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Research article

Study of mathematical model of Hepatitis B under Caputo-Fabrizo derivative

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Abstract: The current work is devoted to bring out a detail analysis including qualitative and semi-analytical study of Hepatitis B model under the Caputo- Fabrizio fractional derivative (CFFD). For the required results, fixed point theory is used to establish the conditions for existence and uniqueness of solution to the considered model. On the other hand, for semi analytical solutions, we use decomposition method of Adomian coupled with integral transform of Laplace. Moreover, the concerned solutions are presented via graphs to analyze the dynamics of different compartments of the model.

Keywords: Hepatitis B; CFFD; existence; fixed point theory; semi analytical results
Mathematics Subject Classification: 26A33, 34A08, 93A30

1. Introduction

A dangerous type of disease that causes by hepatitis virus as known as Hepatitis B. The said disease is a major problem for health all over the world. Liver suffers from serious chronic disease and infection due to the aforesaid disease and thus the lives of the people become at high risk of death. The liver cancer is also mainly caused by the mentioned disease, for detail see [1]. An individual suffers from Hepatitis B infection when the virus is able to enter the blood stream and through which it reaches the liver. Actually the virus of mentioned disease damages liver during its entrance into blood stream (for detail see [2]). Actually the disease has two stages one has a duration of six months, called acute stage which often is clarified by the immune system of human body, while the other stage is called chronic phase which has duration more than six months. Usually adults and children catch such type of infection. HBV is a serious disease and can be transmit from infected individual to healthy people and suffered individuals are chronic carriers. Nearly 240 million people have chronic liver infections around the globe. Due to this dangerous disease, approximately 0.6 million people die each year due to the aforementioned disease.

In previous century, to understand and to predict for future planing , mathematical models were introduced. In same line to understand biological process/phenomenon, mathematical models of infectious disease were also introduced in 1927. The said area has got much attention as with the help of mathematical models, we can properly understand the transmission of a disease in a community. Also one can develop some strategy how to controls the disease in our society and to get information about the cure usually used for the treatment of these disease. In this regard various kinds HBV models were also developed in last few decades. The powerful tools to understand infectious disease and their transmission dynamics are the mathematical modeling. By means of which we can understand about the transmission of the said diseases. For understanding the transmission and control of HBV has been investigated in many articles, see for detail [3–8]. Kamyad et al. [9] have investigated the mathematical model for HBV as:

$$\begin{cases} \dfrac{dS(t)}{dt} = v - [vp_1C - vp_2R + p'(I + \theta C)S + vS + \mu_1 S] + \lambda_4 R, \\ \dfrac{dE(t)}{dt} = p'(I + \theta C)S - (v + \lambda_1)E, \\ \dfrac{dI(t)}{dt} = \lambda_1 E - (v + \lambda_2)I, \\ \dfrac{dC(t)}{dt} = [vp_1 - (v + \lambda_3 - \mu_2)]C + p_3\lambda_2 I, \\ \dfrac{dR(t)}{dt} = (vp_2 - v - \lambda_4)R + (1 - p_3)\lambda_2 I + (\lambda_3 + \mu_2)C + \mu_1 S. \end{cases} \tag{1.1}$$

This model has been very well studied in [10–12]. In the give model, $S(t)$ stands for the density of susceptible, $E(t)$ exposed, $I(t)$ for infection, $C(t)$ for chronic HBV carriers and $R(t)$ recovered individuals respectively. in this model, v is used for per capita birth and death rate, while λ_1, λ_2 and λ_3 are used for the exposed individuals rate, the rate of carrier individuals and rate for to move individuals from carrier to recovered, respectively. Also, θ is the infectiousness rate of carriers relative to acute infections, while p_3 is a proportion value at which acute infected individuals are converted to carriers. According to Law of mass action, the infection transmits horizontally at $p'(1 + \theta C)S$, such that p' is used as a contact rate. Also the same infection transmits vertically with rate p_1 of newborns individuals by the term $vp_1C, (p_1 < 1)$. Also p_2 of newborns from recovered class are immune and it is expressed by $vp_2R, (p_2 < 1)$ [13]. Recently, the researchers uses the tools of fractional calculus for modeling of different dynamical phenomena in nearly in all discipline of applied sciences, because these describe the dynamical behaviors more precisely as compared to natural order derivatives. The researchers initiated the verity of concepts for fractional derivatives. The famous among these concepts are given by Riemann-Liouville, Caputo, Hadamard, etc (see [14, 15]). The aforementioned derivatives are well studied from different aspects, such as existence, stability, approximate solutions, solutions of different biological and physical models, (see [16–18]).

The aforementioned differential operators cannot describe the nonlocal dynamics, due to the involvement of singular kernel. To overcome these complications a new class of fractional operator has been introduced in 2016 as known as CFFD (see detail [19, 20]). The proposed derivative has been newly established having non-local non-singular kernel. The concerned operator has the ability to well describe all those phenomena that suffering from power or exponential decay. It has been observed that the application of proposed derivative is excellently demonstrated in the study of thermal and material sciences, (see for detail [21, 22]). In the concerned theory some time it has to complicated to obtained the exact analytic solutions for each nonlinear problem. In this connection, the researches take keen interest to obtained the approximate solution of proposed problem. There are verities of techniques present in the existence literature, see [23–28]. Probably, an important analytic approximate technique for the solution of non-linear problem is known as Adomain Decomposition Method (ADM), which works more efficiently for both ordinary and fractional differential equations, (see [29, 30]). The aforementioned technique is rarely utilized for the analytic approximate solution of FODEs with involvement of non-singular kernel, (we refer [31]). Inspired from the above mentioned literature, we investigate the series solution for underlying model via CFFD. Under the CFFD, the previous model (1.1) take the form

$$\begin{cases} {}^{CF}\mathcal{D}_t^{\delta}S = v - [vp_1C - vp_2R + p'(I + \theta C)S + vS + \mu_1 S] + \lambda_4 R, \\ {}^{CF}\mathcal{D}_t^{\delta}E = p'I + p'\theta CS - vE - \lambda_1 E, \\ {}^{CF}\mathcal{D}_t^{\delta}I = \lambda_1 E - vI - \lambda_2 I, \\ {}^{CF}\mathcal{D}_t^{\delta}C = [vp_1 - (v + \lambda_3 - \mu_2)]C + p_3\lambda_2 I, \\ {}^{CF}\mathcal{D}_t^{\delta}R = (vp_2 - v - \lambda_4)R + (1 - p_3)\lambda_2 I + (\lambda_3 + \mu_2)C + \mu_1 S \end{cases} \tag{1.2}$$

with given initial conditions/data

$$S(0) \geq 0, E(0) \geq 0, I(0) \geq 0, C(0) \geq 0, R(0) \geq 0.$$

First, we develop some results regarding existence theory by using Banach theorem, which guarantied that the solution of proposed system exists and can be physically interpreted. Further, for the concerned semi-analytical study we use Adomian decomposition method together with Laplace integral transform which is a powerful and efficient technique to handle many nonlinear problems. For graphical presentation, we use Matlab to simulate the results for some already used data available in literature.

2. Preliminaries

Definition 2.1. *[32] Let $\psi \in \mathcal{H}^1(a, b)$, $b > a$, $r \in (0, 1)$, then the CFFD may be expressed as:*

$$ {}^{CF}\mathcal{D}_t^{\delta}(\psi(t)) = \frac{\mathcal{M}(\delta)}{1-\delta}\int_a^t \psi'(\theta)\exp\Big[-\delta\frac{t-\theta}{1-\delta}\Big]d\theta,$$

where the normalizer function $\mathcal{M}(\delta)$ with $\mathcal{M}(0) - 1 = \mathcal{M}(1) - 1 = 0$. In case of failure we use the following derivative $\mathcal{H}^1(a, b)$, reformulated as:

$$^{CF}\mathcal{D}_t^{\delta}(\psi(t)) = \frac{\mathcal{M}(\delta)}{1-\delta} \int_a^t (\psi(t) - \psi(\theta)) \exp\Big[-\delta \frac{t-\theta}{1-\delta}\Big] d\theta.$$

Definition 2.2. *[32] Let $\delta \in]0, 1[$. An integral due to Caputo and Fabrizo with order δ for a function ψ may be recalled as:*

$$^{CF}\mathcal{I}_t^{\delta}[\psi(t)] = \frac{(1-\delta)}{\mathcal{M}(\delta)}\psi(t) + \frac{\delta}{\mathcal{M}(\delta)} \int_0^t \psi(\theta) d\theta, \ t \geq 0.$$

Definition 2.3. *[32] The Laplace transform of CFFD $^{CF}\mathcal{D}_t^{\delta}x(t)$, $\delta \in (0, 1]$ may be described as:*

$$\mathscr{L}[^{CF}\mathcal{D}_t^{\delta}x(t)] = \frac{s\mathscr{L}[x(t)]}{s + \delta(1-s)} - \frac{x(0)}{s + \delta(1-s)}.$$

3. Qualitative study of the proposed model (1.2)

It is natural to ask whether a model we obtain after formulating a physical phenomenon in mathematical form exists or not in real sense. This thing is guaranteed by applying the concept of fixed point theory. In this regard, the well known contraction theorem given by Banach in 1922 is mainly easy and simple to use. Therefore for the concerned model (1.2), we utilize the mentioned theory to prove existence of solution.

$$\begin{aligned}
\psi_1(t, S, E, I, C, R) &= v + \lambda_4 R - [vp_1 C + vp_2 R + p'(I + \theta C)S + vS + \mu_1 S], \\
\psi_2(t, S, E, I, C, R) &= p'(I + \theta C)S - (v + \lambda_1)E, \\
\psi_3(t, S, E, I, C, R) &= \lambda_1 E - (v + \lambda_2)I, \\
\psi_4(t, S, E, I, C, R) &= [vp_1 - (v + \lambda_3 - \mu_2)]C + p_3\lambda_2 I, \\
\psi_5(t, S, E, I, C, R) &= (vp_2 - v - \lambda_4)R + (1 - p_3)\lambda_2 I + (\lambda_3 + \mu_2)C + \mu_1 S.
\end{aligned} \tag{3.1}$$

In this regard applying the operator $^{CF}\mathcal{I}^{\delta}$ to Model (1.2) on both sides yields

$$\begin{cases}
S(t) = S(0) +^{CF} \mathcal{I}^{\delta}[\psi_1(t, S, E, I, C, R)], \\
E(t) = E(0) +^{CF} \mathcal{I}^{\delta}[\psi_2(t, S, E, I, C, R)], \\
I(t) = I(0) +^{CF} \mathcal{I}^{\delta}[\psi_3(t, S, E, I, C, R)], \\
C(t) = C(0) +^{CF} \mathcal{I}^{\delta}[\psi_4(t, S, E, I, C, R)], \\
R(t) = R(0) +^{CF} \mathcal{I}^{\delta}[\psi_5(t, S, E, I, C, R)].
\end{cases} \tag{3.2}$$

Evaluating the right hand side, we have

$$\mathbf{U}(t) = \mathbf{U}_0(t) + \Big[\Psi(t, \mathbf{U}(t)) - \Psi_0(t)\Big]\frac{1-\delta}{\mathcal{M}(\delta)} + \frac{\delta}{\mathcal{M}(\delta)} \int_0^t \Psi(\theta, \mathbf{U}(\theta)) d\theta, \tag{3.3}$$

where

$$\mathbf{U}(t)=\begin{cases}S(t)\\E(t)\\I(t)\\C(t)\\R(t)\end{cases},\quad \mathbf{U}_0(t)=\begin{cases}S(0)\\E(0)\\I(0)\\C(0)\\R(0)\end{cases},\quad \Psi(t,\mathbf{U}(t))=\begin{cases}\psi_1(t,S,E,I,C,R)\\\psi_2(t,S,E,I,C,R)\\\psi_3(t,S,E,I,C,R)\\\psi_4(t,S,E,I,C,R)\\\psi_5(t,S,E,I,C,R).\end{cases}\tag{3.4}$$

Further, we set

$$\mathbf{A}_i=\sup_{t\in[t-d,t+d]}\|\psi_1(t,S,E,I,C,R)\|,\ \text{for } i=1,2,\cdots,5,\tag{3.5}$$

such that

$$C[d,b_i]=[t-d,t+d]\times[u-c_i,u+c_i]=D\times D_i,\ \text{for } i=1,2,\cdots,5.$$

We defined the norm on $C[d,d_i]$, for $i=1,2,\cdots,5$ with help of Banach fixed point theorem as:

$$\|\mathbf{U}\|_\infty=\sup_{t\in[t-d,t+b]}|\phi(t)|.\tag{3.6}$$

Where the Picard's operator is defined as:

$$\mathbf{T}:C(D,D_1,D_2,D_3,D_4,D_5)\to C(D,D_1,D_2,D_3,D_4,D_5).\tag{3.7}$$

Thank to (3.3) and (3.4), in (3.7) the operator may be define as

$$\mathbf{TU}(t)=\mathbf{U}_0(t)+\Psi(t,\mathbf{U}(t))\frac{1-\delta}{\mathcal{M}(\delta)}+\frac{\delta}{\mathcal{M}(\delta)}\int_0^t\Psi(\theta,\mathbf{U}(\theta))d\theta.\tag{3.8}$$

For convince, we write

$$\psi_1(t,S,E,I,C,R)=\mathbf{U}(t),\ \psi_1(t,0,0,0,0,0)=\mathbf{U}_0(t),\ \text{for } i=1,2,\cdots,5.$$

Consider that the proposed problem satisfies the given result

$$\|\mathbf{U}\|_\infty\ \le\ \max\{d_1,d_2,d_3,d_4,d_5\}.\tag{3.9}$$

$$\begin{aligned}\|T\mathbf{U}(t)-\mathbf{U}_0(t)\| &= \sup_{t\in D}\left|\Psi(t,\mathbf{U}(t))\frac{1-\delta}{\mathcal{M}(\delta)}+\frac{\delta}{\mathcal{M}(\delta)}\int_0^t\Psi(\theta,\mathbf{U}(\theta))d\theta\right| \\ &\le \sup_{t\in D}\frac{1-\delta}{\mathcal{M}(\delta)}|\Psi(t,\mathbf{U}(t))|+\sup_{t\in D}\frac{\delta}{\mathcal{M}(\delta)}\int_0^t|\Psi(\theta,\mathbf{U}(\theta))|d\theta \\ &\le \frac{1-\delta}{\mathcal{M}(\delta)}\mathbf{A}+\sup_{t\in D}\frac{\delta}{\mathcal{M}(\delta)}\mathbf{A}t,\ \mathbf{A}=\max\{\mathbf{A}_i\}\text{ for } i=1,2,...,5, t_0=\max\{t\in D\} \\ &< d\mathbf{A}\le\max\{d_1,d_2,d_3,d_4,d_5\}=\bar{d},\end{aligned}\tag{3.10}$$

where define $d=\frac{1+t_0\delta}{\mathcal{M}(\delta)}$ which satisfies the relation as:

$$d<\frac{\bar{d}}{\mathbf{A}}.$$

Further to evaluate the equality given by

$$\|\mathbf{T}\mathbf{U}_1 - \mathbf{T}\mathbf{U}_2\|_\infty = \sup_{t\in D} |\mathbf{U}_1 - \mathbf{U}_2|. \tag{3.11}$$

To compute (3.11), we proceed as:

$$\begin{aligned}
\|\mathbf{T}\mathbf{U}_1 - \mathbf{T}\mathbf{U}_2\| &= \sup_{t\in D}\Bigg|\frac{1-\delta}{\mathcal{M}(\delta)}\Big(\Psi(\theta,\mathbf{U}_1(t)) - \Psi(\theta,\mathbf{U}_2(t))\Big)\\
&+ \frac{\delta}{\mathcal{M}(\delta)}\int_0^t \Big(\Psi(\theta,\mathbf{U}_1(\theta)) - \Psi(\theta,\mathbf{U}_2(\theta))\Big)d\theta\Bigg|\\
&\le \frac{1-\delta}{\mathcal{M}(\delta)}k|\mathbf{U}_1(t) - \mathbf{U}_2(t)| + \frac{\delta k}{\mathcal{M}(\delta)}\int_0^t |\mathbf{U}_1(t) - \mathbf{U}_2(t)|, \text{with } k < 1\\
&\le \left\{\frac{1-\delta}{\mathcal{M}(\delta)}k + \frac{\delta t_0}{\mathcal{M}(\delta)}k\right\}\|\mathbf{U}_1 - \mathbf{U}_2\|\\
&\le dk\|\mathbf{U}_1 - \mathbf{U}_2\|.
\end{aligned} \tag{3.12}$$

As Ψ is contraction so we have $kd < 1$, hence operator $\mathbf{T}$ is contraction. Thus the proposed system (1.2) has a unique solution.

4. General algorithm for semi-analytical results of model (1.2)

This section, is committed to series solution for the suggested system. In order to obtain desired results, we applying *"Laplace transform"* to (1.2), as

$$\begin{cases}
\mathscr{L}[S(t)] - S(0) = \dfrac{s+\delta(1-s)}{s}\mathscr{L}[v + \lambda_4 R - [vp_1C + vp_2R + p'(I+\theta C)S + vS + \mu_1 S]]\\
\mathscr{L}[E(t)] - E(0) = \dfrac{s+\delta(1-s)}{s}\mathscr{L}[p'(I+\theta C)S - (v+\lambda_1)E]\\
\mathscr{L}[I(t)] - I(0) = \dfrac{s+\delta(1-s)}{s}\mathscr{L}[\lambda_1 E - (v+\lambda_2)I]\\
\mathscr{L}[C(t)] - C(0) = \dfrac{s+\delta(1-s)}{s}\mathscr{L}[[vp_1 - (v+\lambda_3-\mu_2)]C + p_3\lambda_2 I]\\
\mathscr{L}[R(t)] - R(0) = \dfrac{s+\delta(1-s)}{s}\mathscr{L}[(vp_2 - v - \lambda_4)R + (1-p_3)\lambda_2 I + (\lambda_3+\mu_2)C + \mu_1 S].
\end{cases} \tag{4.1}$$

Consider the series solution in the form of:

$$\begin{aligned}
&S(t) = \sum_{p=0}^{\infty} S_p(t),\ E(t) = \sum_{p=0}^{\infty} E_p(t),\ I(t) = \sum_{p=0}^{\infty} I_p(t),\\
&C(t) = \sum_{p=0}^{\infty} C_p(t),\ R(t) = \sum_{p=0}^{\infty} R_p(t).
\end{aligned} \tag{4.2}$$

Further, the nonlinear terms $C(t)S(t)$ and $I(t)S(t)$ are decomposed in form of polynomials as:

$$C(t)S(t) = \sum_{p=0}^{\infty} A_p(C,S), \qquad I(t)S(t) = \sum_{p=0}^{\infty} B_p(I,S). \tag{4.3}$$

where the "Adomian polynomial" $A_p(C,S)$ may be defined as:

$$A_p(C,S) = \frac{1}{p!}\frac{d^p}{d\lambda^p}\Big[\sum_{i=0}^{p}\lambda^i C_i(t)\sum_{i=0}^{p}\lambda^i S_i(t)\Big]\Big|_{\lambda=0}.$$

In same way the polynomial B_p may also be defined. The system (4.1) becomes

$$\begin{cases}\mathscr{L}\Big[\sum_{p=0}^{\infty}S_p(t)\Big] = S(0) + \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[v - vp_1\sum_{p=0}^{\infty}C_p(t) - vp_2\sum_{p=0}^{\infty}R_p(t) - p'\sum_{p=0}^{\infty}B_p(I,S) \\ \qquad - p'\theta\sum_{p=0}^{\infty}A_p(C,S) - v\sum_{p=0}^{\infty}S_p(t) - \mu_1\sum_{p=0}^{\infty}S_p(t) + \lambda_4\sum_{p=0}^{\infty}R_p(t)\Big], \\ k\mathscr{L}\Big[\sum_{p=0}^{\infty}E_p(t)\Big] = E(0) + \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[p'\sum_{p=0}^{\infty}B_p(I,S) + p'\theta\sum_{p=0}^{\infty}A_p(C,S) - (v+\lambda_1)\sum_{p=0}^{\infty}E_p(t)\Big], \\ \mathscr{L}\Big[\sum_{p=0}^{\infty}I_p(t)\Big] = I(0) + \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[\lambda_1\sum_{p=0}^{\infty}E_p(t) - (v+\lambda_2)\sum_{p=0}^{\infty}I_p(t)\Big] \\ \mathscr{L}\Big[\sum_{p=0}^{\infty}C_p(t)\Big] = C(0) + \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[vp_1\sum_{p=0}^{\infty}C_p(t) + p_3\lambda_2\sum_{p=0}^{\infty}I_p(t) \\ \qquad - (v+\lambda_3)\sum_{p=0}^{\infty}C_p(t) - \mu_2\sum_{p=0}^{\infty}C_p(t)\Big] \\ \mathscr{L}\Big[\sum_{p=0}^{\infty}R_p(t)\Big] = R(0) + \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[vp_2\sum_{p=0}^{\infty}R_p(t) + (1-p_3)\lambda_2\sum_{p=0}^{\infty}I_p(t) + \lambda_3\sum_{p=0}^{\infty}C_p(t) \\ \qquad - v\sum_{p=0}^{\infty}R_p(t) - \lambda_4\sum_{p=0}^{\infty}R_p(t) + \mu_1\sum_{p=0}^{\infty}S_p(t) + \mu_2\sum_{p=0}^{\infty}C_p(t)\Big].\end{cases} \tag{4.4}$$

Now comparing both sides of (4.4) term by term, we obtain

$$
\begin{cases}
\mathscr{L}[S_0(t)] = S_0,\ \mathscr{L}[E_0(t)] = E_0,\ \mathscr{L}[I_0(t)] = I_0, \mathscr{L}[C_0(t)] = C_0, \mathscr{L}[R_0(t)] = R_0,\\
\mathscr{L}[S_1(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[v - vp_1C_0(t) - vp_2R_0(t) - p'B_0(I,S) - p'\theta A_0(C,S) - vS_0(t)\\
\qquad - \mu_1S_0(t) + \lambda_4R_0(t)\Big],\\
\mathscr{L}[E_1(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[p'B_0(I,S) + p'\theta A_0(C,S) - (v+\lambda_1)E_0(t)\Big],\\
\mathscr{L}[I_1(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[\lambda_1E_0(t) - (v+\lambda_2)I_0(t)\Big],\\
\mathscr{L}[C_1(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[vp_1C_0(t) + p_3\lambda_2I_0(t) - (v+\lambda_3)C_0(t) - \mu_2C_0(t)\Big],\\
\mathscr{L}[R_1(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[vp_2R_0(t) + (1-p_3)\lambda_2I_0(t) + \lambda_3C_0(t) - vR_0(t) - \lambda_4R_0(t)\\
\qquad + \mu_1S_0(t) + \mu_2C_0(t)\Big],\\
\mathscr{L}[S_2(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[v - vp_1C_1(t) - vp_2R_1(t) - p'B_1(I,S) - p'\theta A_1(C,S) - vS_1(t)\\
\qquad - \mu_1S_1(t) + \lambda_4R_1(t)\Big],\\
\mathscr{L}[E_2(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[p'B_1(I,S) + p'\theta A_1(C,S) - (v+\lambda_1)E_1(t)\Big],\\
\mathscr{L}[I_2(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[\lambda_1E_1(t) - (v+\lambda_2)I_1(t)\Big],\\
\mathscr{L}[C_2(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[vp_1C_1(t) + p_3\lambda_2I_1(t) - (v+\lambda_3)C_1(t) - \mu_2C_1(t)\Big],\\
\mathscr{L}[R_2(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[vp_2R_1(t) + (1-p_3)\lambda_2I_1(t) + \lambda_3C_1(t) - vR_1(t) - \lambda_4R_1(t)\\
\qquad + \mu_1S_1(t) + \mu_2C_1(t)\Big],\\
\vdots\\
\mathscr{L}[S_{p+1}(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[v - vp_1C_p(t) - vp_2R_p(t) - p'B_p(I,S) - p'\theta A_p(C,S) - vS_p(t)\\
\qquad - \mu_1S_p(t) + \lambda_4R_p(t)\Big],\\
\mathscr{L}[E_{p+1}(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[p'B_p(I,S) + p'\theta A_p(C,S) - (v+\lambda_1)E_p(t)\Big],\\
\mathscr{L}[I_{p+1}(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[\lambda_1E_p(t) - (v+\lambda_2)I_p(t)\Big],\\
\mathscr{L}[C_{p+1}(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[vp_1C_p(t) + p_3\lambda_2I_p(t) - (v+\lambda_3)C_p(t) - \mu_2C_p(t)\Big],\\
\mathscr{L}[R_{p+1}(t)] = \dfrac{s+\delta(1-s)}{s}\mathscr{L}\Big[vp_2R_p(t) + (1-p_3)\lambda_2I_p(t) + \lambda_3C_p(t) - vR_p(t) - \lambda_4R_p(t)\\
\qquad + \mu_1S_p(t) + \mu_2C_p(t)\Big],\ p \geq 0.
\end{cases}
\tag{4.5}
$$

Exercising the *"Laplace transform"* in (4.5), one has

$$\begin{cases} S_0(t) = S_0,\ E_0(t) = E_0,\ I_0(t) = I_0,\ C_0(t) = C_0,\ R_0(t) = R_0, \\ S_1(t) = \Big[v - vp_1C_0(t) - vp_2R_0(t) - p'I_0(t)S_0(t) - p'\theta C_0(t)S_0(t) - vS_0(t) \\ \qquad - \mu_1 S_0(t) + \lambda_4 R_0(t)\Big](1 + \delta(t-1)), \\ E_1(t) = \Big[p'I_0(t)S_0(t) + p'\theta A_0(C,S) - (v+\lambda_1)E_0(t)\Big](1+\delta(t-1)), \\ I_1(t) = \Big[\lambda_1 E_0(t) - (v+\lambda_2)I_0(t)\Big](1+\delta(t-1)), \\ C_1(t) = \Big[vp_1C_0(t) + p_3\lambda_2 I_0(t) - (v+\lambda_3)C_0(t) - \mu_2 C_0(t)\Big](1+\delta(t-1)), \\ R_1(t) = \Big[vp_2R_0(t) + (1-p_3)\lambda_2 I_0(t) + \lambda_3 C_0(t) - vR_0(t) - \lambda_4 R_0(t) \\ \qquad + \mu_1 S_0(t) + \mu_2 C_0(t)\Big](1+\delta(t-1)), \\ S_2(t) = \Big[v - vp_1C_1(t) - vp_2R_1(t) - p'I_1(t)S_1(t) - p'\theta A_1(C,S) - vS_1(t) \\ \qquad - \mu_1 S_1(t) + \lambda_4 R_1(t)\Big](1+\delta^2(t-1)), \\ E_2(t) = \Big[p'I_1(t)S_1(t) + p'\theta A_1(C,S) - (v+\lambda_1)E_1(t)\Big](1+\delta^2(t-1)), \\ I_2(t) = \Big[\lambda_1 E_1(t) - (v+\lambda_2)I_1(t)\Big](1+\delta^2(t-1)), \\ C_2(t) = \Big[vp_1C_1(t) + p_3\lambda_2 I_1(t) - (v+\lambda_3)C_1(t) - \mu_2 C_1(t)\Big](1+\delta^2(t-1)), \\ R_2(t) = \Big[vp_2R_1(t) + (1-p_3)\lambda_2 I_1(t) + \lambda_3 C_1(t) - vR_1(t) - \lambda_4 R_1(t) \\ \qquad + \mu_1 S_1(t) + \mu_2 C_1(t)\Big](1+\delta^2(t-1)), \end{cases} \tag{4.6}$$

and so on. Proceeding on the same way, we obtain the other terms. The intended solution may be expressed as:

$$\begin{cases} S(t) = \sum_{j=0}^{\infty} S_j(t),\ E(t) = \sum_{j=0}^{\infty} E_j(t),\ I(t) = \sum_{j=0}^{\infty} I_j(t), \\ C(t) = \sum_{j=0}^{\infty} C_j(t),\ \ R(t) = \sum_{j=0}^{\infty} R_j(t), \end{cases} \tag{4.7}$$

5. Graphical presentation and discussion

Here, we provide approximate solution for the considered model. In view of the values given in [9] as $v = 0.0121, p' = 0.820, \theta = 0.1,$, $\lambda_1 = 6$ per year, $\lambda_2 = 4$ per year, $\lambda_3 = 0.025$ per year, $\lambda_4 = 0.06$, $p_1 = 0.11, p_2 = 0.1, p_3 = 0.059, \mu_1 = 0.45, \mu_2 = 0.9$ in (4.6) under the given numerical value, we plot the approximate solution up to first ten terms using Matlab in Figures 1–5.

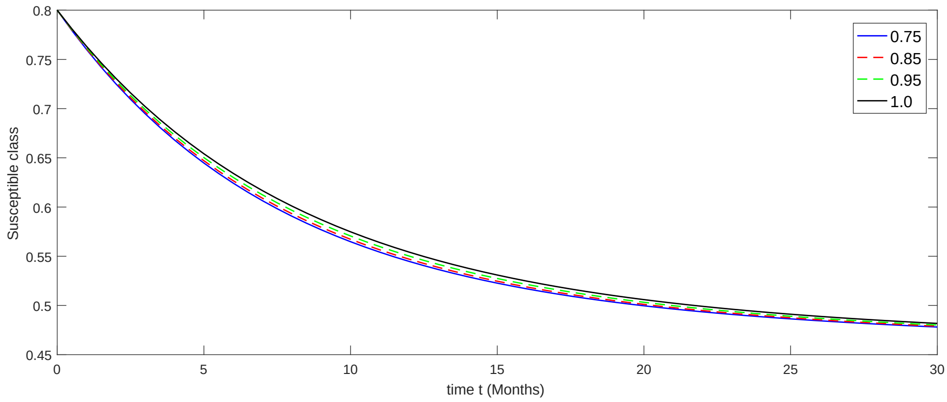

Figure 1. Graphical representation of susceptible individuals corresponding to different fractional order of the proposed model (1.2).

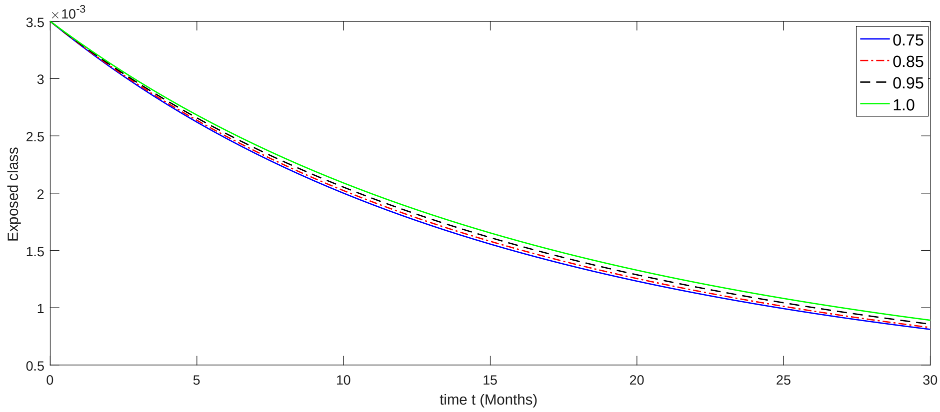

Figure 2. Graphical representation of exposed individuals corresponding to different fractional order of the proposed model (1.2).

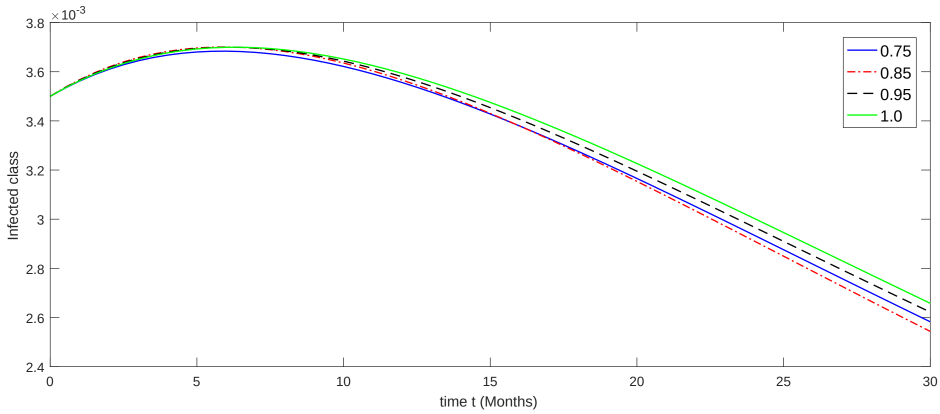

Figure 3. Graphical representation of infected individuals corresponding to different fractional order of the proposed model (1.2).

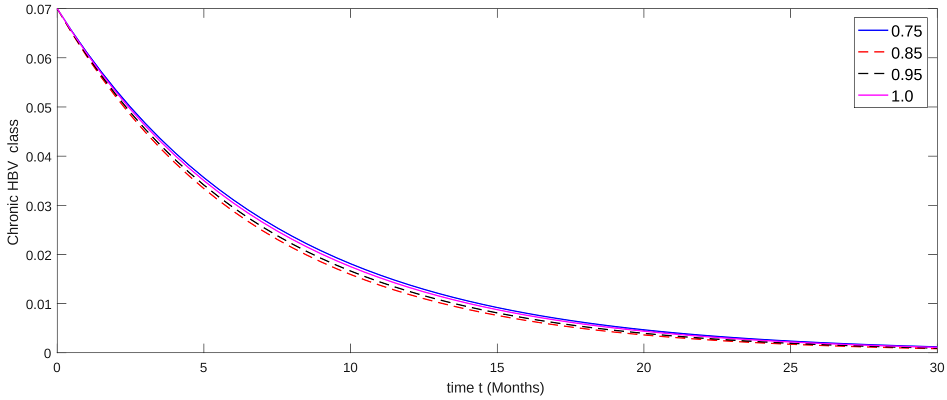

Figure 4. Graphical representation of chronic carriers of HBV corresponding to different fractional order of the proposed model (1.2).

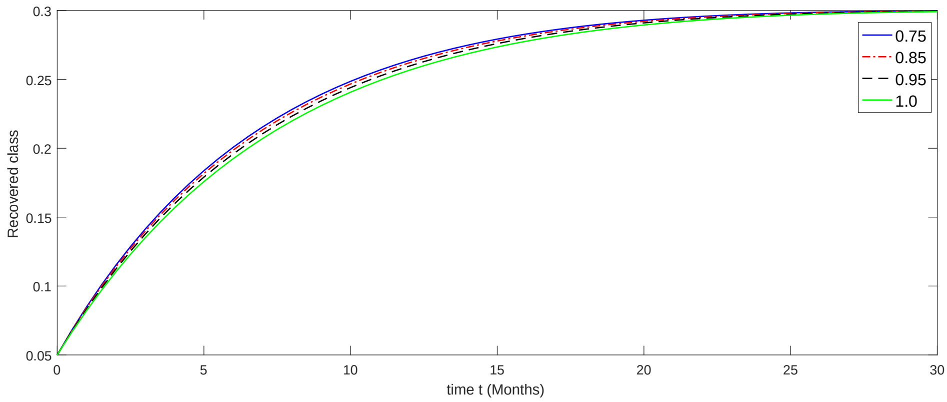

Figure 5. Graphical representation of recovered individuals corresponding to different fractional order of the proposed model (1.2).

From Figure 1, one may observe that the density of susceptible population is decreasing sharply with different rate of corresponding to various fractional order. Upon using vaccination, the density of exposed infected and chronic carriers of HBV are decreasing. The decline is faster at lower fractional order as in Figures 2–4 respectively. As a results the density of recovered corresponding to different fractional order of the proposed model (1.2) is raising up as in Figure 5. Further we compared the obtained solution in (4.7) up to ten terms with the solution of RK4 method as used in [9] for the proposed model corresponding to integer order one. We see from Figures 6–10 that the concerned solution have close agreement with each other.

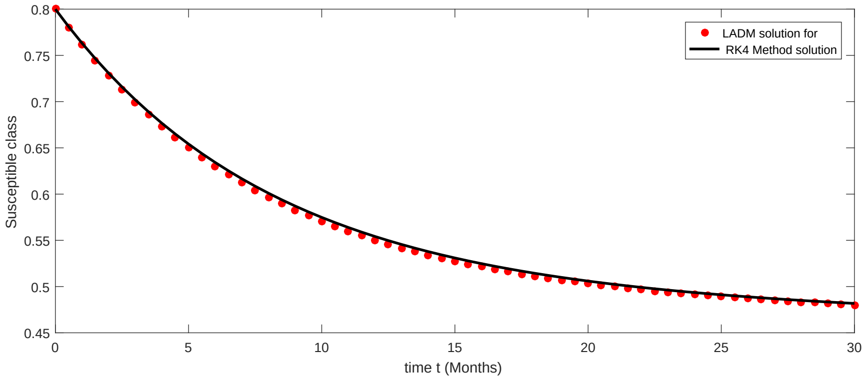

Figure 6. Comparison between our proposed solution up to ten terms with that of RK4 method for susceptible individuals of the proposed model (1.2).

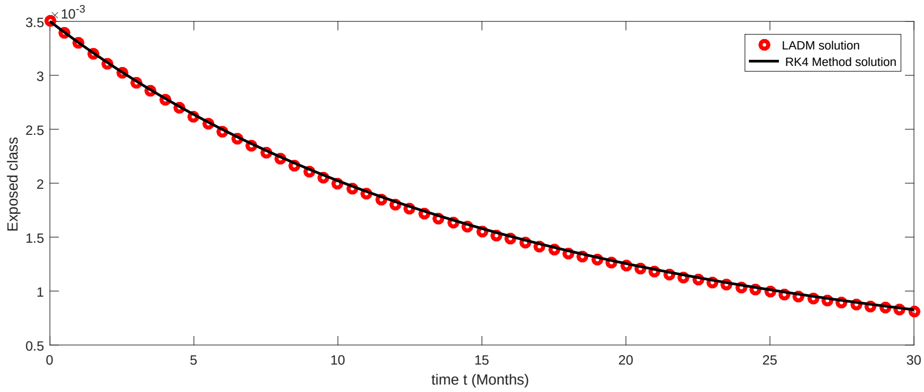

Figure 7. Comparison between our proposed solution up to ten terms with that of RK4 method for exposed individuals of the proposed model (1.2).

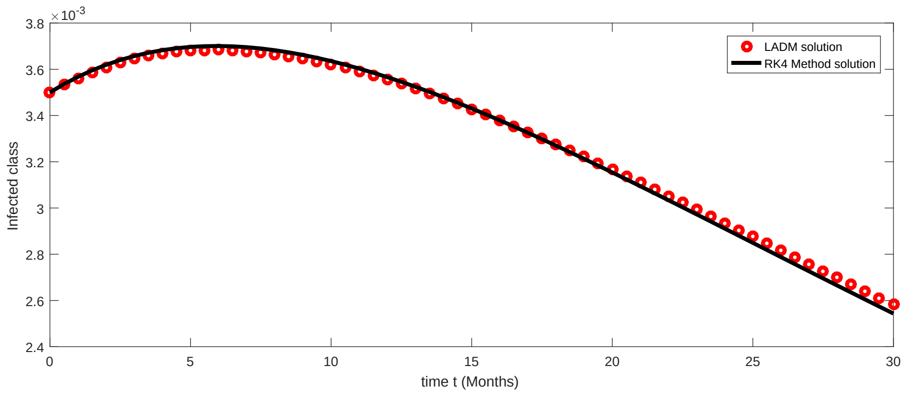

Figure 8. Comparison between our proposed solution up to ten terms with that of RK4 method for infected individuals of the proposed model (1.2).

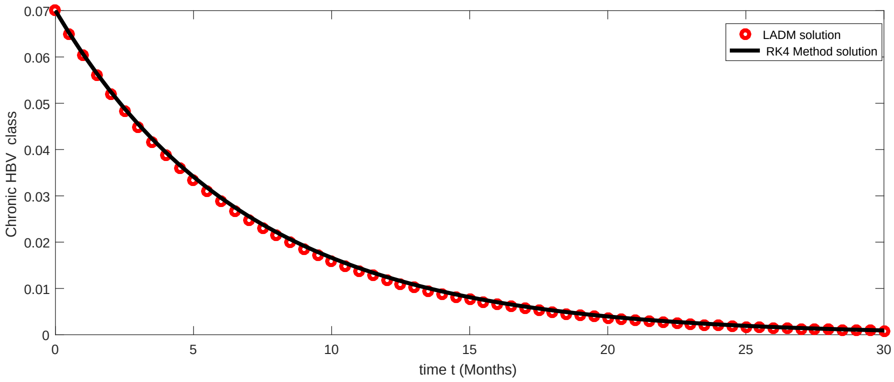

Figure 9. Comparison between our proposed solution up to ten terms with that of RK4 method for chronic carriers of HBV individuals of the proposed model (1.2).

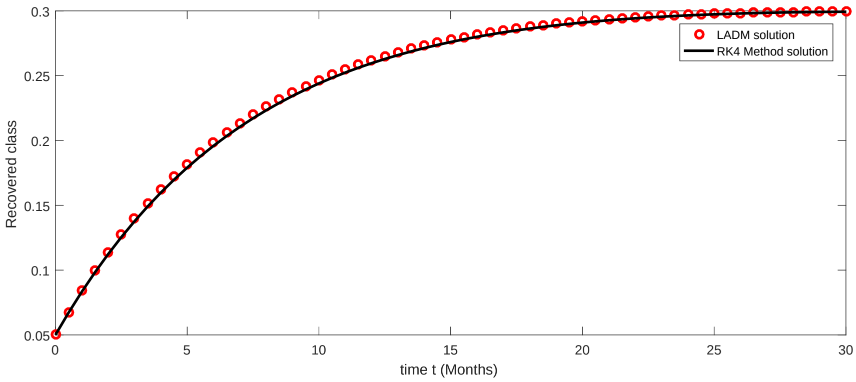

Figure 10. Comparison between our proposed solution up to ten terms with that of RK4 method for recovered individuals of the proposed model (1.2).

6. Conclusion

In this article, we have developed criteria to investigate HBV models from qualitative and analytical aspects. With the help of Picard type contraction operator and using Banach theorem we have proved that our proposed model has solution and physical existence. Further to study in more detail, the transmission and vaticination process of HBV, we have developed an algorithm to investigate semi-analytical solutions for the proposed model. We have presented graphically the approximate solutions up to few terms which has explained the dynamics more comprehensively. In conclusion, we state that CFFD can be also used as powerful tools to study biological models more comprehensively. Further the used method is powerful tool which can give excellent solution closely related to that of RK4 method results. Further the proposed method has been proved in literature a fastest convergent technique as compared to other method.

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Conflict of interest

We have no conflict of interest regarding this paper.

References

1. WHO, Hepatitis B Fact Sheet No. 204, The World Health Organisation, Geneva, Switzerland, 2013. Available from: http://www.who.int/ mediacentre/factsheets/fs204/en/.
2. W. M. Lee, Hepatitis B virus infection, *New Engl. J. Med.*, **337** (1997), 1733–1745.
3. R. M. Anderson, R. M. May, *Infectious disease of humans: dynamics and control*, Oxford: Oxford University Press, 1991.
4. G. F. Medley, N. A. Lindop, W. J. Edmunds, D. J. Nokes, Hepatitis-B virus endemicity: heterogeneity, catastrophic dynamics and control, *Nat. Med.*, **7** (2001), 619–624.
5. J. Mann, M. Roberts, Modelling the epidemiology of hepatitis B in New Zealand, *J. Theor. Biol.*, **269** (2011), 266–272.
6. S. Thornley, C. Bullen, M. Roberts, Hepatitis B in a high prevalence New Zealand population: A mathematical model applied to infection control policy, *J. Theor. Biol.*, **254** (2008), 599–603.
7. S. J. Zhao, Z. Y. Xu, Y. Lu, A mathematical model of hepatitis B virus transmission and its application for vaccination strategy in China, *Int. J. Epidemiol.*, **29** (2000), 744–752.
8. K. Wang, W. Wang, S. Song, Dynamics of an HBV model with diffusion and delay, *J. Theor. Biol.*, **253** (2008), 36–44.
9. A. V. Kamyad, R. Akbari, A. A. Heydari, A. Heydari, Mathematical modeling of transmission dynamics and optimal control of vaccination and treatment for hepatitis B virus, *Comput. Math. Methods Med.*, **2014** (2014), 1–15.
10. H. A. A. El-Saka, The fractional-order SIS epidemic model with variable population size, *J. Egypt. Math. Soc.*, **22** (2014), 50–54.
11. R. Toledo-Hernandez, V. Rico-Ramirez, G. A. Iglesias-Silva, U. M. Diwekar, A fractional calculus approach to the dynamic optimization of biological reactive systems. Part I: Fractional models for biological reactions, *Chem. Eng. Sci.*, **117** (2014), 217–228.
12. J. Pang, J. A. Cui, X. Zhou, Dynamical behavior of a hepatitis B virus transmission model with vaccination, *J. Theor. Biol.*, **265** (2010), 572–578.
13. E. Jung, S. Lenhart, Z. Feng, Optimal control of treatments in a two-strain tuberculosis model, *Discrete Cont. Dyn. B*, **2** (2002), 473–482.
14. I. Podlubny, *Fractional differential equations, mathematics in science and engineering*, New York: Academic Press, 1999.

15. A. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and application of fractional differential equations*, Amsterdam: Elseveir, 2006.
16. S. A. Khan, K. Shah, G. Zaman, F. Jarad, Existence theory and numerical solutions to smoking model under Caputo-Fabrizio fractional derivative, *Chaos*, **29** (2019), 013128.
17. F. Haq, K. Shah, G. Rahman, M. Shahzad, Numerical solution of fractional order smoking model via Laplace Adomian decomposition method, *Alex. Eng. J.*, **57** (2018), 1061–1069.
18. A. Ali, K. Shah, R. A. Khan, Numerical treatment for traveling wave solutions of fractional Whitham-Broer-Kaup equations, *Alex. Eng. J.*, **57** (2018), 1991–1998.
19. M. Caputo, M. Fabrizio, A new definition of fractional derivative with out singular kernel, *Progr. Fract. Diff. Appl.*, **1** (2015), 73–85.
20. T. Abdeljawad, D. Baleanu, On fractional derivatives with exponential kernel and their 28 discrete versions, *Rep. Math. Phys.*, **80** (2017), 11–27.
21. M. Al-Refai, T. Abdeljawad, Analysis of the fractional diffusion equations with fractional 19 derivative of non-singular kernel, *Adv. Differ. Equ.*, **2017** (2017), 315.
22. T. Abdeljawad, Fractional operators with exponential kernels and a Lyapunov type inequality, *Adv. Differ. Equ.*, **2017** (2017), 313.
23. S. J. Liao, *Beyond perturbation: Introduction to the homotopy analysis method*, Boca Raton: Chapman Hall/CRC Press, 2003.
24. M. Rafei, D. D. Ganji, H. Daniali, Solution of the epidemic model by homotopy perturbation method, *Appl. Math. Comput.*, **187** (2007), 1056–1062.
25. M. Rafei, H. Daniali, D. D. Ganji, Variational iteration method for solving the epidemic model and the prey and predator problem, *Appl. Math. Comput.*, **186** (2007), 1701–1709.
26. F. Awawdeh, A. Adawi, Z. Mustafa, Solutions of the SIR models of epidemics using HAM, *Chaos Solit. Frac.*, **42** (2009), 3047–3052.
27. O. A. Arqub, A. El-Ajou, Solution of the fractional epidemic model by homotopy analysis method, *J. King Saud Univ. Sci.*, **25** (2013), 73–81.
28. S. Z. Rida, A. A. M. Arafa, Y. A. Gaber, Solution of the fractional epidimic model by LADM, *Frac. Calc. Appl.*, **7** (2016), 189–195.
29. O. Kiymaz, An algorithm for solving initial value problems using Laplace Adomian decomposition method, *Appl. Math. Sci.*, **3** (2009), 1453–1459.
30. A. S. Khuri, A Laplace decomposition algorithm applied to a class of nonlinear differential equations, *J. Appl. Math.*, **1** (2001), 141–155.
31. A. Shaikh, A. Tassaddiq, K. S. Nisar, D. Baleanu, Analysis of differential equations involving Caputo-Fabrizio fractional operator and its applications to reaction-diffusion equations, *Adv. Differ. Equ.*, **2019** (2019), 178.
32. A. Atangana, B. S. Talkahtani, Extension of the resistance, inductance, capacitance electrical circuit to fractional derivative without singular kernel, *Adv. Mech. Eng.*, **7** (2015), 1–6.