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Research article

The existence of a compact global attractor for a class of competition model

Yanxia Wu*

School of Mathematic and Quantitative Economics, Shandong University of Finance and Economics, Jinan 250014, P.R.China

* **Correspondence:** Email: wuyxmath@163.com; Tel: +86053182617651; Fax: +86053182617651.

Abstract: This paper is concerned with the existence of a compact global attractor for a class of competition model in $n-$dimensional ($n \geq 1$) domains. Using mathematical induction and more detailed interpolation estimates, especially Gagliardo-Nirenberg inequality, we obtain the existence of a compact global attractor, which implies the uniform boundedness of the global solutions. In particular, we get that the Shigesada-Kawasaki-Teramoto competition model has a compact global attractor for $n < 10$. The result of the S-K-T model extends the existence results of compact global attractor in [21] from $n < 8$ to $n < 10$, and extends the uniform boundedness results of the global solutions in [17] to the non-convex domain.

Keywords: competition model; global existence; global attractor; uniform boundedness
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1. Introduction and main results

In population dynamics, N. Shegesada, K. Kawasaki and E. Teromoto [15] proposed the following quasilinear competition model with cross-diffusion,

$$\begin{cases} u_t = \Delta\left[(d_1 + \rho_{11}u + \rho_{12}v)\,u\right] + u\,(a_1 - b_1u - c_1v)\,, & x \in \Omega,\ t > 0, \\ v_t = \Delta\left[(d_2 + \rho_{21}u + \rho_{22}v)\,v\right] + v\,(a_2 - b_2u - c_2v)\,, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geqslant 0, v(x,0) = v_0(x) \geqslant 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where the functions $u,\ v$ are the population densities of the two competing species and the initial values $u_0,\ v_0$ are nonnegative functions, which are not identically zero. Ω is a bounded smooth region in $\mathbb{R}^n$ with ν as its unit outward normal vector to $\partial\Omega$. The constants $a_j, b_j, c_j,\ d_j\,(j = 1,2)$ are all positive, and the constants $\rho_{ij}\,(i, j = 1,2)$ are nonnegative, where d_1 and d_2 are the random diffusion rates, ρ_{11}, ρ_{22} are the self-diffusion rates which represent intraspecific population pressures, and ρ_{12}, ρ_{21} are the so-called cross-diffusion rates which represent the interspecific population pressures.

When $\rho_{ij} = 0\,(i, j = 1, 2)$, (1.1) is reduces to the well-known Lotka-Volterra competition-diffusion system, which has been researched intensively. When ρ_{12} or ρ_{21} is positive, (1.1) is a strongly coupled parabolic system, which has received much attention, since it occurs frequently in biological and chemical models. H. Amann considered a general class of strongly coupled parabolic systems and established the local existence (in time) and uniqueness results in a series of papers [1–3]. Roughly speaking, H. Amann showed that if u_0, v_0 in $W^{1,p}(\Omega)$ with $p > n$, then (1.1) has a unique solution u, v defined in $(0, t_0)$ with $t_0 > 0$ small.

The global existence of nonnegative solutions to (1.1) is considered under some restrictive hypotheses on the smallness of cross-diffusion pressures or on the space dimension. For the case $\rho_{12} > 0, \rho_{21} > 0$, if $\rho_{11} = \rho_{22} = 0$, J. Kim [8] proved the global existence of classical solutions by energy method when $n = 1$ and $d_1 = d_2$. Later, S. Shim [16] improved J. Kim's results and obtained the uniform boundedness of the global solutions in time by interpolated estimates. P. Deuring [6] proved the global existence of classical solutions when $n \geqslant 1$ and ρ_{12}, ρ_{21} are small enough depending on the $C^{2,\alpha}$ norm of initial values u_0, v_0. If the self-diffusion rates ρ_{11} and ρ_{22} are not zero, A. Yagi [22] proved the global existence of solutions when $n = 2$ and $0 < \rho_{12} < 8\rho_{11}$, $0 < \rho_{21} < 8\rho_{22}$, he also proved the same results for $\rho_{22} = \rho_{21} = 0$ and $\rho_{11} > 0$. In addition, Y. Li and C. Zhao [13] obtained the global existence of classical solutions when $n \geqslant 1$, $d_1 = d_2$ and $\frac{\rho_{12}}{\rho_{22}} + \frac{\rho_{21}}{\rho_{11}} = 2$.

For the case of $\rho_{21} = 0$, (1.1) becomes the following system

$$\begin{cases} u_t = \Delta\left[(d_1 + \rho_{11}u + \rho_{12}v)\,u\right] + u\,(a_1 - b_1u - c_1v)\,, & x \in \Omega,\ t > 0, \\ v_t = \Delta\left[(d_2 + \rho_{22}v)\,v\right] + v\,(a_2 - b_2u - c_2v)\,, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geqslant 0, v(x, 0) = v_0(x) \geqslant 0, & x \in \Omega. \end{cases} \tag{1.2}$$

Y. Lou, W. Ni and Y. Wu [14] established a global existence of classical solutions to (1.2) for $n \leqslant 2$ and ρ_{11} is merely assumed nonnegative but ρ_{12} and ρ_{22} are allowed to be positive, which is the only available result for smooth solutions with $\rho_{11} = 0$. When ρ_{11} is positive, Y. Choi, R. Lui and Y. Yamada [4, 5] obtained some results on the global existence of the solutions to (1.2) with the restrictions $n < 6$ and $\rho_{22} > 0$. P. Tuoc [20] showed the global existence of solutions for $n < 10$. The global existence of solutions for arbitrary n under some restrictions on coefficients are investigated (see [7, 9, 11, 19]). For the uniform boundedness of the global solutions, D. Le, L. Nguyen and T. Nguyen [12] using the semi-group techniques obtained the global attractor for $n < 6$, which implies the uniform boundedness of the global solutions. Q. Xu and Y. Zhao [21] obtained the global attractor for $n < 8$. And Y. Tao and M. Winkler [17] showed the boundedness of the solutions for $n < 10$ when $\Omega \in \mathbb{R}^n$ is a bounded convex domain with smooth boundary.

In this paper, we considered the following more general strongly coupled parabolic system

$$\begin{cases} u_t = \nabla \cdot (P(u, v)\nabla u + Q(u, v)\nabla v) + uf(u, v), & x \in \Omega,\ t > 0, \\ v_t = \nabla \cdot (R(v)\nabla v) + vg(u, v), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geqslant 0, v(x, 0) = v_0(x) \geqslant 0, & x \in \Omega. \end{cases} \tag{1.3}$$

Due to the absence of the cross-diffusion term in the $v-$equation, the diffusion matrix of (1.3) is triangular. H. Amann [3] showed that if one can obtain $u \in L^\infty$, $v \in L^\infty$, then the solution of (1.3) exists globally in time. D. Le [10] proved that if $u \in L^n,\ v \in L^\infty$, then the system (1.3) has a global attractor with finite Hausdorff dimension, which attracts all the solutions of (1.3). D. Le, L. Nguyen and T. Nguyen [12] improved the results of that in [10]. In order to state their results, we first introduce the following definition.

Definition 1.1. (see [10], Definition 2.1) *Assume that there exists a solution* (u, v) *of system* (1.3) *defined on a subinterval* I *of* $\mathbb{R}^+$. *Let* $\mathcal{P}$ *be the set of function* ω *on* I *such that there exists a positive constant* C_0, *which may generally depend on the parameters of the system and the* W^{1,p_0} *norm of the initial value* (u_0, v_0) , *such that*

$$\omega(t) \leqslant C_0, \quad \forall t \in I.$$

Furthermore, if $I = (0,\infty)$, *then there exists a positive constant* C_∞ *that depends only on the parameters of the system, but does not depend on the initial value of* (u_0, v_0), *such that*

$$\limsup_{t\downarrow\infty} \omega(t) \leqslant C_\infty.$$

If $\omega \in \mathcal{P}$ *and* $I = (0,\infty)$, *one says* ω *is ultimately uniformly bounded.*

In [12], D. Le, L. Nguyen and T. Nguyen suppose that
(H1) There exist a continuous function Φ and positive constant d such that the differentiable functions $P,\ Q,\ R$ satisfying

$$P(u, v) \geqslant d(1 + u) > 0, \ \ |Q(u, v)| \leqslant \Phi(v)u, \ \ R(v) \geqslant d > 0, \quad \forall u,\ v \geqslant 0.$$

(H2) There exists a nonnegative continuous function $C(v)$ such that

$$|f(u, v)| \leqslant C(v)(1 + u), \quad g(u, v)u^p \leqslant C(v)\left(1 + u^{p+1}\right), \quad \forall u,\ v \geqslant 0,\ p > 0.$$

Under the above hypotheses (H1) and (H2), the authors proved the following results.

Lemma 1.2. (see [12], Theorem 2.4) *Assume* (H1) *and* (H2) *hold. Let* $(u,\ v)$ *be a nonnegative solution to* (1.3) *with its maximal existence interval* I. *If* $\|u\|_{q,r,[t,t+1]\times\Omega} = \left(\int_t^{t+1} \|u(\cdot, s)\|^r_{q,\Omega} ds\right)^{1/r}$ *(as a function in* t*) is in* $\mathcal{P}$ *for some* q, r *satisfying*

$$\tfrac{1}{r} + \tfrac{n}{2q} = 1 - \chi, \quad q \in [\tfrac{n}{2(1-\chi)}, \infty], \quad r \in [\tfrac{1}{1-\chi}, \infty] \ \text{with some } \chi \in (0, 1), \tag{1.4}$$

then there exists an absorbing ball where all solutions will enter eventually. Thus, if the system (1.3) *is autonomous then there is a compact global attractor with finite Hausdorff dimension in* $\mathscr{B}$, *which attracts all solutions, with*

$$\mathscr{B} = \left\{(u, v) \in W^{1,p_0}(\Omega) \times W^{1,p_0}(\Omega) : u(x) \geqslant 0, v(x) \geqslant 0, \forall x \in \Omega\right\}.$$

In this paper, we impose some conditions on the functions $P,\ Q,\ R,\ f,\ g$ in system (1.3) as follows.
(A1) The functions $P,\ Q,\ R$ are differentiable in there variables, and there exist constants $\beta > 0,\ b > 0$ and continuous function $\phi(v) \geqslant 0$ for $v \geqslant 0$, such that

$$P(u, v) \geqslant du^\beta, \ \ |Q(u, v)| \leqslant \phi(v)u, \ \ R(v) \geqslant d, \ \ R'(v) \geqslant 0. \tag{1.5}$$

(A2) For the reaction terms $(f,\ g)$, we assume that there exist positive constants $a,\ b,\ c,\ \alpha$ and nonnegative continuous functions $f_1(u, v),\ \varphi(v)$, such that

$$f(u, v) = a - bu^\alpha - f_1(u, v), \quad g(u, v) \leqslant \varphi(v)(1 - cu)^{\frac{\alpha+1}{2}}. \tag{1.6}$$

Remark 1. *Our assumptions* (A1)–(A2) *on* (P, Q, R, f, g) *in this paper satisfy* (H1)–(H2) *in* [12].

Now, we state our main results.

Theorem 1.3. *Suppose* (A1)–(A2) *hold and* $(u_0, v_0) \in \mathscr{B}$ *with some* $p_0 > n$. *Then* (1.3) *has a compact global attractor with finite Hausdorff dimension in the space* $\mathscr{B}$, *which attracts all the solutions, for any given* $\alpha > 0, \beta > 0$ *and* $n \leqslant 2$, *but we need the following corresponding assumptions in* (B1)–(B2) *for* $n > 2$,
(B1) *For* $0 < \beta \leqslant \frac{n-2}{6}$,
(i) *if* $0 < \alpha \leqslant \frac{n+4}{n-2}\beta$, *then* $n^2 - 2(1 + 4\beta)n + 4\beta < 0$;
(ii) *if* $\frac{n+4}{n-2}\beta < \alpha < \beta + 1$, *then* $n < 2(\alpha + 3\beta)$;
(iii) *if* $\alpha \geqslant \beta + 1$, *then* $n < 2(2\alpha + 2\beta - 1)$.
(B2) *For* $\beta > \frac{n-2}{6}$, *then*
(i) *if* $0 < \alpha \leqslant \beta + 1$, *then* $n^2 - 2(1 + 4\beta)n + 4\beta < 0$;
(ii) *if* $\beta + 1 < \alpha < \frac{n+4}{n-2}\beta$, *then* $n^2 - 2(\alpha + 3\beta)n + 4(\alpha - 1) < 0$;
(iii) *if* $\alpha \geqslant \frac{n+4}{n-2}\beta$, *then* $n < 2(2\alpha + 2\beta - 1)$.

Theorem 1.4. *Assume* $n < 10$ *and* $(u_0, v_0) \in \mathscr{B}$ *with some* $p_0 > n$, *then* (1.2) *has a compact global attractor with finite Hausdorff dimension in the space* $\mathscr{B}$, *which attracts all the solutions.*

Remark 2. *Theorem* 1.3 *and Theorem* 1.4 *imply the uniform boundedness of the global solutions to the systems* (1.3) *and* (1.2), *respectively.*

This paper is organized as follows. In section 2, we shall prove the existence of a compact global attractor with finite Hausdorff dimension to system (1.3). As an application, we consider the Shegesada-Kawasaki-Teromoto competition model (1.2), and get the existence of a global attractor for $n < 10$ in section 3.

2. The existence of a global attractor to (1.3)

We shall first give the uniform Gronwall inequality, which will be frequently used in our proof.

Lemma 2.1. (the uniform Gronwall inequality) (see [18] Chapt. 3, Lemma 1.1). *Suppose positive Lipschitz functions* $y(t)$, $r(t)$, $h(t)$ *defined on* $[t_0, +\infty]$ *satisfy*

$$y'(t) \leqslant r(t)y(t) + h(t),$$

and

$$\int_t^{t+\tau} r(s)ds \leqslant r_0, \quad \int_t^{t+\tau} h(s)ds \leqslant h_0, \quad \int_t^{t+\tau} y(s)ds \leqslant c_0, \quad \forall t \geqslant t_0,$$

with τ, r_0, h_0 *and* c_0 *some positive constants. Then it holds that*

$$y(t + \tau) \leqslant \left(\frac{c_0}{\tau} + h_0\right)e^{r_0}, \quad \forall t \geqslant t_0.$$

For given initial data $u_0(x), v_0(x) \in \mathscr{B}$, it is standard to show that the solutions of (1.3) are still nonnegative. Then using comparison principle for parabolic equation on the $v-$equation of (1.3), it is easy to see

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \in \mathcal{P}. \tag{2.1}$$

For the solution u, it is easy to get the following properties.

Lemma 2.2. *The solution u of* (1.3) *satisfies*

$$\|u(\cdot,t)\|_{L^1(\Omega)} \in \mathcal{P}, \tag{2.2}$$

and

$$\int_t^{t+1}\int_\Omega u^{\alpha+1}dxds \in \mathcal{P}. \tag{2.3}$$

Proof. Integrating the $u-$equation of (1.3) by parts and noting the condition of $f(u,v)$ in (A2), we get

$$\tfrac{d}{dt}\int_\Omega udx \leqslant a\int_\Omega udx - b\int_\Omega u^{\alpha+1}dx, \tag{2.4}$$

which together with Hölder inequality $\|u\|_{L^1(\Omega)} \leqslant \|u\|_{L^{\alpha+1}(\Omega)}\|1\|_{L^{\frac{\alpha+1}{\alpha}}(\Omega)}$ gives

$$\tfrac{d}{dt}\int_\Omega udx \leqslant a\int_\Omega udx - \tfrac{b}{|\Omega|^\alpha}\left(\int_\Omega udx\right)^{\alpha+1}. \tag{2.5}$$

Then the comparison principle of ordinary differential equation implies (2.2) holds. Integrate (2.4) from t to $t+1$ and use (2.2) to yield (2.3). □

For the solution v, we will prove the following result, which plays an important role in the following estimates of u in Theorem 2.4. In the rest of our paper, $C_i\,(i=1,2,\cdots)$ are some positive constants, and we will not point out them one by one.

Lemma 2.3. *For $n \geqslant 1$, the solution v of* (1.3) *satisfies*

$$\int_t^{t+1}\int_\Omega |\nabla v|^4 dxds \in \mathcal{P}. \tag{2.6}$$

Proof. In order to prove (2.6), we first show

$$\int_t^{t+1}\int_\Omega |\nabla\cdot(R(v)\nabla v)|^2 dxds \in \mathcal{P}, \tag{2.7}$$

then we prove

$$\int_t^{t+1}\int_\Omega |R(v)\nabla v|^4 dxds \in \mathcal{P}. \tag{2.8}$$

Recalling the condition of $R(v) \geqslant d$ in (A1), (2.8) ensures (2.6) holds.

Now, we first deal with the proof of (2.7). For this purpose, multiplying the second equation of (1.3) by v and integrating by parts, we have

$$\tfrac{1}{2}\tfrac{d}{dt}\int_\Omega v^2dx + \int_\Omega R(v)|\nabla v|^2dx = \int_\Omega v^2 g(u,v)dx.$$

Integrating the above equation over $[t,t+1]$, we obtain

$$d\int_t^{t+1}\int_\Omega |\nabla v|^2dxds \leqslant \tfrac{1}{2}\|v(t)\|^2_{L^2(\Omega)} + C_1\int_t^{t+1}\int_\Omega (1-cu)^{\frac{\alpha+1}{2}}dxds,$$

by $R(v) \geqslant d$ in (A1), $g(u,v) \leqslant \varphi(v)(1-cu)^{\frac{\alpha+1}{2}}$ in (A2) and the fact (2.1). Therefore, it is known by (2.1) and (2.2) that

$$\int_t^{t+1}\int_\Omega |\nabla v|^2dxds \in \mathcal{P}. \tag{2.9}$$

Next, we multiply the $v-$equation of (1.3) by $R(v)v_t$ and integrate by parts to get

$$\begin{aligned}\int_\Omega R(v)v_t^2dx &= \int_\Omega R(v)v_t\nabla\cdot(R(v)\nabla v)dx+\int_\Omega R(v)v_tvg(u,v)dx\\ &=-\int_\Omega \nabla\left(R(v)v_t\right)\cdot(R(v)\nabla v)dx+\int_\Omega R(v)v_tvg(u,v)dx\\ &=-\tfrac{1}{2}\tfrac{d}{dt}\int_\Omega R^2(v)|\nabla v|^2dx+\int_\Omega R(v)v_tvg(u,v)dx\\ &\leqslant -\tfrac{1}{2}\tfrac{d}{dt}\int_\Omega R^2(v)|\nabla v|^2dx+\tfrac{d}{2}\int_\Omega v_t^2dx+\tfrac{C_2}{2}\int_\Omega(1+u)^{\alpha+1}dx,\end{aligned}$$

here, we use Hölder inequality, the condition of $g(u,v)$ in (A2) and (2.1).

Due to $R(v)\geqslant d$, thus

$$\frac{d}{dt}\int_\Omega R^2(v)|\nabla v|^2dx+d\int_\Omega v_t^2dx\leqslant C_2\int_\Omega(1+u)^{\alpha+1}dx. \tag{2.10}$$

In view of (2.1), (2.3), (2.9) and using the uniform Gronwall inequality on

$$\frac{d}{dt}\int_\Omega R^2(v)|\nabla v|^2dx\leqslant C_2\int_\Omega(1+u)^{\alpha+1}dx,$$

we obtain

$$\int_\Omega R^2(v)|\nabla v|^2dx\in\mathcal{P}. \tag{2.11}$$

Moreover, integrate (2.10) over $[t,\ t+1]$ to know

$$\int_t^{t+1}\int_\Omega v_t^2(x,s)dxds\in\mathcal{P}. \tag{2.12}$$

By the $v-$equation of (1.3) and noting (2.1), we have

$$\begin{aligned}\int_\Omega|\nabla\cdot(R(v)\nabla v)|^2dx &= \int_\Omega\left[v_t-vg(u,v)\right]^2dx\\ &\leqslant 2\int_\Omega v_t^2dx+C_3+C_4\int_\Omega u^{\alpha+1}dx,\end{aligned}$$

this together with (2.12) and (2.3) gives (2.7).

Next, we will prove (2.8). Denote $\xi=R(v)\nabla v$ and note $R'(v)\geqslant 0$ in (A1) to get

$$\begin{aligned}\int_\Omega|\xi|^4dx &= \int_\Omega R(v)|\xi|^2\xi\cdot\nabla vdx\\ &=-\int_\Omega v\nabla\cdot\left(R(v)|\xi|^2\xi\right)dx\\ &=-\int_\Omega v\tfrac{R'(v)}{R(v)}|\xi|^4dx-\int_\Omega vR(v)|\xi|^2\nabla\cdot\xi dx-2\int_\Omega vR(v)\xi\cdot(\nabla\xi\cdot\xi)\,dx\\ &\leqslant-\int_\Omega vR(v)|\xi|^2\nabla\cdot\xi dx-2\int_\Omega vR(v)\xi\cdot(\nabla\xi\cdot\xi)\,dx.\end{aligned}$$

By Hölder inequality, we can get

$$-\int_\Omega vR(v)|\xi|^2\nabla\cdot\xi dx\leqslant\|vR(v)\|_{L^\infty(\Omega)}\|\xi\|^2_{L^4(\Omega)}\|\nabla\cdot\xi\|_{L^2(\Omega)},$$

and

$$-2\int_\Omega vR(v)\xi\cdot(\nabla\xi\cdot\xi)\,dx\leqslant\|vR(v)\|_{L^\infty(\Omega)}\|\xi\|^2_{L^4(\Omega)}\|\nabla\xi\|_{L^2(\Omega)},$$

thus

$$\|\xi\|^2_{L^4(\Omega)} \leqslant \|vR(v)\|_{L^\infty(\Omega)} \left(\|\nabla\cdot\xi\|_{L^2(\Omega)} + 2\|\nabla\xi\|_{L^2(\Omega)}\right). \tag{2.13}$$

Now, we will prove

$$\|\nabla\xi\|_{L^2(\Omega)} \leqslant C_6\|\nabla\cdot\xi\|_{L^2(\Omega)}. \tag{2.14}$$

Noting $\xi = R(v)\nabla v$, then we have

$$\begin{aligned} \nabla\cdot\xi &= \nabla\cdot(R(v)\nabla v) = R'(v)|\nabla v|^2 + R(v)\triangle v, \\ \nabla\xi &= \nabla(R(v)\nabla v) = R'(v)(\nabla v)^T\nabla v + R(v)\nabla^2 v, \end{aligned} \tag{2.15}$$

where we see ∇v as a row vector, $(\nabla v)^T$ is the transpose of ∇v, and $\nabla^2 v$ is a matrix.

By (2.15) and the standard elliptic regularity $\|\nabla^2 v\|_{L^2(\Omega)} \leqslant C_5\|\triangle v\|_{L^2(\Omega)}$, we can prove (2.14) holds.

Therefore, in virtue of (2.13) and (2.14), we obtain

$$\|\xi\|^2_{L^4(\Omega)} \leqslant (1+2C_6)\|vR(v)\|_{L^\infty(\Omega)}\|\nabla\cdot\xi\|_{L^2(\Omega)}.$$

This together with (2.7) and (2.1) indicates that (2.8) holds. This completes the proof of Lemma 2.3. □

Next, we shall give the critical estimates in our paper.

Theorem 2.4. *The solution u of* (1.3) *satisfies*

$$\|u\|_{L^{\bar{p}}(\Omega)} \in \mathcal{P}, \tag{2.16}$$

$$\int_t^{t+1}\int_\Omega u^{\bar{p}+\alpha}dxds \in \mathcal{P}, \tag{2.17}$$

and

$$\int_t^{t+1}\int_\Omega u^{\bar{p}+\beta-2}|\nabla u|^2 dxds \in \mathcal{P}, \tag{2.18}$$

for $\bar{p}$ satisfying (i) $\bar{p} = \alpha + 2\beta$ *or* (ii) $\bar{p} > \alpha + 2\beta$ *and* $(n-2)\bar{p} \leqslant 3n\beta$.

Proof. Multiplying the first equation in (1.3) by u^{p-1} with $p > 1$, and integrating on Ω by parts, we have

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &= -\int_\Omega \nabla u^{p-1}\cdot[P(u,v)\nabla u + Q(u,v)\nabla v]\,dx + \int_\Omega u^p f(u,v)dx \\ &= -(p-1)\int_\Omega u^{p-2}P(u,v)|\nabla u|^2dx - (p-1)\int_\Omega u^{p-2}Q(u,v)\nabla u\cdot\nabla v dx \\ &\quad + \int_\Omega u^p f(u,v)dx. \end{aligned}$$

Recalling the condition of $f(u,v)$ in (A2) and $|Q(u,v)| \leqslant \phi(v)u$ in (A1), we have

$$\int_\Omega u^p f(u,v)dx \leqslant a\int_\Omega u^p dx - b\int_\Omega u^{p+\alpha}dx,$$

and

$$\begin{aligned} \left|-\int_\Omega u^{p-2}Q(u,v)\nabla u\cdot\nabla v dx\right| &\leqslant \int_\Omega u^{p-2}|Q(u,v)||\nabla u\cdot\nabla v|dx \\ &\leqslant \|\phi(v)\|_\infty\int_\Omega |u^{p-1}\nabla u\cdot\nabla v|dx \\ &= \|\phi(v)\|_\infty\int_\Omega |u^{\frac{p+\beta-2}{2}}\nabla u\cdot u^{\frac{p-\beta}{2}}\nabla v|dx \\ &\leqslant \frac{d}{2}\int_\Omega u^{p+\beta-2}|\nabla u|^2dx + \frac{C_7}{p-1}\int_\Omega u^{p-\beta}|\nabla v|^2dx, \end{aligned}$$

by Hölder inequality.
Combining these estimates and $P(u,v) \geqslant du^{\beta}$ in (A1), then

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + \frac{d(p-1)}{2}\int_\Omega u^{p+\beta-2}|\nabla u|^2 dx + b\int_\Omega u^{p+\alpha}dx\\&\leqslant a\int_\Omega u^p dx + C_7\int_\Omega u^{p-\beta}|\nabla v|^2 dx.\end{aligned}\tag{2.19}$$

Case I: $p \leqslant \alpha + 2\beta$. In this case, we have $2(p-\beta) \leqslant p+\alpha$.
Applying Hölder inequality and Young's inequality to the last term of (2.19), we have

$$\begin{aligned}C_7\int_\Omega u^{p-\beta}|\nabla v|^2 dx &\leqslant \frac{b}{2}\int_\Omega u^{2p-2\beta}dx + C_8\int_\Omega |\nabla v|^4 dx\\&\leqslant \frac{b}{2}\int_\Omega u^{p+\alpha}dx + C_8\int_\Omega |\nabla v|^4 dx + C_9.\end{aligned}$$

Consequently, (2.19) becomes

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + \frac{d(p-1)}{2}\int_\Omega u^{p+\beta-2}|\nabla u|^2 dx + \frac{b}{2}\int_\Omega u^{p+\alpha}dx\\&\leqslant a\int_\Omega u^p dx + C_8\int_\Omega |\nabla v|^4 dx + C_9.\end{aligned}\tag{2.20}$$

Obviously, (2.20) entails

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \leqslant a\int_\Omega u^p dx + C_8\int_\Omega |\nabla v|^4 dx + C_9.\tag{2.21}$$

For the above inequality (2.21), if we can show

$$\int_t^{t+1}\int_\Omega u^p dxds \in \mathcal{P},\tag{2.22}$$

then (2.6) and the uniform Gronwall inequality yield

$$\|u\|_{L^p(\Omega)} \in \mathcal{P}.\tag{2.23}$$

Furthermore, integrating (2.20) from t to $t+1$, we can obtain

$$\int_t^{t+1}\int_\Omega u^{p+\alpha} dxds \in \mathcal{P},\tag{2.24}$$

and

$$\int_t^{t+1}\int_\Omega u^{p+\beta-2}|\nabla u|^2 dxds \in \mathcal{P}.\tag{2.25}$$

Now, we will use mathematical induction to prove that (2.22) holds for $p = \alpha+2\beta$. There exists some $k \in N_+$ such that $1 \leqslant \alpha+2\beta-k\alpha \leqslant \alpha+1$. Denote $p_0 = \alpha+2\beta-k\alpha,\ p_m = p_{m-1}+\alpha\,(m = 1, 2, \cdots, k)$. On one hand, using (2.3) and Hölder inequality, it is easy to see that (2.22) holds for $p = p_0$. On the other hand, we suppose (2.22) holds for $p = p_{m-1}$, then (2.24) means that (2.22) holds for $p = p_{m-1}+\alpha = p_m$. Hence the mathematical induction ensures that (2.22) holds for $p = p_k = \alpha + 2\beta$.

Therefore, (2.23)–(2.25) hold for $p = \alpha + 2\beta$, which implies (2.16)–(2.18) hold for $\bar{p} = \alpha + 2\beta$.

Case II: $p > \alpha + 2\beta$. In this case, we assume $(n-4)p \leqslant (3n-4)\beta$.
Let $w_p = u^{\frac{p+\beta}{2}}$ and denote w_p as w sometimes for simplicity, then (2.19) can be written as

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega w^{\frac{2p}{p+\beta}}dx + \frac{2d(p-1)}{(p+\beta)^2}\int_\Omega |\nabla w|^2 dx + b\int_\Omega w^{\frac{2(p+\alpha)}{p+\beta}}dx\\&\leqslant a\int_\Omega w^{\frac{2p}{p+\beta}}dx + C_7\int_\Omega w^{\frac{2(p-\beta)}{p+\beta}}|\nabla v|^2 dx\\&\leqslant a\int_\Omega w^{\frac{2p}{p+\beta}}dx + C_7\|w^{\frac{2(p-\beta)}{p+\beta}}\|_{L^2(\Omega)}\|\nabla v\|^2_{L^4(\Omega)}\\&= a\|w\|^{\frac{2p}{p+\beta}}_{L^{\frac{2p}{p+\beta}}(\Omega)} + C_7\|w\|^{\frac{2(p-\beta)}{p+\beta}}_{L^{\frac{4(p-\beta)}{p+\beta}}(\Omega)}\|\nabla v\|^2_{L^4(\Omega)},\end{aligned}\tag{2.26}$$

by the Hölder inequality.

The conditions $p > \alpha + 2\beta$ and $(n-4)p \leqslant (3n-4)\beta$ implies

$$\frac{2(p+\alpha)}{p+\beta} < \frac{4(p-\beta)}{p+\beta} \leqslant \frac{2n}{n-2},$$

here $\frac{2n}{n-2}$ can be replaced by $+\infty$ for $n = 2$.
It is known by Gagliardo-Nirenberg inequality that

$$\|w\|_{L^{\frac{4(p-\beta)}{p+\beta}}(\Omega)} \leqslant C_{10}\|w\|^{1-\theta}_{L^{\frac{2p}{p+\beta}}(\Omega)} \|\nabla w\|^{\theta}_{L^2(\Omega)} + C_{10}\|w\|_{L^1(\Omega)}, \tag{2.27}$$

with

$$\theta = \frac{n(p+\beta)(p-2\beta)}{2(p-\beta)(2p+n\beta)}.$$

Using (2.27) and Young's inequality, we have

$$\begin{aligned}
&C_7\|w\|^{\frac{2(p-\beta)}{p+\beta}}_{L^{\frac{4(p-\beta)}{p+\beta}}(\Omega)} \|\nabla v\|^2_{L^4(\Omega)}\\
\leqslant\ & C_{11}\|w\|^{\frac{2(p-\beta)(1-\theta)}{(p+\beta)}}_{L^{\frac{2p}{p+\beta}}(\Omega)} \|\nabla w\|^{\frac{2\theta(p-\beta)}{p+\beta}}_{L^2(\Omega)} \|\nabla v\|^2_{L^4(\Omega)} + C_{11}\|w\|^{\frac{2(p-\beta)}{p+\beta}}_{L^1(\Omega)}\|\nabla v\|^2_{L^4(\Omega)}\\
\leqslant\ & \varepsilon\|\nabla w\|^2_{L^2(\Omega)} + C_\varepsilon\|w\|^{m_1}_{L^{\frac{2p}{p+\beta}}(\Omega)} \|\nabla v\|^{m_2}_{L^4(\Omega)} + C_{11}\|w\|^{\frac{4(p-\beta)}{p+\beta}}_{L^1(\Omega)} + C_{11}\|\nabla v\|^4_{L^4(\Omega)},
\end{aligned}$$

with

$$m_1 = \frac{2(p-\beta)(1-\theta)}{p+\beta-\theta(p-\beta)}, \quad m_2 = \frac{2(p+\beta)}{p+\beta-\theta(p-\beta)}.$$

Let $\varepsilon = \frac{d(p-1)}{(p+\beta)^2}$, then (2.26) becomes

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega w^{\frac{2p}{p+\beta}}dx + \frac{d(p-1)}{(p+\beta)^2}\int_\Omega |\nabla w|^2 dx + b\int_\Omega w^{\frac{2(p+\alpha)}{p+\beta}}dx\\
\leqslant\ & a\|w\|^{\frac{2p}{p+\beta}}_{L^{\frac{2p}{p+\beta}}(\Omega)} + C_\varepsilon\|w\|^{m_1}_{L^{\frac{2p}{p+\beta}}(\Omega)} \|\nabla v\|^{m_2}_{L^4(\Omega)} + C_{11}\|w\|^{\frac{4(p-\beta)}{p+\beta}}_{L^1(\Omega)} + C_{11}\|\nabla v\|^4_{L^4(\Omega)}.
\end{aligned} \tag{2.28}$$

Let $y(t) = \|w\|^{\frac{2p}{p+\beta}}_{L^{\frac{2p}{p+\beta}}(\Omega)}$, $h(t) = C_{11}\|w\|^{\frac{4(p-\beta)}{p+\beta}}_{L^1(\Omega)} + C_{11}\|\nabla v\|^4_{L^4(\Omega)}$, then we have

$$\frac{1}{p}\frac{d}{dt}y(t) \quad\leqslant\quad ay(t) + C_\varepsilon y(t)^{\frac{(p+\beta)m_1}{2p}}\|\nabla v\|^{m_2}_{L^4(\Omega)} + h(t). \tag{2.29}$$

For the case of $y(t) \leqslant 1$, obviously we have

$$y(t) = \int_\Omega w^{\frac{2p}{p+\beta}}dx \in \mathcal{P}. \tag{2.30}$$

Since $p > \alpha + 2\beta$, (2.30) implies $\|w\|_{L^1(\Omega)} \in \mathcal{P}$ by Hölder inequality. Let

$$(n-2)p \leqslant 3n\beta,$$

then a direct calculation shows that $m_2 \leqslant 4$. Consequently, (2.6) and Hölder inequality give

$$\|\nabla v\|^{m_2}_{L^4(\Omega)} \in \mathcal{P}. \tag{2.31}$$

Furthermore, integrating (2.28) from t to $t+1$ yields

$$\int_t^{t+1}\int_\Omega w^{\frac{2(p+\alpha)}{p+\beta}}dxds \in \mathcal{P}, \tag{2.32}$$

and

$$\int_t^{t+1}\int_\Omega |\nabla w|^2 dxds \in \mathcal{P}. \tag{2.33}$$

For the case of $y(t) > 1$, denote $r(t) = a + C_\varepsilon\|\nabla v\|^{m_2}_{L^4(\Omega)}$, then

$$\frac{1}{p}\frac{d}{dt}y(t) \leqslant r(t)y(t) + h(t), \tag{2.34}$$

here, we used the fact

$$\frac{(p+\beta)m_1}{2p} = \frac{(3n-4)\beta-(n-4)p}{(4p-np+4n\beta)} < 1.$$

It is easy to see that (2.31) implies $\int_t^{t+1} r(s)ds \in \mathcal{P}$ for $(n-2)p \leqslant 3n\beta$, thus if we can show

$$\int_t^{t+1} y(s)ds \in \mathcal{P}, \quad \int_t^{t+1} h(s)ds \in \mathcal{P}, \tag{2.35}$$

then using the uniform Gronwall inequality to the inequality (2.34), we obtain (2.30). Similarly, integrate (2.28) over $[t, t+1]$ to obtain (2.32) and (2.33).

Now, we prove (2.35) for $(n-2)p = 3n\beta$. In order to get $\int_t^{t+1} h(s)ds \in \mathcal{P}$, recalling (2.6), the key step is to deal with

$$\int_t^{t+1} \|w\|^{\frac{4(p-\beta)}{p+\beta}}_{L^1(\Omega)}ds \in \mathcal{P}, \tag{2.36}$$

since the Minkowski's inequality ensures

$$\int_t^{t+1} h(s)ds \leqslant C_{11}\int_t^{t+1} \|w\|^{\frac{4(p-\beta)}{p+\beta}}_{L^1(\Omega)}ds + C_{11}\int_t^{t+1}\|\nabla v\|^4_{L^4(\Omega)}ds.$$

Now, we will prove $\int_t^{t+1} y(s)ds \in \mathcal{P}$ and (2.36) by mathematical induction, simultaneously. For the case of $n > 2$, there exists some $\bar{k} \in N_+$ such that $\alpha + 2\beta < \frac{3n\beta}{n-2} - \bar{k}\alpha \leqslant 2\alpha + 2\beta$. Denote $q_0 = \frac{3n\beta}{n-2} - \bar{k}\alpha$, $q_m = q_{m-1} + \alpha\,(m = 1, 2, \cdots, \bar{k})$. Since we have proved (2.17) for $\bar{p} = \alpha + 2\beta$, it is easy to get $\int_t^{t+1} y(s)ds \in \mathcal{P}$ for $p = q_0$ by Hölder inequality. In addition, noting that $\frac{q_0+\beta}{2} \leqslant \frac{2\alpha+3\beta}{2} < \alpha + 2\beta$ and (2.16) holds for $\bar{p} = \alpha + 2\beta$, using Hölder inequality we obtain $\|w_{q_0}\|_{L^1(\Omega)} \in \mathcal{P}$, which indicates that the result (2.36) is true for $p = q_0$. On the other hand, assume $\int_t^{t+1} y(s)ds \in \mathcal{P}$ and (2.36) hold for $p = q_{m-1}$, then (2.32) holds for $p = q_{m-1}$. According to the definition $w_p = u^{\frac{p+\beta}{2}}$, it is easy to see

$$(w_{q_{m-1}})^{\frac{2(q_{m-1}+\alpha)}{q_{m-1}+\beta}} = (w_{q_m})^{\frac{2q_m}{q_m+\beta}},$$

thus $\int_t^{t+1} y(s)ds \in \mathcal{P}$ for $p = q_m$. Moreover, the assumption implies (2.30) holds for $p = q_{m-1}$ and hence

$$\|w_{q_m}\|_{L^1(\Omega)} \leqslant C\|w_{q_{m-1}}\|^{\frac{2q_{m-1}}{q_{m-1}+\beta}}_{L^{\frac{2q_{m-1}}{q_{m-1}+\beta}}(\Omega)},$$

by $q_{m-1} > \alpha + 2\beta$ and Hölder inequality with some $C > 0$. And thus (2.36) holds for $p = q_m$.

Above all, for the case of $n > 2$, we have proved (2.35) with $p = \frac{3n\beta}{n-2}$. Therefore, (2.30), (2.32) and (2.33) hold for $p = \frac{3n\beta}{n-2}$, which implies (2.16)–(2.18) hold for $\bar{p} \leqslant \frac{3n\beta}{n-2}$. Similarly, we can prove (2.16)–(2.18) hold for any positive constant $\bar{p} > \alpha + 2\beta$ if $n \leqslant 2$.

This complete the proof of Theorem 2.4. □

Next, we will use Lemma 1.2 and Theorem 2.4 to give the proof of Theorem 1.3.

proof of Theorem 1.3. Let $s = \bar{p} + \alpha$, then by (2.17),

$$\int_t^{t+1} \|u\|_{L^s(\Omega)}^s ds = \int_t^{t+1} \int_\Omega u^{\bar{p}+\alpha} dx ds \in \mathcal{P}.$$

Define

$$1 - \chi = \frac{1}{s} + \frac{n}{2s} = \frac{n+2}{2s} = \frac{n+2}{2(\bar{p}+\alpha)},$$

$$A = s - \frac{n}{2(1-\chi)} = \frac{2s}{n+2} > 0, \quad B = s - \frac{1}{1-\chi} = \frac{ns}{n+2} > 0.$$

By Lemma 1.2, we also need $\chi \in (0,1)$, which is equivalent to

$$n < 2(\bar{p} + \alpha - 1). \tag{2.37}$$

It is known by Gagliardo-Nirenberg inequality that

$$\|w\|_{L^{2^*}(\Omega)} \leqslant C_{12} \left(\|\nabla w\|_{L^2(\Omega)} + \|w\|_{L^1(\Omega)} \right),$$

with $2^* = 2n/(n-2)$.

Let $l = \frac{\bar{p}+\beta}{2}$ and $r = 2l, q = 2^* l$, then $w = u^l$ and

$$\int_t^{t+1} \|u\|_{L^q(\Omega)}^r ds = \int_t^{t+1} \|w\|_{L^{2^*}(\Omega)}^2 ds \leqslant 2C_{12} [\int_t^{t+1} \|\nabla w\|_{L^2(\Omega)}^2 ds + \sup_{[t,t+1]} \|w\|_{L^1(\Omega)}^2].$$

The estimate $\|w\|_{L^1(\Omega)} \in \mathcal{P}$ comes from (2.16) by Hölder inequality, which together with (2.18) indicates

$$\int_t^{t+1} \|u\|_{L^q(\Omega)}^r ds \in \mathcal{P}.$$

Let

$$1 - \chi = \frac{1}{r} + \frac{n}{2q} = \frac{1}{l} \left(\frac{1}{2} + \frac{n}{2 \cdot 2^*} \right) = \frac{n}{4l} = \frac{n}{2(\bar{p}+\beta)},$$

$$A = q - \frac{n}{2(1-\chi)} = \frac{2(\bar{p}+\beta)}{n-2} > 0, \quad B = r - \frac{1}{1-\chi} = \frac{n-2}{n} (\bar{p} + \beta) > 0.$$

By Lemma 1.2, we also need $\chi \in (0,1)$, which means

$$n < 2(\bar{p} + \beta). \tag{2.38}$$

Comparing (2.37) and (2.38), we choose $n < 2(\bar{p} + \alpha - 1)$ if $\alpha > \beta + 1$, otherwise, we choose $n < 2(\bar{p} + \beta)$. In addition, recall $\bar{p}$ satisfies (i) $\bar{p} = \alpha + 2\beta$ or (ii) $\bar{p} > \alpha + 2\beta$ and $(n-2)\bar{p} \leqslant 3n\beta$ in Theorem 2.4, hence we can assign any positive number to $\bar{p}$ for $n \leqslant 2$, but we choose $\bar{p} = \frac{3n\beta}{n-2}$ if $0 < \alpha \leqslant \frac{n+4}{n-2}\beta$, otherwise, we choose $\bar{p} = \alpha + 2\beta$ for $n > 2$. Consequently, combining these analysis we can obtain Theorem 1.3.

□

3. The uniform boundedness of the solutions to S-K-T model for $n < 10$

In this part, we will consider the boundedness of the global solutions to the following S-K-T model

$$\begin{cases} u_t = \Delta\left[(d_1 + \rho_{11} u + \rho_{12} v)\, u\right] + u\,(a_1 - b_1 u - c_1 v)\,, & x \in \Omega,\ t > 0, \\ v_t = \Delta\left[(d_2 + \rho_{22} v)\, v\right] + v\,(a_2 - b_2 u - c_2 v)\,, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geqslant 0, v(x,0) = v_0(x) \geqslant 0, & x \in \Omega. \end{cases} \tag{3.1}$$

Proof of Theorem 1.4. Comparing with the divergence form of system (1.3), we have

$$P(u,v) = d_1 + 2\rho_{11} u + \rho_{12} v,\ \ Q(u,v) = \rho_{12} u,\ \ R(v) = d_2 + 2\rho_{22} v,$$

$$f(u,v) = a_1 - b_1 u - c_1 v,\ \ g(u,v) = a_2 - b_2 u - c_2 v.$$

It is easy to see that P, Q, R and f, g satisfy the conditions in (A1) and (A2), respectively, with $\alpha = \beta = 1$.

Theorem 1.3 gives Theorem 1.4 for $n \leqslant 2$ directly. Moreover, a simple computation shows (B2) (i) in Theorem 1.3 holds for $2 < n < 8$ and (B1) (i) holds for $8 \leqslant n < 10$. This completes the proof of Theorem 1.4. □

Remark 3. *Our result implies the uniform boundedness of the global solutions to the system* (3.1). *This result extends the existence results of global attractor in* [21] *from $n < 8$ to $n < 10$, and extends the uniform boundedness results of the global solutions in* [17] *to the non-convex domain.*

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Conflict of interest

The author declares no conflicts of interest in this paper.

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