



Research article

Multi-valued versions of Nadler, Banach, Branciari and Reich fixed point theorems in double controlled metric type spaces with applications

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Abstract: In the current work, the multi-valued version of well-known theorems of Nadler, Banach, Branciari and Reich are generalized to the scope of double controlled metric space. A double controlled metric space is a metric type space in which the right hand side of the triangle inequality is controlled by two functions. Furthermore, applications to existence of solution to Volterra integral inclusions and singular Fredholm integral inclusions of are obtained.

Keywords: double controlled metric space; multi-valued mappings; Nadler fixed point theorem; Reich fixed point theorem; Branciari fixed point theorem; Volterra integral inclusion; singular Fredholm integral inclusions

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1. Introduction

Numerous applications in engineering and scientific fields can be managed through Fredholm or Volterra integrals. A substantial amount of initial value and boundary value problems can be transformed to Fredholm or Volterra integral equations. Applications of which can be found in areas of mathematical modeling of physics and biology. For instance, one can observe the applications of these equations in biological sciences such as the heat transfer or heat radiation and biological species living together, the Volterra population growth model, scattering in quantum mechanics, kinetic

theory of gases, conformal mapping, hereditary phenomena in biology and physics, propagation of stocked fish in a new lake, electrostatic, population genetics. In fact, biological sciences are among many areas that can be portrayed by integral equations and in certain circumstances problems such as in economics. Similarly, singular integral equations occupy comparable priority in the areas such as X-ray radiography, microscopy, radio astronomy, atomic scattering, electron emission, optical fiber evaluation, radar ranging, seismology, and plasma diagnostics.

Various methods are present in the literature to find the solution to Volterra or Fredholm integral, singular and integro-differential equations (see for example, [2,5,41,42,47]). In several cases a problem lead to the determination of the exact solution. However, in most cases it is very challenging to find the exact solution and sometimes even impossible to acquire the solution of a selected problem in physical sciences and engineering. In some cases researchers try to highlight the quantitative approximation and nature of solution of a problem. One of the techniques among all the numerous strategies is the fixed point theory. Fixed point theory can be utilized as an essential tool to discuss the existence and uniqueness of the solution to several kinds of integral, differential, integro-differential, and fractional integral or differential equations, for instance, see [1, 13, 17, 18, 21, 29, 31, 40, 46, 50] and references therein.

Fixed point theory has a vital position in mathematics as a powerful tool, and is briefly studied by mathematicians. Because of the extra-ordinary applications of fixed point theory many researchers have contributed with their considerable time to the said subject. Along with the innovation of computers, and growth of modern softwares, for rapid and steady figuring, fixed point theory has gained significant value. It has become the topic of scientific study, both in stochastic circumstances, deterministic and fuzzy. Fixed point theory is a versatile subject, applicable to vast dimensions of mathematics and mathematical sciences like, optimization theory, variational inequalities, mathematical economics, game theory, integral equations, and differential equations.

The concept of fixed point theory began in the early 18th century by Poincare in [22], in 1806. Then, Brouwer [28], in 1912, introduced the solution to the equation $g(\zeta) = \zeta$. The fixed point theorem was also extended for a square, a sphere and their n -dimensional counter parts by him which was in addition, extended by Ky Fan [24]. Afterwards, the powerful Banach principle [43] appeared. Banach proposed that a unique fixed point can be obtained with the help of an operator in the sense of complete metric space and is applicable to numerous mappings such as differentiable or more specifically lipschitz continuous. Several articles can be found on such mappings for example [15, 25, 30, 36, 37, 39]. After then, the Banach contraction principle was generalized in every direction (see for instance see [4, 6–12, 14, 16, 19, 20, 27, 33, 52]).

Nadler, [44] was the first to analyze the multi-valued operator then Caristi [34] introduced Caristi fixed point theorem for multi-value operators. Subsequently, Feng and Sanyang [53] generalized the Banach theorem and Caristi fixed point theorem to the area of multi-valued mapping. Their outcomes are the generalization of famous Nadler fixed point theorem as well as Caristi fixed point theorem for multi-valued operators. Reich [48] generalized his famous Reich contraction to multi-valued mapping.

Likewise, metric spaces are generalized by numerous researchers. One of the generalization was analyzed by Bakhtin [23] and Czerwik [45], who presented the innovated b -metric spaces. The b -metric spaces is further extended by Kamran et al. [51] to extended b metric spaces. In this direction more authors have contributed towards the extension of metric spaces such as quasi metric spaces, dislocated metric spaces, generalized metric spaces and partial metric spaces etc. And the current does

not just stop here. Still, authors are finding ways to restrict or control the triangle inequalities and introducing some new metric type spaces. In recent times, Mlaiki et al. [35] introduced the concept of control metric spaces. Which is in fact, the extension of b -metric spaces and extended b -metric spaces. Abdeljawad et al. [49] modified the control metric space via two control functions which generalizes b -metrics, extended b -metrics and control metric spaces.

In this study, inspired theoretically from the above contribution, we establish some multi-valued fixed point theorems in the setting of double controlled metric spaces and then the established results are used to analyze the existence of solution to a Volterra integral inclusion and singular Fredholm integral inclusions of both types. The obtained results can be utilized in many of research problems seeking existence of solution to various kinds of integral equations. For example one can also use these results to obtain the of solution to Riemann- Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$ [54].

2. Preliminaries

Definition 2.1. Let X be a nonempty set. Define a distance function $\rho : X \times X \rightarrow [0, \infty)$, such that for all $\zeta, \eta, \nu \in X$, if ρ satisfies

- (1) $\rho(\zeta, \eta) = 0 \Leftrightarrow \zeta = \eta$;
- (2) $\rho(\zeta, \eta) = \rho(\eta, \zeta)$;
- (3) $\rho(\zeta, \nu) \leq \rho(\zeta, \eta) + \rho(\eta, \nu)$.

Then ρ is a metric on X . The pair (X, ρ) is called a metric space.

Definition 2.2. [45] Let X be a nonempty set. Define a distance function $\rho : X \times X \rightarrow [0, \infty)$ and $S \geq 1$ be a constant such that for all $\zeta, \eta, \nu \in X$, if ρ satisfies

- (1) $\rho(\zeta, \eta) = 0 \Leftrightarrow \zeta = \eta$;
- (2) $\rho(\zeta, \eta) = \rho(\eta, \zeta)$;
- (3) $\rho(\zeta, \nu) \leq S[\rho(\zeta, \eta) + \rho(\eta, \nu)]$.

Then ρ is a b -metric on X . The pair (X, ρ) is called a b -metric space.

Example 2.3. Consider $X = \{a, b, c\}$, let $\rho : X \rightarrow X$ such that

$$\begin{aligned} \rho(a, a) &= \rho(b, b) = \rho(c, c) = 0, \\ \rho(a, b) &= \rho(b, a) = \frac{2}{9}, \\ \rho(a, c) &= \rho(c, a) = \frac{5}{4}, \\ \rho(b, c) &= \rho(c, b) = \frac{1}{7}. \end{aligned}$$

Now for $S = 4$ we have $\frac{5}{4} = \rho(a, c) \leq 4[\rho(a, b) + \rho(b, c)] = 4[\frac{2}{9} + \frac{1}{7}] = 1.46$. Thus (X, ρ) is a b -metric space. However, it is not a usual metric.

Remark 2.4. From the above example it can be concluded that in general a b -metric is not continuous. It is obvious that for $S = 1$, every b -metric is a standard metric.

Definition 2.5. Let (X, ρ) be a b -metric space, such that for all $\zeta, \eta \in X$.

- (1) A sequence $\{\zeta_n\}_{n \in \mathbb{N}} \in X$, is convergent to ζ , written, $\lim_{n \rightarrow \infty} \zeta_n = \zeta$, if and only if $\lim_{n \rightarrow \infty} \rho(\zeta_n, \zeta) = 0$;
- (2) A sequence $\{\zeta_n\}_{n \in \mathbb{N}} \in X$, is known as Cauchy if and only if $\lim_{n \rightarrow \infty} \rho(\zeta_n, \zeta_m) = 0$, where $n \geq m > \mathbb{N}$;
- (3) (X, ρ) is called complete if and only if every Cauchy sequence is convergent in X .

Definition 2.6. [51] Let X be a nonempty set. Define a distance function $\rho : X \times X \rightarrow [0, \infty)$. Let $\psi : X^2 \rightarrow [1, \infty)$, be a function such that for all $\zeta, \eta, \nu \in X$, if ρ satisfies

- (1) $\rho(\zeta, \eta) = 0 \Leftrightarrow \zeta = \eta$;
- (2) $\rho(\zeta, \eta) = \rho(\eta, \zeta)$;
- (3) $\rho(\zeta, \nu) \leq \psi(\zeta, \nu)[\rho(\zeta, \eta) + \rho(\eta, \nu)]$.

Then ρ is an extended b -metric on X . The pair (X, ρ) is called an extended b -metric space.

Example 2.7. [32] Let $X = \{1, 2, 3, \dots\}$. Define $\psi : X \times X \rightarrow [1, \infty)$ and $\rho : X \times X \rightarrow \mathbb{R}^+$, as

$$\psi(\zeta, \eta) = \begin{cases} |\zeta - \eta|^3, & \text{if } \zeta \neq \eta, \\ 1, & \text{if } \zeta = \eta. \end{cases}$$

And

$$\rho(\zeta, \eta) = (\zeta - \eta)^4.$$

Note that the first two axioms in Definition (2.6) hold, trivially.

For the real numbers x, y , with $x = 0$ or $|x| \geq 1$ and $y = 0$, or $|y| \geq 1$ we establish the following relation.

$$(x + y)^4 \leq |x + y|^3 [x^4 + y^4]. \quad (2.1)$$

Case 1: inequality (2.1) is trivially satisfied if $x = 0$ or $y = 0$.

Case 2: For $|x| \geq 1$ or $|y| \geq 1$, we get

$$\begin{aligned} (x + y)^4 &= \frac{(x + y)^4}{(x^4 + y^4)}(x^4 + y^4), \\ &\leq \frac{(x + y)^4}{|x + y|}(x^4 + y^4), \\ &= |x + y|^3 [x^4 + y^4]. \end{aligned}$$

Finally, setting $x = \zeta - \eta$ and $y = \eta - \nu$, where $\zeta, \eta, \nu \in X$. Then $x = 0$ or $|x| \geq 1$ and $y = 0$ or $|y| \geq 1$. Thus the inequality (2.1) implies

$$(\zeta - \nu)^4 \leq |\zeta - \nu|^3 [(\zeta - \eta)^4 + (\eta - \nu)^4].$$

Thus (X, ρ) is an extended b -metric.

Recently, Mlaiki et al. [35] generalized the notion of b -metric space to control metric space.

Definition 2.8. [35] Let X be a nonempty set. Define a distance function $\rho : X \times X \rightarrow [0, \infty)$. Let $\alpha : X^2 \rightarrow [1, \infty)$, be a function such that for all $\zeta, \eta, \nu \in X$, if ρ satisfies

- (1) $\rho(\zeta, \eta) = 0 \Leftrightarrow \zeta = \eta$;
 (2) $\rho(\zeta, \eta) = \rho(\eta, \zeta)$;
 (3) $\rho(\zeta, \nu) \leq \alpha(\zeta, \eta)\rho(\zeta, \eta) + \alpha(\eta, \nu)\rho(\eta, \nu)$.

Then ρ is a control metric on X . The pair (X, ρ) is called a control metric space.

Example 2.9. Let $X = \{0, 1, 2\}$. We define the distance function $\rho : X \times X \rightarrow [0, \infty)$, by

$\rho(\zeta, \eta)$	0	1	2
0	0	1	$\frac{1}{2}$
1	1	0	$\frac{2}{5}$
2	$\frac{1}{2}$	$\frac{2}{5}$	0

And define the function $\alpha : X^2 \rightarrow [1, \infty)$, by

$\alpha(\zeta, \eta)$	0	1	2
0	1	$\frac{11}{10}$	1
1	$\frac{11}{10}$	1	$\frac{5}{4}$
2	1	$\frac{5}{4}$	1

Thus the above metric is a control metric. However, it is not an extended b -metric as

$$1 = \rho(0, 1) > \alpha(0, 1)[\rho(0, 2), \rho(2, 1)] = 0.99.$$

Thabet et al. [49] introduced double controlled metric type space.

Definition 2.10. [49] Let X be a nonempty set. Define a distance function $\rho : X \times X \rightarrow [0, \infty)$. Let $\alpha, \beta : X^2 \rightarrow [1, \infty)$, be two functions such that for all $\zeta, \eta, \nu \in X$, if ρ satisfies

- (1) $(\rho_1) \rho(\zeta, \eta) = 0 \Leftrightarrow \zeta = \eta$;
 (2) $(\rho_2) \rho(\zeta, \eta) = \rho_d(\eta, \zeta)$;
 (3) $(\rho_3) \rho(\zeta, \nu) \leq \alpha(\zeta, \eta)\rho_d(\zeta, \eta) + \beta(\eta, \nu)\rho_d(\eta, \nu)$.

Then ρ is a double controlled metric on X . The pair (X, ρ) is called a double controlled metric space.

Remark 2.11. A controlled metric type is in fact, a double controlled metric. However, the converse is not true as is verified by the examples given below.

Example 2.12. [49] Let $X = \{0, 1, 2\}$. We define the distance function $\rho : X \times X \rightarrow [0, \infty)$, by

$\rho(\zeta, \eta)$	0	1	2
0	0	1	$\frac{1}{2}$
1	1	0	$\frac{2}{5}$
2	$\frac{1}{2}$	$\frac{2}{5}$	0

And define the functions $\alpha, \beta : X^2 \rightarrow [1, \infty)$, by

$\alpha(\zeta, \eta)$	0	1	2
0	1	$\frac{11}{10}$	1
1	$\frac{11}{10}$	1	$\frac{5}{8}$
2	1	$\frac{5}{8}$	1

and

$\beta(\zeta, \eta)$	0	1	2
0	1	$\frac{11}{10}$	$\frac{3}{2}$
1	$\frac{11}{10}$	$\frac{3}{2}$	1
2	$\frac{3}{2}$	$\frac{5}{4}$	1

It is simple to show that the above metric is a double controlled metric. However, we can deduce that ρ is not an extended b metric as

$$1 = \rho(0, 1) > \alpha(0, 1)[\rho(0, 2), \rho(2, 1)] = 0.99.$$

Example 2.13. [49] Let $X = [0, \infty)$. We define the distance function $\rho : X \times X \rightarrow [0, \infty)$, by

$$\rho(\zeta, \eta) = \begin{cases} 0, & \Leftrightarrow \zeta = \eta, \\ \frac{1}{\zeta}, & \text{if } \zeta \geq 1 \text{ and } \eta \in [0, 1), \\ \frac{1}{\eta}, & \text{if } \eta \geq 1 \text{ and } \zeta \in [0, 1), \\ 1, & \text{If not.} \end{cases}$$

And define the functions $\alpha, \beta : X \times X \rightarrow [1, \infty)$ as

$$\alpha(\zeta, \eta) = \begin{cases} \zeta, & \text{if } \zeta, \eta \geq 1, \\ 1, & \text{If not.} \end{cases}$$

And

$$\beta(\zeta, \eta) = \begin{cases} 1, & \text{if } \zeta, \eta < 1, \\ \max\{\zeta, \eta\}, & \text{If not.} \end{cases}$$

It is obvious that ρ_1 and ρ_2 in Definition (2.10) hold. We claim that (ρ_3) is satisfied in Definition (2.10).

- (1) When $\nu = \zeta$ or $\nu = \eta$, (ρ_3) holds;
- (2) Otherwise (ρ_3) is verified in the case that $\zeta = \eta$. Suppose the condition that $\zeta \neq \eta$, thus we get that $\zeta \neq \eta \neq \nu$. In the subcase $(\zeta \geq 1 \text{ and } \eta \in [0, 1))$ and $(\eta \geq 1 \text{ and } \zeta \in [0, 1))$, it is simple to analyze that (ρ_3) holds.

Subcase 1: $\zeta, \eta \geq 1$.

If $\nu \geq 1$ axiom (ρ_3) holds. But, if $\nu \in [0, 1)$ we have,

$$1 \leq \frac{1}{\zeta} + \eta \frac{1}{\eta},$$

i.e., (ρ_3) is satisfied.

Subcase 2: $\zeta, \eta < 1$.

If $\nu \in [0, 1)$ (ρ_3) holds. However, if $\nu \geq 1$ we have,

$$1 \leq \frac{1}{\nu} + \nu \frac{1}{\nu},$$

i.e., (ρ_3) is verified. We conclude that ρ is double controlled metric.

On the opposite side, we have

$$\rho(0, \frac{1}{2}) = 1 > \alpha(0, 3)\rho(0, 3) + \alpha(3, \frac{1}{2})\rho(3, \frac{1}{2}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

So, the above double controlled metric is not a control metric.

Now, we are going to explore the topological concepts of the double controlled type metric space. Consider the definitions follows immediately.

Definition 2.14. [49] Let (X, ρ) be a double controlled type metric space either by one or two functions

- (1) The sequence $\{\zeta_n\}_{n \in \mathbb{N}}$ is convergent to some $\zeta \in X$, if and only if for each $\epsilon > 0$, \exists an integer N_ϵ such that $\rho(\zeta_n, \zeta) < \epsilon$ for each $n > N_\epsilon$. It can be written mathematically, as $\lim_{n \rightarrow \infty} \rho(\zeta_n, \zeta) = 0$;
- (2) The sequence $\{\zeta_n\}_{n \in \mathbb{N}}$ is Cauchy sequence if and only if for each positive number ϵ , $\rho(\zeta_m, \zeta_n) < \epsilon$, $\forall m > n > N_\epsilon$, where N_ϵ is a positive integer;
- (3) (X, ρ) is said to be a complete double controlled type metric if and only if every Cauchy sequence is convergent in X .

Definition 2.15. [49] Let (X, ρ) be a double controlled type metric space either by one or two functions. For $\zeta \in X$ and $c > 0$.

- (1) The open set $O(\zeta, c)$ is defined as

$$O(\zeta, c) = \{\eta \in X, \rho(\zeta, \eta) < c\};$$

- (2) The self operator τ is continuous at $\zeta \in X$ if $\forall \delta > 0$, $\exists c > 0$ such that $\tau(O(\zeta, c)) \subseteq O(\tau\zeta, \delta)$.

It can be concluded that when τ is continuous at ζ in (X, ρ) . Then $\zeta_n \rightarrow \zeta$ implies that $\tau\zeta_n \rightarrow \tau\zeta$, as $n \rightarrow \infty$.

The quest for the existence of fixed points for multi-valued mapping for complete metric spaces was originated by, Nadler, in 1979. He initiated the study of multi-valued version of Banach contraction mapping exercising the idea of Hausdorff metric.

Consider, some of the fundamental concepts from multi-valued fixed point theory that will help us analyzing the present study.

Definition 2.16. [44] Let (X, ρ) be a metric space, then

- (1) $CB(X) = \{ C \mid C \text{ is a closed, non-empty, and bounded subset of } X \}$;
- (2) $2^X = \{ C \mid C \text{ is a non-empty compact subset of } X \}$;
- (3) $B(\epsilon, C) = \{ \zeta \in X \mid \rho(\zeta, c) < \epsilon \text{ for some } c \in C \}$ if $\epsilon > 0$ and $C \in CB(X)$.

Definition 2.17. The Hausdorff metric H on $CB(X)$ is defined as follow

$$H(R, T) = \max \left\{ \sup_{\zeta \in R} \rho(\zeta, T), \sup_{\eta \in T} \rho(\eta, R) \right\}, \quad \forall R, T \in CB(X),$$

where $\rho(\zeta, R) = \inf_{\eta \in R} \rho(\zeta, \eta)$. In metric space $(CB(X), H)$, $\lim_{n \rightarrow \infty} R_n = R$ means that $\lim_{n \rightarrow \infty} (R_n, R) = 0$. Consider, $R_1, R_2 \in CB(X)$. Then for each $\zeta \in R_1$ and $\delta > 0$, there is $\eta \in R_2$, such that

$$\rho(\zeta, \eta) \leq H(R_1, R_2) + \delta.$$

The mapping H is a metric for $CB(X)$ and is called the Hausdorff metric. It can be deduce that the metric H in fact depends on the metric of X and the two equivalent matrices for X can not generate equivalent Haudorff matrices for $CB(X)$.

Example 2.18. (1) Let $X = \mathbb{R}$, $A = [2, 3]$, $B = [4, 5]$, where $\rho : X \times X \rightarrow [0, \infty)$ is a metric on which H depends. Then,

$$\sup_{\zeta \in A} \rho(\zeta, B) = 2, \quad \sup_{\eta \in B} \rho(\eta, A) = 2, \quad \text{and} \quad H(A, B) = 2.$$

(2) Consider $A = B_r(\zeta)$, $B = B_s(\eta)$, for all $\zeta, \eta \in (X, \rho)$, $0 < r \leq s$. Then, $H(A, B) = \rho(\zeta, \eta) + s - r$.

Definition 2.19. Let $A, B \in C(X)$ where $C(X)$ is the accumulation of all non-empty closed subsets of X . Consider,

$$H(A, B) = \max \left\{ \sup_{\zeta \in B} \rho(\zeta, A), \sup_{\eta \in A} \rho(\eta, B) \right\},$$

where $\rho(\zeta, A) = \inf_{\eta \in A} \rho(\zeta, \eta)$. The pair $(C(H), X)$ is known as generalized Hausdorff distance induced by d .

Definition 2.20. [44] Let (X, ρ_1) and (X, ρ_2) be metric spaces. A mapping $F : X \rightarrow CB(Y)$ is said to be a multi-valued Lipschitz mapping of X into Y iff $\forall \zeta, \eta \in X$ we have,

$$H(F(\zeta), F(\eta)) \leq c\rho_1(\zeta, \eta). \quad (2.2)$$

The constant c in (2.2) is called a Lipschitz constant.

Definition 2.21. If the Lipschitz constant c becomes less than 1 i.e., $c < 1$ in the case of (2.2) than F is called multi-valued contraction mapping.

Definition 2.22. A multi-value operator $\tau : X \rightarrow N(X)$ is called upper semi-continuous, at $\zeta_0 \in X$ if for every $\epsilon > 0 \exists$ a neighbourhood U of ζ_0 , such that $\tau(\zeta) \leq \tau(\zeta_0) + \epsilon \forall \zeta \in U$, when $\tau(\zeta_0) > -\infty$, and $\tau(\zeta) \rightarrow -\infty$ as $\zeta \rightarrow \zeta_0$ when $\tau(\zeta_0) = -\infty$. Mathematically it can be expressed as $\limsup_{\zeta \rightarrow \zeta_0} \tau(\zeta) \leq \tau(\zeta_0)$.

Definition 2.23. A multi-value operator $\tau : X \rightarrow N(X)$ is called lower semi-continuous, at $\zeta_0 \in X$ if for every $\epsilon > 0 \exists$ a neighbourhood U of ζ_0 , such that $\tau(\zeta) \geq \tau(\zeta_0) - \epsilon \forall \zeta \in U$, when $\tau(\zeta_0) < +\infty$, and $\tau(\zeta) \rightarrow +\infty$ as $\zeta \rightarrow \zeta_0$ when $\tau(\zeta_0) = +\infty$. Mathematically it can be expressed as $\liminf_{\zeta \rightarrow \zeta_0} \tau(\zeta) \geq \tau(\zeta_0)$.

Definition 2.24. An element $\zeta \in X$ is known to be a fixed point of multi-valued operator $\tau : X \rightarrow N(X)$ if $\zeta \in \tau(\zeta)$. Mathematically, represented by $Fix(\tau) = \{\zeta \in X : \zeta \in \tau(\zeta)\}$.

3. Multi-value fixed point results

In this section some fixed point results is proved in the setting of double controlled metric spaces. Our first result is Nadler fixed point theorem.

Theorem 3.1. Consider a complete double controlled metric space (X, ρ) controlled by the functions $\alpha, \beta : X \times X \rightarrow [1, \infty)$. Furthermore, suppose that $\tau : X \rightarrow CB(X)$ is a continuous multi-valued contraction mapping. If there exists a $\delta \in (0, 1)$, such that

$$H(\tau\zeta, \tau\eta) \leq \delta\rho(\zeta, \eta), \quad \forall \zeta, \eta \in X, \quad (3.1)$$

with the assumption

$$\sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{\alpha(\zeta_{i+1}, \zeta_{i+2})\beta(\zeta_{i+1}, \zeta_m)}{\alpha(\zeta_i, \zeta_{i+1})} < \frac{1}{\delta}, \quad (3.2)$$

Moreover, for each $\zeta \in X$, suppose that,

$$\lim_{n \rightarrow \infty} \alpha(\zeta, \zeta_n) \quad \text{and} \quad \lim_{n \rightarrow \infty} \beta(\zeta_n, \zeta) \quad \text{exists and are finite.} \quad (3.3)$$

Then, τ has a fixed point.

Proof. The result can be proved via iteration. Consider, $\delta < 1$ be a lipschitz constant for τ , (we may presume $\delta > 0$) and consider $\zeta_0 \in X$.

Now, choose $\zeta_1 \in \tau(\zeta_0)$. Since $\tau(\zeta_0), \tau(\zeta_1) \in CB(X)$ and $\zeta_1 \in \tau(\zeta_0)$ there is a point $\zeta_2 \in \tau(\zeta_1)$ and using Definition (2.17), we have

$$\rho(\zeta_1, \zeta_2) \leq H(\tau(\zeta_0), \tau(\zeta_1)) + \delta,$$

now since, $\tau(\zeta_1), \tau(\zeta_2) \in CB(X)$ and $\zeta_2 \in \tau(\zeta_1)$. Then, there is a point $\zeta_3 \in \tau(\zeta_2)$, such that

$$\rho(\zeta_2, \zeta_3) \leq H(\tau(\zeta_1), \tau(\zeta_2)) + \delta^2.$$

Proceeding similarly a sequence $\{\zeta_n\}_{n=1}^{\infty}$ of points of X is produced. Such that, $\zeta_{n+1} \in \tau(\zeta_n)$ and $\rho(\zeta_n, \zeta_{n+1}) \leq H(\tau(\zeta_{n-1}), \tau(\zeta_n)) + \delta^n$ for all $n \geq 1$, we note that

$$\begin{aligned} \rho(\zeta_n, \zeta_{n+1}) &\leq H(\tau(\zeta_{n-1}), \tau(\zeta_n)) + \delta^n \leq \delta\rho(\zeta_{n-1}, \zeta_n) + \delta^n, \\ &\leq \delta[H(\tau(\zeta_{n-2}), \tau(\zeta_{n-1})) + \delta^{n-1}] + \delta^n, \\ &\leq \delta^2\rho(\zeta_{n-2}, \zeta_{n-1}) + 2\delta^n, \\ &\vdots \\ \rho(\zeta_n, \zeta_{n+1}) &\leq \delta^n\rho(\zeta_0, \zeta_1) + n\delta^n \quad \forall \quad n \geq 1. \end{aligned}$$

We further proceed to check if the given sequence is a Cauchy sequence. Let $n, m \in \mathbb{N}$ be integers such

that $n < m$, we have

$$\begin{aligned}
\rho(\zeta_n, \zeta_m) &\leq \alpha(\zeta_n, \zeta_{n+1})\rho(\zeta_n, \zeta_{n+1}) + \beta(\zeta_{n+1}, \zeta_m)\rho(\zeta_{n+1}, \zeta_m), \\
&\leq \alpha(\zeta_n, \zeta_{n+1})\rho(\zeta_n, \zeta_{n+1}) + \beta(\zeta_{n+1}, \zeta_m)\alpha(\zeta_{n+1}, \zeta_{n+2})\rho(\zeta_{n+1}, \zeta_{n+2}) + \\
&\quad \beta(\zeta_{n+1}, \zeta_m)\beta(\zeta_{n+2}, \zeta_m)\rho(\zeta_{n+2}, \zeta_m), \\
&\leq \alpha(\zeta_n, \zeta_{n+1})\rho(\zeta_n, \zeta_{n+1}) + \beta(\zeta_{n+1}, \zeta_m)\alpha(\zeta_{n+1}, \zeta_{n+2})\rho(\zeta_{n+1}, \zeta_{n+2}) + \\
&\quad \beta(\zeta_{n+1}, \zeta_m)\beta(\zeta_{n+2}, \zeta_m)\alpha(\zeta_{n+2}, \zeta_{n+3})\rho(\zeta_{n+2}, \zeta_{n+3}) + \\
&\quad \beta(\zeta_{n+1}, \zeta_m)\beta(\zeta_{n+2}, \zeta_m)\beta(\zeta_{n+3}, \zeta_m)\rho(\zeta_{n+3}, \zeta_m), \\
&\leq \dots \\
&\leq \alpha(\zeta_n, \zeta_{n+1})\rho(\zeta_n, \zeta_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})\rho(\zeta_i, \zeta_{i+1}) + \\
&\quad \prod_{k=n+1}^{m-1} \beta(\zeta_k, \zeta_m)\rho(\zeta_{m-1}, \zeta_m), \\
&\leq \alpha(\zeta_n, \zeta_{n+1})(\delta^n \rho(\zeta_0, \zeta_1) + n\delta^n) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i + \zeta_{i+1})(\delta^i \rho(\zeta_0, \zeta_1) + i\delta^i) + \\
&\quad \prod_{i=n+1}^{m-1} \beta(\zeta_i, \zeta_m)(\delta^{m-1} \rho(\zeta_0, \zeta_1) + (m-1)\delta^{m-1}), \\
&= \alpha(\zeta_n, \zeta_{n+1})(\delta^n \rho(\zeta_0, \zeta_1) + n\delta^n) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})(\delta^i \rho(\zeta_0, \zeta_1) + i\delta^i), \\
&\leq \alpha(\zeta_n, \zeta_{n+1})(\delta^n \rho(\zeta_0, \zeta_1) + n\delta^n) + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})(\delta^i \rho(\zeta_0, \zeta_1) + i\delta^i), \\
&= \alpha(\zeta_n, \zeta_{n+1})\delta^n \rho(\zeta_0, \zeta_1) + \alpha(\zeta_n, \zeta_{n+1})n\delta^n + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})\delta^i \rho(\zeta_0, \zeta_1) + \\
&\quad \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})i\delta^i,
\end{aligned}$$

we used $\alpha(\zeta, \eta) > 1$. Let us define

$$S_p = \sum_{i=0}^p \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})\delta^i.$$

Hence, we have

$$\rho(\zeta_m, \zeta_n) \leq \rho(\zeta_0, \zeta_1)[\alpha(\zeta_n, \zeta_{n+1})(\delta^n + n\delta^n) + (S_{m-1} - S_n)] + (S_{m-1} - S_n) \sum_{i=n+1}^{m-1} i. \quad (3.4)$$

The ratio test together with (3.2) imply that the limit of real number sequence $\{S_n\}$ exists and so, $\{S_n\}$ is a Cauchy sequence. In fact, the ratio test is applied to the term $x_i = \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_j, \zeta_{i+1})$. As $\delta \in (0, 1)$, by letting $m, n \rightarrow \infty$ in (3.4), that is

$$\lim_{m, n \rightarrow \infty} \rho(\zeta_m, \zeta_n) = 0 \quad \forall \quad i, j \geq 1,$$

which follows that the sequence $\{\zeta_n\}_{n=1}^\infty$ is a Cauchy sequence. Since (X, ρ) is a complete double controlled metric space. So, the sequence $\{\zeta_n\}_{n=1}^\infty$ converges to some point $\zeta_0 \in X$. Consequently, by continuity of τ , $\{\tau(\zeta_i)\}_{i=1}^\infty$ converges to $\tau(\zeta_0)$ and since, $\zeta_n \in \tau(\zeta_{n-1})$ for all n . Therefore, it follows that $\zeta_0 \in \tau(\zeta_0)$. Which completes the proof. \square

Example 3.2. Let $X = \{0, 1, 2\}$ we define distances by

$\rho(\zeta, \eta)$	0	1	2
0	0	$\frac{1}{2}$	$\frac{7}{3}$
1	$\frac{1}{2}$	0	$\frac{2}{3}$
2	$\frac{7}{3}$	$\frac{2}{3}$	0

Lets define the functions $\alpha, \beta : X \rightarrow [1, \infty)$ by

$\alpha(\zeta, \eta)$	0	1	2
0	1	$\frac{7}{2}$	$\frac{9}{32}$
1	$\frac{7}{2}$	1	$\frac{1}{3}$
2	$\frac{9}{32}$	$\frac{1}{3}$	1

$\beta(\zeta, \eta)$	0	1	2
0	1	$\frac{3}{2}$	$\frac{5}{9}$
1	$\frac{3}{2}$	1	$\frac{5}{2}$
2	$\frac{5}{9}$	$\frac{2}{2}$	5

As it seems the metric is not a controlled metric. For instance,

$$2.\bar{3} = \frac{7}{3} = \rho(0, 2) > \alpha(0, 1)\rho(0, 1) + \alpha(1, 2)\rho(1, 2) = \left(\frac{2}{5}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 1.97\bar{2},$$

$$2.\bar{3} = \frac{7}{3} = \rho(0, 2) > \beta(0, 1)\rho(0, 1) + \beta(1, 2)\rho(1, 2) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)\left(\frac{2}{3}\right) = 1.58\bar{3}.$$

It can be verified that (X, ρ) is a complete doubled controlled metric space.

Now we define a multi-value mapping $\tau : X \rightarrow CB(X)$ by

$$\tau(\zeta) = \{0, 1\} \quad \forall \zeta \in X.$$

For $c = \frac{2}{9}$ it is clear that the condition (3.1) is satisfied $\forall \zeta \in X$. In addition, (3.2) holds for each $\zeta_0 \in X$. All the conditions of Theorem (3.1) are satisfied. For instance, Let $\zeta_0 = \{0, 1\}$ and $\delta = \frac{2}{9}$, we have $\sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{\alpha(\zeta_{i+1}, \zeta_{i+2})\beta(\zeta_{i+1}, \zeta_m)}{\alpha(\zeta_i, \zeta_{i+1})} = 1 < \frac{1}{\delta} = \frac{9}{2}$. In addition $\lim_{n \rightarrow \infty} \alpha(\zeta, \zeta_n) = \max(\zeta, 1) < \infty$ and $\lim_{n \rightarrow \infty} \alpha(\zeta_n, \zeta) = \max(\zeta, 1) < \infty$. Hence the fixed points are $\{0, 1\}$.

Our second result is extending Banach multi-valued fixed point theorem to double controlled type metric space. Before proceeding to another result consider the following definition, which will later be used in the following result.

Definition 3.3. [53] Consider a multi-valued mapping $\tau : X \rightarrow C(X)$. Let $f : X \rightarrow \mathbb{R}$ be a function defined as $f(\zeta) = \rho(\zeta, \tau(\zeta))$, for all $\zeta \in X$.

For a positive constant $u \in (0, 1)$, define the set $H_u^\zeta \subset X$ as,

$$H_u^\zeta = \{\eta \in \tau(\zeta) | u\rho(\zeta, \eta) \leq \rho(\zeta, \tau(\zeta))\}.$$

Theorem 3.4. Consider (X, ρ) be a complete a double controlled type metric space by two controlled functions $\alpha, \beta : X \times X \rightarrow [1, \infty)$. Let $\tau : X \rightarrow C(X)$ be a multi-valued mapping. If \exists a constant $k \in (0, 1)$ such that for any $\zeta \in X$ there is $\eta \in H_u^\zeta$, satisfying

$$\rho(\eta, \tau(\eta)) \leq k\rho(\zeta, \eta) \quad \forall \quad \zeta, \eta \in X. \quad (3.5)$$

Further assume that

$$\sup_{m \geq 1} \lim_{n \rightarrow \infty} \frac{\alpha(\zeta_{n+1}, \zeta_{i+2})}{\alpha(\zeta_i, \zeta_{i+1})} \beta(\zeta_{i+1}, \zeta_m) < \frac{1}{k}. \quad (3.6)$$

Moreover, consider that for each $\zeta \in X$ that $\lim_{n \rightarrow \infty} \alpha(\zeta, \zeta_n)$ and $\lim_{n \rightarrow \infty} \alpha(\zeta_n, \zeta)$ exist and are finite. Then, τ has a fixed point provided that $k < u$ and f is lower semi-continuous.

Proof. Since $\tau(\zeta) \in C(X)$ for any $\zeta \in X$, and H_u^ζ is a non-empty set for any constant $u \in (0, 1)$. For any initial point $\zeta_0 \in X$, there exists $\zeta_1 \in H_u^{\zeta_0}$, such that

$$\rho(\zeta_1, \tau(\zeta_1)) \leq k\rho(\zeta_0, \zeta_1).$$

For any $\zeta_1 \in X$ there exists $\zeta_2 \in H_u^{\zeta_1}$, satisfying

$$\rho(\zeta_2, \tau(\zeta_2)) \leq k\rho(\zeta_1, \zeta_2).$$

Continuing the current process an iterative sequence $\{\zeta_n\}_{n=0}^\infty$, where $\zeta_{n+1} \in H_u^{\zeta_n}$ can be obtained, satisfying

$$\rho(\zeta_{n+1}, \tau(\zeta_{n+1})) \leq k\rho(\zeta_n, \zeta_{n+1}). \quad \text{for } n = 0, 1, 2, \dots \quad (3.7)$$

On the other hand, for $\zeta_{n+1} \in H_u^{\zeta_n}$, we have

$$u\rho(\zeta_n, \zeta_{n+1}) \leq \rho(\zeta_n, \tau(\zeta_n)), \quad \text{for } n = 0, 1, 2, \dots \quad (3.8)$$

Comparing inequalities (3.7) and (3.8), we get

$$\rho(\zeta_{n+1}, \zeta_{n+2}) \leq \frac{k}{u} \rho(\zeta_n, \tau(\zeta_n)) \quad \text{for } n = 0, 1, 2, \dots$$

Now it is quite simple to get the following result

$$\rho(\zeta_n, \zeta_{n+1}) \leq L^n \rho(\zeta_0, \zeta_1) \quad \text{where } L = \frac{k}{u} \quad \text{for } n = 0, 1, 2, \dots$$

Now what is left of the theorem is that we prove that $\{\zeta_n\}_{n=0}^\infty$ is a Cauchy sequence.

Let m, n be integers such that $n < m$ then from Theorem (3.1) we get to the following.

$$\begin{aligned} \rho(\zeta_n, \zeta_m) &\leq \alpha(\zeta_n, \zeta_{n+1})\rho(\zeta_n, \zeta_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})\rho(\zeta_i, \zeta_{i+1}) + \\ &\quad \prod_{i=n+1}^{m-1} \beta(\zeta_i, \zeta_m)\rho(\zeta_{m-1}, \zeta_m), \\ &\leq \alpha(\zeta_n, \zeta_{n+1})L^n\rho(\zeta_0, \zeta_1) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})L^i\rho(\zeta_0, \zeta_1) + \\ &\quad \sum_{i=n+1}^{m-1} \beta(\zeta_i, \zeta_m)L^{m-1}\rho(\zeta_0, \zeta_1), \\ &= \alpha(\zeta_n, \zeta_{n+1})L^n\rho(\zeta_0, \zeta_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})L^i\rho(\zeta_0, \zeta_1), \\ &\leq \alpha(\zeta_n, \zeta_{n+1})L^n\rho(\zeta_0, \zeta_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})L^i\rho(\zeta_0, \zeta_1)L^i\rho(\zeta_0, \zeta_1). \end{aligned}$$

Let $S_p = \sum_{i=0}^p \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})L^i$. Hence, we have

$$\rho(\zeta_n, \zeta_m) \leq \rho(\zeta_0, \zeta_1)[L^n\alpha(\zeta_n, \zeta_{n+1}) + (S_{m-1} - S_n)]. \quad (3.9)$$

The ratio test together with (3.6) implies that the limit of real number sequence $\{S_n\}$ exists and so, $\{S_n\}$ is a Cauchy sequence. In fact, the ratio test is applied to the term $x_i = \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_j, \zeta_{i+1})$. Letting $m, n \rightarrow \infty$ in (3.9), that is

$$\lim_{m,n \rightarrow \infty} \rho(\zeta_m, \zeta_n) = 0 \quad \forall \quad i, j \geq 1,$$

where $L = \frac{k}{u}$ now we know that $k < u$ so we have $L^n \rightarrow 0$ as $n \rightarrow \infty$.

From which is deduced that the sequence $\{\zeta_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since (X, ρ) is a complete double controlled metric space. So, the sequence $\{\zeta_n\}_{n=1}^{\infty}$ converges to some point $\zeta_0 \in X$. We claim that ζ is a fixed point of T .

As a matter of fact, from the given proof it is deduced that $\{\zeta_n\}_{n=0}^{\infty}$ converges to ζ . While, on the other hand, $\{f(\zeta_n)\}_{n=0}^{\infty} = \{\rho(\zeta_n, \tau(\zeta_n))\}_{n=0}^{\infty}$ is a decreasing sequence and hence, converges to 0. Since, f is lower semi-continuous, we get

$$0 \leq f(\zeta) \leq \lim_{n \rightarrow \infty} f(\zeta_n) = 0.$$

Therefore, $f(\zeta) = 0$. Consequently, the closeness of $\tau(\zeta)$ implies $\zeta \in \tau(\zeta)$. \square

Remark 3.5. Theorem (3.4) is in fact the extension of Nadler multi-valued fixed point theorem. If τ fulfills the conditions of Theorem (3.1), then τ be upper semi continuous as f being a lower semi continuous $\forall \zeta, \eta \in X$.

Example 3.6. Let $X = \{a, b, c\}$ and consider the distances

$\rho(\zeta, \eta)$	a	b	c
a	0	$\frac{5}{9}$	$\frac{7}{2}$
b	$\frac{5}{9}$	0	$\frac{7}{11}$
c	$\frac{7}{2}$	$\frac{7}{11}$	0

And the functions $\alpha, \beta : X \rightarrow [1, \infty)$

$\alpha(\zeta, \eta)$	a	b	c
a	1	$\frac{11}{4}$	$\frac{5}{8}$
b	$\frac{11}{4}$	1	$\frac{9}{5}$
c	$\frac{5}{8}$	$\frac{9}{5}$	1

$\beta(\zeta, \eta)$	a	b	c
a	1	$\frac{5}{7}$	$\frac{4}{5}$
b	$\frac{5}{7}$	1	$\frac{15}{4}$
c	$\frac{4}{5}$	$\frac{15}{4}$	1

The above double controlled metric is not a control metric, as can be seen.

$$3.5 = \frac{7}{2}\rho(a, c) > \alpha(0, 1)\rho(0, 1) + \alpha(1, 2)\rho(1, 2) = \left(\frac{11}{4}\right)\left(\frac{5}{9}\right) + \left(\frac{9}{5}\right)\left(\frac{7}{11}\right) = 2.67\bar{3}2,$$

$$3.5 = \frac{7}{2}\rho(a, c) > \alpha(0, 1)\rho(0, 1) + \alpha(1, 2)\rho(1, 2) = \left(\frac{5}{7}\right)\left(\frac{5}{9}\right) + \left(\frac{15}{4}\right)\left(\frac{7}{11}\right) \approx 2.79$$

It can be analyzed that (X, ρ) is a complete double controlled metric. Consider, the multi-value operator

$$\tau(\zeta) = \begin{cases} \{a\}, & \text{if } \zeta = a; \\ \{b\}, & \text{if } \zeta = b; \\ \{c\}, & \text{if } \zeta = c. \end{cases}$$

Now it is simple to deduce that $f(\zeta) = \rho(\zeta, \tau(\zeta)) = 0$.

We see that $f(\zeta)$ is continuous. In addition, there exists $\eta \in H_{\frac{5}{9}}^x$, such that the condition (3.5) is satisfied. Moreover, for every $\zeta_0 \in X$ the condition (3.6) is satisfied. Thus all the hypothesis of the Theorem (3.4) are fulfilled. Hence $\{a, b, c\}$ are the fixed points of the τ for $k = \frac{2}{3}$.

However, τ is not a contractive mapping in Nadler. For instance,

$$\frac{5}{9} = H(\tau(\zeta), \tau(\eta)) \geq \delta\rho(\zeta, \eta) = \left(\frac{2}{3}\right)\left(\frac{5}{9}\right),$$

which shows that Theorem (3.1) is the generalization of Nadler multi-valued fixed point theorem.

Now proceeding the work to extending Branciari integral fixed point theorem to double controlled type metric space. The theorem is proved in three steps.

Theorem 3.7. Consider (X, ρ) be a complete double controlled type metric space and $\tau : X \rightarrow C(X)$ be a multi-value mapping. Suppose, there exists a constant $k \in (0, 1)$ such that for any $\zeta \in X$ and $\eta \in \tau(\zeta)$, there is $\mu \in \tau(\eta)$, satisfying

$$\int_0^{\rho(\tau(\zeta), \tau(\eta))} \varphi(t) dt \leq k \int_0^{\rho(\zeta, \eta)} \varphi(t) dt, \quad (3.10)$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a Lebesgue integrable mapping which is summable (that is, with finite integral) on each compact subset of $[0, \infty)$ and such that for each $\epsilon > 0$, we have $\int_0^\epsilon \varphi(t) dt > 0$. Moreover, assume that

$$\sup_{m \geq 1} \lim_{n \rightarrow \infty} \frac{\alpha(\zeta_{n+1}, \zeta_{i+2})}{\alpha(\zeta_i, \zeta_{i+1})} \beta(\zeta_{i+1}, \zeta_m) < \frac{1}{k}. \quad (3.11)$$

Further, consider that for each $\zeta \in X$ that $\lim_{n \rightarrow \infty} \alpha(\zeta, \zeta_n)$ and $\lim_{n \rightarrow \infty} \alpha(\zeta_n, \zeta)$ exist and are finite. Then, τ has a fixed point provided that f is lower semi-continuous.

Proof. For any initial point $\zeta_0 \in X$ and $\zeta_1 \in \tau(\zeta_0)$ a sequence $\{\zeta_n\}_{n=0}^\infty$ can be constructed, where $\zeta_{n+1} \in \tau(\zeta_n)$ and

$$\int_0^{\rho(\zeta_{n+1}, \zeta_{n+2})} \varphi(t) dt \leq k \int_0^{\rho(\zeta_n, \zeta_{n+1})} \varphi(t) dt, \quad \text{for } n = 0, 1, 2, \dots$$

Step 1. Consider $f(\zeta_n) \rightarrow 0$ as $n \rightarrow \infty$, it is simple to verify that $\{\rho(\zeta_n, \zeta_{n+1})\}$ is a decreasing sequence and

$$\int_0^{\rho(\zeta_n, \zeta_{n+1})} \varphi(t) dt \leq k^n \int_0^{\rho(\zeta_0, \zeta_1)} \varphi(t) dt, \quad \text{for } n = 0, 1, 2, \dots$$

Consider $A_n = \rho(\zeta_n, \zeta_{n+1})$, we assert that $A_n \rightarrow 0$ as $n \rightarrow \infty$. Indeed, $\{A_n\}_{n=0}^\infty$ is a convergent sequence, because $\{A_n\}_{n=0}^\infty$ is a non-negative and decreasing sequence. Assume $A_n \rightarrow \epsilon$ as $n \rightarrow \infty$, then

$$\int_0^{A_n} \varphi(t) dt \geq \int_0^\epsilon \varphi(t) dt, \quad \text{for } n = 0, 1, 2, \dots$$

Thus, we have

$$\lim_{n \rightarrow \infty} \int_0^{A_n} \varphi(t) dt \geq \int_0^\epsilon \varphi(t) dt > 0.$$

Thus contradiction, as

$$\lim_{n \rightarrow \infty} \int_0^{A_n} \varphi(t) dt = \lim_{n \rightarrow \infty} k^n \int_0^{\rho(\zeta_0, \zeta_1)} \varphi(t) dt.$$

It can be noted that $0 \leq f(\zeta_n) \leq \rho(\zeta_n, \zeta_{n+1})$, so we have $f(\zeta_n) \rightarrow 0$ as $n \rightarrow \infty$.

Step 2. we now show that $\{\zeta_n\}_{n=0}^\infty$ is a Cauchy sequence. From Theorem (3.1) we have

$$\begin{aligned} \rho(\zeta_n, \zeta_m) &\leq \alpha(\zeta_n, \zeta_{n+1})\rho(\zeta_n, \zeta_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})\rho(\zeta_i, \zeta_{i+1}) + \\ &\quad \prod_{k=n+1}^{m-1} \beta(\zeta_k, \zeta_m)\rho(\zeta_{m-1}, \zeta_m), \\ &\leq \alpha(\zeta_n, \zeta_{n+1})k^n \int_0^{\rho(\zeta_0, \zeta_1)} \varphi(t)dt + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})k^i \int_0^{\rho(\zeta_0, \zeta_1)} \varphi(t)dt + \\ &\quad \prod_{k=n+1}^{m-1} \beta(\zeta_k, \zeta_m)k^{m-1} \int_0^{\rho(\zeta_0, \zeta_1)} \varphi(t)dt, \\ &= \alpha(\zeta_n, \zeta_{n+1})k^n \int_0^{\rho(\zeta_0, \zeta_1)} \varphi(t)dt + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})k^i \int_0^{\rho(\zeta_0, \zeta_1)} \varphi(t)dt, \\ &\leq \alpha(\zeta_n, \zeta_{n+1})k^n \int_0^{\rho(\zeta_0, \zeta_1)} \varphi(t)dt + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})k^i \int_0^{\rho(\zeta_0, \zeta_1)} \varphi(t)dt. \end{aligned}$$

Let $S_p = \sum_{i=0}^p \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})k^i$. Hence, we have

$$\rho(\zeta_n, \zeta_m) \leq \int_0^{\rho(\zeta_0, \zeta_1)} \varphi(t)dt [k^n \alpha(\zeta_n, \zeta_{n+1}) + (S_{m-1} - S_n)]. \quad (3.12)$$

Step 3. The ratio test together with (3.11) implies that the limit of real number sequence $\{S_n\}$ exists and so, $\{S_n\}$ is a Cauchy sequence. In fact, the ratio test is applied to the term $x_i = \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_j, \zeta_{j+1})$. Letting $m, n \rightarrow \infty$ in (3.12), that is

$$\lim_{m, n \rightarrow \infty} \rho(\zeta_m, \zeta_n) = 0.$$

Finally, the sequence is a Cauchy sequence and since (X, ρ) is a complete double controlled metric space. Additionally, from the assertion of semi-continuity of f and convergence of $\{\zeta_n\}_{n=0}^\infty$ and closeness of τ implies $\zeta \in \tau(\zeta)$. Which completes the proof. \square

Example 3.8. Consider the doubled controlled metric and a multi-valued operator defined in Example (3.2). Consider,

$$\phi(t) = \int_0^{\rho(\zeta, \eta)} t dt.$$

For $k = \frac{3}{7}$ we see that all the hypothesis of the Theorem (3.7) are fulfilled. Thus $\{0, 1\}$ are fixed points of τ .

Our next result is extending Reich multi-value contraction mapping theorem to doubled controlled type metric space. Before, proceeding to the result consider the following definition.

Definition 3.9. Consider the multi-value mapping $\tau : X \rightarrow C(X)$. The mapping is said to be multi-value Reich contraction mapping if there exists constants $i, j, k \in \mathbb{R}$, such that $i + j + k < 1$ and for any $\zeta \in X$ there exists $\tau(\zeta)$ such that $\zeta \in \tau(\zeta)$, satisfying

$$\rho(\tau(\zeta), \tau(\eta)) \leq i\rho(\zeta, \eta) + j\rho(\zeta, \tau(\zeta)) + k\rho(\eta, \tau(\eta)), \quad \forall \zeta, \eta \in X. \quad (3.13)$$

Theorem 3.10. Let (X, ρ) be a complete doubled controlled type metric space, which is controlled by the functions $\alpha, \beta : X \rightarrow [1, \infty)$. Consider the multi-value Reich contraction $\tau : X \rightarrow C(X)$. Further assume that

$$\sup_{m \geq 1} \lim_{n \rightarrow \infty} \frac{\alpha(\zeta_{n+1}, \zeta_{i+2})}{\alpha(\zeta_i, \zeta_{i+1})} \beta(\zeta_{i+1}, \zeta_m) < h. \quad (3.14)$$

Moreover, consider that for each $\zeta \in X$, the

$$\lim_{n \rightarrow \infty} \alpha(\zeta, \zeta_n) \quad \text{and} \quad \lim_{n \rightarrow \infty} \beta(\zeta_n, \zeta) \quad \text{exists and are finite.} \quad (3.15)$$

Then, τ has a fixed point.

Proof. We begin our proof by constructing a sequence $\{\zeta_n\}_{n \in \mathbb{N}}$ such $\zeta_{n+1} \in \tau(\zeta_n)$, put $\zeta = \zeta_n$ and $\eta = \zeta_{n-1}$ in (3.13), we have

$$\begin{aligned} \rho(\tau(\zeta_n), \tau(\zeta_{n-1})) &\leq i\rho(\zeta_n, \zeta_{n-1}) + j\rho(\zeta_n, \tau(\zeta_n)) + k\rho(\zeta_{n-1}, \tau(\zeta_{n-1})), \\ \rho(\zeta_{n+1}, \zeta_n) - j\rho(\zeta_{n+1}, \zeta_n) &\leq (i+k)\rho(\zeta_n, \zeta_{n-1}), \\ \rho(\zeta_{n+1}, \zeta_n) &\leq \frac{i+k}{1-j}\rho(\zeta_n, \zeta_{n-1}). \end{aligned}$$

From iteration of the sequence it follows that

$$\rho(\zeta_{n+1}, \zeta_n) \leq h^n \rho(\zeta_1, \zeta_0), \quad (3.16)$$

where $h = \frac{i+k}{1-j}$. Note that $h < 1$. Proceeding the theorem, it is verified that the sequence is a Cauchy. For any $m > n \in \mathbb{N}$, from Theorem (3.1), we get to the following

$$\begin{aligned} \rho(\zeta_n, \zeta_m) &\leq \alpha(\zeta_m, \zeta_{n+1})\rho(\zeta_n, \zeta_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})\rho(\zeta_i, \zeta_{i+1}) + \\ &\quad \prod_{i=n+1}^{m-1} \beta(\zeta_i, \zeta_m)\rho(\zeta_{m-1}, \zeta_m), \\ &\leq \alpha(\zeta_n, \zeta_{n+1})h^n \rho(\zeta_0, \zeta_1) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})h^i \rho(\zeta_0, \zeta_1) + \\ &\quad \sum_{i=n+1}^{m-1} \beta(\zeta_i, \zeta_m)h^{m-1} \rho(\zeta_0, \zeta_1), \\ &= \alpha(\zeta_n, \zeta_{n+1})h^n \rho(\zeta_0, \zeta_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})h^i \rho(\zeta_0, \zeta_1), \\ &\leq \alpha(\zeta_n, \zeta_{n+1})h^n \rho(\zeta_0, \zeta_1) + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1})h^i \rho(\zeta_0, \zeta_1). \end{aligned}$$

Let $S_p = \sum_{i=0}^p \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_i, \zeta_{i+1}) h^i$. Hence, we have

$$\rho(\zeta_n, \zeta_m) \leq \rho(\zeta_0, \zeta_1) [L^n \alpha(\zeta_n, \zeta_{n+1}) + (S_{m-1} - S_n)]. \quad (3.17)$$

The ratio test together with (3.17) implies that the limit of real number sequence $\{S_n\}$ exists and so, $\{S_n\}$ is a Cauchy sequence. Applying the ratio test to the term $x_i = \left(\prod_{j=0}^i \beta(\zeta_j, \zeta_m) \right) \alpha(\zeta_j, \zeta_{i+1})$. Letting $m, n \rightarrow \infty$ in (3.17), that is

$$\lim_{m, n \rightarrow \infty} \rho(\zeta_m, \zeta_n) = 0 \quad \forall \quad i, j \geq 1,$$

where $h = \frac{i+k}{1-j}$ now we know that $h < 1$ so we have $h^n \rightarrow 0$ as $n \rightarrow \infty$.

From which is deduced that the sequence $\{\zeta_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since (X, ρ) is a complete double controlled metric space. So, the sequence $\{\zeta_n\}_{n=1}^{\infty}$ converges to some point $\zeta_0 \in X$. We claim that ζ is a fixed point of τ . Certainly, from the given proof it is assured that $\{\zeta_n\}_{n=0}^{\infty}$ converges to ζ . Finally, the closeness of $\tau(\zeta)$ implies that $\zeta \in \tau(\zeta)$. \square

Remark 3.11. As noticed from (3.13) in Theorem (3.10) if $j = k = 0$ then we get the Banach contraction. Similarly, if $i = 0$ and $j = k$ we get Kannan type contraction in multi-valued version.

Example 3.12. Consider, the double controlled metric space defined in Example (3.6). Define a multi-valued operator $\tau : X \rightarrow CB(X)$, as

$$\tau(\zeta) = \{a, b, c\} \forall \zeta \in X.$$

For $i = \frac{1}{4}$, $j = \frac{2}{4}$ and $k = \frac{3}{11}$ it can be analyzed that overall hypothesis of Theorem (3.10) are fulfilled. The fixed points are $\{a, b, c\}$.

4. Applications to integral inclusions

In the current section existence theorems for Volterra type integral inclusion and singular Fredholm integral inclusions are obtained via multi-valued fixed point results.

The Volterra type integral inclusion can be expressed as

$$\phi(\alpha) \in \vartheta(\alpha) + \delta \int_a^x \Gamma(\alpha, \tau, \phi(\tau)) d\tau, \quad (4.1)$$

where $\vartheta(\alpha)$ is a continuous function on the given interval, $\Gamma(\alpha, \tau)$ is the family of non-empty compact and convex sets on the interval, ϕ is the unknown solution belonging to the given inclusion, and $a \leq x \leq b$.

Moreover, consider a multi-valued operator $L : X \rightarrow CB(X)$, defined by

$$L\phi(\alpha) = \left\{ j \in X : j \in \vartheta(\alpha) + \delta \int_a^x \Gamma(\alpha, \tau, \phi(\tau)) d\tau \right\}. \quad (4.2)$$

Then clearly the operator (4.2) is non-empty and closed.

Theorem 4.1. Consider the space of continuous functions $X = C([a, b], \mathbb{R})$ on the interval $[a, b]$. Furthermore, consider the double controlled metric $\rho : X \times X \rightarrow [0, \infty)$ defined by $\rho(\alpha, \beta) = |\alpha - \beta|$ and the controlled functions $\mu, \nu : X \times X \rightarrow [1, \infty)$ defined by $\mu = \alpha + 9\beta + 7$ and $\nu = 6\alpha + 2\beta + 4$. Additionally, (4.2) be the operator from $X \rightarrow CB(X)$. The the integral (4.1) will have some solution provided that the following holds.

$$|\Gamma_1(\alpha, \tau, \phi(\tau)) - \Gamma_1(\alpha, \tau, \psi(\tau))| \leq \frac{\delta}{e^{1+x}(a-x)} |\phi(\alpha) - \psi(\beta)|.$$

Proof. As obvious the given double controlled metric space is a complete space. Consider $\alpha, \beta \in X$ such that $j \in L\phi$ and $k \in L\psi$ then we have $\alpha, \tau \in [a, b]$ such that $k(\alpha) = \vartheta(\alpha) + \delta \int_a^x \Gamma(\alpha, \tau, \phi(\tau)) d\tau$. Thus,

$$\begin{aligned} |j(\alpha) - k(\beta)| &= \left| \vartheta(\alpha) + \delta \int_a^x \Gamma_1(\alpha, \tau, \phi(\tau)) d\tau - \left[\vartheta(\alpha) + \delta \int_a^x \Gamma_2(\alpha, \tau, \psi(\tau)) d\tau \right] \right|, \\ &\leq \int_a^x \frac{\delta}{(a-x)e^{1+x}} |\phi(\alpha) - \psi(\beta)| d\tau, \\ &\leq \frac{\delta}{e^{1+x}} \rho_d(\alpha, \beta). \end{aligned}$$

Finally, interchanging the role of $j \in L\alpha$ and $k \in L\beta$, the following is obtained

$$H(L\alpha, L\beta) \leq \frac{\delta}{e^{1+x}} \rho_d(\alpha, \beta).$$

Accordingly, all the hypothesis of Theorem (3.1) are fulfilled. Consequently, the integral (4.1) possesses some solution. \square

In general there arises two cases of singularity cases in Fredholm and Volterra integral equations [3, 26, 38] to be precise those cases are:

- (1) Dealing with the limit $a \rightarrow -\infty$ and $b \rightarrow \infty$.
- (2) Dealing with the Kernel $\Gamma(\alpha, \tau) = \pm\infty$ at some point on the interval $[a, b]$.

The general form of both the types are given below.

$$\phi(\alpha) \in \vartheta(\alpha) + \delta \int_{-\infty}^{\infty} \Gamma(\alpha, \tau, \phi(\tau)) d\tau, \quad (4.3)$$

$$\phi(\alpha) \in \vartheta(\alpha) + \delta \int_a^b \frac{1}{(\alpha - \tau)^n} \phi(\tau) d\tau, \quad (4.4)$$

where $\vartheta(\alpha)$ is a function continuous on $[a, b]$, $\Gamma(\alpha, \tau)$ is the family of non-empty compact and convex sets on the interval, ϕ is the unknown solution belonging to the given inclusion, and $a \leq x \leq b$ in each case.

As can be observed in the second case that when $\tau \rightarrow \alpha$ the integrand $\rightarrow \infty$. Both the given forms are singular Fredholm integral inclusions. Whenever one of the limit is unknown then (4.3) – (4.4) represents singular Volterra integral inclusions.

The theorems below guarantee the solution of the integrals (4.3) – (4.4).

Theorem 4.2. Consider the space of continuous functions $X = C([a, b], \mathbb{R})$ on the interval $[a, b]$. Furthermore, consider the double controlled metric $\rho : X \times X \rightarrow [0, \infty)$ defined by $\rho(\alpha, \beta) = |\alpha - \beta|$ and the controlled functions $\mu, \nu : X \times X \rightarrow [1, \infty)$ defined by $\mu = \alpha + 9\beta + 7$ and $\nu = 6\alpha + 2\beta + 4$. The singular Fredholm integral inclusion of type I is given by (4.3). Moreover, consider a multi-valued operator $L : X \rightarrow CB(X)$ defined by

$$L\phi(\alpha) = \left\{ j \in X : j \in \vartheta(\alpha) + \delta \int_{-\infty}^{\infty} \Gamma(\alpha, \tau, \phi(\tau)) d\tau \right\}. \quad (4.5)$$

Then, (4.3) will have some solution in the given space provided that the following holds.

$$|\Gamma_1(\alpha, \tau, \phi(\tau)) - \Gamma_1(\alpha, \tau, \psi(\tau))| \leq \frac{\delta}{(a-x)} \Theta(\alpha, \beta),$$

where $\Theta(\alpha, \beta) = a\rho_d(\zeta, \beta) + b\rho_d(\alpha, L(\alpha)) + c\rho_d(\beta, L(\beta))$ with $a + b + c < 1$.

Proof. Since the given double controlled metric space is a complete space. The theorem can be proved easily by following the same steps from Theorem (4.1). Then, by Theorem (3.10) the integral (4.3) possesses a solution. \square

Theorem 4.3. Consider the space of continuous functions $X = C([a, b], \mathbb{R})$ on the interval $[a, b]$. Furthermore, consider the double controlled metric $\rho : X \times X \rightarrow [0, \infty)$ defined by $\rho(\alpha, \beta) = |\alpha - \beta|$ and the controlled functions $\mu, \nu : X \times X \rightarrow [1, \infty)$ defined by $\mu = \alpha + 9\beta + 7$ and $\nu = 6\alpha + 2\beta + 4$. The singular Fredholm integral inclusion of type II is given by (4.4). Moreover, consider a multi-valued operator $L : X \rightarrow CB(X)$ defined by

$$L\phi(\alpha) = \left\{ j \in X : j \in \vartheta(\alpha) + \delta \int_a^b \frac{1}{(\alpha - \tau)^n} \phi(\tau) d\tau \right\}. \quad (4.6)$$

Then, (4.4) will have some solution in the given space provided that the following holds.

$$\left| \frac{1}{(\alpha - \tau)^n} - \frac{1}{(\beta - \tau)^n} \right| \leq \frac{\delta}{(a-x)} \Phi(\alpha, \beta).$$

where $\Phi(\alpha, \beta) = a\rho_d(\zeta, \beta) + b\rho_d(\alpha, L(\alpha)) + c\rho_d(\beta, L(\beta))$ with $a + b + c < 1$.

Proof. Since the given double controlled metric space is a complete space. Then, the theorem can be proved easily by following the same steps from Theorem (4.1). Then, by Theorem (3.10) the integral (4.4) possesses a solution. \square

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Conflict of interest

The authors declare that they have no competing interest regarding this manuscript.

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