4}\|\nabla w_t\|_2^2 + 2\delta_2\|\nabla w\|_2^2 - C_5\int_0^\infty \mu'(s)\|\nabla\zeta^t\|_2^2 \mathrm{d}s. \tag{3.25}$$

Taking derivative of $\psi(t)$ and using (3.17), we derive that

$$\begin{aligned}
\psi'(t) = &-\int_\Omega \Delta w\int_0^\infty \mu(s)\zeta^t(s)\mathrm{d}s\mathrm{d}x - \int_\Omega\int_0^\infty \mu(s)\Delta\zeta^t(s)\mathrm{d}s\int_0^\infty \mu(s)\zeta^t(s)\mathrm{d}s\mathrm{d}x \\
&+ \int_\Omega (f(u)-f(v))\int_0^\infty \mu(s)\zeta^t(s)\mathrm{d}s\mathrm{d}x + \int_\Omega (g(u_t)-g(v_t))\int_0^\infty \mu(s)\zeta^t(s)\mathrm{d}s\mathrm{d}x \\
&+ \int_\Omega \Delta w_t\int_0^\infty \mu(s)\zeta_t^t(s)\mathrm{d}s\mathrm{d}x - \frac{1}{\rho+1}\int_\Omega (|u_t|^\rho u_t - |v_t|^\rho v_t)\int_0^\infty \mu(s)\zeta_t^t(s)\mathrm{d}s\mathrm{d}x \\
= &\sum_{i=1}^6 A_i.
\end{aligned} \tag{3.26}$$

Applying Hölder inequality, Young inequality, Sobolev embedding, estimate (3.7) and (3.16), we get

$$A_1 \leqslant \delta_2\|\nabla w\|_2^2 + \frac{k_0}{4\delta}\|\zeta^t\|_{\mathcal{M}}^2, \tag{3.27}$$

$$A_2 \leqslant k_0\|\zeta^t\|_{\mathcal{M}}^2, \tag{3.28}$$

$$\begin{aligned}
A_3 &\leqslant 3^p(1 + \|u\|_{p+2}^p + \|v\|_{p+2}^p)\|w\|_{p+2}\int_0^\infty \mu(s)\|\zeta^t(s)\|_{p+2}\mathrm{d}s \\
&\leqslant C_B(1 + \|\nabla u\|_2^p + \|\nabla v\|_2^p)\|\nabla w\|_2\int_0^\infty \mu(s)\|\nabla\zeta^t(s)\|_2\mathrm{d}s \\
&\leqslant \delta_2\|\nabla w\|_2^2 + \frac{C_B}{4\delta_2}\|\zeta^t\|_{\mathcal{M}}^2,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
A_4 &\leqslant 2^q(1 + \|u\|_{q+2}^q + \|v\|_{q+2}^q)\|w_t\|_{q+2}\int_0^\infty \mu(s)\|\zeta^t(s)\|_{q+2}\mathrm{d}s \\
&\leqslant C_B(1 + \|\nabla u\|_2^q + \|\nabla v\|_2^q)\|\nabla w_t\|_2\int_0^\infty \mu(s)\|\nabla\zeta^t(s)\|_2\mathrm{d}s \\
&\leqslant \frac{k_0}{4}\|\nabla w_t\|_2^2 + C_B\|\zeta^t\|_{\mathcal{M}}^2.
\end{aligned} \tag{3.30}$$

From (3.17), one can easily see that

$$\int_0^\infty \mu(s)\zeta_t^t(s)\mathrm{d}s = -\int_0^\infty \mu(s)\zeta_s^t(s)\mathrm{d}s + \int_0^\infty \mu(s)w_t\mathrm{d}s = k_0 w_t + \int_0^\infty \mu'(s)\zeta^t(s)\mathrm{d}s.$$

Hence in light of Young inequality we obtain

$$\begin{aligned}
A_5 &\leqslant -k_0\|\nabla w_t\|_2^2 - \int_0^\infty \mu'(s)\|\nabla w_t\|_2\|\nabla\zeta^t(s)\|_2\mathrm{d}s \\
&\leqslant -\frac{3k_0}{4}\|\nabla w_t\|_2^2 - \frac{\mu(0)}{k_0}\int_0^\infty \mu'(s)\|\nabla\zeta^t(s)\|_2^2\mathrm{d}s.
\end{aligned} \tag{3.31}$$

By the monotonicity of function $x \mapsto |x|^\rho x$ ($\rho > 0$), we get

$$-\frac{1}{\rho+1}\int_\Omega (|u_t|^\rho u_t - |v_t|^\rho v_t)w_t \mathrm{d}x \leqslant 0.$$

Using Hölder inequality, Young inequality and estimate (3.16), we obtain

$$\begin{aligned} A_6 \leqslant & -2^\rho \int_0^\infty \mu'(s)(\|u_t\|_{\rho+2}^\rho + \|v_t\|_{\rho+2}^\rho)\|w_t(t)\|_{\rho+2}\|\eta^t(s)\|_{\rho+2} ds \\ \leqslant & -2^\rho \int_0^\infty \mu'(s)(\|\nabla u_t\|_2^\rho + \|\nabla v_t\|_2^\rho)\|\nabla w_t(t)\|_2\|\nabla\eta^t(s)\|_2 \mathrm{d}s \\ \leqslant & \frac{\mu(0)}{4}\|\nabla w_t(t)\|_2^2 - C_B\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|_2^2 \mathrm{d}s. \end{aligned} \tag{3.32}$$

Inserting (3.27)-(3.32) into (3.26), (3.11) we end up with

$$\psi'(t) \leqslant -\frac{k_0}{4}\|\nabla w_t\|_2^2 + 2\delta_2\|\nabla w\|_2^2 - C_5\int_0^\infty \mu'(s)\|\nabla\zeta^t\|_2^2 \mathrm{d}s.$$

Step 5. Combining (3.21), (3.23), (3.25) with (3.19), we have

$$\begin{aligned} \mathcal{G}'(t) =& ME_w'(t) + \varepsilon\phi'(t) + \psi'(t) \\ \leqslant & -\varepsilon E_w(t) - (\frac{k_0}{2} - \varepsilon C_4)\|\nabla w_t\|_2^2 \\ & - (\frac{\varepsilon}{4} - 2\delta_2)\|\nabla w\|_2^2 + \left(\frac{M}{2} - \varepsilon C_4 - C_5\right)\int_0^\infty \mu'(s)\|\nabla\zeta^t\|_2^2 \mathrm{d}s \\ & + C(B, M, \varepsilon)(\|w\|_{\rho+2} + \|w\|_{p+2} + \|w_t\|_{\rho+2} + \|w_t\|_{p+2}). \end{aligned} \tag{3.33}$$

Firstly we fix $\varepsilon > 0$ such that $\varepsilon C_4 < k_0/2$. Then taking $\delta_2 > 0$ such that $\delta_2 < \varepsilon/8$. For fixed ε and δ_2, we choose $M > 0$ so large that $M > 2(\varepsilon C_4 + C_5)$. Then (3.33) along with (3.20) give

$$\begin{aligned} \mathcal{G}'(t) \leqslant & -\varepsilon E_w(t) + C_B(\|w\|_{\rho+2} + \|w\|_{p+2} + \|w_t\|_{\rho+2} + \|w_t\|_{p+2}) \\ \leqslant & -\frac{2\varepsilon}{3M}\mathcal{G}(t) + C_B(\|w\|_{\rho+2} + \|w\|_{p+2} + \|w_t\|_{\rho+2} + \|w_t\|_{p+2}). \end{aligned} \tag{3.34}$$

Integrating (3.34) over $(0, t)$ with respect to t, we get

$$\mathcal{G}(t) \leqslant \mathcal{G}(0)e^{-\frac{2\varepsilon}{3M}t} + C_B\int_0^t e^{-\frac{2\varepsilon}{3M}(t-\tau)}(\|w\|_{\rho+2} + \|w\|_{p+2} + \|w_t\|_{\rho+2} + \|w_t\|_{p+2})\mathrm{d}\tau,$$

which together with (3.20) implies that

$$E_w(t) \leqslant 3E_w(0)e^{-\gamma t} + C_B\int_0^t e^{-\gamma(t-\tau)}(\|w\|_{\rho+2} + \|w\|_{p+2} + \|w_t\|_{\rho+2} + \|w_t\|_{p+2})\mathrm{d}\tau,$$

where $\gamma = 2\varepsilon/3M$ is a positive constant. Notice that the functional $E_w(t)$ is equivalent to the norm of $\mathcal{H}$, the proof is complete. □

3.3. Existence of global attractor

Lemma 3.7 (Asymptotic smoothness)**.** *Under the hypotheses of Theorem 2.2, the dynamical system corresponding to problem* (1.7) *is asymptotic smooth.*

Proof. Let B be a bounded subset of $\mathcal{H}$ positively invariant with respect to $S(t)$. Let $S(t)z_1(0) = (u, u_t, \eta^t)$ and $S(t)z_2(0) = (v, v_t, \xi^t)$ be two solutions for problem (1.7) corresponding to initial data $z_1(0), z_2(0) \in B$. Given $\varepsilon > 0$, we can choose $T > 0$ so large that $C_B e^{-\gamma t} < \varepsilon$. We claim that there exists constant $C_{BT} > 0$ such that

$$\|z_1 - z_2\|_{\mathcal{H}} \leqslant \varepsilon + \Phi_T(z_1(0), z_2(0)), \qquad \forall\, z_1(0), z_2(0) \in B, \tag{3.35}$$

with

$$\Phi_T(z_0^1, z_0^2) = C_{BT}\Big(\int_0^T (\|u(\tau) - v(\tau)\|_{\rho+2}^2 + \|u(\tau) - v(\tau)\|_{p+2}^2 + \|u_t(\tau) - v_t(\tau)\|_{\rho+2}^2 + \|u_t(\tau) - v_t(\tau)\|_{p+2}^2) \mathrm{d}\tau\Big)^{\frac{1}{2}}. \tag{3.36}$$

Indeed, from Lemma 3.6, we have

$$\begin{aligned}\|z_1(T) - z_2(T)\|_{\mathcal{H}} \leqslant & C_B e^{-\gamma T} + C_B\Big(\int_0^T e^{-2\gamma(t-\tau)}\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_0^T (\|u(\tau) - v(\tau)\|_{\rho+2}^2 \\ & + \|u(\tau) - v(\tau)\|_{p+2}^2 + \|u_t(\tau) - v_t(\tau)\|_{\rho+2}^2 + \|u_t(\tau) - v_t(\tau)\|_{p+2}^2)\mathrm{d}\tau\Big)^{\frac{1}{2}} \\ \leqslant & C_B e^{-\gamma T} + C_{BT}\Big(\int_0^T (\|u(\tau) - v(\tau)\|_{\rho+2}^2 + \|u(\tau) - v(\tau)\|_{p+2}^2 \\ & + \|u_t(\tau) - v_t(\tau)\|_{\rho+2}^2 + \|u_t(\tau) - v_t(\tau)\|_{p+2}^2)\mathrm{d}\tau\Big)^{\frac{1}{2}},\end{aligned}$$

and consequently (3.35) and (3.36) hold.

We are left to prove that Φ_T satisfies (3.1). Indeed, given a sequence of initial data $z_n = (u_0^n, u_1^n, \eta_0^n) \in B$, we write $S(t)z_n = (u^n(t), u_t^n(t), \eta^{n,t})$. Since B is invariant by $S(t)$, $t \geqslant 0$, it follows that $(u^n(t), u_t^n(t), \eta^{n,t})$ uniformly bounded in $\mathcal{H}$. Namely,

$$(u^n, u_t^n, \eta^{n,t}) \text{ is bounded in } C([0,T]; H_0^1(\Omega) \times H_0^1(\Omega) \times \mathcal{M}), \quad T > 0.$$

Then by compact embedding $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^{p+2}(\Omega)$, there exists a subsequence $(u^n, u_t^n, \eta^{n,t})$ such that

$$\begin{aligned}&u^n \text{ and } u_t^n \text{converges strongly in } C([0,T]; L^{\rho+2}(\Omega));\\ &u^n \text{ and } u_t^n \text{converges strongly in } C([0,T]; L^{p+2}(\Omega)).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_0^T \Big(\|u^n(\tau) - u^m(\tau)\|_{\rho+2}^2 + \|u_t^n(\tau) - u_t^m(\tau)\|_{\rho+2}^2 + \|u^n(\tau) - u^m(\tau)\|_{p+2}^2 + \|u_t^n(\tau) - u_t^m(\tau)\|_{p+2}^2\Big)\mathrm{d}\tau = 0,$$

which implies (3.1) holds. Then asymptotic smoothness follows from Theorem 3.2. □

Proof of Theorem 2.2. We first note that Lemmas 3.5 and 3.7 imply that $(\mathcal{H}, S(t))$ is a dissipative dynamical system which is asymptotically smooth. Then the existence of a compact global attractor $\mathscr{A}$ to problem (1.7) in the phase space $\mathcal{H}$ follows from Theorem 3.1. □

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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