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Research article

Admissible multivalued hybrid $\mathcal{Z}$-contractions with applications

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Abstract: In this paper, we introduce new concepts, admissible multivalued hybrid $\mathcal{Z}$-contractions and multivalued hybrid $\mathcal{Z}$-contractions in the framework of b-metric spaces and establish sufficient conditions for existence of fixed points for such contractions. A few consequences of our main theorem involving linear and nonlinear contractions are pointed out and discussed by using variants of simulation functions. In the case where our notions are reduced to their single-valued counterparts, the results presented herein complement, unify and generalize a number of significant fixed point theorems due to Branciari, Czerwik, Jachymski, Karapinar and Argawal, Khojasteh, Rhoades, among others. Nontrivial illustrative examples are provided to support the assertions of the obtained results. From application point of view, some fixed point theorems of b-metric spaces endowed with partial ordering and graph are deduced and solvability conditions of nonlinear matrix equations are investigated.

Keywords: b-metric space; fixed point; hybrid contraction; multivalued contraction; simulation function; $\mathcal{Z}$-contraction; matrix equation
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1. Introduction

The theory of set-valued analysis plays a key role in various branches of mathematics because of its applications in areas such as control theory, game theory, biomathematics, qualitative physics, viability theory, and so on. In particular, the idea of multivalued mappings in fixed point theory was initiated by von Neumann in the study of game theory. On the other hand, the notion of multivalued mappings in metric fixed point theory was brought up by Nadler [28] who used the concept of Hausdorff metric to obtain a generalization of Banach contraction principle. Banach fixed point theorem (see [7]) is the earliest, simple and versatile classical result for single-valued mappings in fixed point theory with metric space structure. More than a handful of literature embrace applications and extensions of this principle from different perspectives, for example, by weakening the hypotheses, employing different mappings and various forms of quasi and pseudo-metric spaces. Meanwhile, a number of generalizations in diverse frames of Nadler's fixed point result have also been investigated by several authors; see, for example, [1, 5, 17, 19, 27] and references therein.

The analysis of new spaces and their properties have been an interesting topic among current mathematical research. In this direction, the notion of b-metric spaces is presently thriving. The idea commenced with the work of Bakhtin [6] and Bourbaki [9]. Thereafter, Czerwik [13] gave a postulate which is weaker than the classical triangle inequality and formally established a b-metric space with a view of improving the Banach fixed point theorem. Meanwhile, the notion of b-metric spaces is gaining fast generalizations. For a recent short survey on basic concepts and results in fixed point theory in the framework of b-metric spaces, we refer the interested reader to Karapinar [25]. On similar development, one of the active branches of fixed point theory that is also currently drawing attentions of researchers is the study of hybrid contractions. The concept has been viewed in two directions, viz, first, hybrid contraction deals with contractions involving both single-valued and multivalued mappings and the second merges linear and nonlinear contractions. Recently, Karapinar and Fulga [23] inaugurated a novel notion of b-hybrid contraction in the framework of b-metric space and studied the existence and uniqueness of fixed points for such contractions. Their ideas merged several existing results in the corresponding literature. Similarly, by using the concept of α-admissible mapping due to Samet [33], Chifu and Karapinar [12] improved the main result in [23] by combining the idea of simulation functions of Khojasteh et al. [26]. Interestingly, hybrid fixed point theory has potential applications in functional inclusions, optimization theory, fractal graphics, discrete dynamics for set-valued operators and other areas of nonlinear functional analysis.

In this work, we introduce two notions, admissible multivalued hybrid $\mathcal{Z}$-contractions and multivalued hybrid $\mathcal{Z}$-contractions in the framework of b-metric spaces and establish sufficient conditions for existence of fixed points for such contractions. A few consequences of our main theorem are pointed out by using variants of simulation functions. Overall, the ideas presented herein unify and complement several significant fixed point theorems in the setting of both single-valued and set-valued mappings involving either linear or nonlinear contractions. Nontrivial illustrative examples are provided to authenticate the hypotheses of our main result. From application perspective, some fixed point theorems of b-metric spaces endowed with partial ordering and graph are derived and solvability conditions of nonlinear matrix equations are investigated. In particular, this paper complements the main results of Branciari [11], Chifu and Karapinar [12], Czerwik [13], Jachymski [22], Karapinar and Agarwal [23], Karapinar and Fulga [24], Khojasteh [26], Nadler [28], Rhoades [31], Samet [33] and a few others in the comparable literature.

2. Preliminaries

In this section, we collect important notation, useful definitions and basic results coherent with the literature. Hereafter, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of natural numbers, non-negative reals and real numbers, respectively.

Czerwik [13] formally defined the notion of a b-metric space as follows: Let X be a nonempty set and $\eta \geq 1$ be a constant. Suppose that the mapping $\mu : X \times X \longrightarrow \mathbb{R}_+$ satisfies the following conditions for all $x, y, z \in X$:

(i) $\mu(x, y) = 0$ if and only if $x = y$ (self-distancy);
(ii) $\mu(x, y) = \mu(y, x)$ (symmetry);
(iii) $\mu(x, y) \leq \eta\,[\mu(x, z) + \mu(z, y)]$ (weighted triangle inequality).

Then, the tripled (X, μ, η) is called a b-metric space. It is noteworthy that every metric is a b-metric with the parameter $\eta = 1$. Also, in general, a b-metric is not a continuous functional. Hence, the class of b-metric is larger than the class of classical metric.

Example 2.1. *[8] Let $X = l_p(\mathbb{R})$ with $0 < p < 1$, where*

$$l_p(\mathbb{R}) = \left\{ \{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Define $\mu : X \times X \longrightarrow \mathbb{R}_+$ as

$$\mu(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

where $x = \{x_n\}_{n\in\mathbb{N}}$ and $y = \{y_n\}_{n\in\mathbb{N}}$. Then, μ is a b-metric with parameter $\eta = 2^{\frac{1}{p}}$ and hence $(X, \mu, 2^{\frac{1}{p}})$ is a b-metric space.

Example 2.2. *[18] Let $X = \mathbb{N} \cup \{\infty\}$ and $\mu : X \times X \longrightarrow \mathbb{R}_+$ be defined by*

$$\mu(x, y) = \begin{cases} 0, & \textit{if } x = y \\ \left| \frac{1}{x} - \frac{1}{y} \right|, & \textit{if } x, y \text{ are even or } xy = \infty \\ 5, & \textit{if } x, y \text{ are odd and } x \neq y \\ 2, & \textit{otherwise}. \end{cases}$$

Then, (X, μ, η) is a b-metric space with parameter $\eta = 3$, but μ is not a continuous functional.

Definition 2.3. *[10] Let (X, μ, η) be a b-metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be:*

(i) convergent if and only if there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$, and we write this as $\lim_{n\longrightarrow\infty} \mu(x_n, x) = 0$.
(ii) Cauchy if and only if $\mu(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$.
(iii) complete if every Cauchy sequence in X is convergent.

In a b-metric space, the limit of a sequence is not always unique. However, if a b-metric is continuous, then every convergent sequence has a unique limit.

Definition 2.4. *[10] Let (X, μ, η) be a b-metric space. Then, a subset A of X is called:*

(i) compact if and only if for every sequence of elements of A, there exists a subsequence that converges to an element of A.
(ii) closed if and only if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of A that converges to an element x, we have $x \in A$.

Definition 2.5. *A nonempty subset A of X is called proximal if, for each $x \in X$, there exists $a \in A$ such that $\mu(x, a) = \mu(x, A)$.*

Throughput this paper, we denote by $\mathcal{N}(X)$, $CB(X)$, $\mathcal{P}^r(X)$, and $\mathcal{K}(X)$, the family of nonempty subsets of X, the set of all nonempty closed and bounded subsets of X, the family of all nonempty proximal subsets of X, and the class of nonempty closed and compact subsets of X, respectively.

Let (X, μ, η) be a b-metric space. For $A, B \in \mathcal{K}(X)$, the function $\aleph_b : \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow \mathbb{R}_+$, defined by

$$\aleph_b(A, B) = \begin{cases} \max\left\{\sup_{x\in A} \mu(x, B), \sup_{x\in B} \mu(x, A)\right\}, & \text{if it exists} \\ \infty, & \text{otherwise,} \end{cases}$$

is called generalized Hausdorff b-metric on $\mathcal{K}(X)$ induced by the b-metric μ, where

$$\mu(x, A) = \inf_{y\in A} \mu(x, y).$$

Remark 2.6. *Since every compact set is proximal and every proximal set is closed, we have the inclusions:*

$$\mathcal{K}(X) \subseteq \mathcal{P}^r(X) \subseteq CB(X) \subseteq \mathcal{N}(X).$$

Definition 2.7. *Let (X, μ, η) be a metric space. A set-valued mapping $T : X \longrightarrow \mathcal{N}(X)$ is called a multi-valued mapping. A point $u \in X$ is said to be a fixed point of T if $u \in Tu$.*

Definition 2.8. *Let X be a nonempty. A multivalued mapping $T : X \longrightarrow \mathcal{N}(X)$ is said to be α-admissible with respect to a function $\alpha : X \times X \longrightarrow \mathbb{R}_+$, if for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$.*

Not long ago, a family of auxiliary functions under the name *simulation functions* was introduced by Khojasteh et al. [26] to unify various types of contractions.

Definition 2.9. *[26] A simulation function is a mapping $\xi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying the following axioms:*

- (i) $\xi(0, 0) = 0$;
- (ii) $\xi(t, s) < t - s$ *for all* $t, s > 0$;
- (iii) *if* $\{t_n\}_{n\geq 1}$ *and* $\{s_n\}_{n\geq 1}$ *are sequences in* $(0, \infty)$ *such that* $\lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} s_n > 0$, *then* $\lim_{n\longrightarrow\infty} \sup \xi(t_n, s_n) < 0$.

We denote the family of simulation functions by $\mathcal{Z}$.

Example 2.10. *[26] Let $\xi_i : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ $(i = i, 2, 3)$ be defined by*

- (i) $\xi_1(t, s) = \tau(s) - \phi(t)$ *for all* $t, s \in \mathbb{R}_+$, *where* $\tau, \phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *are continuous functions such that* $\tau(t) = \phi(t) = 0$ *if and only if* $t = 0$ *and* $\tau(t) < t \leq \phi(t)$ *for all* $t > 0$.
- (ii) $\xi_2(t, s) = s - \frac{\Lambda(t,s)}{\Gamma(t,s)} t$ *for all* $t, s \in \mathbb{R}_+$, *where* $\Lambda, \Gamma : \mathbb{R}_+ \longrightarrow (0, \infty)$ *are two continuous functions with respect to each variable such that* $\Lambda(t, s) > \Gamma(t, s)$ *for all* $t, s > 0$.
- (iii) $\xi_3(t, s) = s - \psi(s) - t$ *for all* $t, s \in \mathbb{R}_+$, *where* $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a continuous function such that* $\psi(0) = 0$ *if and only if* $t = 0$.

Then, ξ_i ($i = 1, 2, 3$) are simulation functions.

For more examples of simulation functions, see [2,4,20].

Definition 2.11. *[26] Let (X,μ) be a metric space. A mapping $T : X \longrightarrow X$ is called a $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}$, if*

$$\xi\left(\mu(Tx,Ty),\mu(x,y)\right) \geq 0 \text{ for all } x,y \in X. \tag{2.1}$$

The following is the main result in [26].

Theorem 2.12. *[26] Every $\mathcal{Z}$-contraction on a complete metric space has a unique fixed point.*

An example of a $\mathcal{Z}$-contraction is the Banach contraction which can be obtained by setting $\xi(t,s) = \rho s - t$, where $\rho \in [0,1)$ in (2.1).

The idea of comparison function was initiated by Rus [32] and it has been studied by several researchers in order to obtain more general forms of contraction mappings.

Definition 2.13. *[32] A mapping $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is said to be a comparison function if it is nondecreasing and $\varphi^n(t) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $t \geq 0$.*

Example 2.14. *The following functions $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are comparison functions:*

(i) $\varphi(t) = \varsigma t$ for all $t \geq 0$, where $\varsigma \in (0,1)$.
(ii) $\varphi(t) = \frac{t}{t+1}$ for each $t \geq 0$.

Definition 2.15. *[23,32] A nondecreasing function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called:*

(i) a c-comparison function if $\varphi^n(t) \longrightarrow 0$ as $n \longrightarrow \infty$ for every $t \in \mathbb{R}_+$;
(ii) a b-comparison function if there exist $k_0 \in \mathbb{N}$, $\lambda \in (0,1)$ and a convergent non-negative series $\sum_{n=1}^{\infty} x_n$ such that $\eta^{k+1}\varphi^{k+1}(t) \leq \lambda\eta^k\varphi^k(t) + x_k$, for $\eta \geq 1$, $k \geq k_0$ and any $t \geq 0$, where φ^n denotes the n^{th} iterate of φ.

Denote by Ω_b, the family of functions $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying the following conditions:

(i) φ is a b-comparison function;
(ii) $\varphi(t) = 0$ if and only if $t = 0$.
(iii) φ is continuous.

Remark 2.16. *A b-comparison function is a c-comparison function when $\eta = 1$. Moreover, it can be shown that a c-comparison function is a comparison function, but the converse is not always true. Notice that in Example 2.13, (i) is a c-comparison function. But, (ii) is not a c-comparison function.*

Lemma 2.17. *[32] For a comparison function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, the following properties hold:*

(i) each iterate φ^n, $n \in \mathbb{N}$ is also a comparison function;
(iii) $\varphi(t) < t$ for all $t > 0$.

Lemma 2.18. *[32] Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a b-comparison function. Then, the series $\sum_{k=0}^{\infty}\eta^k\varphi^k(t)$ converges for every $t \in \mathbb{R}_+$.*

Remark 2.19. *[23] In Lemma 2.18, every b-comparison function is a comparison function and thus, in Lemma 2.17, every b-comparison function satisfies $\varphi(t) < t$.*

Lemma 2.20. *([34]) Let (X, σ, η) be a b-metric space. For $A, B \in \mathcal{K}(X)$ and $x, y \in X$, the following conditions hold:*

(i) $\mu(x, B) \leq \aleph_b(A, B)$ *for any* $x \in A$.
(ii) $\mu(x, B) \leq \mu(x, b)$ *for any* $b \in B$.
(iii) $\mu(x, A) \leq \eta\,[\mu(x, y) + \mu(y, A)]$.
(iv) $\mu(x, A) = 0 \iff x \in A$.
(v) $\aleph_b(A, B) = 0 \iff A = B$.
(vi) $\aleph_b(A, B) = \aleph_b(B, A)$.
(vii) $\aleph_b(A, B) \leq \eta\,[\aleph_b(A, C) + \aleph_b(C, B)]$.

3. Main results

The concepts of admissible multivalued hybrid $\mathcal{Z}$-contractions and multivalued hybrid $\mathcal{Z}$-contractions are introduced as follows.

Definition 3.1. *Let (X, μ, η) be a b-metric space. A set-valued map $T : X \longrightarrow \mathcal{K}(X)$ is called an admissible multivalued hybrid $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}$, if there exists a function $\alpha : X \times X \longrightarrow \mathbb{R}_+$ and a b-comparison function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that*

$$\xi\left(\alpha(x, y)\aleph_b(Tx, Ty), \varphi\left(\mathcal{M}_T^r(x, y)\right)\right) \geq 0, \tag{3.1}$$

for all $x, y \in X$, where

$$\mathcal{M}_T^r(x, y) = \begin{cases} [\mathcal{A}(x, y)]^{\frac{1}{r}}, & \text{for } r > 0, x, y \in X \\ \mathcal{B}(x, y), & \text{for } r = 0, x, y \in X, \end{cases}$$

$$\begin{aligned}\mathcal{A}(x, y) = {} & a_1(\mu(x, y))^r + a_2(\mu(x, Tx))^r + a_3(\mu(y, Ty))^r \\ & + a_4\left(\frac{\mu(y, Ty)(1 + \mu(x, Tx))}{1 + \mu(x, y)}\right)^r + a_5\left(\frac{\mu(y, Tx)(1 + \mu(x, Ty))}{1 + \mu(x, y)}\right)^r,\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}(x, y) = {} & (\mu(x, y))^{a_1} \cdot (\mu(x, Tx))^{a_2} \cdot (\mu(y, Ty))^{a_3} \\ & \cdot \left(\frac{\mu(y, Ty)(1 + \mu(x, Tx))}{1 + \mu(x, y)}\right)^{a_4} \cdot \left(\frac{\mu(x, Ty) + \mu(y, Tx)}{2\eta}\right)^{a_5},\end{aligned}$$

with $r \geq 0$ and $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ such that $\sum_{i=1}^{5} a_i = 1$.

Remark 3.2.

(i) In Definition 3.1, if $\alpha(x, y) = 1$, then T is called a multivalued hybrid $\mathcal{Z}$-contraction with respect $\xi \in \mathcal{Z}$.
(ii) If T is an admissible multivalued hybrid $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}$, then for all $x, y \in X$,

$$\alpha(x, y)\aleph_b(Tx, Ty) < \varphi(\mathcal{M}_T^r(x, y)). \tag{3.2}$$

To prove the assertion (*ii*), suppose $x \neq y$, then, $\mu(x,y) > 0$. If $Tx = Ty$, we have $\alpha(x,y)\aleph_b(Tx,Ty) = 0 < \varphi\left(\mathcal{M}_T^r(x,y)\right)$. Otherwise, $\aleph_b(Tx,Ty) > 0$. If $\alpha(x,y) = 0$, then (3.2) holds trivially. So, assume that $\alpha(x,y) > 0$, an using (*ii*) in Definition 2.9, we obtain

$$\begin{aligned} 0 &\leq \xi\left(\alpha(x,y)\aleph_b(x,y), \varphi\left(\mathcal{M}_T^r(x,y)\right)\right) \\ &< \varphi\left(\mathcal{M}_T^r(x,y)\right) - \alpha(x,y)\aleph_b(Tx,Ty). \end{aligned}$$

The following definition is very significant in the proof of our results.

Definition 3.3. *Let (X,μ,η) be a b-metric space. A multivalued mapping $T : X \longrightarrow \mathcal{K}(X)$ is said to be $\aleph$-continuous at $u \in X$, if for any sequence $\{x_n\}_{n\geq 1}$ in X,*

$$\lim_{n\longrightarrow\infty} \mu(x_n,u) = 0 \implies \lim_{n\longrightarrow\infty} \aleph_b(Tx_n,Tu) = 0.$$

We say that T is $\aleph$-continuous if it is continuous at each point of X.

Definition 3.3 can be reformulated as follows:
T is said to be $\aleph$-continuous at a point u, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\mu(x_n,u) < \delta \implies \aleph_b(Tx_n,Tu) < \epsilon.$$

Example 3.4. *Let $X = \mathbb{R}$ and $\mu(x,y) = |x-y|^2$ for all $x,y \in X$. Then, $(X,\mu,\eta = 2)$ is a b-metric space. Define $T : X \longrightarrow \mathcal{K}(X)$ by $Tx = [x, x+5]$ for all $x \in X$. Then, $\aleph_b(Tx,Ty) = |x-y|^2$. For any $\epsilon > 0$, take $\delta = \frac{\epsilon}{7}$. Then, $\mu(x,y) < \delta$ implies $\aleph_b(Tx,Ty) < \epsilon$. Consequently, T is $\aleph$-continuous.*

Theorem 3.5. *Let (X,μ,η) be a complete b-metric space and $T : X \longrightarrow \mathcal{K}(X)$ be an admissible multivalued hybrid $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}$. Suppose also that the following conditions are satisfied:*

(i) T is an α-admissible multivalued mapping;
(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$;
(iii) T is $\aleph$-continuous;
(iv) Tx is proximal for each $x \in X$.

Then, T has at least one fixed point in X.

Proof. By Condition (*ii*), there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If $x_0 = x_1$ (or $x_0 = x = y = x_1$), then from (3.1), we have

$$\begin{aligned} 0 &\leq \xi\left(\alpha(x_0,x_1)\aleph_b(Tx_0,Tx_1), \varphi\left(\mathcal{M}_T^r(x_0,x_1)\right)\right) \\ &< \varphi\left(\mathcal{M}_T^r(x_0,x_1)\right) - \alpha(x_0,x_1)\aleph_b(Tx_0,Tx_1), \end{aligned}$$

which is equivalent to

$$\alpha(x_0,x_1)\aleph_b(Tx_0,Tx_1) \leq \varphi\left(\mathcal{M}_T^r(x_0,x_1)\right). \tag{3.3}$$

Then, for $r > 0$, using the proximality of T, we get

$$\begin{aligned}
\mathcal{M}_T^r(x_0, x_1) &= [A(x_0, x_1)]^{\frac{1}{r}} \\
&= \Bigg[a_1(\mu(x_0, x_1))^r + a_2(\mu(x_0, Tx_0))^r + a_3(\mu(x_1, Tx_1))^r \\
&\quad a_4\left(\frac{\mu(x_1, Tx_1)(1+\mu(x_0, Tx_0))}{1+\mu(x_0, x_1)}\right)^r \\
&\quad + a_5\left(\frac{\mu(x_1, Tx_0)(1+\mu(x_0, Tx_1))}{1+\mu(x_0, x_1)}\right)^r\Bigg]^{\frac{1}{r}} \\
&= \Bigg[a_1(\mu(x_0, x_1))^r + a_2(\mu(x_0, x_1))^r + a_3(\mu(x_1, Tx_1))^r \\
&\quad + a_4\left(\frac{\mu(x_1, Tx_1)(1+\mu(x_0, x_1))}{1+\mu(x_0, x_1)}\right)^r \\
&\quad + a_5\left(\frac{\mu(x_1, x_1)(1+\mu(x_0, Tx_0))}{1+\mu(x_0, x_1)}\right)^r\Bigg]^{\frac{1}{r}} \\
&= \left[a_1(\mu(x_0, x_1))^r + a_2(\mu(x_0, x_1))^r + a_3(\mu(x_1, Tx_1))^r + a_4(\mu(x_1, Tx_1))^r\right]^{\frac{1}{r}} \\
&= \left[(a_1 + a_2)(\mu(x_0, x_1))^r + (a_3 + a_4)(\mu(x_1, Tx_1))^r\right]^{\frac{1}{r}} \\
&= \left[(a_1 + a_2)(\mu(x_1, x_1))^r + (a_3 + a_4)(\mu(x_1, Tx_0))^r\right]^{\frac{1}{r}} \ (\because x_1 = x_0) \\
&= \left[(a_1 + a_2)(0)^r + (a_3 + a_4)(\mu(x_1, x_1))^r\right]^{\frac{1}{r}} = 0.
\end{aligned}$$

Similarly, $\mathcal{B}(x_0, x_1) = 0$. Hence, (3.3) becomes $\alpha(x_0, x_1)\aleph_b(Tx_0, Tx_1) \leq \varphi(0) = 0$, which implies that $Tx_0 = Tx_1$. It follows directly that $x_1 \in Tx_0 = Tx_1$; that is, x_1 is a fixed point of T. So, hereafter, we assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$ so that $\mu(x_1, Tx_1) > 0$. Since $Tx_1 \in \mathcal{K}(X)$ and $x_1 \in Tx_0$, there exists $x_2 \in Tx_1$ with $x_1 \neq x_2$ such that

$$\mu(x_1, x_2) \leq \aleph_b(Tx_0, Tx_1) \leq \alpha(x_0, x_1)\aleph_b(Tx_0, Tx_1). \tag{3.4}$$

Setting $x = x_0$ and $y = x_1$ in (3.1), gives

$$\begin{aligned}
0 &\leq \xi\left(\alpha(x_0, x_1)\aleph_b(Tx_0, Tx_1), \varphi\left(\mathcal{M}_T^r(x_0, x_1)\right)\right) \\
&< \varphi\left(\mathcal{M}_T^r(x_0, x_1)\right) - \alpha(x_0, x_1)\aleph_b(Tx_0, Tx_1),
\end{aligned}$$

which can also be written as

$$\alpha(x_0, x_1)\aleph_b(Tx_0, Tx_1) \leq \varphi\left(\mathcal{M}_T^r(x_0, x_1)\right). \tag{3.5}$$

Combining (3.4) and (3.5), yields

$$\mu(x_1, x_2) \leq \varphi\left(\mathcal{M}_T^r(x_0, x_1)\right). \tag{3.6}$$

Given that T is α-admissible and $x_2 \in Tx_1$, we have $\alpha(x_1, x_2) \geq 1$. If $x_2 \in Tx_2$, then taking $x_1 = x_2$ (or $x_1 = x = y = x_2$), as in previous steps, we find directly that x_2 is a fixed point of T. So, suppose that

$x_2 \notin Tx_2$ so that $\mu(x_2, Tx_2) > 0$. Since $Tx_1, Tx_2 \in \mathcal{K}(X)$ and $x_2 \in Tx_1$, there exists a point $x_3 \in Tx_2$ with $x_2 \neq x_3$ such that

$$\mu(x_2, x_3) \leq \aleph_b(Tx_1, Tx_2) \leq \alpha(x_1, x_2)\aleph_b(Tx_1, Tx_2). \tag{3.7}$$

Putting $x = x_1$ and $y = x_2$ in (3.1), we get

$$\begin{aligned} 0 &\leq \xi\left(\alpha(x_1, x_2)\aleph_b(Tx_1, Tx_2), \varphi\left(\mathcal{M}_T^r(x_1, x_2)\right)\right) \\ &< \varphi\left(\mathcal{M}_T^r(x_1, x_2)\right) - \alpha(x_1, x_1)\aleph_b(Tx_1, Tx_2), \end{aligned}$$

which is equivalent to

$$\alpha(x_1, x_2)\aleph_b(Tx_1, Tx_2) \leq \varphi\left(\mathcal{M}_T^r(x_1, x_2)\right). \tag{3.8}$$

From (3.7) and (3.8), we have

$$\mu(x_2, x_3) \leq \varphi\left(\mathcal{M}_T^r(x_1, x_2)\right). \tag{3.9}$$

Continuing with this iteration, we generate a sequence $\{x_n\}_{n\geq 1}$ in X with $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$, $\alpha(x_n, x_{n+1}) \geq 1$ such that

$$\mu(x_n, x_{n+1}) \leq \varphi\left(\mathcal{M}_T^r(x_{n-1}, x_n)\right). \tag{3.10}$$

Now, we investigate (3.10) under the following cases:
Case 1: $r > 0$. In this case, from (3.1), using the proximality of T, we have

$$\begin{aligned} \mathcal{M}_T^r(x_{n-1}, x_n) &= [A(x_{n-1}, x_n)]^{\frac{1}{r}} \\ &= \Bigg[a_1(\mu(x_{n-1}, x_n))^r + a_2(\mu(x_{n-1}, Tx_{n-1}))^r + a_3(\mu(x_n, Tx_n))^r \\ &\quad + a_4\left(\frac{\mu(x_n, Tx_n)(1 + \mu(x_{n-1}, Tx_{n-1}))}{1 + \mu(x_{n-1}, x_n)}\right)^r \\ &\quad + a_5\left(\frac{\mu(x_n, Tx_{n-1})(1 + \mu(x_{n-1}, Tx_n))}{1 + \mu(x_{n-1}, x_n)}\right)^r\Bigg]^{\frac{1}{r}} \\ &= \Bigg[a_1(\mu(x_{n-1}, x_n))^r + a_2(\mu(x_{n-1}, x_n))^r + a_3(\mu(x_n, x_{n+1}))^r \\ &\quad + a_4\left(\frac{\mu(x_n, x_{n+1})(1 + \mu(x_{n-1}, x_n))}{1 + \mu(x_{n-1}, x_n)}\right)^r \\ &\quad + a_5\left(\frac{\mu(x_n, x_n)(1 + \mu(x_{n-1}, x_{n+1}))}{1 + \mu(x_{n-1}, x_n)}\right)^r\Bigg]^{\frac{1}{r}} \\ &= \left[(a_1 + a_2)(\mu(x_{n-1}, x_n))^r + (a_3 + a_4)(\mu(x_n, x_{n+1}))^r\right]^{\frac{1}{r}}. \end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we get

$$\mu(x_n, x_{n+1}) \leq \varphi\left(\left[(a_1 + a_2)(\mu(x_{n-1}, x_n))^r + (a_3 + a_4)(\mu(x_n, x_{n+1}))^r\right]^{\frac{1}{r}}\right). \tag{3.12}$$

Assume that $\mu(x_{n-1}, x_n) \leq \mu(x_n, x_{n+1})$, then, since φ is nondecreasing and noting that $a_1 + a_2 + a_3 + a_4 \leq 1$, from (3.12), we have

$$\mu(x_n, x_{n+1}) \leq \varphi\left(\left[\mu(x_n, x_{n+1})^r\right]^{\frac{1}{r}}\right)$$

$$= \varphi(\mu(x_n, x_{n+1})) < \mu(x_n, x_{n+1}),$$

a contradiction. Consequently, (3.12) becomes

$$\begin{aligned} \mu(x_n, x_{n+1}) &\leq \varphi(\mu(x_{n-1}, x_n)) \\ &\leq \varphi^2(\mu(x_{n-2}, x_{n-1})) \\ &\vdots \\ &\leq \varphi^n(\mu(x_0, x_1)). \end{aligned} \tag{3.13}$$

Now, let $m, n \in \mathbb{N}$ with $m > n$. Then, by triangular inequality in (X, μ, η), we have

$$\begin{aligned} \mu(x_n, x_m) &\leq \eta\mu(x_n, x_{n+1}) + \eta^2\mu(x_{n+1}, x_{n+2}) + \cdots + \eta^{m-n}\mu(x_{m-1}, x_m) \\ &\leq \eta\varphi^n(\mu(x_0, x_1)) + \eta^2\varphi^{n+1}(\mu(x_0, x_1)) + \cdots + \eta^{m-n+1}\varphi^m(\mu(x_0, x_1)) \\ &= \frac{1}{\eta^{n-1}}\left(\eta^n\varphi^n(\mu(x_0, x_1)) + \eta^{n+1}\varphi^{n+1}(\mu(x_0, x_1)) + \cdots + \eta^m\varphi^m(\mu(x_0, x_1))\right) \\ &= \frac{1}{\eta^{n-1}}\sum_{i=n}^{m}\eta^i\varphi^i(\mu(x_0, x_1)) \\ &\leq \frac{1}{\eta^{n-1}}\sum_{i=0}^{\infty}\eta^i\varphi^i(\mu(x_0, x_1)). \end{aligned} \tag{3.14}$$

Since φ is a b-comparison function, it follows from Lemma 2.18 that the series $\sum_{i=0}^{\infty}\varphi^i(\mu(x_0, x_1))$ is convergent. Setting $S_k = \sum_{i=1}^{k}\varphi^i(\mu(x_0, x_1))$, (3.14) can be written as

$$\mu(x_n, x_m) \leq \frac{1}{\eta^{n-1}}\left(S_{m-1} - S_{n-1}\right). \tag{3.15}$$

Letting $n, m \longrightarrow \infty$ in (3.15), we obtain $\mu(x_n, x_m) \longrightarrow 0$, which proves that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X. Completeness of this space implies that there exists $u \in X$ such that

$$\lim_{n\longrightarrow\infty}\mu(x_n, u) = 0. \tag{3.16}$$

Now, we show that $u \in Tu$. Using the triangle inequality in X, we have

$$\begin{aligned} \mu(u, Tu) &\leq \eta[\mu(u, x_n) + \mu(x_n, Tu)] \\ &\leq \eta\mu(u, x_n) + \eta\aleph_b(Tx_{n-1}, Tu). \end{aligned} \tag{3.17}$$

Since T is $\aleph$-continuous, passing to limit as $n \longrightarrow \infty$ in (3.17), we have $\mu(u, Tu) = 0$, which implies that $u \in Tu$.

Case 2: $r = 0$. For this, take $x = x_{n-1} \in Tx_n$ and $y = x_n \in Tx_{n+1}$ in (3.1), then, by proximality of T, we have

$$
\begin{aligned}
\mathcal{M}_T^r(x_{n-1}, x_n) = & \ \mathcal{B}(x_{n-1}, x_n) \\
= & \ (\mu(x_{n-1}, x_n))^{a_1} \cdot (\mu(x_{n-1}, Tx_{n-1}))^{a_2} \cdot (\mu(x_n, Tx_n))^{a_3} \\
& \cdot \left(\frac{\mu(x_n, Tx_n)(1 + \mu(x_{n-1}, Tx_{n-1}))}{1 + \mu(x_{n-1}, x_n)} \right)^{a_4} \left(\frac{\mu(x_{n-1}, Tx_n) + (x_n, Tx_{n-1})}{2\eta} \right)^{a_5} \\
= & \ (\mu(x_{n-1}, x_n))^{a_1} \cdot (\mu(x_{n-1}, x_n))^{a_2} \cdot (\mu(x_n, x_{n+1}))^{a_3} \\
& \cdot \left(\frac{\mu(x_n, x_{n+1})(1 + \mu(x_{n-1}, x_n))}{1 + \mu(x_{n-1}, x_n)} \right)^{a_4} \cdot \left(\frac{\mu(x_{n-1}, x_{n+1}) + (x_n, x_n)}{2\eta} \right)^{a_5} \\
= & \ (\mu(x_{n-1}, x_n))^{a_1 + a_2} \cdot (\mu(x_n, x_{n+1}))^{a_3 + a_4} \cdot \left(\frac{\mu(x_{n-1}, x_n) + \mu(x_n, x_{n+1})}{2} \right)^{a_5}.
\end{aligned} \tag{3.18}
$$

It is well-known that for any $p, q, l > 0$,

$$
\left(\frac{p + q}{2} \right)^l \leq \frac{p^l + q^l}{2}. \tag{3.19}
$$

Applying (3.19) to (3.18), gives

$$
\begin{aligned}
\mathcal{M}_T^r(x_{n-1}, x_n) \leq & \ (\mu(x_{n-1}, x_n))^{a_1 + a_2} \cdot (\mu(x_n, x_{n+1}))^{a_3 + a_4} \\
& \cdot \left(\frac{(\mu(x_{n-1}, x_n))^{a_5}}{2} + \frac{(\mu(x_n, x_{n+1}))^{a_5}}{2} \right).
\end{aligned} \tag{3.20}
$$

Recall that from (3.1), we get

$$
\begin{aligned}
0 & \leq \xi \left(\alpha(x_{n-1}, x_n) \aleph_b(Tx_{n-1}, Tx_n), \varphi \left(\mathcal{M}_T^r(x_{n-1}, x_n) \right) \right) \\
& < \varphi \left(\mathcal{M}_T^r(x_{n-1}, x_n) \right) - \alpha(x_{n-1}, x_n) \aleph_b(Tx_{n-1}, Tx_n),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\mu(x_n, x_{n+1}) & \leq \alpha(x_{n-1}, x_n) \aleph_b(Tx_{n-1}, Tx_n) \\
& \leq \varphi \left(\mathcal{M}_T^r(x_{n-1}, x_n) \right).
\end{aligned} \tag{3.21}
$$

Assume that $\mu(x_{n-1}, x_n) \leq \mu(x_n, x_{n+1})$, then (3.20) becomes

$$
\mathcal{M}_T^r(x_{n-1}, x_n) \leq (\mu(x_n, x_{n+1}))^{a_1 + a_2 + a_3 + a_4 + a_5} = \mu(x_n, x_{n+1}). \tag{3.22}
$$

Since φ is nondecreasing, from (3.22) and (3.21), we have

$$
\mu(x_n, x_{n+1}) \leq \varphi(\mu(x_n, x_{n+1})) < \mu(x_n, x_{n+1}),
$$

a contradiction. Therefore, (3.21) yields

$$
\begin{aligned}
\mu(x_n, x_{n+1}) & \leq \varphi(\mu(x_{n-1}, x_n)) \leq \varphi^2(\mu(x_{n-2}, x_{n-1})) \\
& \ \vdots \\
& \leq \varphi^n(\mu(x_0, x_1)).
\end{aligned} \tag{3.23}
$$

Following the same procedures as in the Case $r > 0$, it follows from (3.23) that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X, and the completeness of this space implies that there exists $u \in X$ such that

$$\lim_{n\longrightarrow\infty} \mu(x_n, u) = 0. \tag{3.24}$$

To show that $u \in Tu$, consider:

$$\begin{aligned} \mu(u, Tu) &\leq \eta\,[\mu(u, x_n) + \mu(x_n, Tu)] \\ &\leq \eta\mu(x_n, u) + \eta\aleph_b(Tx_{n-1}, Tu). \end{aligned} \tag{3.25}$$

Since T is $\aleph$-continuous, letting $n \longrightarrow \infty$ in (3.25), and using (3.24), we obtain $\mu(u, Tu) = 0$, which implies that $u \in Tu$. □

Theorem 3.6. *Let (X, μ, η) be a complete b-metric space and $T : X \longrightarrow \mathcal{K}(X)$ be a multivalued mapping satisfying the following conditions:*

(i) T is an α-admissible multivalued mapping;
(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iii) T is $\aleph$-continuous;
(i v) Tx is proximal for each $x \in X$.

Assume further that there exist $\xi \in \mathcal{Z}$, $\varphi \in \Omega_b$ and a function $\alpha : X \times X \longrightarrow \mathbb{R}_+$ such that for all $x, y \in X$,

$$\xi\left(\alpha(x, y)\aleph_b(Tx, Ty), \varphi\left([\mathcal{A}(x, y)]^{\frac{1}{r}}\right)\right) \geq 0, \tag{3.26}$$

where $\mathcal{A}(x, y)$ is as given in Definition 3.1.
Then, T has at least one fixed point in X.

The proof follows from Case 1 of Theorem 3.5.

Remark 3.7. *It is necessary to remind the reader that Theorems 3.5 and 3.6 are invalid if we take $CB(X)$ instead of $\mathcal{K}(X)$ (for example, see [28, Theorem 5]).*

Example 3.8. *Let $X = [1, \infty)$ and $\mu(x, y) = |x - y|^2$ for all $x, y \in X$. Then, $(X, \mu, \eta = 2)$ is a complete b-metric space. Note that $(X, \mu, \eta = 2)$ is not a metric space, since for $x = 1,\ y = 4$ and $z = 3$, we have*

$$\mu(x, y) = 9 > 5 = \mu(x, z) + \mu(z, y).$$

Define $T : X \longrightarrow \mathcal{K}(X)$ by

$$Tx = \begin{cases} \{4\}, & \text{if } x \in [1, 3) \\ [1, 4x]\,, & \text{if } x \in [3, \infty), \end{cases}$$

and the functions $\alpha : X \times X \longrightarrow \mathbb{R}_+$, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$\alpha(x, y) = \begin{cases} 5, & \text{if } x, y \in [1, 3) \\ \frac{1}{600}, & \text{if } x = 4, y = 5 \\ 0, & \text{otherwise}, \end{cases}$$

$\varphi(t) = \frac{t}{2}$ for all $t \geq 0$. Also, take $\xi(t, s) = \frac{1}{8}s - t$ for all $s, t \in \mathbb{R}_+$. Clearly, $\xi \in \mathcal{Z}$ and $\varphi \in \Omega_b$. Now, we show that (3.26) *is satisfied under the following cases:*
Case 1: $x, y \in [1, 3)$. For this, $Tx = Ty = \{4\}$ and so, $\aleph_b(Tx, Ty) = 0$ for all $x, y \in X$. Hence,

$$\begin{aligned} \xi\left(\alpha(x,y)\aleph_b(Tx,Ty), \varphi\left([\mathcal{A}(x,y)]^{\frac{1}{r}}\right)\right) &= \xi\left(0, \varphi\left([\mathcal{A}(x,y)]^{\frac{1}{r}}\right)\right) \\ &= \frac{1}{8}\varphi\left([\mathcal{A}(x,y)]^{\frac{1}{r}}\right) \geq 0. \end{aligned}$$

Case 2: $x = 4$ and $y = 5$. For this, we have

$$\xi\left(\alpha(4,5)\aleph_b(T4,T5), \varphi\left([\mathcal{A}(4,5)]^{\frac{1}{r}}\right)\right) = \frac{1}{8}\varphi\left([\mathcal{A}(4,5)]^{\frac{1}{r}}\right) - \alpha(4,5)\aleph_b(T4,T5), \tag{3.27}$$

$\aleph_b(T4, T5) = 16$, and

$$\begin{aligned} [\mathcal{A}(4,5)]^{\frac{1}{r}} = \Bigg[& a_1(\mu(4,5))^r + a_2(\mu(4,T4))^r + a_3(\mu(5,T5))^r \\ & + a_4\left(\frac{\mu(5,T5)(1+\mu(4,T4))}{1+\mu(4,5)}\right)^r + a_5\left(\frac{\mu(5,T4)(1+\mu(4,T5))}{1+\mu(4,5)}\right)^r\Bigg]^{\frac{1}{r}}. \end{aligned} \tag{3.28}$$

By taking $a_1 = a_2 = \frac{1}{2}$ and $a_3 = a_4 = a_5 = 0$ in (3.28)*, we get*

$$\begin{aligned} [\mathcal{A}(4,5)]^{\frac{1}{r}} &= \left[\frac{1}{2}(1)^r + \frac{1}{2}(0)^r\right]^{\frac{1}{r}} \\ &= \left[\frac{1}{2}\right]^{\frac{1}{r}} \longrightarrow 1 \text{ as } r \longrightarrow \infty. \end{aligned}$$

Therefore, (3.27) *becomes*

$$\begin{aligned} \xi\left(\alpha(4,5)\aleph_b(T4,T5), \varphi\left([\mathcal{A}(4,5)]^{\frac{1}{r}}\right)\right) &= \frac{1}{8}\varphi(1) - \frac{16}{600} \\ &= \frac{1}{8}\left(\frac{1}{2} - \frac{16}{75}\right) \geq 0. \end{aligned}$$

Moreover, it is obvious that T is α-admissible, Tx is proximal and $\aleph$-continuous for each $x \in X$. Consequently, all the assumptions of Theorem 3.6 are satisfied. We can see that T has many fixed points in X.

4. Consequences

In this section, we show that several interesting fixed point results can be deduced from our main theorem, especially by availing variants of simulation functions.

Corollary 4.1. *Let (X, μ, η) be a complete b-metric space and $T : X \longrightarrow \mathcal{K}(X)$ be a multivalued mapping satisfying the condition:*

$$\alpha(x,y)\aleph_b(Tx,Ty) \leq \varphi\left(\mathcal{M}_T^r(x,y)\right), \tag{4.1}$$

for all $x, y \in X$, where $\varphi \in \Omega_b$ and $\alpha : X \times X \longrightarrow \mathbb{R}_+$ is a function. Assume also that the following conditions hold:

(i) T is an α-admissible multivalued mapping;
(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iii) T is $\aleph$-continuous;
(i v) Tx is proximal for each $x \in X$.

Then, there exists $u \in X$ such that $u \in Tu$.

Proof. Set $\xi := \xi(t,s) = \varphi(s) - t$ for all $t, s \in \mathbb{R}_+$ in Theorem 3.5. Then, (4.1) follows easily. Note that $\varphi(s) - t \in \mathcal{Z}$. Consequently, Theorem 3.5 can be applied to find $u \in X$ such that $u \in Tu$. □

Corollary 4.2. *(Rhoades type [31]) Let (X, μ, η) be a complete b-metric space and $T : X \longrightarrow \mathcal{K}(X)$ be a multivalued mapping satisfying the following:*

$$\aleph_b(Tx, Ty) \leq \varphi\left(\mathcal{M}_T^r(x,y)\right) - \varphi^2\left(\mathcal{M}_T^r(x,y)\right), \tag{4.2}$$

for all $x, y \in X$, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is lower semicontinuous and $\varphi^{-1}(0) = \{0\}$. Also, assume that the following conditions hold:

(i) T is $\aleph$-continuous;
(ii) Tx is proximal for each $x \in X$.

Then, there exists $u \in X$ such that $u \in Tu$.

Proof. In Theorem 3.5, take $\xi := \xi(t,s) = s - \varphi(s) - t$ for all $t, s \in \mathbb{R}_+$ and $\alpha(x,y) = 1$ for all $x, y \in X$. Then (4.2) follows. Notice that $s - \varphi(s) - t \in \mathcal{Z}$. Hence, by Theorem 3.5, T has a fixed point in X. □

Corollary 4.3. *(Nadler's type ([28])) Let (X, μ, η) be a complete b-metric space and $T : X \longrightarrow \mathcal{K}(X)$ be a multivalued mapping satisfying:*

$$\aleph_b(Tx, Ty) \leq \lambda\mu(x,y), \tag{4.3}$$

for all $x, y \in X$, where $\lambda \in (0,1)$. Assume also that the following assertion hold:

(i) T is $\aleph$-continuous;
(ii) Tx is proximal for each $x \in X$.

Then, T has a fixed point in X.

Proof. Take $\alpha(x,y) = 1$, $\xi := \xi(t,s) = \lambda s - t$ for all $s, t \in \mathbb{R}_+$ and put $\varphi(t) = \lambda t$ for all $t \geq 0$, with $\lambda \in (0,1)$ in Theorem 3.5. Then (4.3) is obtainable. Notice that $\lambda s - t \in \mathcal{Z}$. Hence, it follows from Theorem 3.5 that there exists $u \in X$ such that $u \in Tu$. □

Corollary 4.4. *(Branciari's type [11]) Let (X, μ, η) be a complete b-metric space and $T : X \longrightarrow \mathcal{K}(X)$ be a multivalued mapping satisfying:*

$$\int_0^{\aleph_b(Tx,Ty)} \omega(t)\mu t \leq \varphi\left(\mathcal{M}_T^r(x,y)\right), \tag{4.4}$$

for all $x, y \in X$, where $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a function such that $\int_0^\gamma \omega(t)\mu t > 0$ for each $\gamma > 0$. Also, assume that the following conditions hold:

(i) T is $\aleph$-continuous;
(ii) Tx is proximal for each $x \in X$.

Then, there exists $u \in X$ such that $u \in Tu$.

Proof. Put $\alpha(x, y) = 1$ and $\xi := \xi(t, s) = s - \int_0^t \omega(u)\mu u$ for all $s, t \in \mathbb{R}_+$. Then (4.4) follows easily. Note that $\xi \in \mathcal{Z}$. Thus, by Theorem 3.5, T has a fixed point in X. □

Corollary 4.5. *Let (X, μ, η) be a complete b-metric space and $T : X \longrightarrow \mathcal{K}(X)$ be a multivalued mapping satisfying:*

$$\Lambda\left(\aleph_b(Tx, Ty), \varphi\left(\mathcal{M}_T^r(x, y)\right)\right) \leq \frac{\varphi\left(\mathcal{M}_T^r(x, y)\right) \cdot \Gamma\left(\aleph_b(Tx, Ty), \varphi\left(\mathcal{M}_T^r(x, y)\right)\right)}{\aleph_b(Tx, Ty)}, \tag{4.5}$$

for all $x, y \in X$ with $\aleph_b(Tx, Ty) > 0$, where $\Lambda, \Gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous functions with respect to each arguments such that $\Lambda(t, s) > \Gamma(t, s)$ for all $t, s > 0$. Assume further that the following axioms hold:

(i) T is $\aleph$-continuous;
(ii) Tx is proximal for each $x \in X$.

Then, T has a fixed point in X.

Proof. Setting $\xi := \xi(t, s) = s - \frac{\Lambda(t,s)}{\Gamma(t,s)}t$ for all $t, s \in \mathbb{R}_+$ and $\alpha(x, y) = 1$ in Theorem 3.5, we obtain (4.5). Observe that $\xi \in \mathcal{Z}$. Consequently, Theorem 3.5 can be employed to locate $u \in X$ such that $u \in Tu$. □

Corollary 4.6. *Let (X, μ, η) be a complete b-metric space and $\Theta : X \longrightarrow X$ be a single-valued mapping satisfying the condition:*

$$\mu(\Theta x, \Theta y) \leq \varphi^2\left([\mathcal{A}(x, y)]^{\frac{1}{r}}\right), \tag{4.6}$$

for all $x, y \in X$, where $\varphi \in \Omega_b$ and $[A(x, y)]^{\frac{1}{r}}$ is as given in Definition 3.1. Then, there exists $u \in X$ such that $\Theta u = u$.

Proof. Take $\xi := \xi(t, s) = \varphi(s) - t$ in Theorem 3.5, then $\xi \in \mathcal{Z}$ and (4.6) follows directly. Then, consider a multivalued mapping $T : X \longrightarrow \mathcal{K}(X)$ defined by $Tx = \{\Theta x\}$ for all $x \in X$. Clearly, $\{\Theta x\} \in \mathcal{K}(X)$ for each $x \in X$. Hence, Theorem 3.5 can be applied with $\alpha(x, y) = 1$ for all $x, y \in X$, to find $u \in X$ such that $u \in Tu = \{\Theta u\}$; which further implies that $u = \Theta u$. □

Remark 4.7. *It is obvious that we can obtain more consequences of our results by considering other variants of simulation functions. Also, by taking the parameter $\eta = 1$, all the established results herein reduce to their classical metric space equivalence. In particular, by following the idea of Corollary 4.6, single-valued versions of the rest results presented here can also be pointed out.*

5. Applications

5.1. Fixed point results in ordered b-metric spaces

Fixed point theory in partially ordered sets is one of the highly useful branches of fixed point theory with enormous applications in areas such as matrix equations, boundary value problems, and many more. For some articles in this direction, see [3, 29, 30]. In this section, our main result is applied to deduce its analogue in the framework of ordered b-metric space. Indeed, a b-metric space can be equipped with a partial ordering; that is, if $(X, \preceq)$ is a partially ordered set, then $(X, \mu, \eta, \preceq)$ is known as an ordered b-metric space. Accordingly, we say that $x, y \in X$ are comparable if either $x \preceq y$ or $y \preceq x$ holds. Let $L, M \subseteq X$, then $L \preceq M$ if for each $l \in L$, there exists $m \in M$ such that $l \preceq m$.

Theorem 5.1. *Let $(X, \mu, \eta, \preceq)$ be a complete ordered b-metric space and $T : X \longrightarrow \mathcal{K}(X)$ be a multivalued mapping. Suppose that there exist $\xi \in \mathcal{Z}$, $\varphi \in \Omega_b$ and a function $\alpha : X \times X \longrightarrow \mathbb{R}_+$ such that*

$$\xi\left(\alpha(x, y)\aleph_b(Tx, Ty), \varphi\left(\mathcal{M}_T^r(x, y)\right)\right) \geq 0, \tag{5.1}$$

for all $x, y \in X$ with $Tx \preceq Ty$. Also, assume that the following conditions hold:

(i) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $Tx \preceq Ty$;
(ii) for each $x \in X$ and $y \in Tx$ with $Tx \preceq Ty$, we have $Ty \preceq Tz$ for all $z \in Ty$;
(iii) T is $\aleph$-continuous;
(iv) Tx is proximal for each $x \in X$.

Then, there exists $u \in X$ such that $u \in Tu$.

Proof. Let the function $\alpha : X \times X \longrightarrow \mathbb{R}_+$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } Tx \preceq Ty \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, T is an α-admissible multivalued mapping. In fact, take $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, then $Tx \preceq Ty$; and by hypothesis (ii), $Ty \preceq Tz$ for each $z \in Ty$. It follows that $\alpha(y, z) \geq 1$ for all $z \in Ty$. Moreover, by inequality (5.1), we find that T is an admissible multivalued hybrid $\mathcal{Z}$-contraction with respect to $z \in \mathcal{Z}$. Consequently, all the axioms of Theorem 3.5 are satisfied. Hence, T has a fixed point in X. □

5.2. Fixed point results for graphic contractions

The notion of contraction mapping on a metric space with a graph was introduced by Jachymski [22]. In this subsection, we deduce a fixed point result in the setting of b-metric space endowed with a graph. Following [22], let (X, μ, η) be a metric space and $\mathcal{D}$ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set V_G of its vertices coincides with X, and the set E_G of its edges contains all loops, that is, $(x, x) \in E_G$ for every $x \in V_G$ (or $\mathcal{D} \subseteq E_G$). We presume that G has no parallel edges so that it can be identified with the pair (V_G, E_G). Furthermore, G is taken as a weighted graph (for details, see [22]) by allocating to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path from x to y of length l ($l \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{l}$ of $l + 1$ vertices such that $x = x_0$, $x_l = y$ and $(x_{n-1}, x_n) \in E_G$ for $i = 1, \cdots, l$. A graph G is connected if there is a path between any two of its vertices. For some fixed point results with graphic contractions, we refer [14, 15, 22]. Hereafter, a nonempty set X endowed with a graph G shall be written as (X, G). First, we introduce the following auxiliary concepts.

Definition 5.2. *Let (X, μ, η, G) be a b-metric space. A multivalued mapping $T : X \longrightarrow \mathcal{K}(X)$ is said to be G-continuous at $u \in X$, if given $u \in X$ and a sequence $\{x_n\}_{n\geq 1}$ such that $\mu(x_n, u) \longrightarrow 0$ as $n \longrightarrow \infty$*

and $(x_{n-1}, x_n) \in E_G$ for all $n \in \mathbb{N}$, we have $\aleph_b(Tx_n, Tu) \longrightarrow 0$ as $n \longrightarrow \infty$, for all $(Tx_{n-1}, Tu) \in E_G$, with $x_n \in Tx_{n-1}$. T is G-continuous if it is G-continuous at each point of X.

Definition 5.3. *Let (X, G) be a nonempty set. We say that a multivalued mapping $T : X \longrightarrow \mathcal{K}(X)$ is edge-preserving, if for all $x, y \in X$, $(x, y) \in E_G$ implies $(Tx, Ty) \subseteq E_G$.*

Definition 5.4. *Let (X, μ, η, G) be a metric space. A subset Δ of X is called proximal, if for each $x \in X$, there exists $\kappa \in \Delta$ with $(x, \kappa) \in E_G$ such that $\mu(x, \kappa) = \mu(x, \Delta)$.*

Now, we present the main result of this subsection as follows.

Theorem 5.5. *Let (X, μ, η, G) be a complete b-metric space and $T : X \longrightarrow \mathcal{K}(X)$ be a multivalued mapping. Assume that there exist $\xi \in \mathcal{Z}$, $\varphi \in \Omega_b$ and a function $\alpha : X \times X \longrightarrow \mathbb{R}_+$ such that*

$$\xi\left(\alpha(x, y)\aleph_b(Tx, Ty), \varphi(\mathcal{M}_T^r(x, y))\right) \geq 0, \tag{5.2}$$

for all $x, y \in X$. Moreover, suppose also that the following assertions hold:

(i) for all $x, y \in X$, $(x, y) \in E_G$ implies $(Tx, Ty) \subseteq E_G$;
(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $(x_0, x_1) \in E_G$;
(iii) for each $x \in X$ and $y \in Tx$ with $(Tx, Ty) \subseteq E_G$, we have $(Ty, Tz) \subseteq E_G$ for all $z \in Ty$;
(iv) T is G-continuous;
(v) Tx is proximal for each $x \in X$.

Then, there exists $u \in X$ such that $u \in Tu$.

Proof. Define the function $\alpha : X \times X \longrightarrow \mathbb{R}_+$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E_G \\ 0, & \text{otherwise.} \end{cases}$$

First, we show that T is an admissible multivalued mapping. Let $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, then $(x, y) \in E_G$ and, by Condition (i), $(Tx, Ty) \in E_G$; hence, from (iii), $(Tx, Tz) \in E_G$ for each $z \in Ty$. Therefore, $\alpha(y, z) \geq 1$, which implies that $(y, z) \in E_G$. So, combining the inequality (5.2), we have that T is an admissible multivalued hybrid $\mathcal{Z}$-contraction with respect to $\xi \in \mathcal{Z}$. Finally, it is easy to see that conditions (*iv*) and (*v*) imply the hypotheses (*iii*) and (*iv*), respectively of Theorem 3.5. Consequently, all the assertions of Theorem 3.5 are satisfied and hence T has a fixed point in X. □

6. Existence results of nonlinear matrix equations

It has always been a worthwhile research to investigate an adequate technique to solve matrix equations because the existence of solutions of matrix equations arises in various applications such as in stochastic filtering, system theory, dynamic programming, control theory, statistics, ladder networks, and a host of other branches of sciences. For some results on this line, we refer [16, 21].

Let F_n, Υ_n and Ξ_n denote the set of all $n \times n$ Hermitian, positive definite and positive semi-definite matrices, respectively. $\daleth > 0$ (respectively, $\daleth \geq 0$) means that $\daleth \in \Upsilon_n$ (respectively, $\daleth \in \Xi_n$). Let the spectral norm of a matrix $\beth$ be defined by

$$\|\beth\|_1 = \sqrt{\varrho^+(\beth^*\beth)},$$

where $\varrho^*(\sqsupset\sqsupset^*)$ represents the greatest eigenvalue of the matrix $\sqsupset^*\sqsupset$. The Ky Fan norm is defined as

$$\|\sqsupset\|_1 = \sum_{i=1}^{n} \Xi_i(\sqsupset),$$

where $\{\Xi_1(\sqsupset), \Xi_2(\sqsupset), \cdots, \Xi_n(\sqsupset)\}$ is the set of singular values of $\sqsupset$.
Consider the nonlinear matrix equation:

$$\nabla = \theta + \sum_{i=1}^{m} \sqsupset_i^* \beta(\nabla) \sqsupset_i, \tag{6.1}$$

where $\theta \in \Upsilon_n$, $\sqsupset_i$ $(i = 1, 2, \cdots, m)$ are $n \times n$ matrices and β is a single-valued mapping.

Theorem 6.1. *Let $\beta : F_n \longrightarrow F_n$ maps Υ_n into Υ_n and $\theta \in \Upsilon_n$. Suppose also that the following conditions are satisfied:*

(i) there exists a constant $\mho \in \left(0, \frac{1}{2}\right)$ such that $\sum_{i=1}^{m} \sqsupset_i \sqsupset_i^* \leq \mho^2 I_n$, where I_n is an $n \times n$ matrix;
(ii) $\sum_{i=1}^{m} \sqsupset_i^* \beta(\theta) \sqsupset > 0$;
(iii) for all $\nabla, \triangle \in F_n$,

$$\|\beta(\nabla) - \beta(\triangle)\|_1^2 \leq \frac{1}{\mho^2} \sum_{\mu}(\nabla, \triangle), \tag{6.2}$$

where

$$\begin{aligned}\sum_{\mu}(\nabla, \triangle) = \Bigg[& a_1(\mu(\nabla, \triangle))^r + a_2(\mu(\nabla, \beta(\nabla)))^r + a_3(\mu(\triangle, \beta(\triangle)))^r \\ & + a_4 \left(\frac{\mu(\triangle, \beta(\triangle))(1 + \mu(\nabla, \beta(\nabla)))}{1 + \mu(\nabla, \triangle)} \right)^r \\ & + a_5 \left(\frac{\mu(\triangle, \beta(\nabla))(1 + \mu(\nabla, \beta(\triangle)))}{1 + \mu(\imath, \triangle)} \right)^r \Bigg]^{\frac{1}{r}},\end{aligned}$$

where $r > 0$ and $a_i \geq 0 (i = 1, 2, 3, 4, 5)$ such that $\sum_{i=1}^{5} a_i = 1$.

Then, (6.1) *has a solution in Υ_n.*

Proof. Let $\mu : F_n \times F_n \longrightarrow \mathbb{R}$ be defined by

$$\mu(\nabla, \triangle) = \|\nabla - \triangle\|_1^2 = (tr(\nabla - \triangle))^2,$$

for all $\nabla, \triangle \in F_n$. Then, $(F_n, \mu, \eta = 2)$ is a complete b-metric space. Consider the mappings $\sigma : F_n \longrightarrow F_n$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, respectively defined by

$$\sigma(\nabla) = \theta + \sum_{i=1}^{m} \sqsupset_i^* \beta(\nabla) \sqsupset_i \tag{6.3}$$

and $\varphi(t) = \mho^2 t$ for all $t \geq 0$ and $\mho \in \left(0, \frac{1}{2}\right)$. Then, $\varphi \in \Omega_b$. Also, notice that the solution of (6.1) is a fixed point of (6.3).
Let $\nabla, \triangle \in F_n$ with $\nabla \neq \triangle$ so that $\mu(\nabla, \triangle) > 0$. Then, we have

$$\begin{aligned} \|\sigma(\nabla) - \sigma(\triangle)\|_1 &= tr\,(\sigma(\nabla) - \sigma(\triangle)) \\ &= \sum_{i=1}^{m} tr\,(\beth_i \beth_i^*(\beta(\nabla) - \beta(\triangle))) \\ &= tr\left(\left(\sum_{i=1}^{m} \beth_i \beth_i^*(\beta(\nabla) - \beta(\triangle))\right)\right) \\ &\leq \left\|\sum_{i=1}^{m} \beth_i \beth_i^*\right\| \|\beta(\nabla) - \beta(\triangle)\|_1 \\ &\leq \mho \sqrt{\sum_{\mu}(\nabla, \triangle)}. \end{aligned} \tag{6.4}$$

From (6.4), we have $\|\sigma(\nabla) - \sigma(\triangle)\|_1^2 \leq \mho^2 \sum_\mu(\nabla, \triangle)$, which implies that

$$\mu(\sigma(\nabla), \sigma(\triangle)) \leq \varphi^2\left(\sum_{\mu}(\nabla, \triangle)\right).$$

Hence, all the assertions of Corollary 4.6 are satisfied. Consequently, (6.1) has a solution. □

Example 6.2. *Consider the nonlinear matrix equation:*

$$\nabla = \theta + \sum_{i=1}^{2} \beth_i^* \beta(\nabla) \beth_i, \tag{6.5}$$

where θ, $\beth_1$ and $\beth_2$ are respectively given by

$$\theta = \begin{bmatrix} 0.02 & 0.002 & 0.002 \\ 0.002 & 0.02 & 0.002 \\ 0.002 & 0.002 & 0.02 \end{bmatrix}, \quad \beth_1 = \begin{bmatrix} 0.15 & 0.002 & 0.002 \\ 0.02 & 0.15 & 0.002 \\ 0.02 & 0.002 & 0.15 \end{bmatrix}$$

$$\text{and } \beth_2 = \begin{bmatrix} 0.25 & 0.002 & 0.002 \\ 0.002 & 0.25 & 0.002 \\ 0.002 & 0.002 & 0.25 \end{bmatrix}$$

Also, let the mappings $\beta, \sigma : F_3 \longrightarrow F_3$ be respectively defined by

$$\beta(\nabla) = \frac{\nabla}{3} \text{ and } \sigma(\nabla) = \theta + \sum_{i=1}^{2} \beth_i^* \beta(\nabla) \beth_i.$$

Then, by taking $\mho = \frac{2}{5}$, we find that all the hypotheses of Theorem 6.2 hold.

7. Conclusion

It is well-known that in some abstract spaces, the triangle inequality does not hold. But, by multiplying the constant $\eta \geq 1$ on the right-hand side of the triangle inequality, one can obtain a more useful abstract structure, now called a b-metric space in the literature. Following this orientation, in this work, two new ideas, admissible multivalued hybrid $\mathcal{Z}$-contractions and multivalued hybrid $\mathcal{Z}$-contractions in the framework of b-metric spaces using generalized Hausdorff metric are initiated. The established concept herein unifies several results in one theorem. A few of these special cases are pointed out. Thereafter, to indicate some applications of our results, a few fixed point theorems in the setting of fixed point results of b-metric spaces endowed with partial ordering and graph are deduced and solvability conditions of nonlinear matrix equations are investigated.

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Conflict of interest

The authors declare that they have no competing interests.

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