Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(1): 333–361.
DOI: 10.3934/math.2021021
Received: 14 July 2020
Accepted: 23 September 2020
Published: 12 October 2020

Research article

Stability rate of a thermoelastic laminated beam: Case of equal-wave speed and nonequal-wave speed of propagation

Soh E. Mukiawa[1,*], Tijani A. Apalara[1] and Salim A. Messaoudi[2,3]

[1] Department of Mathematics, University of Hafr Al Batin, Hafr Al Batin 31991, Saudi Arabia
[2] Department of Mathematics and Statistics University of Sharjah Sharjah, United Arab Emirates
[3] Department of Mathematics and Statistics, KFUPM, Dhahran 31261, Saudi Arabia

* **Correspondence:** Email: mukiawa@uhb.edu.sa.

Abstract: In this article, we investigate a one-dimensional thermoelastic laminated beam system with viscoelastic dissipation on the effective rotation angle and through heat conduction in the interfacial slip equations. Under general conditions on the relaxation function and the relationship between the coefficients of the wave propagation speed of the first two equations, we show that the solution energy has an explicit and general decay rate from which the exponential and polynomial stability are just particular cases. Moreover, we establish a weaker decay result in the case of non-equal wave of speed propagation and give some examples to illustrate our results. This new result improves substantially many other results in the literature.

Keywords: general decay; Laminated beam; thermoelasticitty; fourier heat conduction; viscoelasticity; convexity; optimal
Mathematics Subject Classification: 35D30, 35D35, 35B35, 35L51, 74D10, 93D15

1. Introduction

The fundamental work of Hansen and Spies [4] modeled a two-layer beam with a structural damping due to the interfacial slip through the following system

$$\begin{cases}\rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0,\\ I_\rho\left(3w - \psi\right)_{tt} - D\left(3w - \psi\right)_{xx} - G\left(\psi - \varphi_x\right) = 0,\\ I_\rho w_{tt} - Dw_{xx} + 3G\left(\psi - \varphi_x\right) + 4\gamma w + 4\beta w_t = 0,\end{cases} \tag{1.1}$$

where $\varphi = \varphi(x,t)$ is the transverse displacement, $\psi = \psi(x,t)$ is the rotation angle, $w = w(x,t)$ is proportional to the amount of slip along the interface, $3w - \psi$ denotes the effective rotation angle. The physical quantities $\rho, I_\rho, G, D, \beta$ and γ are respectively: the density, mass moment of inertia, shear stiffness, flexural rigidity, adhesive damping and adhesive stiffness. Equation $(1.1)_3$ describes the dynamics of the slip. For $\beta = 0$, system (1.1) describes the coupled laminated beams without structural damping at the interface. In the recent result [1], Apalara considered the thermoelastic-laminated beam system without structural damping, namely

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \varphi_x) = 0, \\ I_\rho s_{tt} - Ds_{xx} + 3G(\psi - \varphi_x) + 4\gamma s + \delta\theta_x = 0, \\ \rho_3\theta_t - \lambda\theta_{xx} + \delta s_{tx} = 0, \end{cases} \tag{1.2}$$

where $(x, t) \in (0, 1) \times (0, +\infty)$, $\theta = \theta(x, t)$ is the difference temperature. The positive quantities $\gamma, \beta, k, \lambda$ are adhesive stiffness, adhesive damping, heat capacity and the diffusivity respectively. The author proved that (1.2) is exponential stable provided

$$\frac{G}{\rho} = \frac{D}{I_\rho}. \tag{1.3}$$

When $\beta > 0$, the adhesion at the interface supplies a restoring force proportion to the interfacial slip. But this is not enough to stabilize system (1.1), see for instance [2]. To achieve exponential or general stabilization of system (1.1), many authors in literature have used additional damping. In this direction, Gang et al. [9] studied the following memory-type laminated beam system

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_\rho(3w - \psi)_{tt} - D(3w - \psi)_{xx} + \displaystyle\int_0^t g(t-s)(3w - \psi)_{xx}(x, s)ds - G(\psi - \varphi_x) = 0 \\ 3I_\rho w_{tt} - 3Dw_{xx} + 3G(\psi - \varphi_x) + 4\gamma w + 4\beta w_t = 0 \end{cases} \tag{1.4}$$

and established a general decay result for more regular solutions and $\frac{G}{\rho} \neq \frac{D}{I_\rho}$. Mustafa [15] also considered the structural damped laminated beam system (1.4) and established a general decay result provided $\frac{G}{\rho} = \frac{D}{I_\rho}$. Feng et al. [8] investigated the following laminated beam system

$$\begin{cases} \rho w_{tt} + G\varphi_x + g_1(w_t) + f_1(w, \xi, s) = h_1, \\ I_\rho\xi_{tt} - G\varphi - D\xi_{xx} + g_2(\xi_t) + f_2(w, \xi, s) = h_2, \\ I_\rho s_{tt} + G\varphi - Ds_{xx} + g_3(s_t) + f_2(w, \xi, s) = h_3 \end{cases} \tag{1.5}$$

and established the well-posedness, smooth global attractor of finite fractal dimension as well as existence of generalized exponential attractors. See also, recent results by Enyi et al. [20]. We refer the reader to [5–7, 11, 13, 14, 17, 18] and the references cited therein for more related results.

In this present paper, we consider a thermoelastic laminated beam problem with a viscoelastic damping

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} + \displaystyle\int_0^t g(t-\tau)(3s - \psi)_{xx}(x, \tau)d\tau - G(\psi - w_x) = 0 \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + \delta\theta_x = 0, \\ k\theta_t - \lambda\theta_{xx} + \delta s_{xt} = 0 \end{cases} \tag{1.6}$$

under initial conditions

$$\begin{cases} w(x,0) = w_0(x),\ \psi(x,0) = \psi_0(x),\ s(x,0) = s_0(x),\ \theta(x,0) = \theta_0(x), & x \in [0,1], \\ w_t(x,0) = w_1(x),\ \psi_t(x,0) = \psi_1(x),\ s_t(x,0) = s_1(x), & x \in [0,1] \end{cases} \tag{1.7}$$

and boundary conditions

$$\begin{cases} w(0,t) = \psi_x(0,t) = s_x(0,t) = \theta(0,t) = 0, & t \in [0,+\infty), \\ w_x(1,t) = \psi(1,t) = s(1,t) = \theta_x(1,t) = 0, & t \in [0,+\infty). \end{cases} \tag{1.8}$$

In the system (1.6), the integral represents the viscoelastic damping, and g is the relaxation function satisfying some suitable assumptions specified in the next section. According to the Boltzmann Principle, the viscoelastic damping (see [21] for details) is represented by a memory term in the form of convolution. It acts as a damper to reduce the internal/external forces like the beam's weight, heavy loads, wind, etc., that cause undesirable vibrations.

In most of the above works, the authors have established their decay result by including the structural damping along with other dampings. So, the natural question that comes to mind.

> Is it possible to obtain general/optimal decay result (decay rates that agrees with that of g) to the thermoelastic laminated beam system (1.6)–(1.8), in the absence of the structural damping.

The novelty of this article is to answer this question in a consenting way, by using the ideas developed in [10] to establish general and optimal decay results for Problem 1.6. Moreover, we establish a weaker decay result in the case of a non-equal wave of speed propagation. To the best of our knowledge, there is no stability result for the latter in the literature.

The rest of work is organized as follows: In Section 2, we recall some preliminaries and assumptions on the memory term. In Section 3, we state and prove the main stability result for the case equal-speed and in the case of non-equal-speed of propagation. We also give some examples to illustrate our findings. Finally, in Section 4, we give the proofs of the lemmas used our main results.

2. Preliminaries

In this section, we recall some useful materials and conditions. Through out this paper, C is a positive constant that may change through lines, $\langle .,. \rangle$ and $\|.\|_2$ denote respectively the inner product and the norm in $L^2(0,1)$. We assume the relaxation function g obeys the assumptions:

(G1). $g : [0,+\infty) \longrightarrow (0,+\infty)$ is a non-inecreasing C^1- function such that

$$g(0) > 0, \quad D - \int_0^\infty g(\tau)d\tau = l_0 > 0. \tag{2.1}$$

(G2). There exist a C^1 function $H : [0,+\infty) \to (0,+\infty)$ which is linear or is strictly convex C^2 function on $(0,\epsilon_0)$, $\epsilon_0 \leq g(0)$, with $H(0) = H'(0) = 0$ and a positive nonincreasing differentiable function $\xi : [0,+\infty) \to (0,+\infty)$, such that

$$g'(t) \leq -\xi(t)H(g(t)), \quad t \geq 0, \tag{2.2}$$

Remark 2.1. *As in [10], we note here that, if H is a strictly increasing convex C^2- function on $(0, r]$, with $H(0) = H'(0) = 0$, then H has an extension $\bar{H}$, which is strictly increasing and strictly convex C^2-function on $(0, +\infty)$. For example, $\bar{H}$ can be defined by*

$$\bar{H}(s) = \frac{H''(r)}{2}s^2 + (H'(r) - H''(r)r)s + H(r) - H'(r)r + \frac{H''(r)}{2}r^2, \ s > r. \tag{2.3}$$

Let

$$H^1_*(0,1) = \{u \in H^1(0,1)/u(0) = 0\}, \quad \bar{H}^1_*(0,1) = \{u \in H^1(0,1)/u(1) = 0\},$$
$$H^2_*(0,1) = \{u \in H^2(0,1)/u_x \in H^1_*(0,1)\}, \quad \bar{H}^2_*(0,1) = \{u \in H^2(0,1)/u_x \in \bar{H}^1_*(0,1)\}.$$

The existence and regularity result of problem (1.6) is the following

Theorem 2.1. Let $(w_0, 3s_0-\psi_0, s_0, \theta_0) \in H^1_*(0,1)\times\bar{H}^1_*(0,1)\times\bar{H}^1_*(0,1)\times H^1_*(0,1)$ and $(w_1, 3s_1-\psi_1, s_1) \in L^2(0,1) \times L^2(0,1) \times L^2(0,1)$ be given. Suppose (G1) and (G2) hold. Then problem (1.6) has a unique global weak solution $(w, 3s - \psi, s, \theta)$ which satisfies

$$w \in C(\mathbb{R}_+, H^1_*(0,1)) \cap C^1(\mathbb{R}_+, L^2(0,1)), \ (3s - \psi) \in C(\mathbb{R}_+, \bar{H}^1_*(0,1)) \cap C^1(\mathbb{R}_+, L^2(0,1)),$$

$$s \in C(\mathbb{R}_+, \bar{H}^1_*(0,1)) \cap C^1(\mathbb{R}_+, L^2(0,1)), \ \theta \in C(\mathbb{R}_+, L^2(0,1)) \cap L^2(\mathbb{R}_+, H^1(0,1)).$$

Furthermore, if $(w_0, (3s_0 - \psi_0), s_0, \theta_0) \in H^2_*(0,1) \times \bar{H}^2_*(0,1) \times \bar{H}^2_*(0,1) \times H^2(0,1) \cap H^1_*(0,1)$ and $(w_1, (3s_1 - \psi_1), s_1) \in H^1_*(0,1) \times \bar{H}^1_*(0,1) \times \bar{H}^1_*(0,1)$, then the solution of (1.6) satisfies

$$w \in C(\mathbb{R}_+, H^2_*(0,1)) \cap C^1(\mathbb{R}_+, H^1_*(0,1)) \cap C^2(\mathbb{R}_+, L^2(0,1)),$$

$$(3s - \psi) \in C(\mathbb{R}_+, \bar{H}^2_*(0,1)) \cap C^1(\mathbb{R}_+, \bar{H}^1_*(0,1)) \cap C^2(\mathbb{R}_+, L^2(0,1)),$$
$$s \in C(\mathbb{R}_+, \bar{H}^2_*(0,1)) \cap C^1(\mathbb{R}_+, \bar{H}^1_*(0,1)) \cap C^2(\mathbb{R}_+, L^2(0,1)),$$
$$\theta \in C(\mathbb{R}_+, H^2(0,1) \cap H^2_*(0,1)) \cap C^1(\mathbb{R}_+, H^1_*(0,1)).$$

The proof of Theorem 2.1 can be established using the Galerkin approximation method as in [16]. Throughout this paper, we denote by $\diamond$ the binary operator, defined by

$$(g \diamond v)(t) = \int_0^t g(t - \tau)\|v(t) - v(\tau)\|_2^2 d\tau, \qquad t \geq 0.$$

We also define $h(t)$ and C_α as follow

$$h(t) = \alpha g(t) - g'(t) \quad \text{and} \quad C_\alpha = \int_0^{+\infty} \frac{g^2(\tau)}{\alpha g(\tau) - g'(\tau)} d\tau.$$

The following lemmas will be applied repeatedly throughout this paper

Lemma 2.1. *For any function $f \in L^2_{loc}([0, +\infty), L^2(0,1))$, we have*

$$\int_0^1 \left(\int_0^t g(t - s)(f(t) - f(s))ds \right)^2 dx \leq (1 - l_0)(g \diamond f)(t), \tag{2.4}$$

$$\int_0^1 \left(\int_0^x f(y,t)dy \right)^2 dx \leq \|f(t)\|_2^2. \tag{2.5}$$

Lemma 2.2. *Let* $v \in H^1_*(0,1)$ *or* $\bar{H}^1_*(0,1)$*, we have*

$$\int_0^1 \left(\int_0^t g(t-s)(v(t)-v(\tau))d\tau \right)^2 dx \le C_p(1-l_0)(g \diamond v)(t), \tag{2.6}$$

where $\mathrm{C_p} > 0$ *is the poincaré constant.*

Lemma 2.3. *Let* $(w, 3s-\psi, s, \theta)$ *be the solution of* (1.6)*. Then, for any* $0 < \alpha < 1$ *we have*

$$\int_0^1 \left(\int_0^t g(t-\tau)\left((3s-\psi)_x(\tau) - (3s-\psi)_x(t)\right) d\tau \right)^2 dx \le C_\alpha \left(h \diamond (3s-\psi)_x\right)(t). \tag{2.7}$$

Proof. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_0^1 \left(\int_0^t g(t-\tau)\left((3s-\psi)_x(\tau) - (3s-\psi)_x(t)\right) d\tau \right)^2 dx \\
&= \int_0^1 \left(\int_0^t \frac{g(t-\tau)}{\sqrt{h(t-\tau)}} \sqrt{h(t-\tau)} \left((3s-\psi)_x(\tau) - (3s-\psi)_x(t)\right) d\tau \right)^2 dx \\
&\le \left(\int_0^{+\infty} \frac{g^2(\tau)}{h(\tau)} ds \right) \int_0^1 \int_0^t h(t-\tau)\left((3s-\psi)_x(\tau) - (3s-\psi)_x(t)\right)^2 d\tau dx \\
&= C_\alpha \left(h \diamond (3s-\psi)_x\right)(t).
\end{aligned} \tag{2.8}$$

□

Lemma 2.4. *[12] Let F be a convex function on the close interval* $[a,b]$, $f, j : \Omega \to [a,b]$ *be integrable functions on* Ω*, such that* $j(x) \ge 0$ *and* $\int_\Omega j(x)dx = \alpha_1 > 0$*. Then, we have the following Jensen inequality*

$$F\left(\frac{1}{\alpha_1} \int_\Omega f(y)j(y)dy \right) \le \frac{1}{\alpha_1} \int_\Omega F(f(y))j(y)dy. \tag{2.9}$$

In particular if $F(y) = y^{\frac{1}{p}}$, $y \ge 0$, $p > 1$*, then*

$$\left(\frac{1}{\alpha_1} \int_\Omega f(y)j(y)dy \right)^{\frac{1}{p}} \le \frac{1}{\alpha_1} \int_\Omega (f(y))^{\frac{1}{p}} j(y)dy. \tag{2.10}$$

Lemma 2.5. *The energy functional* $E(t)$ *of the system* (1.6)-(1.8) *defined by*

$$\begin{aligned}
E(t) &= \frac{1}{2}\left[\rho\|w_t\|_2^2 + 3I_\rho\|s_t\|_2^2 + I_\rho\|3s_t - \psi_t\|_2^2 + 3D\|s_x\|_2^2 + G\|\psi - w_x\|_2^2 \right] \\
&\quad + \frac{1}{2}\left[\left(D - \int_0^t g(\tau)d\tau \right)\|3s_x - \psi_x\|_2^2 + \left(g \diamond (3s_x - \psi_x)\right)(t) + 4\gamma\|s\|_2^2 + k\|\theta\|_2^2 \right],
\end{aligned} \tag{2.11}$$

satisfies

$$\begin{aligned}
E'(t) &= \frac{1}{2}\left(g' \diamond (3s_x - \psi_x)\right)(t) - \frac{1}{2} g(t)\|3s_x - \psi_x\|_2^2 - \lambda\|\theta_x\|_2^2 \\
&\le \frac{1}{2}\left(g' \diamond (3s_x - \psi_x)\right)(t) \le 0, \quad \forall\, t \ge 0.
\end{aligned} \tag{2.12}$$

Proof. Multiplying $(1.6)_1$, $(1.6)_2$, $(1.6)_3$ and $(1.6)_4$, respectively, by w_t, $(3s_t - \psi_t)$, s_t and θ, integrating over $(0, 1)$, and using integration by parts and the boundary conditions (1.7), we arrive at

$$\frac{1}{2}\frac{d}{dt}\left(\rho\|w_t\|_2^2 + G\|\psi - w_x\|_2^2\right) = G\langle(\psi - w_x), \psi_t\rangle, \tag{2.13}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[I_\rho\|3s_t - \psi_t\|_2^2 + \left(D - \int_0^t g(\tau)d\tau\right)\|3s_x - \psi_x\|_2^2 + (g \diamond (3s_x - \psi_x))(t)\right]\\ &= G\langle(\psi - w_x), (3s - \psi)_t\rangle + \frac{1}{2}(g' \diamond (3s_x - \psi_x))(t) - \frac{1}{2}g(t)\|3s_x - \psi_x\|_2^2,\end{aligned} \tag{2.14}$$

$$\frac{1}{2}\frac{d}{dt}\left[3I_\rho\|s_t\|_2^2 + 3D\|s_x\|_2^2 + 4\gamma\|s\|_2^2\right] = -3G\langle(\psi - w_x), s_t\rangle - \delta\langle\theta_x, s_t\rangle, \tag{2.15}$$

and

$$\frac{1}{2}\frac{d}{dt}\left(k\|\theta\|_2^2\right) = -\lambda\|\theta_x\|_2^2 + \delta\langle\theta_x, s_t\rangle. \tag{2.16}$$

Adding the equations (2.13)–(2.16), taking into account $(G1)$ and $(G2)$, we obtain (2.12) for regular solutions. The result remains valid for weak solutions by a density argument. This implies the energy functional is non-increasing and

$$E(t) \le E(0), \quad \forall t \ge 0.$$

□

3. Stability results

This section is subdivided into two. In the first subsection, we prove the stability result for equal-wave-speed of propagation, whereas in the second subsection, we focus on the stability result for non-equal-wave-speed of propagation.

3.1. Equal-wave-speed of propagation

Our aim, in this subsection, is to prove an explicit, general and optimal decay rate of solutions for system (1.6)–(1.8). To achieve this, we define a Lyapunov functional

$$L(t) = NE(t) + \sum_{j=1}^{6} N_j I_j(t), \tag{3.1}$$

where $N,\ N_j,\ j = 1, 2, 3, 4, 5, 6$ are positive constants to be specified later and

$$I_1(t) = -I_\rho \int_0^1 (3s - \psi)_t \int_0^t g(t - \tau)\left((3s - \psi)(t) - (3s - \psi)(\tau)\right) d\tau dx, \qquad t \ge 0,$$

$$I_2(t) = 3I_\rho \int_0^1 s s_t dx + 3\rho \int_0^1 w_t \int_0^x s(y)dydx, \quad I_3(t) = -3kI_\rho \int_0^1 \theta \int_0^x s_t(y)dydx, \quad t \ge 0,$$

$$I_4(t) = -\rho \int_0^1 w_t w dx, \quad I_5(t) = I_\rho \int_0^1 (3s - \psi)(3s - \psi)_t dx, \qquad t \ge 0,$$

$$I_6(t) = 3I_\rho G \int_0^1 (\psi - w_x)s_t dx - 3\rho D \int_0^1 w_t s_x dx, \quad I_7(t) = \int_0^1 \int_0^t J(t-\tau)(3s_x - \psi_x)^2(\tau) d\tau dx, \quad t \geq 0,$$

where

$$J(t) = \int_t^{+\infty} g(\tau) d\tau.$$

The following lemma is very important in the proof of our stability result.

Lemma 3.1. *Suppose $\frac{G}{\rho} = \frac{D}{I_\rho}$. Under suitable choice of $t_0, N, N_j, \ j = 1,2,3,4,5,6$, the Lyapunov functional L satisfies, along the solution of (1.6) – (1.8), the estimate*

$$\begin{aligned} L'(t) \leq & -\beta\left(\|w_t\|_2^2 + \|s_t\|_2^2 + \|3s_t - \psi_t\|_2^2 + \|s_x\|_2^2 + \|3w_x - \psi_x\|_2^2 + \|\psi - w_x\|_2^2\right) \\ & -\beta\left(\|s\|_2^2 + \|\theta_x\|_2^2\right) + \frac{1}{2}\left(g \diamond (3s_x - \psi_x)\right)(t), \forall\ t \geq t_0 \end{aligned} \tag{3.2}$$

and the equivalence relation

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t) \tag{3.3}$$

holds for some $\beta > 0,\ \alpha_1,\ \alpha_2 > 0$.

Proof. By virtue of assumption (3.1) and using $h(t) = \alpha g(t) - g'(t)$, it follows from Lemmas 2.5, 4.1-4.6 (see the Appendix for detailed derivations) that, for all $t \geq t_0 > 0$,

$$\begin{aligned} L'(t) \leq & -\left[N_4\rho - N_2\delta_4\right]\|w_t\|_2^2 - \left[\frac{N_3\delta I_\rho}{2} - N_2 C\left(1 + \frac{1}{\epsilon_2}\right) - N_6 C\left(1 + \frac{1}{\epsilon_1}\right)\right]\|s_t\|_2^2 - 3N_2\gamma\|s\|_2^2 \\ & - \left[N_1 I_\rho g_0 - N_5 I_\rho - N_6\epsilon_1\right]\|3s_t - \psi_t\|_2^2 - \left[3DN_2 - N_3\epsilon_3 - N_4 C - N_6 C\right]\|s_x\|_2^2 \\ & - \left[N_6 G^2 - N_1\epsilon_2 - N_3\epsilon_3 - N_4\frac{C}{\epsilon_4} - N_5 C\right]\|\psi - w_x\|_2^2 - \left[\frac{N_5 l_0}{4} - N_1\epsilon_1 - N_4\epsilon_4\right]\|3s_x - \psi_x\|_2^2 \\ & - \left[\lambda N - N_2 C - N_3 C\left(1 + \frac{1}{\epsilon_3}\right) - N_6 C\right]\|\theta_x\|_2^2 + \frac{N\alpha}{2}\left(g \diamond (3s_x - \psi_x)\right)(t) \\ & - \left[\frac{N}{2} - CC_\alpha\left(N_5 + N_1\left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)\right)\right]\left(h \diamond (3s_x - \psi_x)\right)(t). \end{aligned} \tag{3.4}$$

Now, we choose

$$N_4 = N_5 = 1, \quad \epsilon_4 = \frac{l_0}{8} \tag{3.5}$$

and select N_1 large enough such that

$$\mu_1 := N_1 I_\rho g_0 - I_\rho > 0. \tag{3.6}$$

Next, we choose N_6 large so that

$$\mu_2 := N_6 G^2 - C > 0. \tag{3.7}$$

Also, we select N_2 large enough so that

$$\mu_3 := 3DN_2 - C - N_6 C > 0. \tag{3.8}$$

After fixing N_1, N_2, N_6, and letting $\epsilon_3 = \dfrac{\mu_1}{2N_3}$, we then select ϵ_1, ϵ_2, and δ_4 very small such that

$$\rho - N_2\delta_4 > 0, \ \ \mu_1 - N_6\epsilon_1 > 0, \ \ \ \mu_4 := \frac{\mu_2}{2} - N_1\epsilon_2 > 0 \tag{3.9}$$

and select N_3 large enough so that

$$\frac{N_3\delta I_\rho}{2} - N_2C\left(1 + \frac{1}{\epsilon_2}\right) - N_6C\left(1 + \frac{1}{\epsilon_1}\right) > 0. \tag{3.10}$$

Now, we note that $\frac{\alpha g^2(s)}{h(s)} = \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s)$; thus the dominated convergence theorem gives

$$\alpha C_\alpha = \int_0^{+\infty} \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} ds \to 0 \ \ as \ \ \alpha \to 0. \tag{3.11}$$

Therefore, we can choose some $0 < \alpha_0 < 1$ such that for all $0 < \alpha \le \alpha_0$,

$$\alpha C_\alpha < \frac{1}{4C\left(1 + N_1\left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)\right)}. \tag{3.12}$$

Finally, we select N so large enough and take $\alpha = \frac{1}{N}$ So that

$$\begin{aligned} &\lambda N - N_2C - N_3C\left(1 + \frac{1}{\epsilon_3}\right) - N_6C > 0, \\ &\frac{N}{2} - CC_\alpha\left(1 + N_1\left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)\right) > 0. \end{aligned} \tag{3.13}$$

Combination of (3.6) - (3.13) yields the estimate (3.2). The equivalent relation (3.3) can be obtain easily by using Young's, Cauchy-Schwarz, and Poincaré's inequalities. □

Now, we state and prove our stability result for this subsection.

Theorem 3.1. *Assume $\frac{G}{\rho} = \frac{D}{I_\rho}$ and (G1) and (G2) hold. Then, there exist positive constants a_1 and a_2 such that the energy solution* (2.11) *satisfies*

$$E(t) \le a_2 H_1^{-1}\left(a_1 \int_{t_0}^t \xi(\tau)d\tau\right), \quad \text{where} \ \ H_1(t) = \int_t^r \frac{1}{\tau H'(\tau)} d\tau \tag{3.14}$$

and H_1 is a strictly decreasing and strictly convex function on $(0, r]$, with $\lim_{t\to 0} H_1(t) = +\infty$.

Proof. Using the fact that g and ξ are positive, non-increasing and continuous, and H is positive and continuous, we have that for all $t \in [0, t_0]$

$$0 < g(t_0) \le g(t) \le g(0), \ \ 0 < \xi(t_0) \le \xi(t) \le \xi(0).$$

Thus for some constants $a, b > 0$, we obtain

$$a \le \xi(t)H(g(t)) \le b.$$

Therefore, for any $t \in [0, t_0]$, we get

$$g'(t) \leq -\xi(t)H(g(t)) \leq -\frac{a}{g(0)}g(0) \leq -\frac{a}{g(0)}g(t) \tag{3.15}$$

and

$$\xi(t)g(t) \leq -\frac{g(0)}{a}g'(t). \tag{3.16}$$

From (2.12) and (3.15), it follows that

$$\begin{aligned}
&\int_0^{t_0} g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\
&\quad \leq -\frac{g(0)}{a}\int_0^{t_0} g'(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\
&\quad \leq -CE'(t), \quad \forall t \geq t_0.
\end{aligned} \tag{3.17}$$

From (3.2) and (3.17), we have

$$\begin{aligned}
L'(t) &\leq -\beta E(t) + \frac{1}{2}(g \diamond (3s_x - \psi_x))(t) \\
&= -\beta E(t) + \frac{1}{2}\int_0^{t_0} g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\
&\quad + \frac{1}{2}\int_{t_0}^{t} g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\
&\leq -\beta E(t) - CE'(t) + \frac{1}{2}\int_{t_0}^{t} g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau.
\end{aligned}$$

Thus, we get

$$L_1'(t) \leq -\beta E(t) + \frac{1}{2}\int_{t_0}^{t} g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau, \ \forall t \geq t_0, \tag{3.18}$$

where $L_1 = L + CE \sim E$ by virtue of (3.3). To finish our proof, we distinct two cases:
Case 1: H(t) is linear. In this case, we multiply (3.18) by $\xi(t)$, keeping in mind (2.12) and $(G2)$, to get

$$\begin{aligned}
\xi(t)L_1'(t) &\leq -\beta\xi(t)E(t) + \frac{1}{2}\xi(t)\int_{t_0}^{t} g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\
&\leq -\beta\xi(t)E(t) + \frac{1}{2}\int_{t_0}^{t} \xi(\tau)g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\
&\leq -\beta\xi(t)E(t) - \frac{1}{2}\int_{t_0}^{t} g'(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\
&\leq -\beta\xi(t)E(t) - CE'(t), \ \forall\, t \geq t_0.
\end{aligned} \tag{3.19}$$

Therefore

$$(\xi L_1 + CE)'(t) \leq -\beta\xi(t)E(t), \quad \forall\, t \geq t_0. \tag{3.20}$$

Since ξ is non-increasing and $L_1 \sim E$, we have

$$L_2 = \xi L_1 + CE \sim E. \tag{3.21}$$

Thus, from (3.20), we get for some positive constant α

$$L_2'(t) \leq -\beta\xi(t)E(t) \leq -\alpha\xi(t)L_2(t), \quad \forall\ t \geq t_0. \tag{3.22}$$

Integrating (3.22) over (t_0, t) and recalling (3.21), we obtain

$$E(t) \leq a_1 e^{-a_2 \int_{t_0}^{t} \xi(s)ds} = a_1 H_1^{-1}\left(a_2 \int_{t_0}^{t} \xi(s)ds\right).$$

Case 2: H(t) is nonlinear. In this case, we consider the functional $\mathcal{L}(t) = L(t) + I_7(t)$. From (3.2) and Lemma 4.7 (see the Appendix), we obtain

$$\mathcal{L}'(t) \leq -dE(t), \quad \forall t \geq t_0, \tag{3.23}$$

where $d > 0$ is a positive constant. Therefore,

$$d\int_{t_0}^{t} E(s)ds \leq \mathcal{L}(t_0) - \mathcal{L}(t) \leq \mathcal{L}(t_0).$$

Hence, we get

$$\int_{0}^{+\infty} E(s)ds < \infty. \tag{3.24}$$

Using (3.24), we define $p(t)$ by

$$p(t) := \eta \int_{t_0}^{t} \|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau,$$

where $0 < \eta < 1$ so that

$$p(t) < 1, \forall t \geq t_0. \tag{3.25}$$

Moreover, we can assume $p(t) > 0$ for all $t \geq t_0$; otherwise using (3.18), we obtain an exponential decay rate. We also define $q(t)$ by

$$q(t) = -\int_{t_0}^{t} g'(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau.$$

Then $q(t) \leq -CE'(t)$, $\forall t \geq t_0$. Now, we have that H is strictly convex on $(0, r]$ (where $r = g(t_0)$) and $H(0) = 0$. Thus,

$$H(\sigma\tau) \leq \sigma H(\tau), \quad 0 \leq \sigma \leq 1 \text{ and } \tau \in (0, r]. \tag{3.26}$$

Using (3.26), condition $(G2)$, (3.25), and Jensen's inequality, we get

$$q(t) \quad = \quad \frac{1}{\eta p(t)} \int_{t_0}^{t} p(t)(-g'(\tau))\eta\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau$$

$$\begin{aligned}
&\geq \frac{1}{\eta p(t)}\int_{t_0}^{t} p(t)\xi(\tau)H(g(\tau))\eta\|(3s_x-\psi_x)(t)-(3s_x-\psi_x)(t-\tau)\|_2^2 d\tau\\
&\geq \frac{\xi(t)}{\eta p(t)}\int_{t_0}^{t} H(p(t)g(\tau))\eta\|(3s_x-\psi_x)(t)-(3s_x-\psi_x)(t-\tau)\|_2^2 d\tau\\
&\geq \frac{\xi(t)}{\eta}H\left(\eta\int_{t_0}^{t} g(\tau)\eta\|(3s_x-\psi_x)(t)-(3s_x-\psi_x)(t-\tau)\|_2^2 d\tau\right)\\
&= \frac{\xi(t)}{\eta}\bar{H}\left(\eta\int_{t_0}^{t} g(\tau)\eta\|(3s_x-\psi_x)(t)-(3s_x-\psi_x)(t-\tau)\|_2^2 d\tau\right), \qquad (3.27)
\end{aligned}$$

where $\bar{H}$ is the convex extention of H on $(0,+\infty)$ (see remark 2.1). From (3.27), we have

$$\int_{t_0}^{t} g(\tau)\eta\|(3s_x-\psi_x)(t)-(3s_x-\psi_x)(t-\tau)\|_2^2 d\tau \leq \frac{1}{\eta}\bar{H}^{-1}\left(\frac{\eta q(t)}{\xi(t)}\right).$$

Therefore, (3.18) yields

$$L_1'(t) \leq -\beta E(t) + C\bar{H}^{-1}\left(\frac{\eta q(t)}{\xi(t)}\right), \quad \forall\, t \geq t_0. \qquad (3.28)$$

For $r_0 < r$, we define $L_3(t)$ by

$$L_3(t) := \bar{H}'\left(r_0\frac{E(t)}{E(0)}\right)L_1(t) + E(t) \sim E(t)$$

since $L_1 \sim E$. From (3.28) and using the fact that

$$E'(t) \leq 0,\ \bar{H}'(t) > 0,\ \bar{H}''(t) > 0,$$

we obtain for all $t \geq t_0$

$$\begin{aligned}
L_3'(t) &= r_0\frac{E'(t)}{E(0)}\bar{H}''\left(r_0\frac{E(t)}{E(0)}\right)L_1(t) + \bar{H}'\left(r_0\frac{E(t)}{E(0)}\right)L_1'(t) + E'(t)\\
&\leq -\beta E(t)\bar{H}'\left(r_0\frac{E(t)}{E(0)}\right) + C\bar{H}'\left(r_0\frac{E(t)}{E(0)}\right)\bar{H}^{-1}\left(\eta\frac{q(t)}{\xi(t)}\right) + E'(t). \qquad (3.29)
\end{aligned}$$

Let us consider the convex conjugate of $\bar{H}$ denoted by $\bar{H}^*$ in the sense of Young (see [3] page 61-64). Thus,

$$\bar{H}^*(\tau) = \tau(\bar{H}')^{-1}(\tau) - \bar{H}\left[(\bar{H}')(\tau)\right] \qquad (3.30)$$

and $\bar{H}^*$ satisfies the generalized Young inequality

$$AB \leq \bar{H}^*(A) + \bar{H}(B). \qquad (3.31)$$

Let $A = \bar{H}'\left(r_0\frac{E(t)}{E(0)}\right)$ and $B = \bar{H}^{-1}\left(\mu\frac{z(t)}{\xi(t)}\right)$, It follows from (2.12) and (3.29)-(3.31) that

$$\begin{aligned}
L_3'(t) &\leq -\beta E(t)\bar{H}'\left(r_0\frac{E(t)}{E(0)}\right) + C\bar{H}^*\left(\bar{H}'\left(r_0\frac{E(t)}{E(0)}\right)\right) + C\eta\frac{q(t)}{\xi(t)} + E'(t)\\
&\leq -\beta E(t)\bar{H}'\left(r_0\frac{E(t)}{E(0)}\right) + Cr_0\frac{E(t)}{E(0)}\bar{H}'\left(r_0\frac{E(t)}{E(0)}\right) + C\eta\frac{q(t)}{\xi(t)} + E'(t). \qquad (3.32)
\end{aligned}$$

Next, we multiply (3.32) by $\xi(t)$ and recall that $r_0\frac{E(t)}{E(0)} < r$ and

$$\bar{H}'\left(r_0\frac{E(t)}{E(0)}\right) = H'\left(r_0\frac{E(t)}{E(0)}\right),$$

we arrive at

$$\begin{aligned}\xi(t)L_3'(t) &\le -\beta\xi(t)E(t)H'\left(r_0\frac{E(t)}{E(0)}\right) + Cr_0\frac{E(t)}{E(0)}\xi(t)H'\left(r_0\frac{E(t)}{E(0)}\right) + C\eta q(t) + \xi(t)E'(t)\\ &\le -\beta\xi(t)E(t)H'\left(r_0\frac{E(t)}{E(0)}\right) + Cr_0\frac{E(t)}{E(0)}\xi(t)H'\left(r_0\frac{E(t)}{E(0)}\right) - CE'(t). \end{aligned}\tag{3.33}$$

Let $L_4(t) = \xi(t)L_3(t) + CE(t)$. Since $L_3 \sim E$, it follows that

$$b_0L_4(t) \le E(t) \le b_1L_4(t), \tag{3.34}$$

for some $b_0, b_1 > 0$. Thus (3.33) gives

$$L_4'(t) \le -(\beta E(0) - Cr_0)\xi(t)\frac{E(t)}{E(0)}\xi(t)H'\left(r_0\frac{E(t)}{E(0)}\right),\ \forall t \ge t_0.$$

We select $r_0 < r$ small enough so that $\beta E(0) - Cr_0 > 0$, we get

$$L_4'(t) \le -m\xi(t)\frac{E(t)}{E(0)}\xi(t)H'\left(r_0\frac{E(t)}{E(0)}\right) = -m\xi(t)H_2\left(\frac{E(t)}{E(0)}\right),\ \ \forall t \ge t_0, \tag{3.35}$$

for some constant $m > 0$ and $H_2(\tau) = \tau H'(r_0\tau)$. We note here that

$$H_2'(\tau) = H'(r_0\tau) + r_0tH''(r_0\tau),$$

thus the strict convexity of H on $(0, r]$, yields $H_2(\tau) > 0, H_2'(\tau) > 0$ on $(0, r]$. Let

$$F(t) = b_0\frac{L_4(t)}{E(0)}.$$

From (3.34) and (3.35), we obtain

$$F(t) \sim E(t) \tag{3.36}$$

and

$$F'(t) = a_0\frac{L_4'(t)(t)}{E(0)} \le -m_1\xi(t)H_2(F(t)),\ \forall t \ge t_0. \tag{3.37}$$

Integrating (3.37) over (t_0, t), we arrive at

$$m_1\int_{t_0}^t \xi(\tau)d\tau \le -\int_{t_0}^t \frac{F'(\tau)}{H_2(F(\tau))}d\tau = \frac{1}{r_0}\int_{r_0F(t)}^{r_0F(t_0)} \frac{1}{\tau H'(\tau)}d\tau. \tag{3.38}$$

This implies

$$F(t) \le \frac{1}{r_0}H_1^{-1}\left(\bar{m}_1\int_{t_0}^t \xi(\tau)d\tau\right),\quad \text{where}\ \ H_1(t) = \int_t^r \frac{1}{\tau H'(\tau)}d\tau. \tag{3.39}$$

Using the properties of H, we see easily that H_1 is strictly decreasing function on $(0, r]$ and

$$\lim_{t\longrightarrow 0} H_1(t) = +\infty.$$

Hence, (3.14) follows from (3.36) and (3.39). This completes the proof. □

Remark 3.1. *The stability result in* (3.1) *is general and optimal in the sense that it agrees with the decay rate of g, see [10], Remark* 2.3.

Corollary 3.2. *Suppose* $\frac{G}{\rho} = \frac{D}{I_\rho}$*, and* (G_1)*, and* (G_2) *hold. Let the function H in* (G_2) *be defined by*

$$H(\tau) = \tau^p, \quad 1 \le p < 2, \tag{3.40}$$

then the solution energy (2.11) *satisfies*

$$\begin{aligned} E(t) &\le a_2 \exp\left(-a_1 \int_0^t \xi(\tau)d\tau\right), \quad \text{for } p = 1, \\ E(t) &\le \frac{C}{\left(1 + \int_{t_0}^t \xi(\tau)d\tau\right)^{\frac{1}{p-1}}}, \quad \text{for } 1 < p < 2 \end{aligned} \tag{3.41}$$

for some positive constants a_2, a_1 *and C.*

3.2. Nonequal-wave-speed of propagation

In this subsection, we establish another stability result in the case non-equal speeds of wave propagation. To achieve this, we consider a stronger solution of (1.6). Let $(w, 3s - \psi, s, \theta)$ be the strong solution of problem (1.6)–(1.8), then differentiation of 1.6 with respect to t gives

$$\begin{cases} \rho w_{ttt} + G(\psi - w_x)_{xt} = 0, \\ I_\rho (3s - \psi)_{ttt} - D(3s - \psi)_{xxt} + \int_0^t g(\tau)(3s - \psi)_{xxt}(x, t - \tau)d\tau + g(t)(3s_0 - \psi_0)_{xx} - G(\psi - w_x)_t = 0 \\ 3I_\rho s_{ttt} - 3D s_{xxt} + 3G(\psi - w_x)_t + 4\gamma s_t + \delta\theta_{xt} = 0, \\ k\theta_{tt} - \lambda\theta_{xxt} + \delta s_{xtt} = 0, \end{cases} \tag{3.42}$$

where $(x, t) \in (0, 1) \times (0, +\infty)$ and $(3s - \psi)_{xx}(x, 0) = (3s_0 - \psi_0)_{xx}$. The modified energy functional associated to (3.42) is defined by

$$\begin{aligned} E_1(t) = &\frac{1}{2}\left[\rho\|w_{tt}\|_2^2 + 3I_\rho\|s_{tt}\|_2^2 + I_\rho\|3s_{tt} - \psi_{tt}\|_2^2 + 3D\|s_{xt}\|_2^2 + G\|\psi_t - w_{xt}\|_2^2\right] \\ &+ \frac{1}{2}\left[4\gamma\|s_t\|_2^2 + k\|\theta_t\|_2^2 + \left(D - \int_0^t g(\tau)d\tau\right)\|3s_{xt} - \psi_{xt}\|_2^2 + (g \diamond (3s_{xt} - \psi_{xt}))(t)\right]. \end{aligned} \tag{3.43}$$

Lemma 3.2. *Let* $(w, 3s - \psi, s, \theta)$ *be the strong solution of problem* (1.6)-(1.8)*. Then, the energy functional* (3.43) *satisfies, for all* $t \ge 0$

$$E_1'(t) = \frac{1}{2}(g' \diamond (3s_{xt} - \psi_{xt}))(t) - \frac{1}{2}g(t)\|3s_{xt} - \psi_{xt}\|_2^2 - g(t)\langle(3s_{tt} - \psi_{tt}), (3s_0 - \psi_0)_{xx}\rangle - \lambda\|\theta_{xt}\|_2^2 \tag{3.44}$$

and

$$E_1(t) \le C\left(E_1(0) + \|(3s_0 - \psi_0)_{xx}\|_2^2\right). \tag{3.45}$$

Proof. The proof of (3.44) follows the same steps as in the proof of Lemma 2.5. From (3.44), it is obvious that

$$E_1'(t) \leq -g(t)\langle(3s_{tt} - \psi_{tt}), (3s_0 - \psi_0)_{xx}\rangle.$$

So, using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} E_1'(t) &\leq \frac{I_\rho g(t)}{2}\|3s_{tt} - \psi_{tt}\|_2^2 + \frac{g(t)}{2I_\rho}\|(3s_0 - \psi_0)_{xx}\|_2^2 \\ &\leq g(t)E_1(t) + \frac{g(t)}{2I_\rho}\|(3s_0 - \psi_0)_{xx}\|_2^2. \end{aligned} \tag{3.46}$$

This implies

$$\frac{d}{dt}\left(E_1(t)e^{-\int_0^t g(\tau)d\tau}\right) \leq e^{-\int_0^t g(\tau)d\tau}\frac{g(t)}{2I_\rho}\|(3s_0 - \psi_0)_{xx}\|_2^2 \leq \frac{g(t)}{2I_\rho}\|(3s_0 - \psi_0)_{xx}\|_2^2 \tag{3.47}$$

Integrating (3.47) over $(0, t)$ yields

$$\begin{aligned} E_1(t)e^{-\int_0^{+\infty} g(\tau)d\tau} &\leq E_1(t)e^{-\int_0^t g(\tau)d\tau} \\ &\leq E_1(0) + \frac{1}{2I_\rho}\left(\int_0^t g(\tau)d\tau\right)\|(3s_0 - \psi_0)_{xx}\|_2^2 \\ &\leq E_1(0) + \frac{1}{2I_\rho}\left(\int_0^{+\infty} g(\tau)d\tau\right)\|(3s_0 - \psi_0)_{xx}\|_2^2. \end{aligned} \tag{3.48}$$

Hence, (3.45) follows. □

Remark 3.2. *Using Young's inequality, we observe from* (3.44) *and* (3.45) *that*

$$\begin{aligned} \lambda\|\theta_{xt}\|_2^2 &= -E_1'(t) + \frac{1}{2}\left(g' \diamond (3s_{xt} - \psi_{xt})\right)(t) - \frac{1}{2}g(t)\|3s_{xt} - \psi_{xt}\|_2^2 - g(t)\langle(3s_{tt} - \psi_{tt}), (3s_0 - \psi_0)_{xx}\rangle \\ &\leq -E_1'(t) - g(t)\langle(3s_{tt} - \psi_{tt}), (3s_0 - \psi_0)_{xx}\rangle \\ &\leq -E_1'(t) + g(t)\left(\|3s_{tt} - \psi_{tt}\|_2^2 + \|(3s_0 - \psi_0)_{xx}\|_2^2\right) \\ &\leq -E_1'(t) + g(t)\left(\frac{2}{I_\rho}E_1(t) + \|(3s_0 - \psi_0)_{xx}\|_2^2\right) \\ &\leq C\left(-E_1'(t) + c_1 g(t)\right) \end{aligned} \tag{3.49}$$

for some fixed positive constant c_1. Similarly, we obtain

$$0 \leq -\left(g' \diamond (3s_{xt} - \psi_{xt})\right)(t) \leq C\left(-E_1'(t) + c_1 g(t)\right). \tag{3.50}$$

As in the case of equal-wave-speed of propagation, we define a Lyapunov functional

$$\tilde{L}(t) = \tilde{N}E(t) + \sum_{j=1}^{6}\tilde{N}_j I_j(t) + \tilde{N}_6 I_8(t), \tag{3.51}$$

where $\tilde{N},\ \tilde{N}_j,\ j = 1,2,3,4,5,6,$ are positive constants to be specified later and

$$I_8(t) = \frac{3\lambda}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_x w_x dx.$$

Lemma 3.3. *Suppose* $\frac{G}{\rho} \neq \frac{D}{I_\rho}$. *Then, under suitable choice of* $\tilde{N},\ \tilde{N}_j,\ j = 1,2,3,4,5,6,$ *the Lyapunov functional* $\tilde{L}$ *satisfies, along the solution of* (1.6)*, the estimate*

$$\tilde{L}'(t) \leq -\tilde{\beta}E(t) + \frac{1}{2}\left(g \diamond (3s_x - \psi_x)\right)(t) + C\left(-E_1'(t) + c_1 g(t)\right), \forall\, t \geq t_0, \tag{3.52}$$

for some positive constants $\tilde{\beta}$ *and* c_1.

Proof. Following the proof of Lemma 3.1, we end up with (3.52). □

Lemma 3.4. *Suppose assumptions* $(G1)$ *and* $(G2)$ *hold and the function* H *in* $(G2)$ *is linear. Let* $(w, 3s - \psi, s, \theta)$ *be the strong solution of problem* (1.6)–(1.8)*. Then,*

$$\xi(t)(g \diamond (3s_{xt} - \psi_{xt}))(t) \leq C\left(-E_1'(t) + c_1 g(t)\right),\ \forall\, t \geq 0, \tag{3.53}$$

where c_1 *is a fixed positive constant.*

Proof. Using (3.50) and the fact that ξ is decreasing, we have

$$\begin{aligned}
&\xi(t)(g \diamond (3s_{xt} - \psi_{xt}))(t)\\
&\quad= \xi(t)\int_0^t g(t-\tau)\left(\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(\tau)\|_2^2\right)d\tau\\
&\quad\leq \int_0^t \xi(t-\tau)g(t-\tau)\left(\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(\tau)\|_2^2\right)d\tau\\
&\quad\leq -\int_0^t g'(t-\tau)\left(\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(\tau)\|_2^2\right)d\tau\\
&\quad= -\left(g' \diamond (3s_{xt} - \psi_{xt})\right)(t)\\
&\quad\leq C\left(-E_1'(t) + c_1 g(t)\right).
\end{aligned} \tag{3.54}$$

□

Our stability result of this subsection is

Theorem 3.3. *Assume (G1) and (G2) hold and* $\frac{G}{\rho} \neq \frac{D}{I_\rho}$. *Then, there exist positive constants* a_1, a_2 *and* $t_2 > t_0$ *such that the energy solution* (2.11) *satisfies*

$$E(t) \leq a_2(t - t_0)H_2^{-1}\left(\frac{a_1}{(t-t_0)\int_{t_2}^t \xi(\tau)d\tau}\right), \forall t > t_2, \quad \text{where}\ \ H_2(\tau) = \tau H'(\tau). \tag{3.55}$$

Proof. **Case 1: H is linear.** Multiplying (3.52) by $\xi(t)$ and using $(G1)$, we get

$$\begin{aligned}\xi(t)\tilde{L}'(t) &\le -\tilde{\beta}\xi(t)E(t) + \frac{1}{2}\xi(t)\left(g \diamond (3s_x - \psi_x)\right)(t) + C\xi(t)\left(-E_1'(t) + c_1 g(t)\right)\\ &\le -\tilde{\beta}\xi(t)E(t) - CE'(t) - C\xi(0)E_1'(t) + \xi(0)c_1 g(t),\ \forall\, t \ge t_0\end{aligned}$$

Using the fact that ξ non-increasing, we obtain

$$\left(\xi\tilde{L} + CE + E_1\right)'(t) \le -\tilde{\beta}\xi(t)E(t) + c_2 g(t),\ \forall\, t \ge t_0.$$

for some fixed positive constant c_2. This implies

$$\tilde{\beta}\xi(t)E(t) \le -\left(\xi\tilde{L} + CE + E_1\right)'(t) + c_2 g(t),\ \forall\, t \ge t_0. \tag{3.56}$$

Integrating (3.56) over (t_0, t), using the fact that E is non-increasing and the inequality (3.45), we arrive at

$$\begin{aligned}\tilde{\beta}E(t)\int_{t_0}^{t}\xi(\tau)d\tau &\le \tilde{\beta}\int_{t_0}^{t}\xi(\tau)E(\tau)d\tau\\ &\le -\left(\xi\tilde{L} + CE + E_1\right)(t) + \left(\xi\tilde{L} + CE + E_1\right)(t_0) + c_2\int_{t_0}^{t} g(\tau)d\tau\\ &\le \left(\xi\tilde{L} + CE + E_1\right)(0) + C\|(3s_0 - \psi_0)_{xx}\|_2^2 + c_2\int_0^{\infty} g(\tau)d\tau\\ &= \left(\xi\tilde{L} + CE + E_1\right)(0) + C\|(3s_0 - \psi_0)_{xx}\|_2^2 + c_2(D - l_0).\end{aligned} \tag{3.57}$$

Thus, we have

$$E(t) \le \frac{C}{\displaystyle\int_{t_0}^{t}\xi(\tau)d\tau},\ \forall\, t \ge t_0. \tag{3.58}$$

Case II: H is nonlinear. First, we observe from (3.52) that

$$\begin{aligned}\tilde{L}'(t) &\le -\tilde{\beta}E(t) + \frac{1}{2}\left(g \diamond (3s_x - \psi_x)\right)(t) + C\left(-E_1'(t) + c_1 g(t)\right)\\ &\le -\tilde{\beta}E(t) + C\left(\left(g \diamond (3s_x - \psi_x)\right)(t) + \left(g \diamond (3s_{xt} - \psi_{xt})\right)(t)\right) + C\left(-E_1'(t) + c_1 g(t)\right),\ \forall\, t \ge t_0.\end{aligned} \tag{3.59}$$

From (2.12), (3.16) and (3.50), we have for any $t \ge t_0$

$$\begin{aligned}&\int_0^{t_0} g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau + \int_0^{t_0} g(\tau)\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau\\ &\le \frac{1}{\xi(t_0)}\int_0^{t_0}\xi(\tau)g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau\\ &\quad + \frac{1}{\xi(t_0)}\int_0^{t_0}\xi(\tau)g(\tau)\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau\\ &\le -\frac{g(0)}{a\xi(t_0)}\int_0^{t_0} g'(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau\\ &\quad - \frac{g(0)}{a\xi(t_0)}\int_0^{t_0} g'(\tau)\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau\\ &\le -C\left(E'(t) + E_1'(t)\right) + c_2 g(t),\end{aligned} \tag{3.60}$$

where c_2 is a fixed positive constant. Substituting (3.60) into (3.59), we obtain for any $t \geq t_0$

$$\begin{aligned}\tilde{L}'(t) \leq & -\tilde{\beta}E(t) - C\left(E'(t) + E_1'(t)\right) + c_3 g(t) + C\int_{t_0}^{t} g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\ & + C\int_{t_0}^{t} g(\tau)\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau,\end{aligned} \tag{3.61}$$

where c_3 is a fixed positive constant. Now, we define the functional Φ by

$$\begin{aligned}\Phi(t) =& \frac{\sigma}{t-t_0}\int_{t_0}^{t} \|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\ & + \frac{\sigma}{t-t_0}\int_{t_0}^{t} \|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau, \ \forall\, t > t_0.\end{aligned} \tag{3.62}$$

Using (2.11), (2.12), (3.43) and (3.45), we easily get

$$\begin{aligned}\frac{1}{t-t_0}&\int_{t_0}^{t} \|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau + \frac{1}{t-t_0}\int_{t_0}^{t} \|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau \\ &\leq \frac{2}{t-t_0}\int_{t_0}^{t} \left(\|(3s_x - \psi_x)(t)\|_2^2 + \|(3s_x - \psi_x)(t-\tau)\|_2^2\right) d\tau \\ &\quad + \frac{2}{t-t_0}\int_{t_0}^{t} \left(\|(3s_{xt} - \psi_{xt})(t)\|_2^2 + \|(3s_{xt} - \psi_{xt})(t-\tau)\|_2^2\right) d\tau \\ &\leq \frac{4}{l_0(t-t_0)}\int_{t_0}^{t} \left(E(t) + E(t-\tau) + E_1(t) + E_1(t-\tau)\right) d\tau \\ &\leq \frac{8}{l_0(t-t_0)}\int_{t_0}^{t} \left(E(0) + C\left(E_1(0) + \|(3s_0 - \psi_0)_{xx}\|_2^2\right)\right) d\tau \\ &\leq \frac{8}{l_0}\left(E(0) + C\left(E_1(0) + \|(3s_0 - \psi_0)_{xx}\|_2^2\right)\right) < \infty, \ \forall\, t > t_0.\end{aligned} \tag{3.63}$$

This last inequality allows us to choose $0 < \sigma < 1$ such that

$$\Phi(t) < 1, \ \ \forall\, t > t_0. \tag{3.64}$$

Hence forth, we assume $\Phi(t) > 0$, otherwise, we get immediately from (3.61)

$$E(t) \leq \frac{C}{t-t_0}, \ \ \forall\, t > t_0.$$

Next, we define the functional μ by

$$\begin{aligned}\mu(t) =& -\int_{t_0}^{t} g'(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\ & - \int_{t_0}^{t} g'(\tau)\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau\end{aligned} \tag{3.65}$$

and observe that

$$\mu(t) \leq -C\left(E'(t) + E_1'(t)\right) + c_4 g(t), \quad \forall\, t > t_0, \tag{3.66}$$

where c_4 is a fixed positive constant. The fact that H is strictly convex and $H(0) = 0$ implies

$$H(\nu\tau) \le \nu H(\tau), \quad 0 \le \nu \le 1 \text{ and } \tau \in (0, r]. \tag{3.67}$$

Using assumption $(G1)$, (3.67), Jensen's inequality and (3.64), we get for any $t > t_0$

$$\begin{aligned}
\mu(t) =& -\frac{1}{\Phi(t)}\int_{t_0}^{t} \Phi(t)g'(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\
& - \frac{1}{\Phi(t)}\int_{t_0}^{t} \Phi(t)g'(\tau)\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau \\
\ge& \frac{1}{\Phi(t)}\int_{t_0}^{t} \Phi(t)\xi(\tau)H(g(\tau))\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\
& + \frac{1}{\Phi(t)}\int_{t_0}^{t} \Phi(t)\xi(\tau)H(g(\tau))\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau \\
\ge& \frac{\xi(t)}{\Phi(t)}\int_{t_0}^{t} H(\Phi(t)g(\tau))\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau \\
& + \frac{\xi(t)}{\Phi(t)}\int_{t_0}^{t} H(\Phi(t)g(\tau))\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau \\
\ge& \frac{\xi(t)(t-t_0)}{\sigma} H\left(\frac{\sigma}{t-t_0}\int_{t_0}^{t} g(\tau)(\Omega_1(t-\tau) + \Omega_2(t-\tau))d\tau\right) \\
=& \frac{\xi(t)(t-t_0)}{\sigma} \bar{H}\left(\frac{\sigma}{t-t_0}\int_{t_0}^{t} (\Omega_1(t-\tau) + \Omega_2(t-\tau))\, d\tau\right),
\end{aligned} \tag{3.68}$$

where

$$\begin{aligned}
\Omega_1(t-\tau) &= \|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2, \\
\Omega_2(t-\tau) &= \|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2
\end{aligned}$$

and $\bar{H}$ is the C^2- strictly increasing and convex extension of H on $(0, +\infty)$. This implies

$$\begin{aligned}
\int_{t_0}^{t} g(\tau)\|(3s_x - \psi_x)(t) - (3s_x - \psi_x)(t-\tau)\|_2^2 d\tau + \int_{t_0}^{t} g(\tau)\|(3s_{xt} - \psi_{xt})(t) - (3s_{xt} - \psi_{xt})(t-\tau)\|_2^2 d\tau \\
\le \frac{(t-t_0)}{\sigma}\bar{H}^{-1}\left(\frac{\sigma\mu(t)}{\xi(t)(t-t_0)}\right), \ \forall\, t > t_0.
\end{aligned} \tag{3.69}$$

Thus, the inequality (3.61) becomes

$$\tilde{L}'(t) \le -\tilde{\beta}E(t) - C\left(E'(t) + E_1'(t)\right) + c_3 g(t) + \frac{C(t-t_0)}{\sigma}\bar{H}^{-1}\left(\frac{\sigma\mu(t)}{\xi(t)(t-t_0)}\right), \ \forall\, t > t_0. \tag{3.70}$$

Let $\tilde{L}_1(t) := \tilde{L}(t) + C\left(E(t) + E_1(t)\right)$. Then (3.70) becomes

$$\tilde{L}_1'(t) \le -\tilde{\beta}E(t) + \frac{C(t-t_0)}{\sigma}\bar{H}^{-1}\left(\frac{\sigma\mu(t)}{\xi(t)(t-t_0)}\right) + c_3 g(t), \ \forall\, t > t_0. \tag{3.71}$$

For $0 < r_1 < r$, we define the functional $\tilde{L}_2$ by

$$\tilde{L}_2(t) := \bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\tilde{L}_1(t),, \ \forall\, t > t_0. \tag{3.72}$$

From (3.71) and the fact that

$$E'(t) \leq 0, \ \bar{H}'(t) > 0, \ \bar{H}''(t) > 0,$$

we obtain, for all $t > t_0$,

$$\begin{aligned}\tilde{L}_2'(t) =& \left(-\frac{r_1}{(t-t_0)^2}.\frac{E(t)}{E(0)} + \frac{r_1}{t-t_0}.\frac{E'(t)}{E(0)}\right)\bar{H}''\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\tilde{L}_1(t) + \bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\tilde{L}_1'(t)\\ \leq& -\tilde{\beta}E(t)\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + c_3g(t)\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + \frac{C(t-t_0)}{\sigma}\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\bar{H}^{-1}\left(\frac{\sigma\mu(t)}{\xi(t)(t-t_0)}\right).\end{aligned} \tag{3.73}$$

Let $\bar{H}^*$ be the convex conjugate of $\bar{H}$ as in (3.30) and let

$$A = \bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) \ \text{ and } \ B = \bar{H}^{-1}\left(\frac{\sigma\mu(t)}{\xi(t)(t-t_0)}\right).$$

Then, (3.30), (3.31) and (3.73) yield, for all $t > t_0$,

$$\begin{aligned}\tilde{L}_2'(t) \leq& -\tilde{\beta}E(t)\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + c_3g(t)\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\\ &+ \frac{C(t-t_0)}{\sigma}\bar{H}^*\left(\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\right) + \frac{C(t-t_0)}{\sigma}.\frac{\sigma\mu(t)}{\xi(t)(t-t_0)}\\ \leq& -\tilde{\beta}E(t)\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + c_3g(t)\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + Cr_1\frac{E(t)}{E(0)}\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + C\frac{\mu(t)}{\xi(t)}\\ \leq& -(\tilde{\beta}E(0) - Cr_1)\frac{E(t)}{E(0)}\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + C\frac{\mu(t)}{\xi(t)} + c_3g(t)\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\end{aligned} \tag{3.74}$$

By selecting r_1 small enough so that $(\tilde{\beta}E(0) - Cr_1) > 0$, we arrive at

$$\tilde{L}_2'(t) \leq -\tilde{\beta}_2\frac{E(t)}{E(0)}\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + C\frac{\mu(t)}{\xi(t)} + c_3g(t)\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right), \ \forall\, t > t_0, \tag{3.75}$$

for some positive constant $\tilde{\beta}_2$.

Now, multiplying (3.75) by $\xi(t)$ and recalling that $r_1\frac{E(t)}{E(0)} < r$, we arrive at

$$\begin{aligned}\xi(t)\tilde{L}_2'(t) \leq& -\tilde{\beta}_2\xi(t)\frac{E(t)}{E(0)}\bar{H}'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + C\mu(t) + c_3g(t)\xi(t)H'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\\ \leq& -\tilde{\beta}_2\xi(t)\frac{E(t)}{E(0)}H'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) - C(E'(t) + E_1'(t)) + c_4g(t) + c_3g(t)H'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right), \ \forall\, t > t_0.\end{aligned} \tag{3.76}$$

Since $\frac{r_1}{t-t_0} \longrightarrow 0$ as $t \longrightarrow \infty$, there exists $t_2 > t_0$ such that $\frac{r_1}{t-t_0} < r_1$, whenever $t > t_2$. Using this fact and observing that H' strictly increasing, and E and ξ are non-decreasing, we get

$$H'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) \leq H'(r_1), \ \forall\, t > t_2. \tag{3.77}$$

Using (3.77), it follows from (3.76) that

$$\tilde{L}_3'(t) \leq -\tilde{\beta}_2\xi(t)\frac{E(t)}{E(0)}H'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) + c_5 g(t), \ \forall\, t > t_2, \tag{3.78}$$

where $\tilde{L}_3 = (\xi\tilde{L}_2 + CE + CE_1)$ and $c_5 > 0$ is a constant. Using the non-increasing property of ξ, we have

$$\tilde{\beta}_2\xi(t)\frac{E(t)}{E(0)}H'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right) \leq -\tilde{L}_3'(t) + c_5 g(t), \ \forall\, t > t_2. \tag{3.79}$$

Using the fact that E is non-increasing and $H'' > 0$ we conclude that the map

$$t \longmapsto E(t)H'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)$$

is non-increasing. Therefore, integrating (3.79) over (t_2, t) yields

$$\begin{aligned} \tilde{\beta}_2\frac{E(t)}{E(0)}H'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\int_{t_2}^{t}\xi(\tau)d\tau &\leq \tilde{\beta}_2\int_{t_2}^{t}\xi(\tau)\frac{E(\tau)}{E(0)}H'\left(\frac{r_1}{\tau-t_0}.\frac{E(\tau)}{E(0)}\right)d\tau \\ &\leq -\tilde{L}_3(t) + \tilde{L}_3(t_2) + c_5\int_{t_2}^{t}g(\tau)d\tau \\ &\leq \tilde{L}_3(t_2) + c_5\int_{0}^{\infty}g(\tau)d\tau \\ &= \tilde{L}_3(t_2) + c_5(b-l_0), \ \forall\, t > t_2. \end{aligned} \tag{3.80}$$

Next, we multiply both sides of (3.80) by $\frac{1}{t-t_0}$, for $t > t_2$, we get

$$\frac{\tilde{\beta}_2}{(t-t_0)}.\frac{E(t)}{E(0)}H'\left(\frac{r_1}{t-t_0}.\frac{E(t)}{E(0)}\right)\int_{t_2}^{t}\xi(\tau)d\tau \leq \frac{\tilde{L}_3(t_2) + c_5(b-l_0)}{t-t_0}, \ \forall\, t > t_2. \tag{3.81}$$

Since H' is strictly increasing, then $H_2(\tau) = \tau H'(\tau)$ is a strictly increasing function. It follows from (3.81) that

$$E(t) \leq a_2(t-t_0)H_2^{-1}\left(\frac{a_1}{(t-t_0)\displaystyle\int_{t_2}^{t}\xi(\tau)d\tau}\right), \ \forall\, t > t_2.$$

for some positive constants a_1 and a_2. This completes the proof. □

3.3. Examples

(1). Let $g(t) = ae^{-bt},\ t \geq 0,\ \ a,\ b > 0$ are constants and a is chosen such that (G_1) holds. Then

$$g'(t) = -abe^{-bt} = -bH(g(t)) \ \text{ with } \ H(t) = t.$$

Therefore, from (3.14), the energy function (2.11) satisfies

$$E(t) \leq a_2 e^{-\alpha t},\ \forall\ t \geq 0,\ \text{where}\ \ \alpha = ba_1. \tag{3.82}$$

Also, for $H_2(\tau) = \tau$, it follows from (3.55) that, there exists $t_2 > 0$ such that the energy function (2.11) satisfies

$$E(t) \leq \frac{C}{t - t_2},\ \ \forall\ t > t_2, \tag{3.83}$$

for some positive constant C.

(2). Let $g(t) = ae^{-(1+t)^b},\ t \geq 0,\ \ a > 0,\ 0 < b < 1$ are constants and a is chosen such that (G_1) holds. Then,

$$g'(t) = -ab(1+t)^{b-1}e^{-(1+t)^b} = -\xi(t)H(g(t)),$$

where $\xi(t) = b(1+t)^{b-1}$ and $H(t) = t$. Thus, we get from (3.14) that

$$E(t) \leq a_2 e^{-a_1(1+t)^b},\ \forall\ t \geq 0. \tag{3.84}$$

Likewise, for $H_2(t) = t$, then estimate (3.55) implies there exists $t_2 > 0$ such that the energy function (2.11) satisfies

$$E(t) \leq \frac{C}{(1+t)^b},\ \ \forall\ t > t_2, \tag{3.85}$$

for some positive constant C.

(3). Let $g(t) = \frac{a}{(1+t)^b},\ t \geq 0,\ \ a > 0,\ b > 1$ are constants and a is chosen in such a way that (G_1) holds. We have

$$g'(t) = \frac{-ab}{(1+t)^{b+1}} = -\xi\left(\frac{a}{(1+t)^b}\right)^{\frac{b+1}{b}} = -\xi g^q(t) = -\xi H(g(t)),$$

where

$$H(t) = t^q,\ \ q = \frac{b+1}{b}\ \text{ satisfying }\ 1 < q < 2\ \text{ and }\ \xi = \frac{b}{a^{\frac{1}{b}}} > 0.$$

Hence, we deduce from (3.41) that

$$E(t) \leq \frac{C}{(1+t)^b},\ \forall\ t \geq 0. \tag{3.86}$$

Furthermore, for $H_2(t) = qt^q$, estimate (3.55) implies there exists $t_2 > 0$ such that the energy function (2.11) satisfies

$$E(t) \leq \frac{C}{(1+t)^{(b-1)/(b+1)}},\ \forall\ t > t_2, \tag{3.87}$$

for some positive constant C.

4. Appendix

In this section, we prove the functionals $L_i, i = 1 \cdots 8$, used in the proof of our stability results.

Lemma 4.1. *The functional* $I_1(t)$ *satisfies, along the solution of* (1.6) − (1.8), *for all* $t \geq t_0 > 0$ *and for any* $\epsilon_1, \epsilon_2 > 0$, *the estimate*

$$I_1'(t) \leq -\frac{I_\rho g_0}{2}\|3s_t - \psi_t\|_2^2 + \epsilon_1\|3s_x - \psi_x\|_2^2 + \epsilon_2\|\psi - w_x\|_2^2 + CC_\alpha\left(1 + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right)(h \diamond (3s_x - \psi_x))(t), \tag{4.1}$$

where $g_0 = \int_0^{t_0} g(\tau)d\tau \leq \int_0^t g(\tau)d\tau.$

Proof. Differentiating $I_1(t)$, using $(1.6)_2$ and integrating by part, we have

$$\begin{aligned}
I_1'(t) = &-I_\rho \int_0^1 (3s_t - \psi_t) \int_0^t g'(t-\tau)\left((3s-\psi)(t) - (3s-\psi)(\tau)\right) d\tau dx \\
&+ D(t) \int_0^1 (3s_x - \psi_x) \int_0^t g(t-\tau)\left((3s_x-\psi_x)(t) - (3s_x-\psi_x)(\tau)\right) d\tau dx \\
&+ \int_0^1 \left(\int_0^t g(t-\tau)\left((3s_x-\psi_x)(t) - (3s_x-\psi_x)(\tau)\right) d\tau\right)^2 dx - I_\rho \left(\int_0^t g(\tau)d\tau\right)\int_0^1 (3s_t - \psi_t)^2 dx \\
&- G\int_0^1 (\psi - w_x) \int_0^t g(t-\tau)\left((3s-\psi)(t) - (3s-\psi)(\tau)\right) d\tau dx,
\end{aligned} \tag{4.2}$$

where $D(t) = \left(D - \int_0^t g(\tau)d\tau\right)$. Now, we estimate the terms on the right hand-side of (4.2). Exploiting Young's and Poincaré's inequalities, Lemmas 2.1- 2.6 and performing similar computations as in (2.8), we have for any $\epsilon_1 > 0$,

$$\begin{aligned}
D(t) \int_0^1 (3s_x - \psi_x) \int_0^t g(t-\tau)\left((3s_x-\psi_x)(t) - (3s_x-\psi_x)(\tau)\right) d\tau dx \\
\leq \epsilon_1 \|(3s_x - \psi_x\|_2^2 + \frac{CC_\alpha}{\epsilon_1}\left(h \diamond (3s_x - \psi_x)\right)(t)
\end{aligned} \tag{4.3}$$

and

$$\int_0^1 \left(\int_0^t g(t-\tau)\left((3s_x-\psi_x)(t) - (3s_x-\psi_x)(\tau)\right) d\tau\right)^2 dx \leq C_\alpha \left(h \diamond (3s_x - \psi_x)\right)(t). \tag{4.4}$$

Also, for $\delta_1 > 0$, we have

$$\begin{aligned}
&-I_\rho \int_0^1 (3s_t - \psi_t) \int_0^t g'(t-\tau)\left((3s-\psi)(t) - (3s-\psi)(\tau)\right) d\tau dx \\
&= I_\rho \int_0^1 (3s_t - \psi_t) \int_0^t h(t-\tau)\left((3s-\psi)(t) - (3s-\psi)(\tau)\right) d\tau dx \\
&\quad - I_\rho \alpha \int_0^1 (3s_t - \psi_t) \int_0^t g(t-\tau)\left((3s-\psi)(t) - (3s-\psi)(\tau)\right) d\tau dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \delta_1\|3s_t-\psi_t\|_2^2+\frac{I_\rho^2}{2\delta_1}\int_0^1\left(\int_0^t h(t-\tau)\left((3s-\psi)(t)-(3s-\psi)(\tau)\right)d\tau\right)^2 dx\\
&\quad+\frac{\alpha^2 I_\rho^2}{2\delta_1}\int_0^1\left(\int_0^t g(t-\tau)\left((3s-\psi)(t)-(3s-\psi)(\tau)\right)d\tau\right)^2 dx\\
&\leq \delta_1\|3s_t-\psi_t\|_2^2+\frac{I_\rho^2}{2\delta_1}\left(\int_0^t h(\tau)d\tau\right)\left(h\diamond(3s-\psi)\right)(t)+\frac{\alpha^2 I_\rho^2 C_\alpha}{2\delta_1}\left(h\diamond(3s-\psi)\right)(t)\\
&\leq \delta_1\|3s_t-\psi_t\|_2^2+\frac{C(C_\alpha+1)}{\delta_1}\left(h\diamond(3s-\psi)_x\right)(t). \qquad (4.5)
\end{aligned}
$$

For the last term, we have

$$
-G\int_0^1(\psi-w_x)\int_0^t g(t-\tau)\left((3s-\psi)(t)-(3s-\psi)(\tau)\right)d\tau dx\leq \epsilon_2\|\psi-w_x\|_2^2+\frac{G^2C_\alpha}{4\epsilon_2}\left(h\diamond(3s-\psi)_x\right)(t). \qquad (4.6)
$$

Combination of (4.2)–(4.6) lead to

$$
\begin{aligned}
I_1'(t)\leq &-\left(I_\rho\int_0^t g(\tau)d\tau-\delta_1\right)\|3w_t-\psi_t\|_2^2+\epsilon_1\|3s_x-\psi_x\|_2^2+\epsilon_2\|\psi-w_x\|_2^2\\
&+CC_\alpha\left(1+\frac{1}{\delta_1}+\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right)\left(h\diamond(3s_x-\psi_x)\right)(t). \qquad (4.7)
\end{aligned}
$$

Since $g(0)>0$ and g is continuous. Thus for any $t\geq t_0>0$, we get

$$
\int_0^t g(\tau)d\tau\geq\int_0^{t_0}g(\tau)d\tau=g_0>0. \qquad (4.8)
$$

We select $\delta_1=\dfrac{I_\rho g_0}{2}$ to get (4.1). □

Lemma 4.2. *The functional $I_2(t)$ satisfies, along the solution of* (1.6) − (1.8) *and for any $\delta_4>0$, the estimate*

$$
I_2'(t)\leq -3D\|s_x\|_2^2-3\gamma\|s\|_2^2+\delta_4\|w_t\|_2^2+C\left(1+\frac{1}{\delta_4}\right)\|s_t\|_2^2+C\|\theta_x\|_2^2,\quad \forall t\geq 0. \qquad (4.9)
$$

Proof. Differentiation of $I_2(t)$, using $(1.6)_1$ and $(1.6)_3$ and integration by part, leads to

$$
I_2'(t)=3I_\rho\|s_t\|_2^2-3D\|s_x\|_2^2-4\gamma\|s\|_2^2-\delta\int_0^t s\theta_x dx+3\rho\int_0^1 w_t\int_0^x s_t(y)dydx.
$$

Applying Cauchy-Schwarz and Young's inequalities and (2.5), we get for any $\delta_4>0$,

$$
\begin{aligned}
I_2'(t)&\leq 3I_\rho\|s_t\|_2^2-3D\|s_x\|_2^2-4\gamma\|s\|_2^2+\gamma\|s\|_2^2+\frac{\delta^2}{4\gamma}\|\theta_x\|_2^2+\delta_4\|w_t\|_2^2+\frac{9\rho^2}{4\delta_4}\int_0^1\left(\int_0^x s_t(y)dy\right)^2 dx\\
&\leq -3D\|s_x\|_2^2-3\gamma\|s\|_2^2+\delta_4\|w_t\|_2^2+C\left(1+\frac{1}{\delta_4}\right)\|s_t\|_2^2+C\|\theta_x\|_2^2.
\end{aligned}
$$

This completes the proof. □

Lemma 4.3. *The functional $I_3(t)$ satisfies, along the solution of* (1.6) − (1.8) *and for any* $\epsilon_3 > 0$, *the estimate*

$$I_3'(t) \leq -\frac{\delta I_\rho}{2}\|s_t\|_2^2 + \epsilon_3\|s_x\|_2^2 + \epsilon_3\|\psi - w_x\|_2^2 + C\left(1 + \frac{1}{\epsilon_3}\right)\|\theta_x\|_2^2, \quad \forall t \geq 0. \tag{4.10}$$

Proof. Differentiation of I_3, using $(1.6)_3$, $(1.6)_4$ and integration by parts, yields

$$I_3'(t) = 3\lambda I_\rho \int_0^1 \theta_x s_t dx - 3I_\rho\delta\|s_t\|_2^2 - 3kD\int_0^1 \theta s_x dx + k\delta\|\theta\|_2^2$$
$$+ 3kG\int_0^1 \theta \int_0^x (\psi - w_y)(y)dydx + 4\gamma k\int_0^1 \theta \int_0^t s(y)dydx.$$

Using Cauchy-Schwarz, Young's and Poincaré's inequalities together with Lemmas 2.1– 2.6, we have

$$\begin{aligned} I_3'(t) \;&\leq\; \delta_2\|s_t\|_2^2 + C_{\delta_2}\|\theta_x\|_2^2 - 3I_\rho\delta\|s_t\|_2^2 + \frac{\epsilon_3}{2}\|s_x\|_2^2 + C\left(1 + \frac{1}{\epsilon_3}\right)\|\theta\|_2^2 \\ &\quad +\epsilon_3\int_0^1\left(\int_0^x (\psi - w_y)(y)dy\right)^2 dx + \frac{\epsilon_3}{2}\int_0^1\left(\int_0^x s(y)dy\right)^2 dx \\ &\leq\; \delta_2\|s_t\|_2^2 + C_{\delta_2}\|\theta_x\|_2^2 - 3I_\rho\delta\|s_t\|_2^2 + \epsilon_3\|s_x\|_2^2 + \epsilon_3\|\psi - w_x\|_2^2 + C\left(1 + \frac{1}{\epsilon_3}\right)\|\theta_x\|_2^2. \end{aligned}$$

We choose $\delta_2 = \dfrac{5I_\rho\delta}{2}$ to get (4.10). □

Lemma 4.4. *The functional $I_4(t)$ satisfies, along the solution of* (1.6) − (1.8) *and for any* $\epsilon_4 > 0$, *the estimate*

$$I_4'(t) \leq -\rho\|w_t\|_2^2 + \epsilon_4\|3s_x - \psi_x\|_2^2 + C\|s_x\|_2^2 + C_{\epsilon_4}\|\psi - w_x\|_2^2, \quad \forall t \geq 0. \tag{4.11}$$

Proof. Using $(1.6)_1$ and integration by parts, we have

$$I_4'(t) = -\rho\|w_t\|_2^2 - G\int_0^1 (\psi - w_x)w_x dx.$$

We note that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$ to arrive at

$$I_4'(t) = -\rho\|w_t\|_2^2 + G\|\psi - w_x\|_2^2 + G\int_0^1 (\psi - w_x)(3s - \psi)dx - 3G\int_0^1 (\psi - w_x)s dx.$$

It follows from Young's and Poincaré's inequalities that

$$\begin{aligned} I_4'(t) &\leq -\rho\|w_t\|_2^2 + G\|\psi - w_x\|_2^2 + \epsilon_4\|3s - \psi\|_2^2 + \frac{C}{\epsilon_4}\|\psi - w_x\|_2^2 + \frac{3G}{2}\|\psi - w_x\|_2^2 + \frac{3G}{2}\|s\|_2^2 \\ &\leq -\rho\|w_t\|_2^2 + G\|\psi - w_x\|_2^2 + \epsilon_4\|3s_x - \psi_x\|_2^2 + C\|s_x\|_2^2 + C\left(1 + \frac{1}{\epsilon_4}\right)\|\psi - w_x\|_2^2. \end{aligned}$$

This completes the proof. □

Lemma 4.5. *The functional $I_5(t)$ satisfies, along the solution of* (1.6) − (1.8) *and for any* $0 < \alpha < 1$, *the estimate*

$$I_5'(t) \leq -\frac{l_0}{4}\|3s_x - \psi_x\|_2^2 + I_\rho\|3s_t - \psi_t\|_2^2 + C\|\psi - w_x\|_2^2 + CC_\alpha\left(h \diamond (3s_x - \psi_x)\right)(t). \tag{4.12}$$

Proof. Differentiating I_5, using $(1.6)_2$, we arrive at

$$\begin{aligned} I_5'(t) &= I_\rho\|3s_t-\psi_t\|_2^2-\left(D-\int_0^t g(\tau)d\tau\right)\|3s_x-\psi_x\|_2^2+G\int_0^1(3s-\psi)(\psi-w_x)dx \\ &+\int_0^1(3s_x-\psi_x)\int_0^t g(t-\tau)\left((3s_x-\psi_x)(x,\tau)-(3s_x-\psi_x)(x,t)\right)d\tau dx. \end{aligned}$$

Applying Lemmas 2.1- 2.6, Cauchy-Schwarz, Young's and Poincaré's inequalities, we obtain any $\delta_3 > 0$

$$\begin{aligned} I_5'(t) &\le I_\rho\|3s_t-\psi_t\|_2^2-l_0\|3s_x-\psi_x\|_2^2+\delta_3\|3s_x-\psi_x\|_2^2+\frac{G^2}{4\delta_3}\|\psi-w_x\|_2^2 \\ &+\frac{l_0}{2}\|3s_x-\psi_x\|_2^2+\frac{1}{2l_0}C_\alpha\left(h\diamond(3s_x-\psi_x)\right)(t). \end{aligned} \tag{4.13}$$

We select $\delta_3=\dfrac{l_0}{4}$ and obtain the desired result. □

Lemma 4.6. *The functional $I_6(t)$ satisfies, along the solution of* $(1.6)-(1.8)$ *and for any for any ϵ_1, the estimate*

$$\begin{aligned} I_6'(t) \le -G^2\|\psi-w_x\|_2^2+\epsilon_1\|3s_t-\psi_t\|_2^2+C\left(1+\frac{1}{\epsilon_1}\right)\|s_t\|_2^2 \\ +C\|s_x\|_2^2+C\|\theta_x\|_2^2+3(I_\rho G-\rho D)\int_0^1 w_t s_{xt}dx, \quad \forall t\ge 0. \end{aligned} \tag{4.14}$$

Proof. Differentiating $I_6(t)$, using $(1.6)_1$ and $(1.6)_3$ and integration by parts, we obtain

$$\begin{aligned} I_6'(t) = -3G^2\|\psi-w_x\|_2^2-4\gamma G\int_0^1(\psi-w_x)s dx-\delta G\int_0^1(\psi-w_x)\theta_x dx \\ -3I_\rho G\int_0^t(3s_t-\psi_t)s_t dx+9I_\rho G\|s_t\|_2^2+3(I_\rho G-\rho D)\int_0^1 w_t s_{xt}dx. \end{aligned} \tag{4.15}$$

Young's and Poincaré's inequalities give

$$\begin{aligned} &-4\gamma G\int_0^1(\psi-w_x)s dx \le G^2\|\psi-w_x\|_2^2+4\gamma^2C_p\|s_x\|_2^2, \\ &-\delta G\int_0^1(\psi-w_x)\theta_x dx \le G^2\|\psi-w_x\|_2^2+\frac{\delta^2}{4}\|\theta_x\|_2^2, \\ &-3I_\rho G\int_0^t(3s_t-\psi_t)s_t dx \le \epsilon_1\|3s_t-\psi_t\|_2^2+\frac{(3I_\rho G)^2}{\epsilon_1}\|s_t\|_2^2. \end{aligned} \tag{4.16}$$

Substituting (4.16) into (4.15), we obtain (4.14). This completes the proof. □

Lemma 4.7. *The functional $I_7(t)$ satisfies, along the solution of* $(1.6)-(1.8)$, *the estimate*

$$I_7'(t)\le 3(D-l_0)\|3s_x-\psi_x\|_2^2-\frac{1}{2}(g\diamond(3s_x-\psi_x))(t), \ \forall t\ge 0. \tag{4.17}$$

Proof. Differentiate $I_7(t)$ and use the fact that $J'(t) = -g(t)$ to get

$$\begin{aligned} I_7'(t) =& \int_0^1 \int_0^t J'(t-\tau)(3s_x - \psi_x)^2(\tau)d\tau dx + J(0)\|3s_x - \psi_x\|_2^2 \\ =& -(g \diamond (3s_x - \psi_x))(t) + J(t)\|3s_x - \psi_x\|_2^2 \\ & - 2\int_0^1 (3s_x - \psi_x) \int_0^t g(t-\tau)\left((3s_x - \psi_x)(\tau) - (3s_x - \psi_x)(t)\right) dx. \end{aligned} \tag{4.18}$$

Using Cauchy-Schwarz and (G1), we have

$$\begin{aligned} -2&\int_0^1 (3s_x - \psi_x) \int_0^t g(t-\tau)\left((3s_x - \psi_x)(\tau) - (3s_x - \psi_x)(t)\right) \\ &\leq 2(D - l_0)\|3s_x - \psi_x\|_2^2 + \frac{\int_0^t g(\tau)d\tau}{2(D-l_0)}(g \diamond (3s_x - \psi_x))(t) \\ &\leq 2(D - l_0)\|3s_x - \psi_x\|_2^2 + \frac{1}{2}(g \diamond (3s_x - \psi_x))(t) \end{aligned} \tag{4.19}$$

Thus, we get

$$I_7'(t) \leq 2(D - l_0)\|3s_x - \psi_x\|_2^2 - \frac{1}{2}(g \diamond (3s_x - \psi_x))(t) + J(t)\|3s_x - \psi_x\|_2^2. \tag{4.20}$$

Since J is decreasing ($J'(t) = -g(t) \leq 0$), so $J(t) \leq J(0) = D - l_0$. Hence, we arrive at

$$I_7'(t) \leq 3(D - l_0)\|3s_x - \psi_x\|_2^2 - \frac{1}{2}(g \diamond (3s_x - \psi_x))(t).$$

□

The next lemma is used only in the proof of the stability result for nonequal-wave-speed of propagation.

Lemma 4.8. *Let* $(w, 3s-\psi, s, \theta)$ *be the strong solution of problem* (1.6). *Then, for any positive numbers* $\sigma_1, \sigma_2, \sigma_3$*, the functional* $I_8(t)$ *satisfies*

$$\begin{aligned} I_8'(t) \leq& -3(I_\rho G - \rho D)\int_0^1 w_t s_{xt} dx + \sigma_1\|w_t\|_2^2 + \sigma_2\|\psi - w_x\|_2^2 + \sigma_3\|3s_x - \psi_x\|_2^2 \\ & + C\|s_x\|_2^2 + C\left(1 + \frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1}{\sigma_3}\right)\|\theta_{xt}\|_2^2, \ \forall\, t \geq t_0. \end{aligned} \tag{4.21}$$

Proof. Differentiation of I_8, using integration by part and the boundary condition give

$$\begin{aligned} I_8'(t) =& \frac{3\lambda}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_x w_{xt} dx + \frac{3\lambda}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_{xt} w_x dx \\ =& \frac{3\lambda}{\delta}(I_\rho G - \rho D)\left[-\int_0^1 \theta_{xx} w_t dx\right] + \frac{3\lambda}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_{xt} w_x dx. \end{aligned} \tag{4.22}$$

We note that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$ and from $(1.6)_4$, $\lambda\theta_{xx} = k\theta_t + \delta s_{xt}$. So, (4.22) becomes

$$\begin{aligned} I_8'(t) = &-\frac{3}{\delta}(I_\rho G - \rho D)k\int_0^1 \theta_t w_t dx - 3(I_\rho G - \rho D)\int_0^1 s_{xt} w_t dx + \frac{9\lambda}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_{xt} s dx \\ &- \frac{3\lambda}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_{xt}(\psi - w_x)dx - \frac{3\lambda}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_{xt}(3s - \psi)dx \end{aligned} \tag{4.23}$$

Using Young's and Poincaré's inequalities, we have for any positive numbers $\sigma_1, \sigma_2, \sigma_3$,

$$\begin{aligned} &-\frac{3}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_t w_t dx \le \sigma_1 \|w_t\|_2^2 + \frac{C}{\sigma_1}\|\theta_{xt}\|_2^2, \\ &-\frac{3\lambda}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_{xt}(\psi - w_x)dx \le \sigma_2 \|\psi - w_x\|_2^2 + \frac{C}{\sigma_2}\|\theta_{xt}\|_2^2, \\ &-\frac{3\lambda}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_{xt}(3s - \psi)dx \le \sigma_3 \|3s_x - \psi_x\|_2^2 + \frac{C}{\sigma_3}\|\theta_{xt}\|_2^2, \\ &\frac{9\lambda}{\delta}(I_\rho G - \rho D)\int_0^1 \theta_{xt} s dx \le C\|s_x\|_2^2 + C\|\theta_{xt}\|_2^2. \end{aligned} \tag{4.24}$$

Substituting (4.24) into (4.23), we obtain (4.21). □

5. Conclusions

In this paper, we have established a general and optimal stability estimates for a thermoelastic Laminated system, where the heat conduction is given by Fourier's Law and memory as the only source of damping. Our results are established under weaker conditions on the memory and physical parameters. From our results, we saw that the decay rate is faster provided the wave speeds of the first two equations of the system are equal (see (1.3)). A similar result was established recently in [19] when the heat conduction is given by Maxwell-Cattaneo's Law. An interesting case is when the kernel memory term is couple with the first or third equations in system (1.6). Our expectation is that the stability in both cases will depend on the speed of wave propagation.

Acknowledgments

The authors appreciate the continuous support of University of Hafr Al Batin, KFUPM and University of Sharjah. The first and second authors are supported by University of Hafr Al Batin under project #$G - 106 - 2020$. The third author is sponsored by KFUPM under project #$SB181018$.

Conflict of interest

The authors declare no conflict of interest

References

1. T. A. Apalara, On the stability of a thermoelastic laminated beam, *Acta Mathematica Scientia*, **39** (2019), 1–8.
2. J. M. Wang, G. Q. Xu, S. P. Yung, Exponential stabilization of laminated beams with structural damping and boundary feedback controls, *SIAM J. Control Optim.*, **44** (2005), 1575–1597.
3. V. I. Arnold, *Mathematical Methods of Classical Mechanics*, New York: Springer-Verlag, 1989.
4. R. Spies, Structural damping in a laminated beams due to interfacial slip, *J. Sound Vib.*, **204** (1997), 183–202.
5. X. Cao, D. Liu, G. Xu, Easy test for stability of laminated beams with structural damping and boundary feedback controls, *J. Dyn. Control Syst.*, **13** (2007), 313–336.
6. S. W. Hansen, A model for a two-layered plate with interfacial slip. In: Control and Estimation of Distributed Parameter Systems, *Int. Series Numer. Math.*, **118** (1993), 143–170.
7. J. M. Wang, G. Q Xu, S. P. Yung, Exponential stabilization of laminated beams with structural damping and boundary feedback controls, *SIAM J. Control Optim.*, **44** (2005), 1575–1597.
8. B. Feng, T. F. Ma, R. N. Monteiro, C. A. Raposo, Dynamics of Laminated Timoshenko Beams, *J. Dyn. Diff Equat.*, **30** (2018), 1489–1507.
9. G. Li, X. Kong, W. Liu, General decay for a laminated beam with structural damping and memory, the case of non-equal-wave, *J. Integral. Equations Appl.*, **30** (2018), 95–116.
10. M. I. Mustafa, General decay result for nonlinear viscoelastic equations *J. Math. Anal. Appl.*, **457** (2018), 134–152.
11. A. Guesmia, S. A. Messaoudi, On the stabilization of Timoshenko systems with memory and different speeds of wave propagation, *Appl. Math. Comput.*, **219** (2013), 9424–9437.
12. J. L. Jensen, Sur les fonctions convexes et les inégualités entre les valeurs moyennes, *Acta Math.*, **30** (1906), 175–193.
13. S. E. Mukiawa, T. A. Apalara, S. A. Messaoudi, A general and optimal stability result for a laminated beam, *J. Integral Equations Appl.*, **32** (2020), 341–359.
14. T. A. Apalara, S. A. Messaoudi, An exponential stability result of a Timoshenko system with thermoelasticity with second sound and in the presence of delay, *Appl. Math. Optim.*, **71** (2015), 449–472.
15. M. I. Mustafa, Laminated Timoshenko beams with viscoelastic damping, *J. Math. Anal. Appl.*, **466** (2018), 619–641.
16. J. H. Hassan, S. A. Messaoudi, M. Zahri, Existence and new general decay result for a viscoleastic-type Timoshenko system, *J. Anal. Appl.*, **39** (2020), 185-222.
17. T. A. Apalara, S. A. Messaoudi, A. A. Keddi, On the decay rates of Timoshenko system with second sound, *Math. Methods Appl. Sci.*, **39** (2016), 2671–2684.
18. M. M. Cavalcanti, A. Guesmia, General decay rates of solutions to a nonlinear wave equation with boundary condition of memory type, *Differ. Integral Equ.*, **18** (2005), 583–600.

19. S. E. Mukiawa, T. A. Apalara, S. A. Messaoudi, A stability result for a memory-type Laminated-thermoelastic system with Maxwell-Cattaneo heat conduction, *J. Thermal Stresses*, **43** (2020), 1437–1466,

20. C. D. Enyi, S. E. Mukiawa, Dynamics of a thermoelastic-laminated beam problem, *AIMS Mathematics*, **5** (2020), 5261–5286.

21. E. H. Dill, *Continuum mechanics: elasticity, plasticity, viscoelasticity*, CRC Press, Taylor and Francis Group, New York, (2006).

AIMS Press