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Research article

Blow up at well defined time for a coupled system of one spatial variable Emden-Fowler type in viscoelasticities with strong nonlinear sources

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Abstract: For one spatial variable, a new kind of coupled system for nonlinear wave equations of Emden-Fowler type is considered with boundary value and initial values. Under certain conditions on the initial data and the exponent ρ, we show that the viscoelastic terms lead our problem to be dissipative and that the global solutions cannot exist in L^2 beyond the given finite time i.e.,

$$\int_{r_1}^{r_2} \left(|u_1|^2 + |u_2|^2\right) dx \to +\infty \qquad \text{as } t \to T^*,$$

where

$$\ln T^* = \frac{2}{\rho+1}\Big(\sum_{i=1}^{2}\int_{r_1}^{r_2} |u_{i0}|^2\, dx\Big)\Big(\sum_{i=1}^{2}\int_{r_1}^{r_2} \left(2u_{i0}u_{i1} - |u_{i0}|^2\right)\, dx\Big)^{-1}.$$

Keywords: viscoelastic; Emden-Fowler wave equation; coupled system; blow-up
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1. Introduction

Let $x \in (r_1, r_2), t \in [1, T), T > 0, u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$. We consider a new kind of coupled system of Emden-Fowler type wave equations in viscoelasticities with strong nonlinear terms

$$\begin{cases} t^2\partial_{tt}u_i - u_{ixx} + \int_1^t \mu_i(s)\, u_{ixx}(t-s)\, ds = f_i(u_1, u_2) \quad \text{in } [1, T) \times (r_1, r_2), \\ u_i(1, x) = u_{i0}(x) \in H^2(r_1, r_2) \cap H_0^1(r_1, r_2), \\ \partial_t u_i(1, x) = u_{i1}(x) \in H_0^1(r_1, r_2), \\ u_i(t, r_1) = u_i(t, r_2) = 0 \quad \text{in } [1, T), \end{cases} \tag{1.1}$$

where $i = 1, 2$ and

$$\begin{cases} f_1(\xi_1, \xi_2) = |\xi_1 + \xi_2|^{2(\rho+1)}(\xi_1 + \xi_2) + |\xi_1|^{\rho}\xi_1|\xi_2|^{\rho+2} \\ f_2(\xi_1, \xi_2) = |\xi_1 + \xi_2|^{2(\rho+1)}(\xi_1 + \xi_2) + |\xi_2|^{\rho}\xi_2|\xi_1|^{\rho+2}, \end{cases} \tag{1.2}$$

for $\rho > -1, r_i$ are real numbers and the scalar functions μ_i (so-called relaxation kernels) are assumed only to be nonincreasing $\mu_i \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$\mu_i(0) > 0,\ 1 - \int_0^\infty e^{-s/2}\mu_i(s)\, ds = l > 0. \tag{1.3}$$

Many issues in physics and engineering pose problems that deal with coupled evolution equations. For example, in diffusion theory and some mechanical applications, such evolution equations are in the form of a system of nonlinear hyperbolic equations. An important example of such systems goes back to [13], which introduced a three-dimensional system of space similar to our system, without dissipations. (see [1, 11, 12, 16, 17]).

The Emden-Fowler equation has an impact on many astrophysics evolution phenomena. It has been poorly studied by scientists until now, essentially of the qualitative point of view.

In 1862, Land [15] proposed the well known Lane-Emden equation

$$\partial_t\Big(t^2\partial_t u\Big) + t^2u^p = 0, \tag{1.4}$$

where $p = 1.5$ and 2.5. When $p = 1$, Eq (1.4) has a solution $u = \sin t/t$, and when $p = 5$, the explicit solution is given by $u = 1/\sqrt{1 + t^2/3}$ ([2, 3, 14]).

The generalization of such equation is given by

$$\partial_t\Big(t^\rho\partial_t u\Big) + t^\sigma u^\gamma = 0, \quad t \geq 0.$$

It was considered by Fowler [4–7] in a series of four papers during 1914–1931.

Next, the generalized Emden-Fowler equation

$$\partial_{tt}u + a(t)|u|^\gamma \operatorname{sgn} u = 0, \quad t \geq 0,$$

was studied by Atkinson et al.

Recently, M. R. Li in [8] considered and studied the blow-up phenomena of solutions to the Emden-Fowler type semilinear wave equation

$$t^2\partial_{tt}u - u_{xx} = u^p \quad \text{in } [1,T)\times(r_1,r_2).$$

The present research aims to extend the study of Emden-Fowler type wave equation to the case when the viscoelastic term is injected in domain $[r_1, r_2]$ where there is no result about this topic as equations and as coupled systems. Thus, a wider class of phenomena can be modeled.

The main results here are to exhibit the role of viscoelasticities, which makes our system (1.1) dissipative. When the energy is negative, the blow up of solutions in L^2 at finite time given by

$$\ln T^* = \frac{2}{\rho+1}\left(\sum_{i=1}^{2}\int_{r_1}^{r_2}|u_{i0}|^2\,dx\right)\left(\sum_{i=1}^{2}\int_{r_1}^{r_2}\left(2u_{i0}u_{i1}-|u_{i0}|^2\right)dx\right)^{-1},$$

will be the main result in Theorem 3.1.

The plan of this article is as follows. We present some notations and assumptions needed for our results in section 2. Section 3 is devoted to the blow up result of solutions.

2. Preliminaries and position of problem

Under some suitable transformations, we can get the local existence of solutions to Eq (1.1). Taking the first transform

$$\tau = \ln t, \qquad v_i(\tau,x) = u_i(t,x),$$

for $i = 1, 2$, we have

$$u_{ixx} = v_{ixx}, \quad \partial_t u_i = t^{-1}\partial_\tau v_i, \quad t^2\partial_{tt}u_i = -\partial_\tau v_i + \partial_{\tau\tau}v_i.$$

Then problem (1.1) takes the form

$$\begin{cases} \partial_{\tau\tau}v_i - v_{ixx} + \int_0^\tau \mu_i(s)v_{ixx}(\tau-s)\,ds = \partial_\tau v_i + f_i(v_1,v_2) \quad \text{in } [0,\ln T)\times(r_1,r_2), \\ v_i(0,x) = u_{i0}(x) \quad \text{in } (r_1,r_2), \\ \partial_\tau v_i(0,x) = u_{i1}(x) \quad \text{in } (r_1,r_2), \\ v_i(\tau,r_1) = v_i(\tau,r_2) = 0 \quad \text{in } [0,\ln T). \end{cases} \tag{2.1}$$

To make the second transformation, let

$$v_i(\tau,x) = e^{\tau/2}w_i(\tau,x).$$

Since we have

$$\partial_\tau v_i(\tau,x) = e^{\tau/2}\partial_\tau w_i(\tau,x) + \frac{1}{2}e^{\tau/2}w_i(\tau,x),$$
$$\partial_{\tau\tau}v_i(\tau,x) = e^{\tau/2}\partial_{\tau\tau}w_i(\tau,x) + e^{\tau/2}\partial_\tau w_i(\tau,x) + \frac{1}{4}e^{\tau/2}w_i(\tau,x),$$

then Eq in (2.1) can be rewritten as

$$e^{\tau/2}\partial_{\tau\tau}w_i - e^{\tau/2}w_{ixx} + \int_0^\tau e^{(\tau-s)/2}\mu_i(s)w_{ixx}(\tau-s)\,ds = \frac{1}{4}e^{\tau/2}w_i + f_i(e^{\tau/2}w_1, e^{\tau/2}w_2),$$

and converted to

$$\partial_{tt}w_i - w_{ixx} + \int_0^t e^{-s/2}\mu_i(s)w_{ixx}(t-s)\,ds = \frac{1}{4}w_i + e^{-t/2}f_i(e^{t/2}w_1, e^{t/2}w_2), \tag{2.2}$$

with the corresponding initial and boundary conditions. Throughout this paper, we shall write $w_i = w_i(t,x)$ where no confusion occurs. The following technical Lemma will play an important role.

Lemma 2.1. *For any $y \in C^1\left(0, \ln T; H_0^1(r_1,r_2)\right)$ and $i = 1, 2$, we have*

$$\begin{aligned}
&\int_{r_1}^{r_2}\int_0^t e^{-s/2}\mu_i(s)y_{xx}(t-s)\partial_t y(t)\,ds\,dx \\
= \ & \frac{1}{2}\partial_t\left(\int_0^t e^{-(t-p)/2}\mu_i(t-p)\int_{r_1}^{r_2}|y_x(p)-y_x(t)|^2\,dx\,dp\right) \\
&-\frac{1}{2}\partial_t\left(\int_0^t e^{-s/2}\mu_i(s)\,ds\int_{r_1}^{r_2}|y_x(t)|^2\,dx\right) \\
&+\frac{1}{4}\int_0^t e^{-(t-p)/2}\mu_i(t-p)\int_{r_1}^{r_2}|y_x(p)-y_x(t)|^2\,dx\,dp \\
&-\frac{1}{2}\int_0^t e^{-(t-p)/2}\partial_t\mu_i(t-p)\int_{r_1}^{r_2}|y_x(p)-y_x(t)|^2\,dx\,dp \\
&+\frac{1}{2}e^{-t/2}\mu_i(t)\int_{r_1}^{r_2}|y_x(t)|^2\,dx.
\end{aligned}$$

Proof. It's not hard to see

$$\begin{aligned}
&\int_{r_1}^{r_2}\int_0^t e^{-s/2}\mu_i(s)y_{xx}(t-s)\partial_t y(t)\,ds\,dx \\
= \ & -\int_0^t e^{-s/2}\mu_i(s)\int_{r_1}^{r_2} y_x(t-s)\partial_t y_x(t)\,dx\,ds \\
= \ & -\int_0^t e^{-(t-p)/2}\mu_i(t-p)\int_{r_1}^{r_2} y_x(p)\partial_t y_x(t)\,dx\,dp \\
= \ & -\int_0^t e^{-(t-p)/2}\mu_i(t-p)\int_{r_1}^{r_2}\big(y_x(p)-y_x(t)\big)\partial_t y_x(t)\,dx\,dp \\
&-\int_0^t e^{-s/2}\mu_i(s)\,ds\int_{r_1}^{r_2} y_x(t)\partial_t y_x(t)\,dx.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\int_{r_1}^{r_2}\int_0^t e^{-s/2}\mu_i(s)y_{xx}(t-s)\partial_t y(t)\,ds\,dx \\
= \;& \frac{1}{2}\int_0^t e^{-(t-p)/2}\mu_i(t-p)\partial_t\bigg(\int_{r_1}^{r_2}|y_x(p)-y_x(t)|^2\;dx\bigg)dp \\
&-\frac{1}{2}\int_0^t e^{-s/2}\mu_i(s)\,ds\partial_t\bigg(\int_{r_1}^{r_2}|y_x(t)|^2\;dx\bigg),
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{r_1}^{r_2}\int_0^t e^{-s/2}\mu_i(s)y_{xx}(t-s)\partial_t y(t)\,ds\,dx \\
= \;& \frac{1}{2}\partial_t\bigg(\int_0^t e^{-(t-p)/2}\mu_i(t-p)\int_{r_1}^{r_2}|y_x(p)-y_x(t)|^2\,dx\,dp\bigg) \\
&+\frac{1}{4}\int_0^t e^{-(t-p)/2}\mu_i(t-p)\int_{r_1}^{r_2}|y_x(p)-y_x(t)|^2\,dx\,dp \\
&-\frac{1}{2}\int_0^t e^{-(t-p)/2}\partial_t\mu_i(t-p)\int_{r_1}^{r_2}|y_x(p)-y_x(t)|^2\,dx\,dp \\
&-\frac{1}{2}\partial_t\bigg(\int_0^t e^{-s/2}\mu_i(s)\,ds\int_{r_1}^{r_2}|y_x(t)|^2\,dx\bigg) \\
&+\frac{1}{2}e^{-t/2}\mu_i(t)\int_{r_1}^{r_2}|y_x(t)|^2\,dx.
\end{aligned}$$

This completes the proof. □

The modified energy associated to problem (2.2) is introduced as

$$\begin{aligned}
2E_w(t) = & \sum_{i=1}^{2}\int_{r_1}^{r_2}|\partial_t w_i|^2\,dx + \sum_{i=1}^{2}\bigg(1-\int_0^t e^{-s/2}\mu_i(s)\,ds\bigg)\int_{r_1}^{r_2}|w_{ix}|^2\,dx \\
&+\sum_{i=1}^{2}\int_0^t e^{-(t-p)/2}\mu_i(t-p)\int_{r_1}^{r_2}|w_{ix}(p)-w_{ix}(t)|^2\;dx\,dp \\
&-\frac{1}{4}\sum_{i=1}^{2}\int_{r_1}^{r_2}|w_i|^2\,dx - \frac{1}{\rho+2}e^{(\rho+1)t}\int_{r_1}^{r_2}\Big(|w_1+w_2|^{2(\rho+2)}+2|w_1w_2|^{\rho+2}\Big)dx,
\end{aligned} \tag{2.3}$$

and

$$2E_w(0) = \sum_{i=1}^{2}\int_{r_1}^{r_2}\Big(u_{i1}-\frac{1}{2}u_{i0}\Big)^2dx + \sum_{i=1}^{2}\int_{r_1}^{r_2}|u_{i0x}|^2\,dx - \frac{1}{4}\sum_{i=1}^{2}\int_{r_1}^{r_2}|u_{i0}|^2\,dx$$

$$-\frac{1}{\rho+2}\int_{r_1}^{r_2}\Big(|u_{10}+u_{20}|^{2(\rho+2)}+2|u_{10}u_{20}|^{\rho+2}\Big)\,dx.$$

For this energy, Lemma 2.3 leads to

$$\partial_t E_w(t) \le 0.$$

As in [10], one can easily verify that

$$e^{-t/2}\sum_{i=1}^{2}\partial_t w_i f_i(e^{t/2}w_1, e^{t/2}u_2) = \frac{1}{2(\rho+2)}e^{(\rho+1)t}\partial_t\Big(|w_1+w_2|^{2(\rho+2)}+2|w_1w_2|^{\rho+2}\Big),$$

and

$$e^{-t/2}\sum_{i=1}^{2} w_i f_i(e^{t/2}w_1, e^{t/2}w_2)\,dx = e^{(\rho+1)t}\Big(|w_1+w_2|^{2(\rho+2)}+2|w_1w_2|^{\rho+2}\Big). \tag{2.4}$$

Next, we introduce the Dirichlet-Poincaré's inequality in one spatial variable.

Lemma 2.2. *For any $v \in H_0^1(r_1, r_2)$, we have*

$$\int_{r_1}^{r_2}|v|^2\,dx \le \frac{(r_2-r_1)^2}{2}\int_{r_1}^{r_2}|v_x|^2\,dx.$$

Proof. From (2.1), we have $w(t, r_1) = w(t, r_2) = 0$. By the Fundamental Theorem of Calculus

$$w(s) = \int_{r_1}^{s} w_x dx. \tag{2.5}$$

Therefore

$$|w(s)| \le \int_{r_1}^{s}|w_x|dx. \tag{2.6}$$

Recall the Cauchy-Schwar's inequality

$$\int fgdx \le \Big(\int f^2dx\Big)^{1/2}\Big(\int g^2dx\Big)^{1/2}.$$

Apply this with $f = 1, g = |w_x|$ to get

$$\begin{aligned}|w(s)| &\le \Big(\int_{r_1}^{s}|w_x|^2dx\Big)^{1/2}(s-r_1)^{1/2}\\ &\le \Big(\int_{r_1}^{r_2}|w_x|^2dx\Big)^{1/2}(r_2-r_1)^{1/2}.\end{aligned}$$

Squaring both sides gives

$$|w(s)|^2 \le \Big(\int_{r_1}^{r_2}|w_x|^2dx\Big)(r_2-r_1),$$

and finally we integrate over $[r_1, r_2]$ to give the required. □

Now, in order to deal with nonlinear terms (1.2) which are considered as a sources of dissipativity in (2.2), we need the next important Lemma. This Lemma shows that the energy functional is decreasing.

Lemma 2.3. *Suppose that $v \in C^1(0, \ln T; H_0^1(r_1, r_2)) \cap C^2(0, \ln T; L^2(r_1, r_2))$ is a solution of the semi-linear wave equation (2.2). Then for $t \geq 0$, we have*

$$E_w(t) \leq E_w(0) - \frac{\rho+1}{2(\rho+2)} \int_0^t e^{(\rho+1)s} \int_{r_1}^{r_2} \left(|w_1 + w_2|^{2(\rho+2)} + 2|w_1 w_2|^{\rho+2} \right) dx\, ds. \tag{2.7}$$

Proof. Taking the L^2 product of $(2.2)_i$ with $\partial_t w_i$ yields

$$\begin{aligned} &\int_{r_1}^{r_2} \partial_{tt} w_i \partial_t w_i \, dx - \int_{r_1}^{r_2} w_{ixx} \partial_t w_i \, dx + \int_{r_1}^{r_2} \int_0^t e^{-s/2} \mu_i(s) w_{ixx}(t-s)\, ds \partial_t w_i \, dx \\ &= \frac{1}{4} \int_{r_1}^{r_2} w_i \partial_t w_i \, dx + e^{-t/2} \int_{r_1}^{r_2} f_i(e^{t/2} w_1, e^{t/2} w_2) \partial_t w_i \, dx, \end{aligned}$$

for $i = 1, 2$. Adding each other, we have

$$\begin{aligned} &\frac{1}{2} \sum_{i=1}^{2} \partial_t \int_{r_1}^{r_2} \left[|\partial_t w_i|^2 + |w_{ix}|^2 - \frac{1}{4}|w_i|^2 \right] dx \\ &+ \sum_{i=1}^{2} \int_{r_1}^{r_2} \int_0^t e^{-s/2} \mu_i(s) w_{ixx}(t-s) \partial_t w_i(t)\, ds\, dx \\ &= e^{-t/2} \sum_{i=1}^{2} \int_{r_1}^{r_2} \partial_t w_i f_i(e^{t/2} w_1, e^{t/2} w_2)\, dx. \end{aligned}$$

Thus, by Lemma 2.1 and (2.4), we obtain

$$\begin{aligned} &\frac{1}{2} \sum_{i=1}^{2} \partial_t \int_{r_1}^{r_2} \left[|\partial_t w_i|^2 + |w_{ix}|^2 - \frac{1}{4}|w_i|^2 \right] dx \\ &+ \frac{1}{2} \sum_{i=1}^{2} \partial_t \left(\int_0^t e^{-(t-p)/2} \mu_i(t-p) \int_{r_1}^{r_2} |w_{ix}(p) - w_{ix}(t)|^2 \, dx\, dp \right) \\ &- \frac{1}{2} \sum_{i=1}^{2} \partial_t \left(\int_0^t e^{-s/2} \mu_i(s)\, ds \int_{r_1}^{r_2} |w_{ix}(t)|^2 \, dx \right) \\ &+ \frac{1}{4} \sum_{i=1}^{2} \int_0^t e^{-(t-p)/2} \mu_i(t-p) \int_{r_1}^{r_2} |w_{ix}(p) - w_{ix}(t)|^2 \, dx\, dp \\ &- \frac{1}{2} \sum_{i=1}^{2} \int_0^t e^{-(t-p)/2} \partial_t \mu_i(t-p) \int_{r_1}^{r_2} |w_{ix}(p) - w_{ix}(t)|^2 \, dx\, dp \\ &+ \frac{1}{2} \sum_{i=1}^{2} e^{-t/2} \mu_i(t) \int_{r_1}^{r_2} |w_{ix}(t)|^2 \, dx \end{aligned}$$

$$= \frac{1}{2(\rho+2)}\partial_t\Big(e^{(\rho+1)t}\int_{r_1}^{r_2}\Big[|w_1+w_2|^{2(\rho+2)}+2|w_1w_2|^{\rho+2}\Big]\Big)dx$$
$$-\frac{\rho+1}{2(\rho+2)}e^{(\rho+1)t}\int_{r_1}^{r_2}\Big[|w_1+w_2|^{2(\rho+2)}+2|w_1w_2|^{\rho+2}\Big]dx.$$

Dropping the positive terms from the left-hand side, we have

$$\partial_t E_w(t) \le -\frac{\rho+1}{2(\rho+2)}e^{(\rho+1)t}\int_{r_1}^{r_2}\Big[|w_1+w_2|^{2(\rho+2)}+2|w_1w_2|^{\rho+2}\Big]dx,$$

which gives the conclusion by integrating both sides with respect to t. □

Remark 2.4. *Concerning the local existence, we can follow the steps of results in [9] as equations, with some modifications imposed by the existence of the memories terms, where we replace the operator $\partial_{tt}-\Delta$ by $\partial_{tt}-\Delta\big(1-\int_0^t \mu_i(t-s)\,ds\big)$ with some conditions on the exponent ρ. The local existence results for one equation still valid for a coupled system in the same type.*

3. Blow up result

We prove that (u_1,u_2) blows up in L^2 at finite time T^* in the following Theorem.

Theorem 3.1. *Let $\rho > -1$ and $(r_2-r_1)^2 < 8$. Suppose that*

$$(u_1,u_2) \in \Big(C^1(0,T;H_0^1(r_1,r_2)) \cap C^2(0,T;L^2(r_1,r_2))\Big)^2,$$

is a weak solution of (1.1) with

$$e(0) := \sum_{i=1}^{2}\int_{r_1}^{r_2} u_{i0}\left(u_{i1}-\frac{1}{2}u_{i0}\right)dx > 0 \quad \text{and} \quad E_w(0) \le 0.$$

Assume that l satisfies

$$\frac{2\,(4\rho+9)+(\rho+1)(\rho+2)(r_2-r_1)^2}{2\,(4\rho^2+16\rho+17)} \le l.$$

Then there exists T^* such that

$$\sum_{i=1}^{2}\int_{r_1}^{r_2}|u_i|^2\,dx \to +\infty \quad \text{as } t \to T^*,$$

where

$$\ln T^* = \frac{2}{\rho+1}\Big(\sum_{i=1}^{2}\int_{r_1}^{r_2}|u_{i0}|^2\,dx\Big)\Big(\sum_{i=1}^{2}\int_{r_1}^{r_2}\big(2u_{i0}u_{i1}-|u_{i0}|^2\big)\,dx\Big)^{-1}.$$

Remark 3.2. *Let*

$$g(x) \equiv \frac{2\,(4x+9)+(x+1)(x+2)(r_2-r_1)^2}{2\,(4x^2+16x+17)}.$$

Now that $(r_2 - r_1)^2 < 8$ *holds, we have*

$$\partial_x g(x) = \frac{4x^2 + 18x + 19}{2\left(4x^2 + 16x + 17\right)^2}\left((r_2 - r_1)^2 - 8\right) < 0.$$

Thus by the monotonicity and $\rho > -1$*, we obtain*

$$\frac{(r_2 - r_1)^2}{8} = \lim_{t \to +\infty} g(t) < g(x) < g(-1) = 1.$$

Hence the assumptions of Theorem make sense.

Remark 3.3. *In the case of* $r_1 = 0$ *and* $r_2 = 1$*, we introduce the example of* $\mu_i(t)$ *satisfying* (1.3) *and assumptions of Theorem 3.1. Let*

$$\rho > -1 \quad and \quad \mu_i(t) = e^{-kt} \quad for \quad k > \frac{9}{14}.$$

Then we have

$$l = 1 - \int_0^{\infty} e^{-(k+1/2)t}\, dt = \frac{2k - 1}{2k + 1} \in \left(\frac{1}{8}, 1\right).$$

The condition $g(\rho) \leq l$ *is equivalent to*

$$k \geq \frac{9}{14} + \frac{8\rho + 18}{7\left(\rho + 1\right)\left(\rho + 2\right)}.$$

We need to state and prove the next intermediate Lemma.

Lemma 3.4. *Under the assumptions in Theorem 3.1, we have*

$$\begin{aligned} & e^{(\rho+1)t} \int_{r_1}^{r_2} \left(|w_1 + w_2|^{2(\rho+2)} + 2|w_1 w_2|^{\rho+2}\right) dx \\ & \geq (\rho + 2) \sum_{i=1}^{2} \int_{r_1}^{r_2} \left(|\partial_t w_i|^2 + l|w_{ix}|^2 - \frac{1}{4}|w_i|^2\right) dx \\ & + (\rho + 2) \sum_{i=1}^{2} \int_{r_1}^{r_2} \int_0^t e^{-(t-p)/2} \mu_i(t - p) \left|w_{ix}(p) - w_{ix}(t)\right|^2 \, dp\, dx. \end{aligned}$$

Proof. Let

$$L(t) \equiv e^{(\rho+1)t} \int_{r_1}^{r_2} \left(|w_1 + w_2|^{2(\rho+2)} + 2|w_1 w_2|^{\rho+2}\right) dx.$$

We have

$$L(t) \quad = \quad (\rho + 2) \sum_{i=1}^{2} \int_{r_1}^{r_2} \left[|\partial_t w_i|^2 + \left(1 - \int_0^t e^{-p/2} \mu_i(p)\, dp\right)|w_{ix}|^2 - \frac{1}{4}|w_i|^2\right] dx$$

$$+(\rho+2)\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-(t-p)/2}\mu_i(t-p)\,|w_{ix}(p)-w_{ix}(t)|^2\,dp\,dx$$
$$-2(\rho+2)\,E_w(t),$$

by (2.3). Since

$$E_w(t)\le 0,$$

holds by (2.7), we have

$$\begin{aligned} L(t) &\ge (\rho+2)\sum_{i=1}^{2}\int_{r_1}^{r_2}\left(|\partial_t w_i|^2+l|w_{ix}|^2-\frac{1}{4}|w_i|^2\right)dx \\ &\quad +(\rho+2)\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-(t-p)/2}\mu_i(t-p)\,|w_{ix}(p)-w_{ix}(t)|^2\,dp\,dx, \end{aligned}$$

by (1.3) and Lemma 2.3. □

We are now ready to prove Theorem 3.1

Proof. (of Theorem 3.1)
Let

$$A(t):=\sum_{i=1}^{2}\int_{r_1}^{r_2}|w_i(t,x)|^2\,dx.$$

Then we have

$$\partial_t A(t)=2\sum_{i=1}^{2}\int_{r_1}^{r_2}w_i(t,x)\partial_t w_i(t,x)\,dx,$$

and

$$\begin{aligned} \partial_{tt}A(t) &= 2\sum_{i=1}^{2}\int_{r_1}^{r_2}w_i(t,x)\partial_{tt}w_i(t,x)\,dx+2\sum_{i=1}^{2}\int_{r_1}^{r_2}|\partial_t w_i(t,x)|^2\,dx \\ &= 2\sum_{i=1}^{2}\int_{r_1}^{r_2}\left(w_iw_{ixx}-w_i\int_0^t e^{-p/2}\mu_i(p)w_{ixx}(t-p)\,dp+\frac{1}{4}|w_i|^2+|\partial_t w_i|^2\right)dx \\ &\quad +2\sum_{i=1}^{2}\int_{r_1}^{r_2}e^{-t/2}w_if_i(e^{t/2}w_1,e^{t/2}w_2)\,dx \\ &= 2\sum_{i=1}^{2}\int_{r_1}^{r_2}\left(-|w_{ix}|^2+\frac{1}{4}|w_i|^2+|\partial_t w_i|^2\right)dx \\ &\quad +2\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-p/2}\mu_i(p)w_{ix}(t)w_{ix}(t-p)\,dp\,dx+2L(t). \end{aligned}$$

By Lemma 2.4 and similar computation to Lemma 2.1 with Young's inequality

$$ab\le\frac{a^2}{2\theta}+\frac{\theta b^2}{2},$$

for $a, b \geq 0$ and $\theta > 0$, we have

$$\begin{aligned}
&2\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-p/2}\mu_i(p)w_{ix}(t)w_{ix}(t-p)\,dp\,dx\\
&\geq -2\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-p/2}\mu_i(p)|w_{ix}(t)||w_{ix}(t-p)-w_{ix}(t)|\,dp\,dx\\
&+2\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-p/2}\mu_i(p)|w_{ix}(t)|^2\,dp\,dx\\
&\geq \left(2-\frac{1}{\theta}\right)\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-p/2}\mu_i(p)|w_{ix}(t)|^2\,dp\,dx\\
&-\theta\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-(t-p)/2}\mu_i(t-p)|w_{ix}(p)-w_{ix}(t)|^2\,dp\,dx,
\end{aligned}$$

where θ is a positive constant to be chosen later. This estimate implies that

$$\begin{aligned}
\partial_{tt}A(t) \;\geq\;& 2\sum_{i=1}^{2}\int_{r_1}^{r_2}\Big(-|w_{ix}|^2+\frac{1}{4}|w_i|^2+|\partial_t w_i|^2\Big)\,dx\\
&-\frac{1}{\theta}\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-p/2}\mu_i(p)|w_{ix}(t)|^2\,dp\,dx\\
&-\theta\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-(t-p)/2}\mu_i(t-p)|w_{ix}(p)-w_{ix}(t)|^2\,dp\,dx\\
&+2\,(\rho+2)\sum_{i=1}^{2}\int_{r_1}^{r_2}\Big(|\partial_t w_i|^2+l|w_{ix}|^2-\frac{1}{4}|w_i|^2\Big)\,dx\\
&+2\,(\rho+2)\int_{r_1}^{r_2}\int_0^t e^{-(t-p)/2}\mu_i(t-p)\,|w_{ix}(p)-w_{ix}(t)|^2\,dp\,dx,
\end{aligned}$$

by using Lemma 3.4. Hence we choose $\theta = 2(\rho+2)$ to obtain

$$\begin{aligned}
\partial_{tt}A(t) \;\geq\;& 2\sum_{i=1}^{2}\int_{r_1}^{r_2}\Big(-|w_{ix}|^2+\frac{1}{4}|w_i|^2+|\partial_t w_i|^2\Big)\,dx\\
&-\frac{1}{2(\rho+2)}\sum_{i=1}^{2}\int_{r_1}^{r_2}\int_0^t e^{-p/2}\mu_i(p)|w_{ix}(t)|^2\,dp\,dx\\
&+2\,(\rho+2)\sum_{i=1}^{2}\int_{r_1}^{r_2}\Big(|\partial_t w_i|^2+l|w_{ix}|^2-\frac{1}{4}|w_i|^2\Big)\,dx\\
\geq\;& 2\,(\rho+3)\sum_{i=1}^{2}\int_{r_1}^{r_2}|\partial_t w_i|^2\,dx\\
&+\left(2\,(\rho+2)\,l-2-\frac{1-l}{2(\rho+2)}\right)\sum_{i=1}^{2}\int_{r_1}^{r_2}|w_{ix}|^2\,dx
\end{aligned}$$

$$-\frac{(r_2-r_1)^2}{4}(\rho+1)\sum_{i=1}^{2}\int_{r_1}^{r_2}|w_{ix}|^2\,dx$$

$$\geq\ 2(\rho+3)\sum_{i=1}^{2}\int_{r_1}^{r_2}|\partial_t w_i|^2\,dx+\frac{4\rho^2+16\rho+17}{2(\rho+2)}(l-g(\rho))\sum_{i=1}^{2}\int_{r_1}^{r_2}|w_{ix}|^2\,dx,$$

by Lemma 2.2, we have

$$\partial_{tt}A(t)\geq 2(\rho+3)\sum_{i=1}^{2}\int_{r_1}^{r_2}|\partial_t w_t|^2\,dx, \tag{3.1}$$

where $g(\rho)$ is defined in Remark 3.2. Now under the assumption $e(0)>0$, (3.1) yields

$$\partial_t A(t)\geq\partial_t A(0)>0,$$

and

$$A(t)\geq A(0)+\partial_t A(0)t\geq A(0)>0.$$

Here, thanks to $e(0)>0$, $A(0)>0$ follows. Hence we have just showed that $A(t)$ blows up. To complete the proof, we'll prove that the blow-up time T_1^* is finite. As in [8], let us now set

$$J(t):=A(t)^{-k},\qquad 2k=\rho+1>0.$$

We have only to show that $J(t)$ reaches 0 in finite time. Then we have

$$\partial_t J(t)=-kA(t)^{-k-1}\partial_t A(t)<0,$$

and

$$\begin{aligned}\partial_{tt}J(t)&=-kA(t)^{-k-2}\Big[A(t)\partial_{tt}A(t)-(k+1)\partial_t A(t)^2\Big]\\&\leq -kA(t)^{-k-1}\Big[\partial_{tt}A(t)-2(\rho+3)\sum_{i=1}^{2}\int_{r_1}^{r_2}|\partial_t w_i|^2\,dx\Big]\\&\leq 0,\end{aligned} \tag{3.2}$$

by using Cauchy-Schwarz and Hölder's inequalities. Integrating (3.2) twice, we have

$$0<J(t)\leq J(0)+\partial_t J(0)t.$$

Noting that $J'(0)<0$, we take

$$T_1^*=-\frac{J(0)}{J'(0)}=\frac{1}{\rho+1}\frac{\sum_{i=1}^{2}\int_{r_1}^{r_2}|w_{i0}|^2\,dx}{\sum_{i=1}^{2}\int_{r_1}^{r_2}w_{i0}w_{i1}\,dx}=\frac{e(0)^{-1}}{\rho+1}\sum_{i=1}^{2}\int_{r_1}^{r_2}|u_{i0}|^2\,dx>0,$$

so that $J(t)\to 0$ as $t\to T_1^*$. Thus for a solution w_i of (2.2), we obtain

$$A(t)=\sum_{i=1}^{2}\int_{r_1}^{r_2}|w_i(t,x)|^2\,dx\to+\infty,$$

as $t\to T_1^*$. Since

$$\sum_{i=1}^{2}\int_{r_1}^{r_2}|u_i(t,x)|^2\,dx=e^{\tau}A(\tau)=tA(\ln t),$$

holds for all $t\in[1,\exp T_1{}^*)$ by denoting $w_i=w_i(\tau,x)$, the conclusion follows right away together with $T^*=\exp T_1^*$. □

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Conflict of interest

The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

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