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Research article

Necessary and sufficient conditions on the Schur convexity of a bivariate mean

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Abstract: In the paper, the authors find and apply necessary and sufficient conditions for a bivariate mean of two positive numbers with three parameters to be Schur convex or Schur harmonically convex respectively.

Keywords: necessary and sufficient condition; bivariate mean; Schur convex function; Schur harmonically convex function; inequality; majorization
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1. Preliminaries

Throughout the paper, denote $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_0 = [0, \infty)$, and $\mathbb{R}_+ = (0, \infty)$.

We recall the following definitions.

Definition 1.1 ([5, 13]). Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

1. A n-tuple $\boldsymbol{x}$ is said to be strictly majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge y_{[2]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.
2. A set $\Omega \subseteq \mathbb{R}^n$ is said to be convex if $(\alpha x_1 + \beta y_1, \dots, \alpha x_n + \beta y_n) \in \Omega$ for all $\boldsymbol{x}$ and $\boldsymbol{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.
3. Let $\Omega \subseteq \mathbb{R}^n$. Then a function $\varphi : \Omega \to \mathbb{R}$ is said to be Schur convex (or Schur concave respectively) on Ω if the majorization $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies the inequality $\varphi(\boldsymbol{x}) \lesseqgtr \varphi(\boldsymbol{y})$.

Definition 1.2 ([1,18]). Let $\Omega \subseteq \mathbb{R}_+^n$.

1. A set Ω is said to be harmonically convex if $\frac{\boldsymbol{x}\boldsymbol{y}}{\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, where $\boldsymbol{x}\boldsymbol{y} = \sum_{i=1}^n x_i y_i$ and $\frac{1}{\boldsymbol{x}} = (\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n})$.
2. A function $\varphi : \Omega \to \mathbb{R}$ is said to be Schur harmonically convex (or Schur harmonically concave respectively) on Ω if $\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \lesseqgtr \varphi(\boldsymbol{y})$.
3. A function $\psi : \Omega \to \mathbb{R}$ is called to be Schur geometrically convex on Ω if the relation $\ln \boldsymbol{\nu} \le \ln \boldsymbol{\vartheta}$ on Ω implies the inequality $\psi(\boldsymbol{\nu}) \le \psi(\boldsymbol{\vartheta})$.

For more information on the theory of majorization, various Schur convexity, and their applications, please refer to the papers [6–8, 11, 12, 14–17, 19] and closely related references therein.

In 2009, Kuang defined in [4, p. 61] a bivariate mean $K_{\omega_1,\omega_2;p}(a,b)$ of positive numbers a, b with three real parameters ω_1, ω_2, p as

$$K_{\omega_1,\omega_2;p}(a,b) = \begin{cases} \left[\dfrac{\omega_1 A(a^p,b^p) + \omega_2 G(a^p,b^p)}{\omega_1+\omega_2}\right]^{1/p}, & p \ne 0; \\ G(a,b), & p = 0, \end{cases}$$

where $(a,b) \in \mathbb{R}_+^2$, $\omega_1, \omega_2 \in \mathbb{R}_0$ with $\omega_1 + \omega_2 \ne 0$, $A(a,b) = \frac{a+b}{2}$, and $G(a,b) = \sqrt{ab}$. In particular, if $\omega_1 = 2$ and $\omega_2 = \omega \ge 0$, then

$$K_{2,\omega;p}(a,b) = H_{p,\omega}(a,b) = \begin{cases} \left[\dfrac{a^p + \omega(ab)^{p/2} + b^p}{\omega+2}\right]^{1/p}, & p \ne 0 \\ G(a,b), & p = 0 \end{cases}$$

is the generalized Heronian mean which was introduced in [10] in 2008.

2. Backgrounds and motivations

We recall the following known conclusions.

Theorem 2.1 ([2, Theorem 1.1]). *Let $p \in \mathbb{R}$ and $\omega_1, \omega_2 \in \mathbb{R}_0$ with $\omega_1 + \omega_2 \ne 0$.*

1. *When $\omega_1\omega_2 \ne 0$, if $p \ge 2$ and $p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \ge 0$, then $K_{\omega_1,\omega_2;p}(a,b)$ is Schur convex with respect to $(a,b) \in \mathbb{R}_+^2$; if $1 \le p < 2$ and $p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \le 0$, then $K_{\omega_1,\omega_2;p}(a,b)$ is Schur concave with respect to $(a,b) \in \mathbb{R}_+^2$; if $p < 1$, then $K(\omega_1, w\omega_2, p; a, b)$ is Schur concave with respect to $(a,b) \in \mathbb{R}_+^2$.*
2. *When $\omega_1 = 0$ and $\omega_2 \ne 0$, the mean $K_{\omega_1,\omega_2;p}(a,b)$ is Schur concave with respect to $(a,b) \in \mathbb{R}_+^2$.*
3. *When $\omega_1 \ne 0$ and $\omega_2 = 0$, if $p \ge 1$, then $K_{\omega_1,\omega_2;p}(a,b)$ is Schur convex with respect to $(a,b) \in \mathbb{R}_+^2$; if $p < 1$, then $K_{\omega_1,\omega_2;p}(a,b)$ is Schur concave with respect to $(a,b) \in \mathbb{R}_+^2$.*

Theorem 2.2 ([2, Theorem 1.2]). *Let $p \in \mathbb{R}$ and $\omega_1, \omega_2 \in \mathbb{R}_0$ with $\omega_1 + \omega_2 \ne 0$. If $p \ge 0$, then $K_{\omega_1,\omega_2;p}(a,b)$ is Schur geometrically convex with respect to $(a,b) \in \mathbb{R}_+^2$. If $p < 0$, then $K_{\omega_1,\omega_2;p}(a,b)$ is Schur geometrically concave with respect to $(a,b) \in \mathbb{R}_+^2$.*

Theorem 2.3 ([2, Theorem 1.3]). *Let $p \in \mathbb{R}$ and $\omega_1, \omega_2 \in \mathbb{R}_0$ with $\omega_1 + \omega_2 \ne 0$. If $p \ge -1$, then $K_{\omega_1,\omega_2;p}(a,b)$ is Schur harmonically convex with respect to $(a,b) \in \mathbb{R}_+^2$. If $-2 < p < -1$ and*

$\omega_1(p+1)+\omega_2(\frac{p}{2}+1)\geq 0$, *then the mean* $K_{\omega_1,\omega_2;p}(a,b)$ *is Schur harmonically convex with respect to* $(a,b)\in\mathbb{R}^2_+$. *If* $p\leq -2$ *and* $\omega_1(\frac{p}{2}+1)+\omega_2=0$, *then* $K_{\omega_1,\omega_2;p}(a,b)$ *is Schur harmonically concave with respect to* $(a,b)\in\mathbb{R}^2_+$.

In [10], the Schur convexity, Schur geometric convexity, and Schur harmonic convexity of the mean $H_{p,\omega}(a,b)$ are studied. In [3], sufficient and necessary conditions for $H_{p,\omega}(a,b)$ to be Schur convex and Schur harmonically convex are studied.

Theorem 2.4 ([3, Theorem 2]). *Let* $p\in\mathbb{R}$ *and* $\omega\geq 0$. *Then* $H_{p,\omega}(a,b)$ *is*

1. *Schur convex on* $\mathbb{R}^2_+$ *if and only if* $(p,\omega)\in E_1$,
2. *Schur concave on* $\mathbb{R}^2_+$ *if and only if* $(p,\omega)\in E_2$,

where E_1 *and* E_2 *are given by*

$$E_1=\{(p,\omega): 2\leq p, 0\leq\omega\leq 2(p-1)\}\cup\{(p,\omega): 1<p\leq 2, \omega=0\}$$

and

$$E_2=\{(p,\omega): p\leq 2, \max\{0,2(p-1)\}\leq\omega\}.$$

Theorem 2.5 ([3, Theorem 3]). *Let* $p\in\mathbb{R}$ *and* $\omega\geq 0$. *Then* $H_{p,\omega}(a,b)$ *is*

1. *Schur harmonically convex on* $\mathbb{R}^2_+$ *if and only if* $(p,\omega)\in F_1$,
2. *Schur harmonically concave on* $\mathbb{R}^2_+$ *if and only if* $(p,\omega)\in F_2$,

where F_1 *and* F_2 *are given by*

$$F_1=\{(p,\omega): -2\leq p, \max\{0,-2(p+1)\}\leq\omega\}$$

and

$$F_2=\{(p,\omega): p\leq -2, 0\leq\omega\leq -2(p+1)\}\cup\{(p,\omega): p\leq -1, \omega=0\}.$$

The main purpose of this paper is to discover necessary and sufficient conditions for the bivariate mean $K_{\omega_1,\omega_2;p}(a,b)$ to be Schur convex with respect to $(a,b)\in\mathbb{R}^2_+$ for $p\in\mathbb{R}$ and to be Schur harmonically convex with respect to $(\omega_1,\omega_2)\in\mathbb{R}_0$ with $\omega_1+\omega_2>0$. These new results strengthen those sufficient conditions stated in Theorems 2.1 and 2.3 mentioned above.

3. Necessary and sufficient conditions

Now we concretely state our necessary and sufficient conditions and prove them.

Theorem 3.1. *Let* $p\in\mathbb{R}$ *and* $\omega_1,\omega_2\in\mathbb{R}_0$ *with* $\omega_1+\omega_2>0$. *Then*

1. *the bivariate mean* $K_{\omega_1,\omega_2;p}(a,b)$ *is Schur convex with respect to* $(a,b)\in\mathbb{R}^2_+$ *if and only if* $(\omega_1,\omega_2;p)\in S_1$,
2. *the bivariate mean* $K_{\omega_1,\omega_2;p}(a,b)$ *is Schur concave with respect to* $(a,b)\in\mathbb{R}^2_+$ *if and only if* $(\omega_1,\omega_2;p)\in S_2$,

where S_1 and S_2 are given by

$$S_1 = \{(\omega_1, \omega_2; p) : 2 \le p, 0 < \omega_1, 0 \le \omega_2 \le \omega_1(p-1)\} \cup \{(\omega_1, \omega_2; p) : 1 \le p < 2, \omega_2 = 0 < \omega_1\} \tag{3.1}$$

and

$$S_2 = \{(\omega_1, \omega_2; p) : p \le 2, 0 < \omega_1, \max\{0, \omega_1(p-1)\} \le \omega_2\} \cup \{(\omega_1, \omega_2; p) : p \in \mathbb{R}, 0 = \omega_1 < \omega_2\}. \tag{3.2}$$

Proof. We divide our proof into three cases.

1. When $p = 0$, we have that $K_{\omega_1,\omega_2;0}(a, b) = \sqrt{ab}$ which is Schur concave with respect to $(a, b) \in \mathbb{R}^2_+$ for $\{(\omega_1, \omega_2; p) : p = 0, \omega_1, \omega_2 \in \mathbb{R}_0, \omega_1 + \omega_2 > 0\}$.
2. When $p \neq 0$ and $\omega_1 = 0$, we have that $K_{0,\omega_2;p}(a, b) = \sqrt{ab}$ which is Schur concave with respect to $(a, b) \in \mathbb{R}^2_+$ for $\{(\omega_1, \omega_2; p) : p \neq 0, \omega_2 > \omega_1 = 0\}$.
3. When $p \neq 0$ and $\omega_1 > 0$, if we let $\omega = \frac{2\omega_2}{\omega_1}$, then

$$K_{\omega_1,\omega_2;p}(a, b) = \left[\frac{a^p + \omega(ab)^{p/2} + b^p}{\omega + 2}\right]^{1/p} = H_{p,\omega}(a, b). \tag{3.3}$$

By Theorem 2.4, we obtain that

(a) the mean $K_{\omega_1,\omega_2;p}(a, b)$ is Schur convex with respect to $(a, b) \in \mathbb{R}^2_+$ if and only if

$$\{(\omega_1, \omega_2; p) : 2 \le p, 0 < \omega_1, 0 \le \omega_2 \le \omega_1(p-1)\} \cup \{(\omega_1, \omega_2; p) : 1 \le p < 2, 0 = \omega_2 < \omega_1\};$$

(b) the mean $K_{\omega_1,\omega_2;p}(a, b)$ is Schur concave with respect to $(a, b) \in \mathbb{R}^2_+$ if and only if

$$\{(\omega_1, \omega_2; p) : p \le 2, 0 < \omega_1, \max\{0, 0 < \omega_1(p-1)\} \le \omega_2\}.$$

The proof of Theorem 3.1 is complete. □

Remark 3.1. If we let

$$S_{11} = \left\{(\omega_1, \omega_2; p) : 2 \le p, 0 < \frac{p}{2}\omega_2 \le \omega_1(p-1)\right\} \cup \{(\omega_1, \omega_2; p) : p \ge 1, 0 = \omega_2 < \omega_1\}$$

and

$$\begin{aligned} S_{21} = &\left\{(\omega_1, \omega_2; p) : 1 \le 2 < p, 0 < \omega_1(p-1) \le \frac{p}{2}\omega_2\right\} \cup \{(\omega_1, \omega_2; p) : p \neq 0, 0 = \omega_1 < \omega_2\} \\ &\cup \{(\omega_1, \omega_2; p) : p < 1, p \neq 0, \omega_1 > 0, \omega_2 > 0\} \cup \{(\omega_1, \omega_2; p) : p < 1, p \neq 0, 0 = \omega_2 < \omega_1\}, \end{aligned}$$

then $S_{11} \subseteq S_1$ and $S_{21} \subseteq S_2$. This means that [2, Theorem 1.1] recited in Theorem 2.1 just only put forward a sufficient condition for $K_{\omega_1,\omega_2;p}(a, b)$ to be Schur convex. In other words, our Theorem 3.1 strengths [2, Theorem 1.1].

Theorem 3.2. *Let $p \in \mathbb{R}$ and $\omega_1, \omega_2 \in \mathbb{R}_0$ with $\omega_1 + \omega_2 > 0$. Then*

1. *the bivariate mean $K_{\omega_1,\omega_2;p}(a, b)$ is Schur harmonically convex with respect to $(a, b) \in \mathbb{R}^2_+$ if and only if $(\omega_1, \omega_2; p) \in H_1$,*

2. *the bivariate mean* $K_{\omega_1,\omega_2;p}(a,b)$ *is Schur harmonically concave with respect to* $(a,b) \in \mathbb{R}^2_+$ *if and only if* $(\omega_1,\omega_2;p) \in H_2$,

where H_1 *and* H_2 *are given by*

$$H_1 = \{(\omega_1,\omega_2;p) : -2 \le p, \max\{0, -\omega_1(p+1)\} \le \omega_2\} \tag{3.4}$$

and

$$H_2 = \{(\omega_1,\omega_2;p) : p \le -2, 0 \le \omega_2 \le -\omega_1(p+1)\} \cup \{(\omega_1,\omega_2;p) : p \le -1, \omega_2 = 0 < \omega_1\}. \tag{3.5}$$

Proof. We divide our proof into two cases.

1. If $p = 0$ and $\omega_1, \omega_2 \in \mathbb{R}_0$ with $\omega_1+\omega_2 > 0$, or if $p \ne 0$ and $\omega_1 = 0 < \omega_2$, then $K_{\omega_1,\omega_2;p}(a,b) = \sqrt{ab}$ is Schur harmonically convex with respect to $(a,b) \in \mathbb{R}^2_+$.
2. If $p \ne 0$ and $\omega_1 > 0$, then letting $\omega = \frac{2\omega_2}{\omega_1}$, using (3.3), and employing Theorem 2.5, we obtain
   (a) the mean $K_{\omega_1,\omega_2;p}(a,b)$ is Schur harmonically convex with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if
   $$\{(\omega_1,\omega_2;p) : -p \le 2, p \ne 0, 0 < \omega_1, \max\{0, -\omega_1(p+1)\} \le \omega_2\}.$$
   (b) the mean $K_{\omega_1,\omega_2;p}(a,b)$ is Schur harmonically concave with respect to $(a,b) \in \mathbb{R}^2_+$ if and only if
   $$\{(\omega_1,\omega_2;p) : p \le -2, 0 \le \omega_2 \le -\omega_1(p+1)\} \cup \{(\omega_1,\omega_2;p) : p \le -1, \omega_2 = 0 < \omega_1\}.$$

The proof of Theorem 3.2 is complete. □

Remark 3.2. If letting

$$\begin{aligned} H_{11} &= \{(\omega_1,\omega_2;p) : p \ge -1, \omega_1, \omega_2 \in \mathbb{R}_0, \omega_1+\omega_2 > 0\} \\ &\cup \left\{(\omega_1,\omega_2;p) : -2 < p < -1, \omega_1, \omega_2 \in \mathbb{R}_0, -\omega_1(p+1) \le \omega_2\left(\frac{p}{2}+1\right)\right\} \end{aligned}$$

and

$$H_{21} : = \left\{(\omega_1,\omega_2;p) : p \le -2, \omega_1, \omega_2 \in \mathbb{R}_0, \omega_2 = -\omega_1\left(\frac{p}{2}+1\right)\right\},$$

then $H_{11} \subseteq H_1$ and $H_{21} \subseteq H_2$. This means that [2, Theorem 1.3] recited in Theorem 2.3 just only put forward a sufficient condition for $K_{\omega_1,\omega_2;p}(a,b)$ to be Schur harmonically convex. In other words, our Theorem 3.2 strengths [2, Theorem 1.3].

4. Applications

Now we apply Theorems 3.1 and 3.2 to construct several inequalities.

Theorem 4.1. *Suppose* $p \in \mathbb{R}$ *and* $\omega_1, \omega_2 \in \mathbb{R}_0$ *with* $\omega_1+\omega_2 > 0$. *Let* $(a,b) \in \mathbb{R}^2_+$, $u(t) = tb + (1-t)a$, *and* $v(t) = ta + (1-t)b$ *for* $t \in [0,1]$. *Assume* $\frac{1}{2} \le t_2 \le t_1 \le 1$ *or* $0 \le t_1 \le t_2 \le \frac{1}{2}$.

1. *If* $(\omega_1, \omega_2; p) \in S_1$*, then*
$$\frac{a+b}{2} \le K_{\omega_1,\omega_2;p}(u(t_2), v(t_2)) \le K_{\omega_1,\omega_2;p}(u(t_1), v(t_1)) \le K_{\omega_1,\omega_2;p}(a, b) \le \left(\frac{\omega_1}{\omega_1+\omega_2}\right)^{1/p}(a+b),$$
where S_1 *is given by* (3.1).
2. *If* $(\omega_1, \omega_2; p) \in S_2$*, then*
$$K_{\omega_1,\omega_2;p}(a, b) \le K_{\omega_1,\omega_2;p}(u(t_1), v(t_1)) \le K_{\omega_1,\omega_2;p}(u(t_2), v(t_2)) \le \frac{a+b}{2},$$
where S_2 *is given by* (3.2).

Proof. In [9], it was obtained that, if $a \le b$, then
$$(u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b). \tag{4.1}$$
Then, by a similar derivation as in the proof of [2, Theorem 4.1], one can verify required inequalities straightforwardly. □

Remark 4.1. In Theorem 4.1, if $(\omega_1, \omega_2; p) \in S_1$ and $\omega_2 > 0$, then $(\frac{\omega_1}{\omega_1+\omega_2})^{1/p} > \frac{1}{2}$.

Theorem 4.2. *Suppose* $p \in \mathbb{R}$ *and* $\omega_1, \omega_2 \in \mathbb{R}_0$ *with* $\omega_1 + \omega_2 > 0$*. Let* $(a, b) \in \mathbb{R}^2_+$.

1. *If* $(\omega_1, \omega_2; p) \in H_1$*, then*
$$\frac{2ab}{a+b} \le K_{\omega_1,\omega_2;p}\left(\frac{ab}{tb+(1-t)a}, \frac{ab}{ta+(1-t)b}\right) \le K_{\omega_1,\omega_2;p}(a, b), \tag{4.2}$$
where H_1 *is given by* (3.4).
2. *If* $(\omega_1, \omega_2; p) \in H_2$*, the double inequality* (4.2) *is reversed, where* H_2 *is given by* (3.5).

Proof. Considering the relations in (4.1) and imitating the proof of [2, Theorem 4.6] arrive at required inequalities. The proof is complete. □

5. Conclusions

In this paper, we have found necessary and sufficient conditions in Theorems 3.1 and 3.2 for the bivariate mean $K_{\omega_1,\omega_2;p}(a, b)$ of positive numbers a, b with three real parameters ω_1, ω_2, p to be Schur convex or Schur harmonically convex respectively. These necessary and sufficient conditions have also been applied in Theorems 4.1 and 4.2.

In mathematics, necessary and sufficient condition is the best conclusion.

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Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

Conflict of interest

The authors declare that they have no conflict of interest.

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