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Research article

Meromorphic harmonic univalent functions related with generalized (p,q)-post quantum calculus operators

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Abstract: In this paper, we introduce certain subclasses of meromorphic harmonic univalent functions, which are defined by using generalized (p, q)-post quantum calculus operators as well as subordination relationship. Sufficient coefficient conditions, extreme points, distortion bounds and convolution properties for functions belonging to the subclasses are obtained.

Keywords: meromorphic harmonic univalent function; subordination; convolution; generalized (p,q)-post quantum calculus operator
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1. Introduction and preliminaries

An analytic function $s : \mathbb{U} = \{z : |z| < 1\} \to \mathbb{C}$ is subordinate to an analytic function $t : \mathbb{U} \to \mathbb{C}$ and write $s(z) \prec t(z)$, if there exists a complex value function ω which maps $\mathbb{U}$ into itself with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$ such that $s(z) = t(\omega(z))$ $(z \in \mathbb{U})$. Furthermore, if the function t is univalent in $\mathbb{U}$, then we have the following equivalence (see [1]):

$$s(z) \prec t(z) \Longleftrightarrow s(0) = t(0) \text{ and } s(\mathbb{U}) \subset t(\mathbb{U}).$$

Let $\mathcal{A}$ define the class of functions f that are analytic in the open unit disc $\mathbb{U}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

The theory of (p, q)-calculus (or post quantum calculus [2]) operators are used in various areas of science and also in geometric function theory. For $0 < q \leq p \leq 1$ and $f \in \mathcal{A}$, Chakrabarti and

Jagannathan [2] defined the (p,q)-derivative operator $D_{p,q}:\mathcal{A}\to\mathcal{A}$ by

$$D_{p,q}f(z)=\begin{cases}\frac{f(pz)-f(qz)}{(p-q)z}, & p\neq q, z\neq 0,\\ \lim\limits_{q\to p^-}\frac{f(pz)-f(qz)}{(p-q)z}, & p=q, z\neq 0,\end{cases} \tag{1.1}$$

where

$$D_{p,q}f(z)=1+\sum_{k=2}^{\infty}[k]_{p,q}a_k z^{k-1} \tag{1.2}$$

and

$$[k]_{p,q}=\frac{p^k-q^k}{p-q}=\begin{cases}\sum_{\ell=1}^{k}p^{\ell-1}q^{(k-1)-(\ell-1)}, & p\neq q,\\ kp^{k-1}, & p=q.\end{cases} \tag{1.3}$$

From (1.1), we have

$$\lim_{z\to 0}D_{p,q}f(z)=1 \text{ and } \lim_{p\to 1^-}D_{p,p}f(z)=D_{1,1}f(z)=f'(z).$$

Next, we introduce the (p,q)-derivative operator in the class of meromorphic functions.

Suppose $\mathcal{M}$ be the class of functions f that are meromorphic analytic in the punctured disk $\mathbb{U}^*=\mathbb{U}\setminus\{0\}=\{z:0<|z|<1\}$ of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k. \tag{1.4}$$

Now, we define the (p,q)-post quantum derivative operator $\tilde{d}_{p,q}:\mathcal{M}\to\mathcal{M}$ by

$$\tilde{d}_{p,q}f(z)=\begin{cases}\frac{f(pz)-f(qz)}{(p-q)z}, & p\neq q, z\in\mathbb{U}^*,\\ \lim\limits_{q\to p^-}\frac{f(pz)-f(qz)}{(p-q)z}, & p=q, z\in\mathbb{U}^*.\end{cases} \tag{1.5}$$

Using (1.4) and (1.5), we have

$$\tilde{d}_{p,q}f(z)=-\frac{1}{pqz^2}+\sum_{k=1}^{\infty}[k]_{p,q}a_k z^{k-1} \quad (k\in\mathbb{N}), \tag{1.6}$$

where $0<q\leq p\leq 1$ and $[k]_{p,q}$ is defined by (1.3).

Let $\lambda\geq 0, 0<q\leq p\leq 1,\ m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $f(z)\in\mathcal{M}$, we introduce the generalized (p,q)-post quantum calculus operator $\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda}:\mathcal{M}\to\mathcal{M}$ as follows,

$$\widetilde{\mathfrak{D}}_{p,q}^{0,0}f(z)=f(z),$$

$$\widetilde{\mathfrak{D}}_{p,q}^{1,\lambda}f(z)=(1-\lambda)pqz\tilde{d}_{p,q}f(z)+\lambda pqz(z\tilde{d}_{p,q}f(z)))'+\frac{2(\lambda+1)}{z}=\widetilde{\mathfrak{D}}_{p,q}^{\lambda}f(z), \tag{1.7}$$

$$\widetilde{\mathfrak{D}}_{p,q}^{2,\lambda}f(z)=\widetilde{\mathfrak{D}}_{p,q}^{\lambda}(\widetilde{\mathfrak{D}}_{p,q}^{\lambda}f(z)) \tag{1.8}$$

and in general,

$$\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda}f(z)=\widetilde{\mathfrak{D}}_{p,q}^{\lambda}(\widetilde{\mathfrak{D}}_{p,q}^{m-1,\lambda}f(z)) \quad (m\geq 1, z\in\mathbb{U}^*). \tag{1.9}$$

After a simple calculation, we can obtain the following conclusion

$$\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left\{[(k-1)\lambda + 1]pq[k]_{p,q}\right\}^m a_k z^k, \tag{1.10}$$

where $[k]_{p,q}$ is defined by (1.3). For simple of notation, we let

$$\omega_k(\lambda, p, q) := [(k-1)\lambda + 1]pq[k]_{p,q}. \tag{1.11}$$

Obviously, for $\lambda = 0$, the operator $\widetilde{\mathfrak{D}}_{p,q}^{m,0} f(z) = L_{p,q}^m f(z)$ reduces to the (p,q)-Sălăgean operator [3].

A complex valued harmonic function f in a simply connected domain $\mathbb{D} \subset \mathbb{C}$ has the canonical representation $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ and $g(z_0) = 0$ for some prescribed point $z_0 \in \mathbb{D}$. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ (see [4,5]).

Denote by $\mathcal{M}_H$ the class of meromorphic univalent and harmonic functions f that are sense preserving in $\mathbb{U}^*$ and have the following form

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}, \ |b_1| < 1, \tag{1.12}$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{U}^*$ and $\mathbb{U}$ respectively. The class $\mathcal{M}_H$ was studied in [6–10].

Let $\lambda \geq 0, 0 < q \leq p \leq 1, m \in \mathbb{N}_0$ and $f \in \mathcal{M}_H$, we now define the operator $\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} : \mathcal{M}_H \to \mathcal{M}_H$ as

$$\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} f(z) = \widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} h(z) + \overline{\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} g(z)}, \tag{1.13}$$

where

$$\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \omega_k(\lambda, p, q) a_k z^k, \ \ \widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} g(z) = \sum_{k=1}^{\infty} \omega_k(\lambda, p, q) b_k z^k, \tag{1.14}$$

with $\omega_k(\lambda, p, q)$ defined by (1.11).

Assume that F be fixed meromorphic harmonic function given by

$$F(z) = H(z) + \overline{G(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^k}, \ \ |B_1| < 1. \tag{1.15}$$

For f given by (1.12) and F given by (1.15), we define the convolution (or Hadamard product) of F and f by

$$(f * F)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k} = (F * f)(z). \tag{1.16}$$

Also, we denote by $\mathcal{T}(\mathcal{T} \subset \mathcal{M}_H)$ the class of meromorphic harmonic functions f of the following form

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \overline{z}^k \ \ (z \in \mathbb{U}^*). \tag{1.17}$$

Throughout this paper, we shall assume $\lambda \geq 0, 0 < q \leq p \leq 1, m \in \mathbb{N}_0$ and $-1 \leq B < A \leq 1$.

Let

$$\phi(z) = \frac{1}{z} + \sum_{k=1}^{\infty} u_k z^k + \sum_{k=1}^{\infty} \overline{v_k z^k} \tag{1.18}$$

be harmonic in $\mathbb{U}^*$ with $u_k > 0$ and $v_k > 0$.

Taking

$$\mathcal{L}_H f(z) = zh'(z) - \overline{zg'(z)}, \quad \mathcal{L}_H^2 f(z) = \mathcal{L}_H(\mathcal{L}_H f(z)), \quad f \in \mathcal{M}_H.$$

Now, using the operator $\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda}$ and subordination relationship, we define the following two classes.

Definition 1. *Let the function $f \in \mathcal{M}_H$ of the form (1.12). The function $f \in \mathcal{M}_\phi^{p,q}(\lambda, m, A, B)$ if and only if*

$$-\frac{\mathcal{L}_H(\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} f * \phi)(z)}{(\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} f * \phi)(z)} \prec \frac{1 + Az}{1 + Bz} \tag{1.19}$$

and also the function $f \in \mathcal{K}_\phi^{p,q}(\lambda, m, A, B)$ if and only if

$$-\frac{\mathcal{L}_H^2(\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} f * \phi)(z)}{\mathcal{L}_H(\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda} f * \phi)(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{1.20}$$

where

$$\widetilde{\mathfrak{D}}_{p,q}^{m,\lambda}(f * \phi)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \omega_k^m(\lambda; p, q) u_k a_k z^k + \sum_{k=1}^{\infty} \omega_k^m(\lambda; p, q) v_k \overline{b_k z^k} \tag{1.21}$$

with $\omega_k(\lambda; p, q)$ given by (1.11).

We let

$$\widetilde{\mathcal{M}}_\phi^{p,q}(\lambda, m, A, B) = \mathcal{T} \cap \mathcal{M}_\phi^{p,q}(\lambda, m, A, B)$$

and

$$\widetilde{\mathcal{K}}_\phi^{p,q}(\lambda, m, A, B) = \mathcal{T} \cap \mathcal{K}_\phi^{p,q}(\lambda, m, A, B).$$

The classes $\mathcal{M}_\phi^{p,q}(\lambda, m, A, B)$ and $\mathcal{K}_\phi^{p,q}(\lambda, m, A, B)$ reduce to the well-known subclasses of $\mathcal{M}_H$ as well as many new ones. For example, let $\phi(z) = \frac{1}{z} + \sum_{k=1}^{\infty}(z^k + \bar{z}^k)$, we have

$$\mathcal{M}_\phi^{1,1}(0, 1, 1 - 2\gamma, -1) = MHS^*(\gamma) = \left\{ f \in \mathcal{M}_H : \operatorname{Re}\left[-\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right] > \gamma \right\}$$

and

$$\mathcal{K}_\phi^{1,1}(0, 1, 1 - 2\gamma, -1) = MCH(\gamma) = \left\{ f \in \mathcal{M}_H : \operatorname{Re}\left[-\frac{zh''(z) + h'(z) + \overline{zg''(z) + g'(z)}}{h'(z) - \overline{g'(z)}} \right] > \gamma \right\},$$

where $\gamma \in [0, 1)$.

The classes $MHS^*(\gamma)$ and $MCH(\gamma)$ were studied by Jahangiri [9].

In particular, the classes $MHS^*(0) = MHS^*$ (Meromorphically harmonic starlike functions) and $MCH(0) = MCH$ (Meromorphically harmonic convex functions) were studied by Jahangiri and Silverman [10].

In this paper, the sufficient and necessary conditions of coefficients are discussed. As what we have hoped, distortion estimates, extreme points and convolution properties for the above-defined classes are also obtained.

2. Basic properties

First of all, we provide the sufficient conditions of coefficients for the classes defined in Definition 1.

Theorem 1. *Let $f = h + \overline{g}$ be given by (1.12) and $\omega_k(\lambda; p, q)$ given by (1.11).*

(i) The sufficient condition for f to be sense-preserving and meromorphic harmonic univalent in $\mathbb{U}^*$ and $f \in \mathcal{M}_{\phi}^{p,q}(\lambda, m, A, B)$ is

$$\sum_{k=1}^{\infty} \left[\xi_{k,m}(p,q)|a_k| + \mu_{k,m}(p,q)|b_k|\right] \leq 1, \tag{2.1}$$

where

$$\begin{cases} k \leq \xi_{k,m}(p,q) := \frac{[k(1-B)+(1-A)]u_k\omega_k^m(\lambda;p,q)}{A-B}, \\ k \leq \mu_{k,m}(p,q) := \frac{[k(1-B)-(1-A)]v_k\omega_k^m(\lambda;p,q)}{A-B}. \end{cases} \tag{2.2}$$

(ii) The sufficient condition for f to be sense-preserving and meromorphic harmonic univalent in $\mathbb{U}^*$ and $f \in \mathcal{K}_{\phi}^{p,q}(\lambda, m, A, B)$ is

$$\sum_{k=1}^{\infty} k\left[\xi_{k,m}(p,q)|a_k| + \mu_{k,m}(p,q)|b_k|\right] \leq 1, \tag{2.3}$$

where $\xi_{k,m}(p,q)$ and $\mu_{k,m}(p,q)$ are given by (2.2).

Proof. (i) For $0 < |z_1| \leq |z_2| < 1$, we obtain

$$\begin{aligned} \left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| \\ &= 1 - \left|\frac{z_1 z_2 \sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_1 - z_2) - z_1 z_2 \sum_{k=1}^{\infty} a_k(z_1^k - z_2^k)}\right| \\ &> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=1}^{\infty} k|a_k|} \\ &\geq 1 - \frac{\sum_{k=1}^{\infty} \mu_{k,m}(p,q)|b_k|}{1 - \sum_{k=1}^{\infty} \xi_{k,m}(p,q)|a_k|} \\ &\geq 0, \end{aligned}$$

which proves univalence. Note that f is sense-preserving harmonic in $\mathbb{U}^*$. This is because

$$|h'(z)| \geq \frac{1}{|z|^2} - \sum_{k=1}^{\infty} k|a_k||z|^{k-1} > 1 - \sum_{k=1}^{\infty} \xi_{k,m}(p,q)|a_k| \geq \sum_{k=1}^{\infty} \mu_{k,m}(p,q)|b_k| > \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.$$

Next, we show that if the inequality (2.1) holds, then the required condition (1.19) is satisfied.

By means of Definition 1 and relationship of subordination, the function $f \in \mathcal{M}_{\phi}^{p,q}(\lambda, m, A, B)$ iff there exists an analytic function $\varpi(z)$ satisfying $\varpi(0) = 0$, $|\varpi(z)| < 1$ $(z \in \mathbb{U})$ such that

$$-\frac{\mathcal{L}_H(\mathfrak{D}_{p,q}^{m,\lambda} f * \phi)(z)}{\mathfrak{D}_{p,q}^{m,\lambda} f * \phi(z)} = \frac{1 + A\varpi(z)}{1 + B\varpi(z)},$$

or equivalently

$$\left|\frac{\mathcal{L}_H(\mathfrak{D}^{m,\lambda}_{p,q}f*\phi)(z)+\mathfrak{D}^{m,\lambda}_{p,q}f*\phi(z)}{A\mathfrak{D}^{m,\lambda}_{p,q}f*\phi(z)+B\mathcal{L}_H(\mathfrak{D}^{m,\lambda}_{p,q}f*\phi)(z)}\right|<1.$$

We only need to show that

$$|A\mathfrak{D}^{m,\lambda}_{p,q}f*\phi(z)+B\mathcal{L}_H(\mathfrak{D}^{m,\lambda}_{p,q}f*\phi)(z)|-|\mathcal{L}_H(\mathfrak{D}^{m,\lambda}_{p,q}f*\phi)(z)+\mathfrak{D}^{m,\lambda}_{p,q}f*\phi(z)|>0\ (z\in\mathbb{U}^*). \tag{2.4}$$

Letting

$$\begin{cases}\sigma_{k,j}=(A+(-1)^{j-1}kB)\omega^m_k(\lambda;p,q), & j=1,2,\\ \theta_{k,j}=(k+(-1)^{j-1})\omega^m_k(\lambda;p,q), & j=1,2.\end{cases} \tag{2.5}$$

Therefore, from (2.1) we get

$$\begin{aligned}
&|A\mathfrak{D}^{m,\lambda}_{p,q}f*\phi(z)+B\mathcal{L}_H(\mathfrak{D}^{m,\lambda}_{p,q}f*\phi)(z)|-|\mathcal{L}_H(\mathfrak{D}^{m,\lambda}_{p,q}f*\phi)(z)+\mathfrak{D}^{m,\lambda}_{p,q}f*\phi(z)|\\
&=\left|(A-B)\frac{1}{z}+\sum_{k=1}^{\infty}\sigma_{k,1}u_ka_kz^k+\sum_{k=1}^{\infty}\sigma_{k,2}v_k\overline{b_kz^k}\right|-\left|-\sum_{k=1}^{\infty}\theta_{k,1}u_ka_kz^k+\sum_{k=1}^{\infty}\theta_{k,2}v_k\overline{b_kz^k}\right|\\
&\geq(A-B)\frac{1}{|z|}+\sum_{k=1}^{\infty}\sigma_{k,1}u_k|a_k||z|^k-\sum_{k=1}^{\infty}\sigma_{k,2}v_k|b_k||z|^k-\sum_{k=1}^{\infty}\theta_{k,1}u_k|a_k||z|^k-\sum_{k=1}^{\infty}\theta_{k,2}v_k|b_k||z|^k\\
&=(A-B)\frac{1}{|z|}\left[1-\sum_{k=1}^{\infty}\xi_{k,m}(p,q)|a_k||z|^{k+1}-\sum_{k=1}^{\infty}\mu_{k,m}(p,q)|b_k||z|^{k+1}\right]\\
&>(A-B)\frac{1}{|z|}\left[1-\sum_{k=1}^{\infty}\xi_{k,m}(p,q)|a_k|-\sum_{k=1}^{\infty}\mu_{k,m}(p,q)|b_k|\right]\\
&\geq 0.
\end{aligned}$$

Hence, we complete the proof of (i). Also, applying the same method as (i), we can obtain (ii).

The harmonic univalent function

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{A-B}{u_k\omega^m_k(\lambda,p,q)[k(1-B)+(1-A)]}x_kz^k+\frac{A-B}{v_k\omega^m_k(\lambda,p,q)[k(1-B)-(1-A)]}\overline{y_kz^k}, \tag{2.6}$$

where $\sum_{k=1}^{\infty}(|x_k|+|y_k|)=1$, shows that the coefficient bound given by (2.1) is sharp. □

Theorem 2. *Let $f=h+\overline{g}$ be given by (1.17). Then*
(i) $f\in\widetilde{\mathcal{M}}^{p,q}_{\phi}(\lambda,m,A,B)$ iff (2.1) holds true.
(ii) $f\in\widetilde{\mathcal{K}}^{p,q}_{\phi}(\lambda,m,A,B)$ iff (2.3) holds true.

Proof. (i) It appears from (1.17) that $\widetilde{\mathcal{M}}^{p,q}_{\phi}(\lambda,m,A,B)\subset\mathcal{M}^{p,q}_{\phi}(\lambda,m,A,B)$. In view of Theorem 1, it is straightforward to show that if $f\in\widetilde{\mathcal{M}}^{p,q}_{\phi}(\lambda,m,A,B)$, then (2.1) holds true. Next, we use the method in [11] to prove.

Let $f\in\widetilde{\mathcal{M}}^{p,q}_{\phi}(\lambda,m,A,B)$, then it satisfies (1.19) or equivalently

$$\left|\frac{\sum_{k=1}^{\infty}\theta_{k,1}u_k|a_k|z^k+\sum_{k=1}^{\infty}\theta_{k,2}v_k|b_k|\overline{z}^k}{(A-B)\frac{1}{z}+\sum_{k=1}^{\infty}\sigma_{k,1}u_k|a_k|z^k-\sum_{k=1}^{\infty}\sigma_{k,2}v_k|b_k|\overline{z}^k}\right|<1\qquad(z\in\mathbb{U}^*). \tag{2.7}$$

From (2.7), we get

$$\operatorname{Re}\left\{\frac{\sum_{k=1}^{\infty}\theta_{k,1}u_k|a_k|z^k+\sum_{k=1}^{\infty}\theta_{k,2}v_k|b_k|\bar{z}^k}{(A-B)\frac{1}{z}+\sum_{k=1}^{\infty}\sigma_{k,1}u_k|a_k|z^k-\sum_{k=1}^{\infty}\sigma_{k,2}v_k|b_k|\bar{z}^k}\right\}<1, \tag{2.8}$$

which holds for all $z\in\mathbb{U}^*$. Setting $z=r$ $(0<r<1)$ in (2.8), we get

$$\frac{\sum_{k=1}^{\infty}\theta_{k,1}u_k|a_k|r^{k+1}+\sum_{k=1}^{\infty}\theta_{k,2}v_k|b_k|r^{k+1}}{(A-B)+\sum_{k=1}^{\infty}\sigma_{k,1}u_k|a_k|r^{k+1}-\sum_{k=1}^{\infty}\sigma_{k,2}v_k|b_k|r^{k+1}}<1. \tag{2.9}$$

Thus, from (2.9) we have

$$\sum_{k=1}^{\infty}[\xi_{k,m}(p,q)|a_k|+\mu_{k,m}(p,q)|b_k|]r^{k+1}<1\ (0<r<1), \tag{2.10}$$

where $\xi_{k,m}(p,q)$ and $\mu_{k,m}(p,q)$ are given by (2.2).

Putting

$$S_n=\sum_{k=1}^{n}(\xi_{k,m}(p,q)|a_k|+\mu_{k,m}(p,q)|b_k|).$$

For the series $\sum_{k=1}^{\infty}[\xi_{k,m}(p,q)|a_k|+\mu_{k,m}(p,q)|b_k|]$, $\{S_n\}$ is the nondecreasing sequence of partial sums of it. Moreover, by (2.10) it is bounded by 1. Therefore, it is convergent and

$$\sum_{k=1}^{\infty}(\xi_{k,m}(p,q)|a_k|+\mu_{k,m}(p,q)|b_k|)=\lim_{n\to\infty}S_n\le 1.$$

Thus, we get the inequality (2.1). Similarly, it is easy to prove (ii) of Theorem 2. □

Clearly, from Theorem 2, we have

$$\widetilde{\mathcal{K}}_{\phi}^{p,q}(\lambda,m,A,B)\subset\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda,m,A,B). \tag{2.11}$$

Next, we give the extreme points of these classes.

Theorem 3. *Let $X_k\ge 0, Y_k\ge 0, \sum_{k=0}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1, \xi_{k,m}(p,q)$ and $\mu_{k,m}(p,q)$ be given by (2.2).*

(i) If $f\in\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda,m,A,B)$. Then $f\in clco\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda,m,A,B)$ iff

$$f(z)=\sum_{k=0}^{\infty}X_kh_k+\sum_{k=1}^{\infty}Y_kg_k\quad(z\in\mathbb{U}^*), \tag{2.12}$$

where

$$\begin{cases} h_0=\frac{1}{z},\quad h_k=\frac{1}{z}+\frac{1}{\xi_{k,m}(p,q)}z^k, & k\ge 1,\\ g_k=\frac{1}{z}-\frac{1}{\mu_{k,m}(p,q)}\bar{z}^k, & k\ge 1. \end{cases} \tag{2.13}$$

(ii) If $f\in\widetilde{\mathcal{K}}_{\phi}^{p,q}(\lambda,m,A,B)$. Then $f\in clco\widetilde{\mathcal{K}}_{\phi}^{p,q}(\lambda,m,A,B)$ iff the condition (2.12) holds and

$$\begin{cases} h_0=\frac{1}{z},\quad h_k=\frac{1}{z}+\frac{1}{k\xi_{k,m}(p,q)}z^k, & k\ge 1,\\ g_k=\frac{1}{z}-\frac{1}{k\mu_{k,m}(p,q)}\bar{z}^k, & k\ge 1. \end{cases} \tag{2.14}$$

Proof. From (2.12) we get

$$f(z)=\left(X_0+\sum_{k=1}^{\infty}[X_k+Y_k]\right)\frac{1}{z}+\sum_{k=1}^{\infty}\frac{1}{\xi_{k,m}(p,q)}X_kz^k-\sum_{k=1}^{\infty}\frac{1}{\mu_{k,m}(p,q)}Y_k\overline{z^k}.$$

Since $0\leq X_k\leq 1$ $(k=0,1,2,\cdots)$, we obtain

$$\sum_{k=1}^{\infty}\xi_{k,m}(p,q)\frac{1}{\xi_{k,m}(p,q)}X_k+\sum_{k=1}^{\infty}\mu_{k,m}(p,q)\frac{1}{\mu_{k,m}(p,q)}Y_k=\sum_{k=1}^{\infty}X_k+Y_k=1-X_0\leq 1.$$

It follows, from (i) of Theorem 2, that $f\in\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda,m,A,B)$.

Conversely, if $f\in\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda,m,A,B)$, then

$$|a_k|\leq\frac{1}{\xi_{k,m}(p,q)}\quad\text{and}\quad|b_k|\leq\frac{1}{\mu_{k,m}(p,q)}.$$

Putting $X_k=\xi_{k,m}(p,q)|a_k|$, $Y_k=\mu_{k,m}(p,q)|b_k|$ and $X_0=1-\sum_{k=1}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k\geq 0$, we obtain

$$\begin{aligned}f(z)&=\frac{1}{z}+\sum_{k=1}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z}^k\\&=\left(\sum_{k=0}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k\right)\frac{1}{z}+\sum_{k=1}^{\infty}\frac{1}{\xi_{k,m}(p,q)}X_kz^k-\sum_{k=1}^{\infty}\frac{1}{\mu_{k,m}(p,q)}Y_k\overline{z}^k\\&=\sum_{k=0}^{\infty}h_k(z)X_k+\sum_{k=1}^{\infty}g_k(z)Y_k.\end{aligned}$$

Thus f can be expressed in the form of (2.12). The remainder of the proof is analogous to (i) in Theorem 3 and so we omit. □

Next, using Theorem 2, we proceed to discuss the distortion theorems for functions of these classes.

Theorem 4. *Let $f=h+\overline{g}$ be of the form (1.17), $|z|=r\in(0,1)$, $\xi_{k,m}(p,q)$ and $\mu_{k,m}(p,q)$ are defined by (2.2), $\{\xi_{k,m}(p,q)\}$ and $\{\mu_{k,m}(p,q)\}$ are non-decreasing sequences. If $f\in\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda,m,A,B)$, then*

$$\frac{1}{r}-\frac{r}{\min\{\xi_{1,m}(p,q),\mu_{1,m}(p,q)\}}\leq|f(z)|\leq\frac{1}{r}+\frac{r}{\min\{\xi_{1,m}(p,q),\mu_{1,m}(p,q)\}}.$$

Proof. For $f\in\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda,m,A,B)$, using Theorem 2 and (2.1), we have

$$\begin{aligned}|f(z)|&=\left|\frac{1}{z}+\sum_{k=1}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z^k}\right|\\&\leq\frac{1}{r}+\frac{1}{\min\{\xi_{1,m}(p,q),\mu_{1,m}(p,q)\}}\sum_{k=1}^{\infty}\left(\xi_{k,m}(p,q)|a_k|+\mu_{k,m}(p,q)|b_k|\right)r\\&\leq\frac{1}{r}+\frac{1}{\min\{\xi_{1,m}(p,q),\mu_{1,m}(p,q)\}}r\end{aligned}$$

and

$$|f(z)| \geq \frac{1}{r} - \left(\sum_{k=1}^{\infty} |a_k| + \sum_{k=1}^{\infty} |b_k| \right) r \geq \frac{1}{r} - \frac{1}{\min\{\xi_{1,m}(p,q), \mu_{1,m}(p,q)\}} r.$$

The result is sharp and the extremal function is

$$f(z) = \frac{1}{z} - \frac{1}{\min\{\xi_{1,m}(p,q), \mu_{1,m}(p,q)\}} z.$$

So, we complete the proof of Theorem 4. □

By virtue of Theorem 4, we obtain the following covering result.

Theorem 5. *Let $\xi_{k,m}(p,q)$ and $\mu_{k,m}(p,q)$ be given by (2.2). If $f \in \widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda, m, A, B)$, then*

$$\left\{ w : |w| < 1 - \frac{1}{\min\{\xi_{1,m}(p,q), \mu_{1,m}(p,q)\}} \right\} \subset f(\mathbb{U}^*).$$

Theorem 6. *The classes $\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda, m, A, B)$ and $\widetilde{\mathcal{K}}_{\phi}^{p,q}(\lambda, m, A, B)$ are closed under convex combinations.*

Remark 1. *By taking the special values of the parameters λ, p, q, m, A, B and ϕ in Theorems 1-6, it is easy to show the corresponding results for the classes $MHS^*(\gamma)$ and $MCH(\gamma)$ which are defined in Section 1.*

Especially, let $\lambda = 0, p = q = m = 1, A = 1 - 2\gamma, B = -1, 0 \leq \gamma < 1$ and $\phi(z) = \frac{1}{z} + \sum_{k=1}^{\infty}(z^k + \overline{z}^k)$ in Theorem 2, we can obtain the results of Theorems 1 and 7 in [9].

Corollary 1. *Let $f = h + \overline{g}$ be given by (1.17). Then*

(i) $f \in MHS^*(\gamma)$ iff

$$\sum_{k=1}^{\infty} \frac{k+\gamma}{1-\gamma}|a_k| + \frac{k-\gamma}{1-\gamma}|b_k| \leq 1.$$

(ii) $f \in MCH^*(\gamma)$ iff

$$\sum_{k=1}^{\infty} \frac{k(k+\gamma)}{1-\gamma}|a_k| + \frac{k(k-\gamma)}{1-\gamma}|b_k| \leq 1.$$

3. Convolution properties

Next, in order to obtain the convolution properties of functions belonging to the classes $\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda, m, A, B)$ and $\widetilde{\mathcal{K}}_{\phi}^{p,q}(\lambda, m, A, B)$, we now introduce a new class of harmonic functions.

Definition 2. *Let $\delta \geq 0$, the function $f = h + \overline{g}$ of the form (1.17) belongs to the class $\widetilde{\mathcal{L}}_{\phi}^{\delta,p,q}(\lambda, m, A, B)$ if and only if*

$$\sum_{k=1}^{\infty} k^{\delta} \xi_{k,m}(p,q)|a_k| + \sum_{k=1}^{\infty} k^{\delta} \mu_{k,m}(p,q)|b_k| \leq 1, \tag{3.1}$$

where $\xi_{k,m}(p,q)$ and $\mu_{k,m}(p,q)$ are defined by (2.2).

Obviously, for any positive integer δ, we have the following inclusion relation:

$$\widetilde{\mathcal{L}}_{\phi}^{\delta,p,q}(\lambda, m, A, B) \subset \widetilde{\mathcal{K}}_{\phi}^{p,q}(\lambda, m, A, B) \subset \widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda, m, A, B). \tag{3.2}$$

Let the harmonic functions f_t $(t = 1, 2, \cdots, \rho)$ and F_l $(l = 1, 2, \cdots, \eta)$ of the following form

$$f_t(z) = h_t(z) + \overline{g_t(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} |a_{k,t}| z^k - \sum_{k=1}^{\infty} |b_{k,t}| \overline{z}^k, |b_{1,t}| < 1 \tag{3.3}$$

and

$$F_l(z) = H_l(z) + \overline{G_l(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} |A_{k,l}| z^k - \sum_{k=1}^{\infty} |B_{k,l}| \overline{z}^k, |B_{1,\ell}| < 1. \tag{3.4}$$

We define the Hadamard product (or convolution) of f_t and F_ℓ by

$$(f_t * F_l)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} |a_{k,t}||A_{k,l}| z^k - \sum_{k=1}^{\infty} |b_{k,t}||B_{k,l}| \overline{z}^k =: (F_l * f_t)(z), \tag{3.5}$$

where $t = 1, 2, \cdots, \rho$ and $l = 1, 2, \cdots, \eta$.

Using Theorem 2, we obtain the following results.

Theorem 7. *Let f_t of the form (3.3) be in the class $\widetilde{\mathcal{K}}_{\phi}^{p,q}(\lambda, m, A, B)(t = 1, 2, \cdots, \rho)$ and F_l of the form (3.4) be in the class $\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda, m, A, B)(l = 1, 2, \cdots, \eta)$. Then the Hadamard product $(f_1 * f_2 * \cdots * f_\rho * F_1 * F_2 * \cdots * F_\eta)(z)$ belongs to the class $\widetilde{\mathcal{L}}_{\phi}^{\delta,p,q}(\lambda, m, A, B)$, where $\delta = 2\rho + \eta - 1$.*

Proof. Using the method in [8] to prove the theorem. Putting

$$\chi(z) = (f_1 * f_2 * \cdots * f_\rho * F_1 * F_2 * \cdots * F_\eta)(z). \tag{3.6}$$

From (3.6) we have

$$\chi(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\prod_{t=1}^{\rho} |a_{k,t}| \prod_{l=1}^{\eta} |A_{k,l}| \right) z^k - \sum_{k=1}^{\infty} \left(\prod_{t=1}^{\rho} |b_{k,t}| \prod_{l=1}^{\eta} |B_{k,l}| \right) \overline{z}^k. \tag{3.7}$$

According to Definition 2, we only need to show that

$$\sum_{k=1}^{\infty} k^{2\rho+\eta-1} \xi_{k,m}(p,q) \Big(\prod_{t=1}^{\rho} |a_{k,t}| \prod_{l=1}^{\eta} |A_{k,l}| \Big) + \sum_{k=1}^{\infty} k^{2\rho+\eta-1} \mu_k(p,q) \Big(\prod_{t=1}^{\rho} |b_{k,t}| \prod_{l=1}^{\eta} |B_{k,l}| \Big) \leq 1, \tag{3.8}$$

where $\xi_{k,m}(p,q)$ and $\mu_{k,m}(p,q)$ are defined by (2.2).

For $f_t \in \widetilde{\mathcal{K}}_{\phi}^{p,q}(\lambda, m, A, B)$, we obtain

$$\sum_{k=1}^{\infty} k \xi_{k,m}(p,q) |a_{k,t}| + \sum_{k=1}^{\infty} k \mu_{k,m}(p,q) |b_{k,t}| \leq 1, \tag{3.9}$$

for every $t = 1, 2, \cdots, \rho$. Therefore

$$k \xi_{k,m}(p,q) |a_{k,t}| \leq 1 \text{ and } k \mu_{k,m}(p,q) |b_{k,t}| \leq 1. \tag{3.10}$$

Further, by $\xi_{k,m}(p,q) \geq k$ and $\mu_{k,m}(p,q) \geq k$, we have

$$|a_{k,t}| \leq k^{-2} \text{ and } |b_{k,t}| \leq k^{-2} \ (t = 1, 2, \cdots, \rho). \tag{3.11}$$

Also, since $F_l \in \widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda, m, A, B)$, we have

$$\sum_{k=1}^{\infty} \xi_{k,m}(p,q)|A_{k,l}| + \sum_{k=1}^{\infty} \mu_{k,m}(p,q)|B_{k,l}| \leq 1 \ (l = 1, 2, \cdots, \eta). \tag{3.12}$$

Hence we obtain

$$|A_{k,l}| \leq k^{-1} \text{ and } |B_{k,l}| \leq k^{-1} \ (l = 1, 2, \cdots, \eta). \tag{3.13}$$

Using (3.11) for $t = 1, 2, \cdots, \rho$, (3.13) for $l = 1, 2, \cdots, \eta - 1$ and (3.12) for $l = \eta$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty} k^{2\rho+\eta-1} \xi_{k,m}(p,q) \left(\prod_{t=1}^{\rho} |a_{k,t}| \prod_{l=1}^{\eta-1} |A_{k,l}| \right) |A_{k,\eta}| + \sum_{k=1}^{\infty} k^{2\rho+\eta-1} \mu_{k,m}(p,q) \left(\prod_{t=1}^{\rho} |b_{k,t}| \prod_{l=1}^{\eta-1} |B_{k,l}| \right) |B_{k,\eta}| \\
&\leq \sum_{k=2}^{\infty} k^{2\rho+\eta-1} (\xi_{k,m}(p,q) k^{-2\rho} k^{-(\eta-1)}) |A_{k,\eta}| + \sum_{k=1}^{\infty} k^{2\rho+\eta-1} (\mu_{k,m}(p,q) k^{-2\rho} k^{-(\eta-1)}) |B_{k,\eta}| \\
&= \sum_{k=1}^{\infty} \xi_{k,m}(p,q)|A_{k,l}| + \sum_{k=1}^{\infty} \mu_{k,m}(p,q)|B_{k,l}| \leq 1,
\end{aligned}$$

and therefore $\chi(z) \in \widetilde{\mathcal{L}}_{\phi}^{\delta,p,q}(\lambda, m, A, B)$, $\delta = 2\rho + \eta - 1$. We note that the required estimate can also be obtained by using (3.11) for $t = 1, 2, \cdots, \eta - 1$; (3.13) for $l = 1, 2, \cdots, \eta$ and (3.9) for $t = \rho$. □

Taking into account the Hadamard product of functions $f_1 * f_2 * \cdots * f_\rho$ only, in the proof of Theorem 3.3, and using (3.11) for $t = 1, 2, \ldots, \rho - 1$; and relation (3.9) for $t = \rho$, we are led to

Corollary 2. *Let the functions f_t defined by (3.3) be in the class $\widetilde{\mathcal{K}}_{\phi}^{p,q}(\lambda, m, A, B)$ for every $t = 1, 2, \ldots, \rho$. Then the Hadamard product $(f_1 * f_2 * \cdots * f_\rho)(z)$ belongs to the class $\widetilde{\mathcal{L}}_{\phi}^{2\rho-1,p,q}(\lambda, m, A, B)$.*

Also, taking into account the Hadamard product of functions $F_1 * F_2 * \cdots * F_\eta$ only, in the proof of Theorem 3.3, and using (3.13) for $l = 1, 2, \ldots, \eta - 1$; and relation (3.12) for $l = \eta$, we are led to

Corollary 3. *Let the functions $F_{m,l}$ defined by (3.4) be in the class $\widetilde{\mathcal{M}}_{\phi}^{p,q}(\lambda, m, A, B)$ for every $l = 1, 2, \ldots, \eta$. Then the Hadamard product $(F_1 * F_2 * \cdots * F_\eta)(z)$ belongs to the class $\widetilde{\mathcal{L}}_{\phi}^{\eta-1,p,q}(\lambda, m, A, B)$.*

Remark 2. *For different choices of the parameters λ, p, q, m, A, B and ϕ in Theorem 7, we can deduce some new results for each of the following univalent harmonic function classes $MHS^*(\gamma)$ and $MCH(\gamma)$ which are defined in Section 1.*

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Conflict of interest

The authors agree with the contents of the manuscript, and there is no conflict of interest among the authors.

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