AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(1): 518–531.
DOI:10.3934/math.2021031
Received: 8 October 2020
Accepted: 11 October 2020
Published: 19 October 2020

Research article

Existence of a unique solution to an elliptic partial differential equation when the average value is known

Diane Denny*

Department of Mathematics and Statistics, Texas A&M University-Corpus Christi, 6300 Ocean Drive, Corpus Christi, TX 78412, USA

* **Correspondence:** Email: diane.denny@tamucc.edu; Tel: +13618253485; Fax: +13618255789.

Abstract: The purpose of this paper is to prove the existence of a unique classical solution $u(\mathbf{x})$ to the quasilinear elliptic partial differential equation $\nabla \cdot (a(u)\nabla u) = f$ for $\mathbf{x} \in \Omega$, which satisfies the condition that the average value $\frac{1}{|\Omega|}\int_\Omega u d\mathbf{x} = u_0$, where u_0 is a given constant and $\frac{1}{|\Omega|}\int_\Omega f d\mathbf{x} = 0$. Periodic boundary conditions will be used. That is, we choose for our spatial domain the N-dimensional torus $\mathbb{T}^N$, where $N = 2$ or $N = 3$. The key to the proof lies in obtaining a priori estimates for u.

Keywords: elliptic; existence; uniqueness
Mathematics Subject Classification: 35A01

1. Introduction

In this paper, we consider the existence of a unique, classical solution $u(\mathbf{x})$ to the quasilinear elliptic equation

$$\nabla \cdot (a(u)\nabla u) = f \tag{1.1}$$

for $\mathbf{x} \in \Omega$, which satisfies the condition that the average value

$$\frac{1}{|\Omega|}\int_\Omega u d\mathbf{x} = u_0 \tag{1.2}$$

where u_0 is a given constant and $\frac{1}{|\Omega|}\int_\Omega f d\mathbf{x} = 0$. Periodic boundary conditions will be used. That is, we choose for our spatial domain the N-dimensional torus $\mathbb{T}^N$, where $N = 2$ or $N = 3$.

The purpose of this paper is to prove the existence of a unique classical solution u to (1.1), (1.2). The proof of the existence theorem uses the method of successive approximations in which an iteration scheme, based on solving a linearized version of Eq (1.1), will be defined and then convergence of the sequence of approximating solutions to a unique solution satisfying the quasilinear equation will be proven. The key to the proof lies in obtaining a priori estimates for u. To the best of our knowledge, no other researcher has proven the existence and uniqueness of the solution to this partial differential equation when the given data is the average value of the solution.

The paper is organized as follows. The main result, Theorem 2.1, is presented and proven in the next section. The existence of a solution to the linearized equation used in the iteration scheme is proven in Appendix A. Appendix B presents lemmas supporting the proof of the theorem.

2. Existence theorem

We will be working with the Sobolev space $H^s(\Omega)$ (where $s \geq 0$ is an integer) of real-valued functions in $L^2(\Omega)$ whose distribution derivatives up to order s are in $L^2(\Omega)$. The norm is $\|u\|_s^2 = \sum_{0\leq|\alpha|\leq s}\int_\Omega |D^\alpha u|^2 d\mathbf{x}$. We are using the standard multi-index notation. We define $|F|_{r,\overline{G}_0} = \max\{\left|\frac{d^jF}{du^j}(u_*)\right| : u_* \in \overline{G}_0, 0 \leq j \leq r\}$, where F is a function of u and where $\overline{G}_0 \subset \mathbb{R}$ is a closed, bounded interval. Also, we let both ∇u and Du denote the gradient of u. And $C^k(\Omega)$ is the set of real-valued functions having all derivatives of order $\leq k$ continuous in Ω (where k = integer ≥ 0 or $k = \infty$). The purpose of this paper is to prove the following theorem:

Theorem 2.1. *Let a be a smooth, positive function of u. Let $f \in H^2(\Omega)$ and let $\frac{1}{|\Omega|}\int_\Omega f d\mathbf{x} = 0$. Let the domain $\Omega = \mathbb{T}^N$, the N-dimensional torus, where $N = 2$ or $N = 3$.*

There exists a constant C_1 which depends only on N, Ω such that if

$$\frac{1}{(\min_{u_*\in\overline{G}_0} a(u_*))^4}\left|\frac{da}{du}\right|^2_{0,\overline{G}_0}\|\nabla f\|_0^2 \leq C_1$$

and if

$$\left|\frac{d^2a}{du^2}\right|_{0,\overline{G}_0} \leq \frac{1}{(\min_{u_*\in\overline{G}_0} a(u_*))}\left|\frac{da}{du}\right|^2_{0,\overline{G}_0} \tag{2.1}$$

where $\overline{G}_0 \subset \mathbb{R}$ is a closed, bounded interval, then there exists a unique solution $u \in C^2(\Omega)$ to the equation

$$\nabla\cdot(a(u)\nabla u) = f \tag{2.2}$$

which satisfies the condition that the average value

$$\frac{1}{|\Omega|}\int_\Omega u d\mathbf{x} = u_0 \tag{2.3}$$

where u_0 is a given constant.

Proof.

We begin by using the following change of variables:

$$v = \Big(\frac{a_0}{\|\nabla f\|_0}\Big)u$$
$$b(v) = \Big(\frac{1}{a_0}\Big)a\Big(\frac{\|\nabla f\|_0}{a_0}v\Big)$$

$$g = \Big(\frac{1}{||\nabla f||_0}\Big) f \tag{2.4}$$

where the constant $a_0 = \min_{u_* \in \overline{G}_0} a(u_*)$ and $\overline{G}_0 \subset \mathbb{R}$ is a closed, bounded interval.

Under this change of variables the equation (2.2) becomes

$$\nabla \cdot (b(v)\nabla v) = g \tag{2.5}$$

And under this change of variables, (2.3) becomes

$$\frac{1}{|\Omega|}\int_\Omega v d\mathbf{x} = v_0 = \frac{a_0}{||\nabla f||_0} u_0 \tag{2.6}$$

We fix closed, bounded intervals $\overline{G}_0 \subset \mathbb{R}$ and $\overline{G}_1 \subset \mathbb{R}$ by defining $\overline{G}_0 = \{u_* \in \mathbb{R} \quad : \quad |u_* - u_0|_{L^\infty} \leq \frac{R||\nabla f||_0}{a_0}\}$ and $\overline{G}_1 = \{v_* \in \mathbb{R} \quad : \quad |v_* - v_0|_{L^\infty} \leq R\}$, where R is a constant to be defined later. We will prove that $v(\mathbf{x}) \in \overline{G}_1$ for $\mathbf{x} \in \Omega$. It follows that $u(\mathbf{x}) \in \overline{G}_0$ for $\mathbf{x} \in \Omega$.

We will construct the solution of (2.5), (2.6) through an iteration scheme. To define the iteration scheme, we will let the sequence of approximate solutions be $\{v^k\}$. Set the initial iterate $v^0 = v_0$. For $k = 0, 1, 2, \ldots$, construct v^{k+1} from the previous iterate v^k by solving the linear equation

$$\nabla \cdot (b(v^k)\nabla v^{k+1}) = g \tag{2.7}$$

which satisfies the condition that the average value

$$\frac{1}{|\Omega|}\int_\Omega v^{k+1} d\mathbf{x} = v_0 \tag{2.8}$$

and using periodic boundary conditions.

The existence of a unique solution $v^{k+1} \in C^2(\Omega)$ to the linear equation (2.7) for fixed k which satisfies (2.8) is proven in Appendix A. Lemmas supporting the proof are presented in Appendix B. We proceed now to prove convergence of the iterates as $k \to \infty$ to a unique, classical solution v of (2.5), (2.6), which therefore produces a unique, classical solution $u = \frac{||\nabla f||_0}{a_0} v$ of (2.2), (2.3).

We begin by proving the following proposition:

Proposition 1. *Assume that the hypotheses of Theorem 2.1 hold. Then there exist constants C_2, C_3, and R such that the following inequalities hold for $k = 1, 2, 3 \ldots$:*

$$||\nabla v^k||_2^2 \leq C_2 \tag{2.9}$$

$$||v^k||_4^2 \leq C_3 \tag{2.10}$$

$$|v^k - v_0|_{L^\infty} \leq R \tag{2.11}$$

$$||\nabla(v^{k+1} - v^k)||_0^2 \leq \Big(\frac{1}{2}\Big)^k C_2 \tag{2.12}$$

where the constants C_2, R depend on N and Ω, and where the constant C_3 depends on R, u_0, a_0, $||\nabla f||_0$, $||\nabla f||_1$, $\left|\frac{da}{du}\right|_{2,\overline{G}_0}$, N, and Ω. From (2.11) it follows that $v^k(\mathbf{x}) \in \overline{G}_1$ for $\mathbf{x} \in \Omega$ and for $k = 1, 2, 3 \ldots$.

Proof. The proof is by induction on k. We prove in Lemma B.2 in Appendix B that if v^k satisfies (2.9) and (2.11), then v^{k+1} satisfies (2.9) and (2.10). See Lemma B.2 in Appendix B for the detailed proof. It only remains to prove inequalities (2.11) for $v^{k+1} - v_0$ and (2.12) for $\nabla(v^{k+1} - v^k)$.

In the estimates below, we will let C denote a generic constant whose value may change from one relation to the next.

Estimate for $|v^{k+1} - v_0|_{L^\infty}$**:**

Lemma B.2 in Appendix B presents the proof that $\|\nabla v^{k+1}\|_2^2 \leq C_2$. Then by using standard Sobolev space inequalities we obtain the inequality:

$$\begin{aligned} |v^{k+1} - v_0|_{L^\infty} &\leq C\|v^{k+1} - v_0\|_2 \\ &\leq C\|\nabla(v^{k+1} - v_0)\|_1 \\ &= C\|\nabla v^{k+1}\|_1 \\ &\leq C\sqrt{C_2} \\ &= R \end{aligned}$$

where the constants C and C_2 depend on Ω, N. Here we used the fact that $|v^{k+1} - v_0|_{L^\infty} \leq C\|v^{k+1} - v_0\|_2$ by Sobolev's Lemma. Since $\frac{1}{|\Omega|}\int_\Omega v^{k+1} d\mathbf{x} = v_0$ by (2.8), it follows that $v^{k+1} - v_0$ is a zero-mean function and $\|v^{k+1} - v_0\|_0 \leq C\|\nabla(v^{k+1} - v_0)\|_0$ by Poincaré's inequality. Therefore $\|v^{k+1} - v_0\|_2 \leq C\|\nabla(v^{k+1} - v_0)\|_1$. We define $R = C\sqrt{C_2}$. Then inequality (2.11) of Proposition 1 holds for $v^{k+1} - v_0$.

Estimate for $\|\nabla(v^{k+1} - v^k)\|_0^2$**:**

From successive iterates of Eq (2.7) we obtain the following:

$$\begin{aligned} &\nabla \cdot (b(v^k)\nabla(v^{k+1} - v^k)) = \nabla \cdot (b(v^k)\nabla v^{k+1}) - \nabla \cdot (b(v^k)\nabla v^k) \\ =\;& g - \nabla \cdot ((b(v^k) - b(v^{k-1}))\nabla v^k) - \nabla \cdot (b(v^{k-1})\nabla v^k) \\ =\;& -\nabla \cdot ((b(v^k) - b(v^{k-1}))\nabla v^k) \end{aligned} \tag{2.13}$$

In the estimates that follow, we use the notation $(h_1, h_2) = \int_\Omega h_1 h_2 d\mathbf{x}$ for the L^2 inner product of functions h_1, h_2. Note that $v^k - v^{k-1}$ is a zero-mean function because $v^k - v^{k-1} = (v^k - v_0) - (v^{k-1} - v_0)$ and $v^k - v_0$, $v^{k-1} - v_0$ are zero-mean functions by successive iterates of (2.8).

We define the constant $b_0 = \min_{v_* \in \overline{G}_1} b(v_*)$, where $\overline{G}_1 \subset \mathbb{R}$ is a closed, bounded interval. Note that $b_0 = 1$ by the definition of the function b in (2.4). Then integration by parts and using Eq (2.13) yields

$$\begin{aligned} \|\nabla(v^{k+1} - v^k)\|_0^2 &= (\nabla(v^{k+1} - v^k), \nabla(v^{k+1} - v^k)) \\ &\leq \frac{1}{b_0}(b(v^k)\nabla(v^{k+1} - v^k), \nabla(v^{k+1} - v^k)) \\ &= -\frac{1}{b_0}(\nabla \cdot (b(v^k)\nabla(v^{k+1} - v^k)), (v^{k+1} - v^k)) \\ &= \frac{1}{b_0}(\nabla \cdot ((b(v^k) - b(v^{k-1}))\nabla v^k), (v^{k+1} - v^k)) \\ &= -\frac{1}{b_0}((b(v^k) - b(v^{k-1}))\nabla v^k, \nabla(v^{k+1} - v^k)) \\ &\leq \frac{1}{b_0}\|(b(v^k) - b(v^{k-1}))\nabla v^k\|_0 \|\nabla(v^{k+1} - v^k))\|_0 \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{b_0}\Big|\frac{db}{dv}\Big|_{0,\overline{G}_1}\|v^k - v^{k-1}\|_0|\nabla v^k|_{L^\infty}\|\nabla(v^{k+1} - v^k)\|_0 \\
&\leq C(\frac{1}{b_0})^2\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_1}\|\nabla(v^k - v^{k-1})\|_0^2\|\nabla v^k\|_2^2 \\
&+ \frac{1}{2}\|\nabla(v^{k+1} - v^k)\|_0^2
\end{aligned}
\tag{2.14}
$$

where C is a constant which depends on N, Ω. Here we used the fact that $|\nabla v^k|_{L^\infty} \leq C\|\nabla v^k\|_2$ by Sobolev's Lemma. And we used Poincaré's inequality to obtain $\|v^k - v^{k-1}\|_0 \leq C\|\nabla(v^k - v^{k-1})\|_0$, since $v^k - v^{k-1}$ is a zero-mean function.

Using the facts that $\frac{1}{b_0} = 1$ and that $\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_1} = \frac{1}{a_0^4}\|\nabla f\|_0^2\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0} \leq C_1$ by the definition of $b(v)$ in (2.4) and by the statement of the theorem, and using the fact that $\|\nabla v^k\|_2^2 \leq C_2$ by the induction hypothesis, we obtain from re-arranging terms in (2.14) the inequality

$$
\begin{aligned}
\|\nabla(v^{k+1} - v^k)\|_0^2 &\leq C(\frac{1}{b_0})^2\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_1}\|\nabla v^k\|_2^2\|\nabla(v^k - v^{k-1})\|_0^2 \\
&\leq CC_1C_2\|\nabla(v^k - v^{k-1})\|_0^2 \\
&\leq \frac{1}{2}\|\nabla(v^k - v^{k-1})\|_0^2
\end{aligned}
\tag{2.15}
$$

where we define the constant C_1 to be sufficiently small so that $CC_1C_2 \leq \frac{1}{2}$. And the constants C, C_1, C_2 depend on N, Ω.

By repeatedly applying inequality (2.15) it follows that

$$
\begin{aligned}
\|\nabla(v^{k+1} - v^k)\|_0^2 &\leq (\frac{1}{2})^k\|\nabla(v^1 - v^0)\|_0^2 \\
&= (\frac{1}{2})^k\|\nabla v^1\|_0^2 \\
&\leq (\frac{1}{2})^kC_2
\end{aligned}
\tag{2.16}
$$

where the initial iterate $v^0 = v_0$, which is a constant, and where $\|\nabla v^1\|_0^2 \leq \|\nabla v^1\|_2^2 \leq C_2$ by Lemma B.2 in Appendix B. Therefore inequality (2.12) of Proposition 1 holds for $\nabla(v^{k+1} - v^k)$.

This completes the proof of Proposition 1. □

We now complete the proof of Theorem 2.1. By (2.16), $\|\nabla(v^{k+1} - v^k)\|_0 \to 0$ as $k \to \infty$. By Poincaré's inequality, $\|v^{k+1} - v^k\|_0^2 \leq C\|\nabla(v^{k+1} - v^k)\|_0^2$. It follows that $\|v^{k+1} - v^k\|_0 \to 0$ as $k \to \infty$. We next use the standard interpolation inequality $\|v^{k+1} - v^k\|_r \leq C\|v^{k+1} - v^k\|_0^{\beta}\|v^{k+1} - v^k\|_4^{1-\beta}$, where $\beta = \frac{4-r}{4}$, and $0 < r < 4$. Then since $\|v^{k+1} - v^k\|_4^2 \leq C(\|v^{k+1}\|_4^2 + \|v^k\|_4^2) \leq CC_3$ by (2.10) in Proposition 1, it follows that $\|v^{k+1} - v^k\|_r \to 0$ as $k \to \infty$ for $0 < r < 4$.

Therefore there exists $v \in H^r(\Omega)$, where $r < 4$, such that $\|v^k - v\|_r \to 0$ as $k \to \infty$. The fact that $v \in H^4(\Omega)$ can be deduced using boundedness in high norm and a standard compactness argument (see, for example, Embid [2], Majda [6]). Sobolev's Lemma implies that $v \in C^2(\Omega)$.

From Lemma A.1 in Appendix A, $v^{k+1} \in C^2(\Omega)$ is a solution of the linear equation $\nabla\cdot(b(v^k)\nabla v^{k+1}) = g$ for each $k \geq 0$, and v^{k+1} satisfies the condition that $\frac{1}{|\Omega|}\int_\Omega v^{k+1}d\mathbf{x} = v_0$. It follows that v is a classical solution of the equation $\nabla \cdot (b(v)\nabla v) = g$, and v satisfies the condition that $\frac{1}{|\Omega|}\int_\Omega v d\mathbf{x} = v_0$. The uniqueness of the solution follows by a standard proof using estimates similar to the estimates used in the proof of inequality (2.12). Therefore, there exists a unique classical solution $u = \left(\frac{\|\nabla f\|_0}{a_0}\right)v$ of $\nabla \cdot (a(u)\nabla u) = f$ which satisfies the condition that $\frac{1}{|\Omega|}\int_\Omega u d\mathbf{x} = u_0$. This completes the proof of the theorem. $\square$

3. Conclusion

We have proven that if

$$\frac{1}{(\min_{u_* \in \overline{G}_0} a(u_*))^4}\left|\frac{da}{du}\right|^2_{0,\overline{G}_0}\|\nabla f\|_0^2 \leq C_1$$

and if

$$\left|\frac{d^2a}{du^2}\right|_{0,\overline{G}_0} \leq \frac{1}{(\min_{u_* \in \overline{G}_0} a(u_*))}\left|\frac{da}{du}\right|^2_{0,\overline{G}_0}$$

where $\overline{G}_0 \subset \mathbb{R}$ is a closed, bounded interval and where the constant C_1 depends on N, Ω, then there exists a unique solution $u \in C^2(\Omega)$ to the equation

$$\nabla \cdot (a(u)\nabla u) = f$$

which satisfies the condition that the average value

$$\frac{1}{|\Omega|}\int_\Omega u d\mathbf{x} = u_0$$

where u_0 is a given constant, under periodic boundary conditions. We remark that in the trivial case in which $\nabla f = 0$ (and therefore $f = 0$), it follows that $u = u_0$ is the unique solution.

Conflict of interest

The author confirms that there is no conflict of interest.

References

1. D. Denny, Existence of a unique solution to a quasilinear elliptic equation, *J. Math. Anal. Appl.*, **380** (2011), 653–668.
2. P. Embid, On the Reactive and Non-diffusive Equations for Zero Mach Number Flow, *Commun. Part. Differ. Eq.*, **14** (1989), 1249–1281.
3. L. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, Rhode Island, 1998.
4. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

5. S. Klainerman, A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.*, **34** (1981), 481–524.
6. A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag: New York, 1984.
7. J. Moser, A rapidly convergent iteration method and non-linear differential equations, *Ann. Scuola Norm. Sup., Pisa*, **20** (1966), 265–315.

A. Existence for the linear equation

In this appendix, we present the proof of the existence of a unique, classical solution to the linear problem (2.7), (2.8).

Lemma A.1. *Let b be a smooth positive function of w. Let $w \in C^2(\Omega)$, let $g \in H^2(\Omega)$, and let $\frac{1}{|\Omega|}\int_\Omega g d\mathbf{x} = 0$. Let the domain $\Omega = \mathbb{T}^N$, the N-dimensional torus, where $N = 2$ or $N = 3$. Then there exists a unique solution $v \in C^2(\Omega)$ of the equation*

$$\nabla \cdot (b(w)\nabla v) = g \tag{A.1}$$

which satisfies the condition

$$\frac{1}{|\Omega|}\int_\Omega v d\mathbf{x} = v_0 \tag{A.2}$$

where v_0 is a given constant.

Proof.

We define the zero-mean function

$$\overline{v} = v - \frac{1}{|\Omega|}\int_\Omega v d\mathbf{x} \tag{A.3}$$

The existence of a unique zero-mean solution $\overline{v} \in C^2(\Omega)$ to equation (A.1) under periodic boundary conditions is a well-known result from the standard theory of elliptic equations (see, e.g., Embid [2], Evans [3], Gilbarg and Trudinger [4]).

It follows that the function v defined by

$$v(\mathbf{x}) = \overline{v}(\mathbf{x}) + v_0 \tag{A.4}$$

is the unique solution to equation (A.1) which satisfies the condition (A.2) that $\frac{1}{|\Omega|}\int_\Omega v d\mathbf{x} = v_0$.

This completes the proof of the lemma.

□

B. A priori estimates

In this appendix, we present lemmas supporting the proof of the theorem.

We begin by listing several standard Sobolev space inequalities.

Lemma B.1. ***(Standard Sobolev Space Inequalities)***
(a) Let b be a smooth function of w, and let $w(\mathbf{x})$ be a continuous function such that $w(\mathbf{x}) \in \overline{G}_1$ for $\mathbf{x} \in \Omega$ where $\overline{G}_1 \subset \mathbb{R}$ is a closed, bounded interval. And let $w \in H^{r+1}(\Omega)$ where $r \geq 0$.

Then

$$||D(b(w))||_r^2 \leq C\left|\frac{db}{dw}\right|^2_{r,\bar{G}_1}(1 + |w|_{L^\infty})^{2r}||\nabla w||_r^2 \tag{B.1}$$

where $|\frac{db}{dw}|_{r,\overline{G}_1} = \max\{\left|\frac{d^{j+1}b}{dw^{j+1}}(w_*)\right| : w_* \in \overline{G}_1, 0 \leq j \leq r\}$. And the constant C depends on r, N, Ω.

(b) If $f \in H^n(\Omega)$, where $\Omega \subset \mathbb{R}^N$, and $r = \beta m + (1-\beta)n$, with $0 \leq \beta \leq 1$ and $m < n$, then

$$||f||_r \leq C||f||_m^\beta ||f||_n^{1-\beta} \tag{B.2}$$

Here C is a constant which depends on m, n, N, Ω.

(c) If $f \in H^{s_0}(\Omega)$ where $\Omega \subset \mathbb{R}^N$, $N = 2$ or $N = 3$, and $s_0 = [\frac{N}{2}] + 1$, then

$$|f|_{L^\infty} \leq C||f||_{s_0} \tag{B.3}$$

Here C is a constant which depends on N, Ω.

(d) If $Df \in H^{r_1}(\Omega)$, $g \in H^{r-1}(\Omega)$, where $r \geq 1$ and where $r_1 = \max\{r-1, s_0\}$ and $s_0 = [\frac{N}{2}] + 1$, then:

$$||D^\alpha(fg) - fD^\alpha g||_0 \leq C||Df||_{r_1}||g||_{r-1}, \tag{B.4}$$

where $|\alpha| = r$ and where the constant C depends on r, N, Ω.

These inequalities are well-known. Proofs may be found, for example, in [5], [7]. These inequalities also appear in [1], [2].

Lemma B.2. *Let the function $w \in C^2(\Omega)$ satisfy (2.9), (2.11) in Proposition 1 and let the hypotheses in the statement of Theorem 2.1 hold. Let b be a smooth, positive function of w. Let $g \in H^2(\Omega)$ and let $\frac{1}{|\Omega|}\int_\Omega g d\mathbf{x} = 0$. Let (2.4) define the functions b, g. Let the domain $\Omega = \mathbb{T}^N$, the N-dimensional torus, where $N = 2$ or $N = 3$.*

Let v be the solution from Lemma A.1 in Appendix A of

$$\nabla \cdot (b(w)\nabla v) = g \tag{B.5}$$

which satisfies the condition

$$\frac{1}{|\Omega|}\int_\Omega v d\mathbf{x} = v_0, \tag{B.6}$$

where v_0 is a given constant.

Then ∇v and v satisfy the following inequalities:

$$||\nabla v||_2^2 \leq C_2$$
$$||v||_4^2 \leq C_3$$

where the constant C_2 depends on N and Ω and where the constant C_3 depends on R, u_0, a_0, $||\nabla f||_0$, $||\nabla f||_1$, $\left|\frac{da}{du}\right|_{2,\overline{G}_0}$, N, and Ω.

Proof.

In the estimates below, we will let C denote a generic constant whose value may change from one relation to the next. We use the notation $(h_1, h_2) = \int_\Omega h_1 h_2 d\mathbf{x}$ for the L^2 inner product of two functions h_1, h_2. And we use the notation $h_\alpha = D^\alpha h$ for differentiation with a multi-index α.

Estimate for $\|\nabla v\|_0^2$**:**

Using integration by parts and then substituting equation (B.5) yields

$$\begin{aligned}
\|\nabla v\|_0^2 &= (\nabla v, \nabla v) \\
&\le \frac{1}{b_0}(b(w)\nabla v, \nabla v) \\
&= -\frac{1}{b_0}(\nabla \cdot (b(w)\nabla v), v - \frac{1}{|\Omega|}\int_\Omega v d\mathbf{x}) \\
&= -\frac{1}{b_0}(g, v - \frac{1}{|\Omega|}\int_\Omega v d\mathbf{x}) \\
&\le \frac{1}{b_0}\|g\|_0 \|v - \frac{1}{|\Omega|}\int_\Omega v d\mathbf{x}\|_0 \\
&\le \frac{C}{b_0}\|\nabla g\|_0 \|\nabla v\|_0 \\
&= C\|\nabla v\|_0
\end{aligned} \tag{B.7}$$

where $b_0 = \min_{w_* \in \overline{G}_1} b(w_*) = 1$ by definition of the function b, and $\|\nabla g\|_0 = 1$ by definition of the function g. Here we used the fact that g and $v - \frac{1}{|\Omega|}\int_\Omega v d\mathbf{x}$ are zero-mean functions and we used Poincaré's inequality for a zero-mean function h, namely $\|h\|_0 \le C\|\nabla h\|_0$. The constant C depends on N, Ω.

It follows that

$$\|\nabla v\|_0^2 \le \tilde{C} \tag{B.8}$$

where the generic constant $\tilde{C}$ depends on N, Ω.

Estimate for $\|\nabla v\|_1^2$**:** To begin, let $|\alpha| \ge 1$. Using integration by parts and then substituting equation (B.5) yields

$$\begin{aligned}
\|\nabla v_\alpha\|_0^2 &= (\nabla v_\alpha, \nabla v_\alpha) \\
&\le \frac{1}{b_0}(b(w)\nabla v_\alpha, \nabla v_\alpha) \\
&= \frac{1}{b_0}((b(w)\nabla v)_\alpha, \nabla v_\alpha) - \frac{1}{b_0}((b(w)\nabla v)_\alpha - b(w)\nabla v_\alpha, \nabla v_\alpha) \\
&= -\frac{1}{b_0}(\nabla \cdot (b(w)\nabla v)_\alpha, v_\alpha) - \frac{1}{b_0}((b(w)\nabla v)_\alpha - b(w)\nabla v_\alpha, \nabla v_\alpha) \\
&= -\frac{1}{b_0}(g_\alpha, v_\alpha) - \frac{1}{b_0}((b(w)\nabla v)_\alpha - b(w)\nabla v_\alpha, \nabla v_\alpha)
\end{aligned} \tag{B.9}$$

where $b_0 = \min_{w_* \in \overline{G}_1} b(w_*)$. If $|\alpha| = 1$ in (B.9) then

$$\|\nabla v_\alpha\|_0^2 \le -\frac{1}{b_0}(g_\alpha, v_\alpha) - \frac{1}{b_0}((b(w)\nabla v)_\alpha - b(w)\nabla v_\alpha, \nabla v_\alpha)$$

$$
\begin{aligned}
&= -\frac{1}{b_0}(g_\alpha, v_\alpha) - \frac{1}{b_0}(b(w)_\alpha \nabla v, \nabla v_\alpha) \\
&\le (\frac{1}{b_0})\|g_\alpha\|_0\|v_\alpha\|_0 + (\frac{1}{b_0})\|b(w)_\alpha \nabla v\|_0\|\nabla v_\alpha\|_0 \\
&\le \frac{1}{2}(\frac{1}{b_0})^2\|g_\alpha\|_0^2 + \frac{1}{2}\|v_\alpha\|_0^2 + (\frac{1}{b_0})\|\frac{db}{dw}w_\alpha \nabla v\|_0\|\nabla v_\alpha\|_0 \\
&\le \frac{1}{2}(\frac{1}{b_0})^2\|g_\alpha\|_0^2 + \frac{1}{2}\|v_\alpha\|_0^2 \\
&+ \frac{1}{2}(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|w_\alpha \nabla v\|_0^2 + \frac{1}{2}\|\nabla v_\alpha\|_0^2 \qquad \text{(B.10)}
\end{aligned}
$$

Re-arranging the terms in (B.10) and adding the resulting inequality over $|\alpha| = 1$ yields

$$
\begin{aligned}
\sum_{|\alpha|=1}\|\nabla v_\alpha\|_0^2 &\le (\frac{1}{b_0})^2\|\nabla g\|_0^2 + \|\nabla v\|_0^2 + (\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}|\nabla w|^2_{L^\infty}\|\nabla v\|_0^2 \\
&\le 1 + \tilde{C} + C\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|\nabla w\|_2^2\tilde{C} \\
&\le 1 + \tilde{C} + CC_1C_2\tilde{C} \qquad \text{(B.11)}
\end{aligned}
$$

where the generic constants C, $\tilde{C}$ depend on N, Ω. Here we used the facts that $\frac{1}{b_0} = 1$, $\|\nabla g\|_0 = 1$, and $\|\nabla w\|_2^2 \le C_2$. And $\|\nabla v\|_0^2 \le \tilde{C}$ from (B.8). And we used the fact that $\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1} = \Big(\frac{\|\nabla f\|_0^2}{a_0^4}\Big)\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0} \le C_1$ by definition of the function b in (2.4) and by the statement of Theorem 2.1.

From (B.8), (B.11) it follows that

$$
\begin{aligned}
\|\nabla v\|_1^2 &= \sum_{0\le|\alpha|\le 1}\|\nabla v_\alpha\|_0^2 = \|\nabla v\|_0^2 + \sum_{|\alpha|=1}\|\nabla v_\alpha\|_0^2 \\
&\le 1 + 2\tilde{C} + CC_1C_2\tilde{C} \qquad \text{(B.12)}
\end{aligned}
$$

Estimate for $\|\nabla v\|_2^2$:

Letting $|\alpha| = 2$ in inequality (B.9) and then using integration by parts with $|\gamma| = 1$ produces

$$
\begin{aligned}
&\|\nabla v_\alpha\|_0^2 \le -\frac{1}{b_0}(g_\alpha, v_\alpha) - \frac{1}{b_0}((b(w)\nabla v)_\alpha - b(w)\nabla v_\alpha, \nabla v_\alpha) \\
&= (\frac{1}{b_0})(g_{\alpha-\gamma}, v_{\alpha+\gamma}) - \frac{1}{b_0}(b(w)_\alpha \nabla v, \nabla v_\alpha) \\
&- \frac{1}{b_0}(b(w)_\gamma \nabla v_{\alpha-\gamma}, \nabla v_\alpha) - \frac{1}{b_0}(b(w)_{\alpha-\gamma}\nabla v_\gamma, \nabla v_\alpha) \\
&= (\frac{1}{b_0})(g_{\alpha-\gamma}, v_{\alpha+\gamma}) - \frac{1}{b_0}((\frac{d^2b}{dw^2}w_{\alpha-\gamma}w_\gamma)\nabla v, \nabla v_\alpha) - \frac{1}{b_0}((\frac{db}{dw}w_\alpha)\nabla v, \nabla v_\alpha) \\
&- \frac{1}{b_0}(\frac{db}{dw}w_\gamma \nabla v_{\alpha-\gamma}, \nabla v_\alpha) - \frac{1}{b_0}(\frac{db}{dw}w_{\alpha-\gamma}\nabla v_\gamma, \nabla v_\alpha) \\
&\le (\frac{1}{b_0})\|g_{\alpha-\gamma}\|_0\|v_{\alpha+\gamma}\|_0 + (\frac{1}{b_0})\Big|\frac{d^2b}{dw^2}\Big|_{0,\overline{G}_1}|w_{\alpha-\gamma}|_{L^\infty}|w_\gamma|_{L^\infty}\|\nabla v\|_0\|\nabla v_\alpha\|_0
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad (\frac{1}{b_0})\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}\|w_\alpha\|_0|\nabla v|_{L^\infty}\|\nabla v_\alpha\|_0 \\
&+ \quad (\frac{1}{b_0})\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}|w_\gamma|_{L^\infty}\|\nabla v_{\alpha-\gamma}\|_0\|\nabla v_\alpha\|_0 \\
&+ \quad (\frac{1}{b_0})\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}|w_{\alpha-\gamma}|_{L^\infty}\|\nabla v_\gamma\|_0\|\nabla v_\alpha\|_0 \\
&\leq \quad \frac{C}{\epsilon}(\frac{1}{b_0})^2\|g_{\alpha-\gamma}\|_0^2 + \epsilon\|\nabla v_\alpha\|_0^2 \\
&+ \quad \frac{1}{4\epsilon}(\frac{1}{b_0})^2\Big|\frac{d^2b}{dw^2}\Big|^2_{0,\overline{G}_1}|w_{\alpha-\gamma}|^2_{L^\infty}|w_\gamma|^2_{L^\infty}\|\nabla v\|_0^2 + \epsilon\|\nabla v_\alpha\|_0^2 \\
&+ \quad \frac{1}{4\epsilon}(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|w_\alpha\|_0^2|\nabla v|^2_{L^\infty} + \epsilon\|\nabla v_\alpha\|_0^2 \\
&+ \quad \frac{1}{4\epsilon}(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}|w_\gamma|^2_{L^\infty}\|\nabla v_{\alpha-\gamma}\|_0^2 + \epsilon\|\nabla v_\alpha\|_0^2 \\
&+ \quad \frac{1}{4\epsilon}(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}|w_{\alpha-\gamma}|^2_{L^\infty}\|\nabla v_\gamma\|_0^2 + \epsilon\|\nabla v_\alpha\|_0^2 \\
&\leq \quad \frac{C}{\epsilon}(\frac{1}{b_0})^2\|g_{\alpha-\gamma}\|_0^2 + \frac{C}{\epsilon}(\frac{1}{b_0})^2\Big|\frac{d^2b}{dw^2}\Big|^2_{0,\overline{G}_1}\|w_{\alpha-\gamma}\|_2^2\|w_\gamma\|_2^2\|\nabla v\|_0^2 \\
&+ \quad \frac{C}{\epsilon}(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|D^\gamma w_{\alpha-\gamma}\|_0^2\|\nabla v\|_2^2 + \frac{C}{\epsilon}(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|w_\gamma\|_2^2\|\nabla v_{\alpha-\gamma}\|_0^2 \\
&+ \quad \frac{C}{\epsilon}(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|w_{\alpha-\gamma}\|_2^2\|\nabla v_\gamma\|_0^2 + 5\epsilon\|\nabla v_\alpha\|_0^2
\end{aligned}
\tag{B.13}
$$

where we used Cauchy's inequality with ϵ and we define $\epsilon = \frac{1}{10}$. We also used Sobolev's Lemma, i.e., $|h|_{L^\infty} \leq C\|h\|_2$.

Re-arranging terms in (B.13), and then adding the resulting inequality over $|\alpha| = 2$ and $|\gamma| = 1$, produces

$$
\begin{aligned}
\sum_{|\alpha|=2}\|\nabla v_\alpha\|_0^2 &\leq C(\frac{1}{b_0})^2\|\nabla g\|_0^2 + C(\frac{1}{b_0})^2\Big|\frac{d^2b}{dw^2}\Big|^2_{0,\overline{G}_1}\|\nabla w\|_2^4\|\nabla v\|_0^2 \\
&+ \quad C(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|\nabla w\|_1^2\|\nabla v\|_2^2 + C(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|\nabla w\|_2^2\|\nabla v\|_1^2 \\
&\leq \quad C(\frac{1}{b_0})^2\|\nabla g\|_0^2 + C(\frac{1}{b_0})^2\Big|\frac{d^2b}{dw^2}\Big|^2_{0,\overline{G}_1}\|\nabla w\|_2^4\|\nabla v\|_2^2 \\
&+ \quad C(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|\nabla w\|_2^2\|\nabla v\|_2^2 \\
&\leq \quad C + (CC_1^2C_2^2 + CC_1C_2)\|\nabla v\|_2^2 \\
&\leq \quad C + CC_1C_2^2\|\nabla v\|_2^2
\end{aligned}
\tag{B.14}
$$

where we can assume that $C_1 < 1$ and that $C_2 > 1$. Here we used the fact that $\|\nabla w\|_2^2 \leq C_2$. And we used the fact that $\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1} = \Big(\frac{\|\nabla f\|_0^2}{a_0^4}\Big)\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0} \leq C_1$. And we used the fact that $\Big|\frac{d^2b}{dw^2}\Big|^2_{0,\overline{G}_1} = \Big(\frac{\|\nabla f\|_0^4}{a_0^6}\Big)\Big|\frac{d^2a}{du^2}\Big|^2_{0,\overline{G}_0}$

$\leq \left(\frac{\|\nabla f\|_0^4}{a_0^8}\right)\left|\frac{da}{du}\right|_{0,\overline{G}_0}^4 \leq C_1^2$ by the definition of $b(v)$ in (2.4) and by the statement of the theorem. And we used the facts that $\|\nabla g\|_0 = 1$ and that $\frac{1}{b_0} = 1$.

From (B.14) and from inequality (B.12) for $\|\nabla v\|_1^2$, it follows that

$$\begin{aligned}
\|\nabla v\|_2^2 &= \sum_{0\leq|\alpha|\leq 2} \|\nabla v_\alpha\|_0^2 \\
&= \|\nabla v\|_1^2 + \sum_{|\alpha|=2} \|\nabla v_\alpha\|_0^2 \\
&\leq 1 + 2\tilde{C} + CC_1C_2\tilde{C} + C + CC_1C_2^2\|\nabla v\|_2^2 \\
&\leq 1 + 2\tilde{C} + \frac{1}{2} + C + \frac{1}{2}\|\nabla v\|_2^2 \qquad \text{(B.15)}
\end{aligned}$$

where the generic constants $C, \tilde{C}$ depend on N, Ω, and where C_1 is sufficiently small so that $CC_1C_2\tilde{C} \leq \frac{1}{2}$ and so that $CC_1C_2^2 \leq \frac{1}{2}$.

Re-arranging terms in (B.15) yields

$$\begin{aligned}
\|\nabla v\|_2^2 &\leq 4\tilde{C} + C \\
&= C_2 \qquad \text{(B.16)}
\end{aligned}$$

where we define $C_2 = 4\tilde{C} + C$, and where the constant C_2 depends on N, Ω.

Estimate for $\|\nabla v\|_3^2$:

Letting $|\alpha| = 3$ in inequality (B.9) and then using integration by parts with $|\gamma| = 1$ produces

$$\begin{aligned}
\|\nabla v_\alpha\|_0^2 &\leq -\frac{1}{b_0}(g_\alpha, v_\alpha) - \frac{1}{b_0}(((b(w)\nabla v)_\alpha - b(w)\nabla v_\alpha), \nabla v_\alpha) \\
&= \frac{1}{b_0}(g_{\alpha-\gamma}, v_{\alpha+\gamma}) - \frac{1}{b_0}(((b(w)\nabla v)_\alpha - b(w)\nabla v_\alpha), \nabla v_\alpha) \\
&\leq (\frac{1}{b_0})\|g_{\alpha-\gamma}\|_0\|v_{\alpha+\gamma}\|_0 + (\frac{1}{b_0})\|(b(w)\nabla v)_\alpha - b(w)\nabla v_\alpha\|_0\|\nabla v_\alpha\|_0 \\
&\leq (\frac{1}{b_0})\|g_{\alpha-\gamma}\|_0\|v_{\alpha+\gamma}\|_0 + C(\frac{1}{b_0})\|Db\|_2\|\nabla v\|_2\|\nabla v_\alpha\|_0 \\
&\leq \frac{C}{\epsilon}(\frac{1}{b_0})^2\|g_{\alpha-\gamma}\|_0^2 + \epsilon\|\nabla v_\alpha\|_0^2 + \frac{C}{\epsilon}(\frac{1}{b_0})^2\|Db\|_2^2\|\nabla v\|_2^2 \\
&+ \epsilon\|\nabla v_\alpha\|_0^2 \qquad \text{(B.17)}
\end{aligned}$$

where $\epsilon = \frac{1}{4}$ and where we used the Sobolev space inequality (B.4) from Lemma B.1 with $r = |\alpha| = 3$ and $r_1 = 2$.

Re-arranging the terms in (B.17) and then adding the resulting inequality over $|\alpha| = 3$ and $|\gamma| = 1$ yields

$$\sum_{|\alpha|=3} \|\nabla v_\alpha\|_0^2 \leq C(\frac{1}{b_0})^2\|g\|_2^2 + C(\frac{1}{b_0})^2\|Db\|_2^2\|\nabla v\|_2^2$$

$$
\begin{aligned}
&\leq C(\frac{1}{b_0})^2\|g\|_2^2 + C(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{2,\overline{G}_1}(1+|w|_{L^\infty})^4\|\nabla w\|_2^2\|\nabla v\|_2^2 \\
&= C(\frac{1}{b_0})^2\|g\|_2^2 + C(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{2,\overline{G}_1}(1+|w-v_0+v_0|_{L^\infty})^4\|\nabla w\|_2^2\|\nabla v\|_2^2 \\
&\leq C(\frac{1}{b_0})^2\|g\|_2^2 + C(\frac{1}{b_0})^2\Big|\frac{db}{dw}\Big|^2_{2,\overline{G}_1}(1+|w-v_0|_{L^\infty}+|v_0|)^4\|\nabla w\|_2^2\|\nabla v\|_2^2 \\
&\leq C\|\nabla g\|_1^2 + C\Big|\frac{db}{dw}\Big|^2_{2,\overline{G}_1}(1+R+|v_0|)^4C_2\|\nabla v\|_2^2
\end{aligned}
\tag{B.18}
$$

where we used the Sobolev space inequality (B.1) from Lemma B.1 for $\|Db\|_2^2$. We also used the facts that $|w-v_0|_{L^\infty} \leq R$, $\|\nabla w\|_2^2 \leq C_2$, and $b_0 = 1$. And we used the fact that g is a zero-mean function, so that $\|g\|_0 \leq C\|\nabla g\|_0$ by Poincaré's inequality.

From (B.18) and from inequality (B.16) for $\|\nabla v\|_2^2$, it follows that

$$
\begin{aligned}
\|\nabla v\|_3^2 &= \sum_{0\leq|\alpha|\leq 3}\|\nabla v_\alpha\|_0^2 \\
&= \sum_{0\leq|\alpha|\leq 2}\|\nabla v_\alpha\|_0^2 + \sum_{|\alpha|=3}\|\nabla v_\alpha\|_0^2 \\
&= \|\nabla v\|_2^2 + \sum_{|\alpha|=3}\|\nabla v_\alpha\|_0^2 \\
&\leq C_2 + C\|\nabla g\|_1^2 \\
&+ C\Big|\frac{db}{dw}\Big|^2_{2,\overline{G}_1}(1+R+|v_0|)^4C_2^2
\end{aligned}
\tag{B.19}
$$

Estimate for $\|v\|_4^2$:

From (B.19) it follows that

$$
\begin{aligned}
\|v\|_4^2 &= \|v-v_0+v_0\|_4^2 \\
&\leq C\|v-v_0\|_4^2 + C\|v_0\|_4^2 \\
&\leq C\|\nabla(v-v_0)\|_3^2 + C|v_0|^2|\Omega| \\
&= C\|\nabla v\|_3^2 + C|v_0|^2|\Omega| \\
&\leq CC_2 + C\|\nabla g\|_1^2 + C\Big|\frac{db}{dw}\Big|^2_{2,\overline{G}_1}(1+R+|v_0|)^4C_2^2 \\
&+ C|v_0|^2|\Omega| \\
&= CC_2 + C\frac{\|\nabla f\|_1^2}{\|\nabla f\|_0^2} \\
&+ C\Big(\frac{\|\nabla f\|_0^2}{a_0^4}\Big)\Big|\frac{da}{du}\Big|^2_{2,\overline{G}_0}(1+R+\Big(\frac{a_0}{\|\nabla f\|_0}\Big)|u_0|)^4C_2^2 \\
&+ C\Big(\frac{a_0}{\|\nabla f\|_0}\Big)^2|u_0|^2|\Omega| \\
&= C_3
\end{aligned}
\tag{B.20}
$$

Here we used Poincaré's inequality for the zero-mean function $v - v_0$, where the constant $v_0 = \frac{1}{|\Omega|} \int_\Omega v d\mathbf{x}$. And we used the definition of the functions v, b, g from (2.4). This completes the proof of the lemma. □