AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(1): 168–194.
DOI: 10.3934/math.2021012
Received: 09 May 2020
Accepted: 21 September 2020
Published: 09 October 2020

Research article

Existence and Hyers-Ulam type stability results for nonlinear coupled system of Caputo-Hadamard type fractional differential equations

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Abstract: This paper aims to present the existence, uniqueness, and Hyers-Ulam stability of the coupled system of nonlinear fractional differential equations (FDEs) with multipoint and nonlocal integral boundary conditions. The fractional derivative of the Caputo-Hadamard type is used to formulate the FDEs, and the fractional integrals described in the boundary conditions are due to Hadamard. The consequence of existence is obtained employing the alternative of Leray-Schauder, and Krasnoselskii's, whereas the uniqueness result, is based on the principle of Banach contraction mapping. We examine the stability of the solutions involved in the Hyers-Ulam type. A few examples are presented as an application to illustrate the main results. Finally, it addresses some variants of the problem.

Keywords: coupled system; Caputo-Hadamard derivatives; Hadamard integrals; multi-points
Mathematics Subject Classification: 26A33, 34A08, 34B10, 34B15

1. Introduction

Recently fractional differential equations (FDEs) have been used in various fields of physics, bioengineering, biology, aerodynamics, chemistry, applied sciences etc. We refer the reader to the articles and books of [2, 6, 11–17, 19, 20, 23, 24, 28] for certain foundational concepts in the theory of fractional calculus and FDEs, and the references cited therein. The majority of the works on the FDEs are based on fractional derivatives in the types of Riemann-Liouville, Caputo, and Hadamard.

In 2012, Jarad et al. modified the fractional derivative of Hadamard type into a more suitable one with physically interpretable initial conditions comparable to the singles in the Caputo setting and named it fractional derivative Caputo-Hadamard type. Refer to [8] for defining the properties of the modified derivative. Coupled systems of differential equations of fractional order with different boundary conditions have received considerable attention. These structures were used in many real-world experiments, such as modeling of infections [7], controlling chaotic systems [30], etc. A series of papers [5, 9, 10, 22, 27] and references cited therein include some recent studies on coupled fractional-order BVPs. A coupled fractional BVPs have recently begun to be studied by a few authors. The existence of solutions of the following BVP of Hadamard type FDEs with integral boundary conditions was studied by Muthaiah *et al.* [18]:

$$\begin{cases} {}^{H}\mathcal{D}^{\varrho}y(\tau) = g(\tau, y(\tau)), \\ y(1) = 0, \;\; y^{'}(1) = 0, {}^{H}\mathcal{D}^{\varsigma}y(T) = \omega {}^{H}\mathcal{I}^{\gamma}y(\varphi), \\ 1 < \;\tau < T, \;\; 2 < \varrho \leq 3 \end{cases}$$

${}^{H}\mathcal{D}^{\varrho}, {}^{H}\mathcal{D}^{\varsigma}$ denote the Hadamard fractional derivatives (HFDs) of order ϱ, ς, $g\colon [1, T]\times\mathbb{R} \to \mathbb{R}$ is a given continuous function, and ω is a positive real constant. The results are obtained through the applying of various fixed-point theorems. The nonlinear coupled system of Hadamard FDEs

$$\begin{cases} \mathcal{D}^{\alpha}u(t) = f(t, u(t), v(t)), \\ \mathcal{D}^{\beta}v(t) = g(t, u(t), v(t)), \\ u(1) = 0, \;\; u(e) = \mathcal{I}^{\gamma}u(\sigma_1), \;\; \sigma_1 \in (1, e), \\ v(1) = 0, \;\; v(e) = \mathcal{I}^{\gamma}v(\sigma_2), \;\; \sigma_2 \in (1, e), \\ 1 < t < e, \;\; 1 < \alpha,\beta \leq 2, \;\; \gamma > 0, \end{cases}$$

has been discussed in [4], where $\mathcal{D}^{\alpha},\mathcal{D}^{\beta}$ denote the HFDs of order α,β, $f, g\colon [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The existence and uniqueness of solutions are proved by Leray-Schauder alternative and contraction mapping principle. Agarwal *et al.* [1] addressed the consequences of the existence of coupled fractional-order systems with discrete and integral boundary conditions. Subramanian *et al.* [25] studied coupled non-local slit-strip conditions in fractional BVP involving the Caputo derivatives. Similarly, Ahmad *et al.* [3] analyzed the coupled system of sequential fractional BVP, under periodic/antiperiodic boundary conditions. Recently, Subramanian *et al.* [26] investigated the existence of solutions involving Caputo derivative with integral sub-strips and multi-point BVP for coupled FDEs.

We are investigating a new BVP of Caputo Hadamard type FDEs in this article:

$$\begin{cases} {}^{C}\mathcal{D}^{\varrho}y(\tau) = f(\tau, y(\tau), z(\tau)), \;\; \tau \in [1, T] := \mathcal{K}, \\ {}^{C}\mathcal{D}^{\varsigma}z(\tau) = g(\tau, y(\tau), z(\tau)), \;\; \tau \in [1, T] := \mathcal{K}, \end{cases} \tag{1.1}$$

enhanced with boundary conditions defined by:

$$\begin{cases} y(1) = 0, \;\; y^{'}(1) = 0, \;\; y(T) = \alpha_1 \sum_{j=1}^{k-2} \xi_j z(\zeta_j) + \beta_1 {}^{H}\mathcal{I}^{\varsigma_1} z(\vartheta), \\ z(1) = 0, \;\; z^{'}(1) = 0, \;\; z(T) = \alpha_2 \sum_{j=1}^{k-2} \upsilon_j y(\omega_j) + \beta_2 {}^{H}\mathcal{I}^{\varrho_1} y(\varphi), \\ 1 < \vartheta < \varphi < \zeta_1 < \omega_1 < \zeta_2 < \omega_2 < \cdots < \zeta_{k-2} < \omega_{k-2} < T, \end{cases} \tag{1.2}$$

where ${}^{C}\mathcal{D}^{(\cdot)}$ denote the Caputo-Hadamard fractional derivatives (CHFDs) of order $(\cdot)$, $2 < \varrho, \varsigma \leq 3$, ${}^{H}\mathcal{I}^{(\cdot)}$ denote the Hadamard fractional integrals (HFIs) of order $(\cdot)$, $0 < \varrho_1, \varsigma_1 < 1$, $f, g : \mathcal{K} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, α_1, α_2, β_1 and β_2 are real constants and ξ_j, υ_j, $j = 1, 2, ..., k-2$ are positive real constants. Additionally, we are studying the system (1.1) under the condition:

$$\begin{cases} y(1) = 0, \;\; y^{'}(1) = 0, \;\; y(T) = \alpha_1 \sum\limits_{j=1}^{k-2} \xi_j z(\zeta_j) + \beta_1 {}^{H}\mathcal{I}^{\varsigma_1} z(\vartheta), \\ z(1) = 0, \;\; z^{'}(1) = 0, \;\; z(T) = \alpha_2 \sum\limits_{j=1}^{k-2} \upsilon_j y(\zeta_j) + \beta_2 {}^{H}\mathcal{I}^{\varrho_1} y(\vartheta), \\ 1 < \vartheta < \zeta_1 < \zeta_2 < \cdots < \zeta_{k-2} < T. \end{cases} \tag{1.3}$$

Remember that the conditions (1.2) contain the strips of the different lengths, while the one found in (1.3) is of the same length $(1, \vartheta)$. On the other side, as opposed to the multi-point boundary conditions in (1.3), the multi-point boundary conditions in (1.2) contain different multi-points. The rest of the article is formed as follows: Section 2 focuses primarily on certain basic concepts of fractional calculus with the related basic lemmas. The consequences of existence and uniqueness can be addressed using the Leray-Schauder, Krasnoselskii's, and Banach fixed-point theorems in Section 3. Examples are given in Section 4 for verification of the results. Section 5 discusses the stability of the Hyers-Ulam solutions and establishes sufficient conditions of stability. In Section 6, the stability result is well illustrated with the aid of an example. The existence, uniqueness, and stability results for the problem (1.1)–(1.3) are presented in Section 7.

2. Preliminaries

Here we remember some preliminary ideas of the fractional calculus of Hadamard and Caputo-Hadamard relevant to our research. We are also proving lemmas, which plays a vital role in turning the given problem into a fixed point problem [8, 11, 19].

Definition 2.1. *Let $0 \leq b \leq c \leq \infty$ be finite or infinite interval of the half-axis $\mathbb{R}^{+}$. The HFIs of order $\varrho \in \mathbb{C}$ are defined by*

$$\begin{aligned} (\mathcal{I}^{\varrho}_{b+} h)(\tau) &= \frac{1}{\Gamma(\varrho)} \int_b^{\tau} \Big(\log \frac{\tau}{\theta}\Big)^{\varrho-1} h(\theta) \frac{d\theta}{\theta}, \;\; b < \tau < c, \;\; \textit{and} \\ (\mathcal{I}^{\varrho}_{c-} h)(\tau) &= \frac{1}{\Gamma(\varrho)} \int_{\tau}^{c} \Big(\log \frac{\theta}{\tau}\Big)^{\varrho-1} h(\theta) \frac{d\theta}{\theta}, \;\; b < \tau < c. \end{aligned}$$

Definition 2.2. *The left and right-sided Hadamard fractional derivatives of order $\varrho \in \mathbb{C}$ with $\mathbb{R}(\varrho) \geq 0$ on (b, c) and $b < \tau < c$ are defined by*

$$\begin{aligned} (\mathcal{D}^{\varrho}_{b+} h)(\tau) &= \Big(\tau \frac{d}{d\tau}\Big)^{n} \frac{1}{\Gamma(n-\varrho)} \int_b^{\tau} \Big(\log \frac{\tau}{\theta}\Big)^{n-\varrho-1} h(\theta) \frac{d\theta}{\theta}, \;\; \textit{and} \\ (\mathcal{D}^{\varrho}_{c-} h)(\tau) &= \Big(-\tau \frac{d}{d\tau}\Big)^{n} \frac{1}{\Gamma(n-\varrho)} \int_{\tau}^{c} \Big(\log \frac{\theta}{\tau}\Big)^{n-\varrho-1} h(\theta) \frac{d\theta}{\theta}, \end{aligned}$$

where $n = [\mathbb{R}(\varrho)] + 1$.

Lemma 2.3. *If* $\mathbb{R}(\varrho) > 0$, $\mathbb{R}(\varsigma) > 0$ *and* $0 < b < c < \infty$, *then we have*

$$\Big(\mathcal{I}^{\varrho}_{b+}\Big(\log\frac{\theta}{b}\Big)^{\varsigma-1}\Big)(\tau) = \frac{\Gamma(\varsigma)}{\Gamma(\varsigma+\varrho)}\Big(\log\frac{\tau}{b}\Big)^{\varsigma+\varrho-1},$$
$$\Big(\mathcal{I}^{\varrho}_{c-}\Big(\log\frac{c}{\theta}\Big)^{\varsigma-1}\Big)(\tau) = \frac{\Gamma(\varsigma)}{\Gamma(\varsigma+\varrho)}\Big(\log\frac{c}{\tau}\Big)^{\varsigma+\varrho-1}.$$

Definition 2.4. *Let* $0 < b < c < \infty$, $\mathbb{R}(\varrho) \geq 0$, $n = [\mathbb{R}(\varrho) + 1]$. *The left and right CHFDs of order* ϱ *are respectively defined by*

$$({}^{C}\mathcal{D}^{\varrho}_{b+}h)(\tau) \quad = \quad \mathcal{D}^{\varrho}_{b+}\Bigg[h(\theta) - \sum_{k=0}^{n-1}\frac{\delta^k h(b)}{k!}\Big(\log\frac{\theta}{b}\Big)^k\Bigg](\tau),$$

and

$$({}^{C}\mathcal{D}^{\varrho}_{c-}h)(\tau) \quad = \quad \mathcal{D}^{\varrho}_{c-}\Bigg[h(\theta) - \sum_{k=0}^{n-1}\frac{(-1)^k\delta^k h(c)}{k!}\Big(\log\frac{c}{\theta}\Big)^k\Bigg](\tau).$$

Lemma 2.5. *Let* $\mathbb{R}(\varrho) > 0$, $n = [\mathbb{R}(\varrho)] + 1$ *and* $h \in C[b, c]$. *If* $\mathbb{R}(\varrho) \neq 0$ *or* $\varrho \in \mathbb{N}$, *then*

$${}^{C}\mathcal{D}^{\varrho}_{b+}(\mathcal{I}^{\varrho}_{b+}h)(\tau) = h(\tau), \quad {}^{C}\mathcal{D}^{\varrho}_{c-}(\mathcal{I}^{\varrho}_{c-}h)(\tau) \; = h(\tau).$$

Lemma 2.6. *Let* $h \in \mathcal{AC}^n_\delta[b, c]$ *or* $C^n_\delta[b, c]$ *and* $\varrho \in \mathbb{C}$, *then*

$$\mathcal{I}^{\varrho}_{b+}({}^{C}\mathcal{D}^{\varrho}_{b+}h)(\tau) = h(\tau) - \sum_{j=0}^{n-1}\frac{\delta^j h(b)}{j!}\Big(\log\frac{\tau}{b}\Big)^j,$$
$$\mathcal{I}^{\varrho}_{c-}({}^{C}\mathcal{D}^{\varrho}_{c-}h)(\tau) = h(\tau) - \sum_{j=0}^{n-1}\frac{\delta^j h(c)}{j!}\Big(\log\frac{c}{\tau}\Big)^j.$$

Lemma 2.7. *Let* $\hat{f}, \hat{g} \in \mathcal{AC}^n_\delta[1, T]$. *Then, the linear system solution of FDEs*

$$\begin{cases} {}^{C}\mathcal{D}^{\varrho}y(\tau) = \hat{f}(\tau), \\ {}^{C}\mathcal{D}^{\varsigma}z(\tau) = \hat{g}(\tau), \end{cases} \tag{2.1}$$

enhanced with the boundary conditions:

$$\begin{cases} y(1) = 0, \;\; y'(1) = 0, \;\; y(T) = \alpha_1\displaystyle\sum_{j=1}^{k-2}\xi_j z(\zeta_j) + \beta_1{}^{H}\mathcal{I}^{\varsigma_1}z(\vartheta), \\ z(1) = 0, \;\; z'(1) = 0, \;\; z(T) = \alpha_2\displaystyle\sum_{j=1}^{k-2}\upsilon_j y(\omega_j) + \beta_2{}^{H}\mathcal{I}^{\varrho_1}y(\varphi), \\ 1 < \vartheta < \varphi < \zeta_1 < \omega_1 < \zeta_2 < \omega_2 < \cdots < \zeta_{k-2} < \omega_{k-2} < T, \end{cases} \tag{2.2}$$

is given by

$$y(\tau) = {}^H\mathcal{I}^{\varrho}\hat{f}(\tau) + \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j{}^H\mathcal{I}^{\varsigma}\hat{g}(\zeta_j) + \beta_1{}^H\mathcal{I}^{\varsigma+\varsigma_1}\hat{g}(\vartheta) - {}^H\mathcal{I}^{\varrho}\hat{f}(T)\Bigg\}$$
$$+\nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j{}^H\mathcal{I}^{\varrho}\hat{f}(\omega_j) + \beta_2{}^H\mathcal{I}^{\varrho+\varrho_1}\hat{f}(\varphi) - {}^H\mathcal{I}^{\varsigma}\hat{g}(T)\Bigg\}\Bigg] \tag{2.3}$$

and

$$z(\tau) = {}^H\mathcal{I}^{\varsigma}\hat{g}(\tau) + \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j{}^H\mathcal{I}^{\varrho}\hat{f}(\omega_j) + \beta_2{}^H\mathcal{I}^{\varrho+\varrho_1}\hat{f}(\varphi) - {}^H\mathcal{I}^{\varsigma}\hat{g}(T)\Bigg\}$$
$$+\nu_2\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j{}^H\mathcal{I}^{\varsigma}\hat{g}(\zeta_j) + \beta_1{}^H\mathcal{I}^{\varsigma+\varsigma_1}\hat{g}(\vartheta) - {}^H\mathcal{I}^{\varrho}\hat{f}(T)\Bigg\}\Bigg] \tag{2.4}$$

where

$$\nu_1 = (\log T)^2, \ \ \nu_2 = \alpha_2\sum_{j=1}^{k-2}\upsilon_j(\log\omega_j)^2 + \frac{2\beta_2(\log\varphi)^{2+\varrho_1}}{\Gamma(3+\varrho_1)}, \tag{2.5}$$

$$\nu_3 = \alpha_1\sum_{j=1}^{k-2}\xi_j(\log\zeta_j)^2 + \frac{2\beta_1(\log\vartheta)^{2+\varsigma_1}}{\Gamma(3+\varsigma_1)}, \ \ \nu = \nu_1^2 - \nu_2\nu_3. \tag{2.6}$$

Proof. Solving the FDEs (2.1) in a standard manner, we get

$$y(\tau) = {}^H\mathcal{I}^{\varrho}\hat{f}(\tau) + a_0 + a_1\log\tau + a_2(\log\tau)^2, \tag{2.7}$$
$$z(\tau) = {}^H\mathcal{I}^{\varsigma}\hat{g}(\tau) + b_0 + b_1\log\tau + b_2(\log\tau)^2, \tag{2.8}$$

where $a_i, b_i \in \mathbb{R}$, $i = 0, 1, 2$, are arbitrary constants. Using the boundary conditions (2.2) in (2.7) and (2.8), we obtain $a_0 = a_1 = 0$, $b_0 = b_1 = 0$, and

$$a_2\nu_1 - b_2\nu_3 = \alpha_1\sum_{j=1}^{k-2}\xi_j{}^H\mathcal{I}^{\varsigma}\hat{g}(\zeta_j) + \beta_1{}^H\mathcal{I}^{\varsigma+\varsigma_1}\hat{g}(\vartheta) - {}^H\mathcal{I}^{\varrho}\hat{f}(T), \tag{2.9}$$

$$b_2\nu_1 - a_2\nu_2 = \alpha_2\sum_{j=1}^{k-2}\upsilon_j{}^H\mathcal{I}^{\varrho}\hat{f}(\omega_j) + \beta_2{}^H\mathcal{I}^{\varrho+\varrho_1}\hat{f}(\varphi) - {}^H\mathcal{I}^{\varsigma}\hat{g}(T). \tag{2.10}$$

Solving the system (2.9)–(2.10) for a_2, b_2, we get

$$\begin{aligned} a_2 &= \nu_1\Big(\alpha_1\sum_{j=1}^{k-2}\xi_j{}^H\mathcal{I}^{\varsigma}\hat{g}(\zeta_j)+\beta_1{}^H\mathcal{I}^{\varsigma+\varsigma_1}\hat{g}(\vartheta)-{}^H\mathcal{I}^{\varrho}\hat{f}(T)\Big)\\ &\quad+\nu_3\Big(\alpha_2\sum_{j=1}^{k-2}\upsilon_j{}^H\mathcal{I}^{\varrho}\hat{f}(\omega_j)+\beta_2{}^H\mathcal{I}^{\varrho+\varrho_1}\hat{f}(\varphi)-{}^H\mathcal{I}^{\varsigma}\hat{g}(T)\Big), \qquad (2.11)\\ b_2 &= \nu_1\Big(\alpha_2\sum_{j=1}^{k-2}\upsilon_j{}^H\mathcal{I}^{\varrho}\hat{f}(\omega_j)+\beta_2{}^H\mathcal{I}^{\varrho+\varrho_1}\hat{f}(\varphi)-{}^H\mathcal{I}^{\varsigma}\hat{g}(T)\Big)\\ &\quad+\nu_2\Big(\alpha_1\sum_{j=1}^{k-2}\xi_j{}^H\mathcal{I}^{\varsigma}\hat{g}(\zeta_j)+\beta_1{}^H\mathcal{I}^{\varsigma+\varsigma_1}\hat{g}(\vartheta)-{}^H\mathcal{I}^{\varrho}\hat{f}(T)\Big), \qquad (2.12)\end{aligned}$$

where ν_1,ν_2,ν_3,ν are given by (2.5) and (2.6) respectively. Substituting the values of a_2, b_2 in (2.7) and (2.8), we obtain the solutions (2.3) and (2.4). □

3. Existence results for the problem (1.1) and (1.2)

We define spaces $\mathcal{Y}=\{y(\tau):y(\tau)\in C(\mathcal{K},\mathbb{R})\}$ endowed with the norm $\|y\|=\sup\{|y(\tau)|,\tau\in\mathcal{K}\}$. Obviously $(\mathcal{Y},\|\cdot\|)$ is a Banach space. Also $\mathcal{Z}=\{z(\tau):z(\tau)\in C(\mathcal{K},\mathbb{R})\}$ endowed with the norm $\|z\|=\sup\{|z(\tau)|,\tau\in\mathcal{K}\}$ is a Banach space. Then the product space $(\mathcal{Y}\times\mathcal{Z},\|(y,z)\|)$ is also a Banach space equipped with norm $\|(y,z)\|=\|y\|+\|z\|$.

We implement operator $\Upsilon:\mathcal{Y}\times\mathcal{Z}\to\mathcal{Y}\times\mathcal{Z}$ using Lemma 2.7 as follows:

$$\Upsilon(y,z)(\tau)=(\Upsilon_1(y,z)(\tau),\Upsilon_2(y,z)(\tau)), \qquad (3.1)$$

where

$$\begin{aligned}\Upsilon_1(y,z)(\tau)&=\frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\ &+\frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\ &+\beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\ &-\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\Bigg\}\\ &+\nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\ &+\beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\ &-\frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\Bigg\}\Bigg], \qquad (3.2)\end{aligned}$$

$$\begin{aligned}
\Upsilon_2(y,z)(\tau) = & \frac{1}{\Gamma(\varsigma)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varsigma-1} g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta} \\
& + \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1} f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta} \\
& + \beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1} f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta} \\
& - \frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1} g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\Bigg\} \\
& + \nu_2\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1} g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta} \\
& + \beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1} g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta} \\
& - \frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1} f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\Bigg\}\Bigg]. \qquad (3.3)
\end{aligned}$$

For the convenience of computation, we set

$$\mathcal{P}_1 = \frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \frac{(\log T)^2}{\nu}\Bigg[\frac{\nu_1(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)} + \beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Bigg\}\Bigg] \qquad (3.4)$$

$$\mathcal{Q}_1 = \frac{(\log T)^2}{\nu}\Bigg[\frac{\nu_3(\log T)^{\varsigma}}{\Gamma(\varsigma+1)} + \nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)} + \beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Bigg\}\Bigg] \qquad (3.5)$$

$$\mathcal{P}_2 = \frac{(\log T)^2}{\nu}\Bigg[\frac{\nu_2(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \nu_1\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)} + \beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Bigg\}\Bigg] \qquad (3.6)$$

$$\mathcal{Q}_2 = \frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)} + \frac{(\log T)^2}{\nu}\Bigg[\frac{\nu_1(\log T)^{\varsigma}}{\Gamma(\varsigma+1)} + \nu_2\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)} + \beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Bigg\}\Bigg]. \qquad (3.7)$$

$$\Delta = \min\{1-[\lambda_1(\mathcal{P}_1+\mathcal{P}_2)+\widehat{\lambda_1}(\mathcal{Q}_1+\mathcal{Q}_2)], 1-[\lambda_2(\mathcal{P}_1+\mathcal{P}_2)+\widehat{\lambda_2}(\mathcal{Q}_1+\mathcal{Q}_2)]\}. \qquad (3.8)$$

Next, in the sequel, we enlist the premises we need. Let $f, g : \mathcal{K}\times\mathbb{R}^2 \to \mathbb{R}$ functions be continuous.

$(\mathcal{E}_1)$ $\exists$ real constants $\lambda_i, \widehat{\lambda_i} \geq 0$ $(i = 1, 2)$ and $\lambda_0 > 0, \widehat{\lambda_0} > 0$ such that

$$\begin{aligned}
&|f(\tau, y_1, y_2)| \leq \lambda_0 + \lambda_1|y_1| + \lambda_2|y_2|, \\
&|g(\tau, y_1, y_2)| \leq \widehat{\lambda_0} + \widehat{\lambda_1}|y_1| + \widehat{\lambda_2}|y_2|, \forall\, y_i \in \mathbb{R},\ i = 1, 2.
\end{aligned}$$

$(\mathcal{E}_2)$ $\exists$ positive constants $\kappa_i, \widehat{\kappa_i}$ $(i = 1, 2)$ such that

$$\begin{aligned}
&|f(\tau, y_1, y_2) - f(\tau, z_1, z_2)| \leq \kappa_1|y_1 - z_1| + \kappa_2|y_2 - z_2|, \\
&|g(\tau, y_1, y_2) - g(\tau, z_1, z_2)| \leq \widehat{\kappa_1}|y_1 - z_1| + \widehat{\kappa_2}|y_2 - z_2|, \forall\, \tau \in \mathcal{K},\ y_i, z_i \in \mathbb{R},\ i = 1, 2.
\end{aligned}$$

Theorem 3.1. *Suppose that ($\mathcal{E}_1$) hold. If*

$$\lambda_1(\mathcal{P}_1+\mathcal{P}_2)+\widehat{\lambda}_1(\mathcal{Q}_1+\mathcal{Q}_2)<1,\ \ \lambda_2(\mathcal{P}_1+\mathcal{P}_2)+\widehat{\lambda}_2(\mathcal{Q}_1+\mathcal{Q}_2)<1. \tag{3.9}$$

Then there exists at least one solution for problem (1.1) and (1.2) on $\mathcal{K}$, where $\mathcal{P}_1$, $\mathcal{Q}_1$, $\mathcal{P}_2$, and $\mathcal{Q}_2$ are given by (3.4)–(3.7) respectively.

Proof. We define in the first step that operator $\Upsilon:\mathcal{Y}\times\mathcal{Z}\to\mathcal{Y}\times\mathcal{Z}$ is completely continuous. This implies that the operators Υ_1 and Υ_2 are continuous by the continuity of the functions f and g. Accordingly, operator Υ is continuous. To demonstrate the uniformly bounded of operator Υ, let $\Lambda\subset\mathcal{Y}\times\mathcal{Z}$ be a bounded set. Then $\exists$ positive constants $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $|f(\tau,y(\tau),z(\tau))|\le\mathcal{M}_1$, $|g(\tau,y(\tau),z(\tau))|\le\mathcal{M}_2$, $\forall\ (y,z)\in\Lambda$. Then, for any $(y,z)\in\Lambda$, we have

$$\begin{aligned}
|\Upsilon_1(y,z)(\tau)| \le{}& \frac{\mathcal{M}_1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta}+\frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{\mathcal{M}_2}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}\frac{d\theta}{\theta}\\
&+\beta_1\frac{\mathcal{M}_2}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}\frac{d\theta}{\theta}+\frac{\mathcal{M}_1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta}\Bigg\}\\
&+\nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{\mathcal{M}_1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta}+\beta_2\frac{\mathcal{M}_1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}\frac{d\theta}{\theta}\\
&+\frac{\mathcal{M}_2}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}\frac{d\theta}{\theta}\Bigg\}\Bigg]\\
\le{}& \frac{(\log T)^2}{\nu}\Bigg\{\mathcal{M}_2\Bigg[\frac{\nu_3(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+\nu_1\Bigg(\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)}+\beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Bigg)\Bigg]\\
&+\frac{\nu\mathcal{M}_1(\log T)^{\varrho}}{(\log T)^2\Gamma(\varrho+1)}+\mathcal{M}_1\Bigg[\frac{\nu_1(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\nu_3\Bigg(\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)}+\beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Bigg)\Bigg]\Bigg\},
\end{aligned}$$

that yields when taking $\tau\in\mathcal{K}$ norm and using (3.4) and (3.5),

$$\|\Upsilon_1(y,z)\|\le\mathcal{P}_1\mathcal{M}_1+\mathcal{Q}_1\mathcal{M}_2. \tag{3.10}$$

Similarly, using (3.6) and (3.7), we obtain

$$\begin{aligned}
|\Upsilon_2(y,z)(\tau)| \le{}& \frac{(\log T)^2}{\nu}\Bigg\{\mathcal{M}_1\Bigg[\frac{\nu_2(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\nu_1\Bigg(\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)}+\beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Bigg)\Bigg]\\
&+\frac{\nu\mathcal{M}_2(\log T)^{\varsigma}}{(\log T)^2\Gamma(\varsigma+1)}+\mathcal{M}_2\Bigg[\frac{\nu_1(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+\nu_2\Bigg(\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)}+\beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Bigg)\Bigg]\Bigg\}\\
\le{}& \mathcal{P}_2\mathcal{M}_1+\mathcal{Q}_2\mathcal{M}_2.
\end{aligned} \tag{3.11}$$

We deduce from the inequalities (3.10) and (3.11) that Υ_1 and Υ_2 are uniformly bounded, implying operator Υ is uniformly bounded. Next we demonstrate the equicontinuous of Υ. Let $\tau_1,\tau_2\in\mathcal{K}$ with $\tau_1<\tau_2$. Then we have

$$
\begin{aligned}
|\Upsilon_1(y,z)(\tau_2)-\Upsilon_1(y,z)(\tau_1)| \le \frac{1}{\Gamma(\varrho)}\Bigg| & \int_1^{\tau_1}\Big[\Big(\log\frac{\tau_2}{\theta}\Big)^{\varrho-1}-\Big(\log\frac{\tau_1}{\theta}\Big)^{\varrho-1}\Big]|f(\theta,y(\theta),z(\theta))|\frac{d\theta}{\theta} \\
& + \int_{\tau_1}^{\tau_2}\Big(\log\frac{\tau_2}{\theta}\Big)^{\varrho-1}|f(\theta,y(\theta),z(\theta))|\frac{d\theta}{\theta}\Bigg| \\
& + \frac{(\log\tau_2)^2-(\log\tau_1)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}|g(\theta,y(\theta),z(\theta))|\frac{d\theta}{\theta} \\
& + \beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}|g(\theta,y(\theta),z(\theta))|\frac{d\theta}{\theta} \\
& + \frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}|f(\theta,y(\theta),z(\theta))|\frac{d\theta}{\theta}\Bigg\} \\
& + \nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}|f(\theta,y(\theta),z(\theta))|\frac{d\theta}{\theta} \\
& + \beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}|f(\theta,y(\theta),z(\theta))|\frac{d\theta}{\theta} \\
& + \frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}|g(\theta,y(\theta),z(\theta))|\frac{d\theta}{\theta}\Bigg\}\Bigg] \\
\to 0 \text{ as } \tau_2\to\tau_1,
\end{aligned}
$$

independent of (y,z) with respect to $|f(\tau,y(\tau),z(\tau))| \le \mathcal{M}_1$ and $|g(\tau,y(\tau),z(\tau))| \le \mathcal{M}_2$. Analogously, we can do that $|\Upsilon_2(y,z)(\tau_2)-\Upsilon_2(y,z)(\tau_1)| \to 0$ as $\tau_2\to\tau_1$ independent of (y,z) with respect to the boundedness of f and g. Thus the operator Υ is equicontinuous in view of equicontinuity of Υ_1 and Υ_2. Thus, the operator Υ is compact by Lemma (see Lemma 1.2 [29]). At last, the set $\Omega(\Upsilon)=\{(y,z)\in\mathcal{Y}\times\mathcal{Z}:(y,z)=\varepsilon\Upsilon(y,z);0<\varepsilon<1\}$ is shown to be bounded. Let $(y,z)\in\Omega(\Upsilon)$. Then $(y,z)=\varepsilon\Upsilon(y,z)$. For any $\tau\in\mathcal{K}$, we have $y(\tau)=\varepsilon\Upsilon_1(y,z)(\tau)$, $z(\tau)=\varepsilon\Upsilon_2(y,z)(\tau)$. Utilizing $(\mathcal{E}_1)$ in (3.2), we get

$$
\begin{aligned}
|y(\tau)| \le & \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\Big(\lambda_0+\lambda_1|y(\theta)|+\lambda_2|z(\theta)|\Big)\frac{d\theta}{\theta} \\
& + \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}\Big(\widehat{\lambda}_0+\widehat{\lambda}_1|y(\theta)|+\widehat{\lambda}_2|z(\theta)|\Big)\frac{d\theta}{\theta} \\
& + \beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}\Big(\widehat{\lambda}_0+\widehat{\lambda}_1|y(\theta)|+\widehat{\lambda}_2|z(\theta)|\Big)\frac{d\theta}{\theta} \\
& + \frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\Big(\lambda_0+\lambda_1|y(\theta)|+\lambda_2|z(\theta)|\Big)\frac{d\theta}{\theta}\Bigg\} \\
& + \nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}\Big(\lambda_0+\lambda_1|y(\theta)|+\lambda_2|z(\theta)|\Big)\frac{d\theta}{\theta} \\
& + \beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}\Big(\lambda_0+\lambda_1|y(\theta)|+\lambda_2|z(\theta)|\Big)\frac{d\theta}{\theta} \\
& + \frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}\Big(\widehat{\lambda}_0+\widehat{\lambda}_1|y(\theta)|+\widehat{\lambda}_2|z(\theta)|\Big)\frac{d\theta}{\theta}\Bigg\}\Bigg],
\end{aligned}
$$

that yields when taking the norm for $\tau \in \mathcal{K}$,

$$\|y\| \leq \Big(\lambda_0 + \lambda_1|y(\theta)| + \lambda_2|z(\theta)|\Big)\mathcal{P}_1 + \Big(\widehat{\lambda_0} + \widehat{\lambda_1}|y(\theta)| + \widehat{\lambda_2}|z(\theta)|\Big)\mathcal{Q}_1. \tag{3.12}$$

Indeed, we can obtain that

$$\|z\| \leq \Big(\lambda_0 + \lambda_1|y(\theta)| + \lambda_2|z(\theta)|\Big)\mathcal{P}_2 + \Big(\widehat{\lambda_0} + \widehat{\lambda_1}|y(\theta)| + \widehat{\lambda_2}|z(\theta)|\Big)\mathcal{Q}_2. \tag{3.13}$$

From (3.12) and (3.13), we get

$$\begin{aligned}\|y\| + \|z\| = {} & \lambda_0\Big(\mathcal{P}_1 + \mathcal{P}_2\Big) + \widehat{\lambda_0}\Big(\mathcal{Q}_1 + \mathcal{Q}_2\Big) + \|y\|\Big[\lambda_1\Big(\mathcal{P}_1 + \mathcal{P}_2\Big) + \widehat{\lambda_1}\Big(\mathcal{Q}_1 + \mathcal{Q}_2\Big)\Big] \\ & + \|z\|\Big[\lambda_2\Big(\mathcal{P}_1 + \mathcal{P}_2\Big) + \widehat{\lambda_2}\Big(\mathcal{Q}_1 + \mathcal{Q}_2\Big)\Big],\end{aligned}$$

which yields, with $\|y, z\| = \|y\| + \|z\|$,

$$\|y, z\| \leq \frac{\lambda_0\Big(\mathcal{P}_1 + \mathcal{P}_2\Big) + \widehat{\lambda_0}\Big(\mathcal{Q}_1 + \mathcal{Q}_2\Big)}{\Delta}.$$

It implies $\Omega(\Upsilon)$ is bounded. So Theorem (see Theorem [27]) is valid and Υ has at least one fixed point. This indicates the BVP (1.1) and (1.2) have at least one solution on $\mathcal{K}$. □

Theorem 3.2. *Suppose that* $(\mathcal{E}_2)$ *hold. Then the BVP (1.1) and (1.2) has a unique solution on* $\mathcal{K}$*, provided that*

$$(\mathcal{P}_1 + \mathcal{P}_2)(\kappa_1 + \kappa_2) + (\mathcal{Q}_1 + \mathcal{Q}_2)(\widehat{\kappa_1} + \widehat{\kappa_2}) < 1, \tag{3.14}$$

where $\mathcal{P}_1$*,* $\mathcal{Q}_1$*,* $\mathcal{P}_2$*, and* $\mathcal{Q}_2$ *are given by (3.4)-(3.7).*

Proof. Let's set $\rho \geq \dfrac{\mathcal{G}_1(\mathcal{P}_1 + \mathcal{P}_2) + \mathcal{G}_2(\mathcal{Q}_1 + \mathcal{Q}_2)}{1 - (\mathcal{P}_1 + \mathcal{P}_2)(\kappa_1 + \kappa_2) - (\mathcal{Q}_1 + \mathcal{Q}_2)(\widehat{\kappa_1} + \widehat{\kappa_2})}$, and show that $\Upsilon\mathcal{B}_\rho \subset \mathcal{B}_\rho$, when operator Υ is given by (3.1) and $\mathcal{B}_\rho = \{(y, z) \in \mathcal{Y} \times \mathcal{Z} : \|(y, z)\| \leq \rho\}$. For $(y, z) \in \mathcal{B}_\rho$, $\tau \in \mathcal{K}$, we have

$$\begin{aligned}|f(\tau, y(\tau), z(\tau))| &\leq \kappa_1|y(\tau)| + \kappa_2|z(\tau)| + \mathcal{G}_1 \\ &\leq \kappa_1\|y\| + \kappa_2\|z\| + \mathcal{G}_1,\end{aligned}$$

and

$$|g(\tau, y(\tau), z(\tau))| \leq \widehat{\kappa_1}\|y\| + \widehat{\kappa_2}\|z\| + \mathcal{G}_2.$$

This guides to

$$\begin{aligned}|\Upsilon_1(y, z)(\tau)| \leq {} & \frac{1}{\Gamma(\varrho)} \int_1^\tau \Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1} |f(\theta, y(\theta), z(\theta)) - f(\theta, 0, 0)| + |f(\theta, 0, 0)|\frac{d\theta}{\theta} \\ & + \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1 \sum_{j=1}^{k-2} \xi_j \frac{1}{\Gamma(\varsigma)} \int_1^{\zeta_j} \Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1} |g(\theta, y(\theta), z(\theta)) - g(\theta, 0, 0)| + |g(\theta, 0, 0)|\frac{d\theta}{\theta} \\ & + \beta_1 \frac{1}{\Gamma(\varsigma + \varsigma_1)} \int_1^\vartheta \Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1} |g(\theta, y(\theta), z(\theta)) - g(\theta, 0, 0)| + |g(\theta, 0, 0)|\frac{d\theta}{\theta}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\varrho)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}|f(\theta,y(\theta),z(\theta))-f(\theta,0,0)|+|f(\theta,0,0)|\frac{d\theta}{\theta}\Big\}\\
&+\nu_3\Big\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}|f(\theta,y(\theta),z(\theta))-f(\theta,0,0)|+|f(\theta,0,0)|\frac{d\theta}{\theta}\\
&+\beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}|f(\theta,y(\theta),z(\theta))-f(\theta,0,0)|+|f(\theta,0,0)|\frac{d\theta}{\theta}\\
&+\frac{1}{\Gamma(\varsigma)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}|g(\theta,y(\theta),z(\theta))-g(\theta,0,0)|+|g(\theta,0,0)|\frac{d\theta}{\theta}\Big\}\Big]\\
\le{}&(\kappa_1\|y\|+\kappa_2\|z\|+\mathcal{G}_1)\Big[\frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\frac{(\log T)^2}{\nu}\Big\{\frac{\nu_1(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\nu_3\Big(\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)}\\
&+\beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Big)\Big]+(\widehat{\kappa_1}\|y\|+\widehat{\kappa_2}\|z\|+\mathcal{G}_2)\Big[\frac{\nu_3(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}\\
&+\nu_1\Big(\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)}+\beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Big)\Big\}\Big]\\
\le{}&(\kappa_1\|y\|+\kappa_2\|z\|+\mathcal{G}_1)\mathcal{P}_1+(\widehat{\kappa_1}\|y\|+\widehat{\kappa_2}\|z\|+\mathcal{G}_2)\mathcal{Q}_1.
\end{aligned}\tag{3.15}
$$

Equivalently, we obtain

$$
\begin{aligned}
\Upsilon_2(y,z)(\tau)\le{}&\frac{1}{\Gamma(\varsigma)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varsigma-1}|g(\theta,y(\theta),z(\theta))-g(\theta,0,0)|+|g(\theta,0,0)|\frac{d\theta}{\theta}\\
&+\frac{(\log\tau)^2}{\nu}\Big[\nu_1\Big\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}|f(\theta,y(\theta),z(\theta))-f(\theta,0,0)|+|f(\theta,0,0)|\frac{d\theta}{\theta}\\
&+\beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}|f(\theta,y(\theta),z(\theta))-f(\theta,0,0)|+|f(\theta,0,0)|\frac{d\theta}{\theta}\\
&+\frac{1}{\Gamma(\varsigma)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}|g(\theta,y(\theta),z(\theta))-g(\theta,0,0)|+|g(\theta,0,0)|\frac{d\theta}{\theta}\Big\}\\
&+\nu_2\Big\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}|g(\theta,y(\theta),z(\theta))-g(\theta,0,0)|+|g(\theta,0,0)|\frac{d\theta}{\theta}\\
&+\beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}|g(\theta,y(\theta),z(\theta))-g(\theta,0,0)|+|g(\theta,0,0)|\frac{d\theta}{\theta}\\
&+\frac{1}{\Gamma(\varrho)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}|f(\theta,y(\theta),z(\theta))-f(\theta,0,0)|+|f(\theta,0,0)|\frac{d\theta}{\theta}\Big\}\Big]\\
\le{}&(\widehat{\kappa_1}\|y\|+\widehat{\kappa_2}\|z\|+\mathcal{G}_2)\mathcal{Q}_2+(\kappa_1\|y\|+\kappa_2\|z\|+\mathcal{G}_1)\mathcal{P}_2.
\end{aligned}\tag{3.16}
$$

Therefore, (3.15) and (3.16) follows that $\|\Upsilon(y,z)\|\le\rho$, and therefore $\Upsilon\mathcal{B}_\rho\subset\mathcal{B}_\rho$. Now, for (y_1,z_1), $(y_2,z_2)\in\mathcal{Y}\times\mathcal{Z}$ and any $\tau\in\mathcal{K}$, we get

$$
\begin{aligned}
&|\Upsilon_1(y_1,z_1)(\tau)-\Upsilon_1(y_2,z_2)(\tau)|\\
&\le \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}|f(\theta,y_1(\theta),z_1(\theta))-f(\theta,y_2(\theta),z_2(\theta))|\frac{d\theta}{\theta}\\
&\quad+\frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}|g(\theta,y_1(\theta),z_1(\theta))-g(\theta,y_2(\theta),z_2(\theta))|\frac{d\theta}{\theta}\\
&\qquad+\beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}|g(\theta,y_1(\theta),z_1(\theta))-g(\theta,y_2(\theta),z_2(\theta))|\frac{d\theta}{\theta}\\
&\qquad+\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}|f(\theta,y_1(\theta),z_1(\theta))-f(\theta,y_2(\theta),z_2(\theta))|\frac{d\theta}{\theta}\Bigg\}\\
&\quad+\nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}|f(\theta,y_1(\theta),z_1(\theta))-f(\theta,y_2(\theta),z_2(\theta))|\frac{d\theta}{\theta}\\
&\qquad+\beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}|f(\theta,y_1(\theta),z_1(\theta))-f(\theta,y_2(\theta),z_2(\theta))|\frac{d\theta}{\theta}\\
&\qquad+\frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}|g(\theta,y_1(\theta),z_1(\theta))-g(\theta,y_2(\theta),z_2(\theta))|\frac{d\theta}{\theta}\Bigg\}\Bigg]\\
&\le(\kappa_1\|y_1-y_2\|+\kappa_2\|z_1-z_2\|)\Bigg[\frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\frac{(\log T)^2}{\nu}\Bigg\{\frac{\nu_1(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\nu_3\Bigg(\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)}\\
&\qquad+\beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Bigg)\Bigg\}\Bigg]+\Bigg[\frac{(\log T)^2}{\nu}\Bigg\{\frac{\nu_3(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+\nu_1\Bigg(\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)}+\beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Bigg)\Bigg\}\Bigg]\\
&\qquad\times(\widehat{\kappa_1}\|y_1-y_2\|+\widehat{\kappa_2}\|z_1-z_2\|)\\
&\le(\mathcal{P}_1(\kappa_1+\kappa_2)+\mathcal{Q}_1(\widehat{\kappa_1}+\widehat{\kappa_2}))(\|y_1-y_2\|+\|z_1-z_2\|).
\end{aligned}
$$

Likewise, we obtain

$$
\begin{aligned}
&|\Upsilon_2(y_1,z_1)(\tau)-\Upsilon_2(y_2,z_2)(\tau)|\\
&\le(\widehat{\kappa_1}\|y_1-y_2\|+\widehat{\kappa_2}\|z_1-z_2\|)\Bigg[\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+\frac{(\log T)^2}{\nu}\Bigg\{\frac{\nu_1(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+\nu_2\Bigg(\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)}\\
&\qquad+\beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Bigg)\Bigg\}\Bigg]+\frac{(\log T)^2}{\nu}\Bigg[\Bigg\{\frac{\nu_2(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\nu_1\Bigg(\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)}+\beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Bigg)\Bigg\}\Bigg]\\
&\qquad\times(\kappa_1\|y_1-y_2\|+\kappa_2\|z_1-z_2\|)\\
&\le(\mathcal{P}_2(\kappa_1+\kappa_2)+\mathcal{Q}_2(\widehat{\kappa_1}+\widehat{\kappa_2}))(\|y_1-y_2\|+\|z_1-z_2\|).
\end{aligned}
$$

So we obtain

$$\|\Upsilon_1(y_1,z_1)-\Upsilon_1(y_2,z_2)\| \le (\mathcal{P}_1(\kappa_1+\kappa_2)+\mathcal{Q}_1(\widehat{\kappa_1}+\widehat{\kappa_2}))(\|y_1-y_2\|+\|z_1-z_2\|). \tag{3.17}$$

In the same way,

$$\|\Upsilon_2(y_1,z_1)-\Upsilon_2(y_2,z_2)\| \le (\mathcal{P}_2(\kappa_1+\kappa_2)+\mathcal{Q}_2(\widehat{\kappa_1}+\widehat{\kappa_2}))(\|y_1-y_2\|+\|z_1-z_2\|). \tag{3.18}$$

So, from (3.17) and (3.18) we conclude that

$$\|\Upsilon(y_1,z_1)-\Upsilon(y_2,z_2)\| \le (\mathcal{P}_1+\mathcal{P}_2)(\kappa_1+\kappa_2)+(\mathcal{Q}_1+\mathcal{Q}_2)(\widehat{\kappa_1}+\widehat{\kappa_2})[\|y_1-y_2\|+\|z_1-z_2\|].$$

Therefore, it follows from condition $(\mathcal{P}_1+\mathcal{P}_2)(\kappa_1+\kappa_2)+(\mathcal{Q}_1+\mathcal{Q}_2)(\widehat{\kappa_1}+\widehat{\kappa_2})<1$, that Υ is a contraction operator. Thus we conclude by Theorem (see Theorem 1.2.2 [21]) that operator Υ has a unique fixed point, which is the unique solution to the problem (1.1) and (1.2). □

Theorem 3.3. *Suppose that ($\mathcal{E}_2$) hold. In addition, $\exists$ positive constants $\mathcal{T}_1$, $\mathcal{T}_2$ such that $\forall\ \tau\in\mathcal{K}$ and $y,z\in\mathbb{R}$,*

$$|f(\tau,y,z)|\le\mathcal{T}_1,\quad |g(\tau,y,z)|\ \le\mathcal{T}_2. \tag{3.19}$$

Then the BVP (1.1) and (1.2) has at least one solution on $\mathcal{K}$, if

$$\frac{(\log T)^{\varrho}(\kappa_1+\kappa_2)}{\Gamma(\varrho+1)}+\frac{(\log T)^{\varsigma}(\widehat{\kappa_1}+\widehat{\kappa_2})}{\Gamma(\varsigma+1)}<1. \tag{3.20}$$

Proof. Let us define a ball $\mathcal{B}_\rho=\{(y,z)\in\mathcal{Y}\times\mathcal{Z}:\|(y,z)\|\le\rho\}$ closed as follows:

$$\begin{aligned}
\Upsilon_{1,1}(y,z)(\tau) &= \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\
&\quad+\beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\
&\quad-\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\Bigg\}\\
&\quad+\nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\
&\quad+\beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\
&\quad-\frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\Bigg\}\Bigg],\\
\Upsilon_{1,2}(y,z)(\tau) &= \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta},
\end{aligned}$$

and

$$\Upsilon_{2,1}(y,z)(\tau) = \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}$$
$$+\beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}$$
$$-\frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\Bigg\}$$
$$+\nu_2\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}$$
$$+\beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}$$
$$-\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\Bigg\}\Bigg],$$
$$\Upsilon_{2,2}(y,z)(\tau) = \frac{1}{\Gamma(\varsigma)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varsigma-1}g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}.$$

Note that $\Upsilon_1(y,z)(\tau) = \Upsilon_{1,1}(y,z)(\tau) + \Upsilon_{1,2}(y,z)(\tau)$ and $\Upsilon_2(y,z)(\tau) = \Upsilon_{2,1}(y,z)(\tau) + \Upsilon_{2,2}(y,z)(\tau)$ on $\mathcal{B}_\rho$ is a closed, bounded, and convex subset of Banach space $\mathcal{Y}\times\mathcal{Z}$ and that the Ball $\mathcal{B}_\rho$. Now let us choose $\rho \geq \max\{\mathcal{P}_1\mathcal{T}_1 + \mathcal{Q}_1\mathcal{T}_2, \mathcal{P}_2\mathcal{T}_1 + \mathcal{Q}_2\mathcal{T}_2\}$, and demonstrate the $\Upsilon\mathcal{B}_\rho \subset \mathcal{B}_\rho$ to test Theorem's (see Theorem 4.4.1 [21]) condition (*i*), if we set $y = (y_1, y_2)$, $z = (z_1, z_2) \in \mathcal{B}_\rho$, and using condition (3.19), we get

$$|\Upsilon_{1,1}(y,z)(\tau) + \Upsilon_{1,2}(y,z)(\tau)| \leq \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\mathcal{T}_1\frac{d\theta}{\theta}$$
$$+\frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}\mathcal{T}_2\frac{d\theta}{\theta}$$
$$+\beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}\mathcal{T}_2\frac{d\theta}{\theta}$$
$$+\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\mathcal{T}_1\frac{d\theta}{\theta}\Bigg\}$$
$$+\nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}\mathcal{T}_1\frac{d\theta}{\theta}$$
$$+\beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}\mathcal{T}_1\frac{d\theta}{\theta}$$
$$+\frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}\mathcal{T}_2\frac{d\theta}{\theta}\Bigg\}\Bigg]$$
$$\leq \mathcal{P}_1\mathcal{T}_1 + \mathcal{Q}_1\mathcal{T}_2 \leq \rho.$$

Similarly, we find that

$$|\Upsilon_{2,1}(y,z)(\tau) + \Upsilon_{2,2}(y,z)(\tau)| \leq \mathcal{P}_2\mathcal{T}_1 + \mathcal{Q}_2\mathcal{T}_2 \leq \rho.$$

The two above inequalities contribute to the assumption that $\Upsilon_1(y,z)+\Upsilon_2(\hat{y},\hat{z})\in\mathcal{B}_\rho$ does. So we define that operator $(\Upsilon_{1,2},\Upsilon_{2,2})$ is a condition (*iii*) of Theorem (see Theorem 4.4.1 [21]) that satisfies contraction. For $(y_1,z_1),(y_2,z_2)\in\mathcal{B}_\rho$, we have

$$\begin{aligned}|\Upsilon_{1,2}(y_1,z_1)(\tau)-\Upsilon_{1,2}(y_2,z_2)(\tau)| &\le \frac{1}{\Gamma(\varrho)}\int_1^\tau\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}|f(\theta,y_1(\theta),z_1(\theta))-f(\theta,y_2(\theta),z_2(\theta))|\frac{d\theta}{\theta}\\ &\le \frac{(\log T)^\varrho}{\Gamma(\varrho+1)}(\kappa_1\|y_1-y_2\|+\kappa_2\|z_1-z_2\|),\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}|\Upsilon_{2,2}(y_1,z_1)(\tau)-\Upsilon_{2,2}(y_2,z_2)(\tau)| &\le \frac{1}{\Gamma(\varsigma)}\int_1^\tau\Big(\log\frac{\tau}{\theta}\Big)^{\varsigma-1}|g(\theta,y_1(\theta),z_1(\theta))-g(\theta,y_2(\theta),z_2(\theta))|\frac{d\theta}{\theta}\\ &\le \frac{(\log T)^\varsigma}{\Gamma(\varsigma+1)}(\widehat{\kappa_1}\|y_1-y_2\|+\widehat{\kappa_2}\|z_1-z_2\|).\end{aligned}\tag{3.22}$$

From (3.21) and (3.22) it follows that

$$\begin{aligned}&|(\Upsilon_{1,2},\Upsilon_{2,2})(y_1,z_1)(\tau)-(\Upsilon_{1,2},\Upsilon_{2,2})(y_2,z_2)(\tau)|\\ &\le\left(\frac{(\log T)^\varrho(\kappa_1+\kappa_2)}{\Gamma(\varrho+1)}+\frac{(\log T)^\varsigma(\widehat{\kappa_1}+\widehat{\kappa_2})}{\Gamma(\varsigma+1)}\right)(\|y_1-y_2\|+\|z_1-z_2\|),\end{aligned}$$

which is a contraction by (3.20). Hence Theorem's (see Theorem 4.4.1 [21]) condition (*iii*) is satisfied. Next we can demonstrate that the operator $(\Upsilon_{1,1},\Upsilon_{2,1})$ fulfills the Theorem's (see Theorem 4.4.1 [21]) condition (*ii*). By applying the continuity of the $f,g:\mathcal{K}\times\mathbb{R}\times\mathbb{R}$ functions, we can infer that the $(\Upsilon_{1,1},\Upsilon_{2,1})$ operator is continuous. For each $(y,z)\in\mathcal{B}_\rho$ we have

$$\begin{aligned}|\Upsilon_{1,1}(y,z)(\tau)| \le \frac{(\log\tau)^2}{\nu}\Bigg[&\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}\mathcal{T}_2\frac{d\theta}{\theta}\\ &+\beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}\mathcal{T}_2\frac{d\theta}{\theta}\\ &+\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\mathcal{T}_1\frac{d\theta}{\theta}\Bigg\}\\ &+\nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}\mathcal{T}_1\frac{d\theta}{\theta}\\ &+\beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}\mathcal{T}_1\frac{d\theta}{\theta}\\ &+\frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}\mathcal{T}_2\frac{d\theta}{\theta}\Bigg\}\Bigg]\\ =\widehat{\Lambda}_1,&\end{aligned}$$

and

$$
\begin{aligned}
|\Upsilon_{2,1}(y,z)(\tau)| \leq \frac{(\log\tau)^2}{\nu}\Bigg[& \nu_1\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}\mathcal{T}_1\frac{d\theta}{\theta} \\
& +\beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}\mathcal{T}_1\frac{d\theta}{\theta} \\
& +\frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}\mathcal{T}_2\frac{d\theta}{\theta}\Bigg\} \\
& +\nu_2\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}\mathcal{T}_2\frac{d\theta}{\theta} \\
& +\beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}\mathcal{T}_2\frac{d\theta}{\theta} \\
& +\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\mathcal{T}_2\frac{d\theta}{\theta}\Bigg\}\Bigg] \\
= {} & \widehat{\Lambda}_2,
\end{aligned}
$$

that leads to this

$$\|(\Upsilon_{1,1},\Upsilon_{2,1})(y,z)\| \leq \widehat{\Lambda}_1+\widehat{\Lambda}_2.$$

Therefore the set $(\Upsilon_{1,1},\Upsilon_{2,1})\mathcal{B}_\rho$ is bounded uniformly. We'll be demonstrating in the next phase that the $(\Upsilon_{1,1},\Upsilon_{2,1})\mathcal{B}_\rho$ set is equicontinuous. For $\tau_1,\tau_2\in\mathcal{K}$ with $\tau_1<\tau_2$ and for any $(y,z)\in\mathcal{B}_\rho$ we obtain

$$
\begin{aligned}
& |\Upsilon_{1,1}(y,z)(\tau_2)-\Upsilon_{1,1}(y,z)(\tau_1)| \\
& \leq \frac{(\log\tau_2)^2-(\log\tau_1)^2}{\nu}\Bigg\{\mathcal{T}_1\Bigg[\frac{\nu_1(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\nu_3\Bigg(\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)}+\beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Bigg)\Bigg] \\
& \quad +\mathcal{T}_2\Bigg[\frac{\nu_3(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+\nu_1\Bigg(\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)}+\beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Bigg)\Bigg]\Bigg\}.
\end{aligned}
$$

In a similar manner, we can get

$$
\begin{aligned}
& |\Upsilon_{2,1}(y,z)(\tau_2)-\Upsilon_{2,1}(y,z)(\tau_1)| \\
& \leq \frac{(\log\tau_2)^2-(\log\tau_1)^2}{\nu}\Bigg\{\mathcal{T}_2\Bigg[\frac{\nu_1(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}+\nu_2\Bigg(\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)}+\beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Bigg)\Bigg] \\
& \quad +\mathcal{T}_1\Bigg[\frac{\nu_2(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\nu_1\Bigg(\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)}+\beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Bigg)\Bigg]\Bigg\}.
\end{aligned}
$$

Thus $|(\Upsilon_{1,1},\Upsilon_{2,1})(y,z)(\tau_2)-(\Upsilon_{1,1},\Upsilon_{2,1})(y,z)(\tau_1)|$ tends to zero as $\tau_1\to\tau_1$ independent of $(y,z)\in\mathcal{B}_\rho$. Therefore the set $(\Upsilon_{1,1},\Upsilon_{2,1})\mathcal{B}_\rho$ is equicontinuous. Therefore it implies from the lemma (see Lemma 1.2 [29]) that the operator $(\Upsilon_{1,1},\Upsilon_{2,1})$ is compact on $\mathcal{B}_\rho$. We conclude from Theorem's (see Theorem 4.4.1 [21]) statement that the problem (1.1) and (1.2) has at least one solution on $\mathcal{K}$. □

4. Examples

Example 4.1. *Consider the following coupled system of Caputo-Hadamard type FDEs:*

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{57}{20}} y(\tau) = f(\tau, y(\tau), z(\tau)), & \tau \in [1,2], \\ {}^{C}\mathcal{D}^{\frac{49}{20}} z(\tau) = g(\tau, y(\tau), z(\tau)), & \tau \in [1,2], \end{cases} \tag{4.1}$$

equipped with coupled boundary conditions:

$$\begin{cases} y(1) = 0, \;\; y'(1) = 0, \;\; y(T) = \dfrac{17}{400} \displaystyle\sum_{j=1}^{4} \xi_j z(\zeta_j) + \dfrac{13}{250} {}^{H}\mathcal{I}^{\frac{9}{20}} z(\tfrac{93}{50}), \\ z(1) = 0, \;\; z'(1) = 0, \;\; z(T) = \dfrac{8}{125} \displaystyle\sum_{j=1}^{4} \upsilon_j y(\omega_j) + \dfrac{3}{40} {}^{H}\mathcal{I}^{\frac{17}{20}} y(\tfrac{73}{50}). \end{cases} \tag{4.2}$$

Here, $\varrho = \frac{57}{20}$, $\varsigma = \frac{49}{20}$, $\varrho_1 = \frac{17}{20}$, $\varsigma_1 = \frac{9}{20}$, $\xi_1 = \frac{1}{8}$, $\xi_2 = \frac{9}{40}$, $\xi_3 = \frac{13}{40}$, $\xi_4 = \frac{17}{40}$, $\upsilon_1 = \frac{3}{25}$, $\upsilon_2 = \frac{11}{50}$, $\upsilon_3 = \frac{8}{25}$, $\upsilon_4 = \frac{21}{50}$, $\zeta_1 = \frac{36}{25}$, $\zeta_2 = \frac{79}{50}$, $\zeta_3 = \frac{44}{25}$, $\zeta_4 = \frac{97}{50}$, $\omega_1 = \frac{63}{50}$, $\omega_2 = \frac{34}{25}$, $\omega_3 = \frac{83}{50}$, $\omega_4 = \frac{47}{25}$, $\alpha_1 = \frac{17}{400}$, $\alpha_2 = \frac{8}{125}$, $\beta_1 = \frac{13}{250}$, $\beta_2 = \frac{3}{40}$, $T = 2$, $\vartheta = \frac{93}{50}$, $\varphi = \frac{73}{50}$,

$$f(\tau, y(\tau), z(\tau)) = \frac{1}{4(\tau^2 + 16)}\left(2\tau + \frac{|y(\tau)|}{1 + |y(\tau)|} + \frac{1}{4}\sin(z(\tau))\right)$$

$$g(\tau, y(\tau), z(\tau)) = \frac{1}{25\tau}\left(\frac{\sqrt{\tau}}{4} + \sin(y(\tau)) + \frac{1}{3}\frac{|z(\tau)|}{1 + |z(\tau)|}\right).$$

The functions f and g obviously satisfy the $(\mathcal{E}_1)$ *condition with* $\kappa_0 = \frac{1}{34}$, $\kappa_1 = \frac{1}{68}$, $\kappa_2 = \frac{1}{272}$, $\widehat{\kappa_0} = \frac{1}{100}$, $\widehat{\kappa_1} = \frac{1}{25}$, $\widehat{\kappa_2} = \frac{1}{75}$. *With the data given, we find that* $\nu_1 \approx 0.4804530139182014$, $\nu_2 \approx 0.01960022943737283$, $\nu_3 \approx 0.02532387378450272$, $\nu \approx 0.23033874484666408$, $\mathcal{P}_1 \approx 0.1414146436570378$, $\mathcal{Q}_1 \approx 0.13001906168784882$, $\mathcal{P}_2 \approx 0.005190521521769147$, $\mathcal{Q}_2 \approx 0.25950439692408167$.

With $\lambda_1\big(\mathcal{P}_1 + \mathcal{P}_2\big) + \widehat{\lambda_1}\big(\mathcal{Q}_1 + \mathcal{Q}_2\big) \approx 0.017736896655930263 < 1$, $\lambda_2\big(\mathcal{P}_1 + \mathcal{P}_2\big) + \widehat{\lambda_2}\big(\mathcal{Q}_1 + \mathcal{Q}_2\big) \approx 0.0057326356926890015 < 1$, *all of the Theorem 3.1 requirements are fulfilled. Problem (4.1) and (4.2) therefore have a solution on* $[1,2]$.

Example 4.2. *Consider the following coupled system of Caputo-Hadamard type FDEs:*

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{83}{30}} y(\tau) = \dfrac{\tau}{2} + \dfrac{1}{80}\sin(y(\tau)) + \dfrac{9}{400}\dfrac{|z(\tau)|}{1 + |z(\tau)|}, & \tau \in [1,2], \\ {}^{C}\mathcal{D}^{\frac{47}{20}} z(\tau) = \dfrac{1 + \sqrt{\tau}}{3} + \dfrac{8}{10(\tau + 24)}\dfrac{|y(\tau)|}{1 + |y(\tau)|} + \dfrac{21}{500}\sin(z(\tau)), & \tau \in [1,2], \end{cases} \tag{4.3}$$

equipped with coupled boundary conditions:

$$\begin{cases} y(1) = 0, \;\; y^{'}(1) = 0, \;\; y(T) = \dfrac{17}{400}\displaystyle\sum_{j=1}^{4} \xi_j z(\zeta_j) + \dfrac{13}{250}{}^{H}\mathcal{I}^{\frac{9}{20}} z(\dfrac{93}{50}), \\ z(1) = 0, \;\; z^{'}(1) = 0, \;\; z(T) = \dfrac{8}{125}\displaystyle\sum_{j=1}^{4} \upsilon_j y(\omega_j) + \dfrac{3}{40}{}^{H}\mathcal{I}^{\frac{17}{20}} y(\dfrac{73}{50}). \end{cases} \tag{4.4}$$

Here, $\varrho = \frac{83}{30}$, $\varsigma = \frac{47}{20}$, $\varrho_1 = \frac{17}{20}$, $\varsigma_1 = \frac{9}{20}$, $\xi_1 = \frac{1}{8}$, $\xi_2 = \frac{9}{40}$, $\xi_3 = \frac{13}{40}$, $\xi_4 = \frac{17}{40}$, $\upsilon_1 = \frac{3}{25}$, $\upsilon_2 = \frac{11}{50}$, $\upsilon_3 = \frac{8}{25}$, $\upsilon_4 = \frac{21}{50}$, $\zeta_1 = \frac{36}{25}$, $\zeta_2 = \frac{79}{50}$, $\zeta_3 = \frac{44}{25}$, $\zeta_4 = \frac{97}{50}$, $\omega_1 = \frac{63}{50}$, $\omega_2 = \frac{34}{25}$, $\omega_3 = \frac{83}{50}$, $\omega_4 = \frac{47}{25}$, $\alpha_1 = \frac{17}{400}$, $\alpha_2 = \frac{8}{125}$, $\beta_1 = \frac{13}{250}$, $\beta_2 = \frac{3}{40}$, $T = 2$, $\vartheta = \frac{93}{50}$, $\varphi = \frac{73}{50}$,

$$|f(\tau, y_1(\tau), z_1(\tau)) - f(\tau, y_2(\tau), z_2(\tau))| = \Big(\frac{1}{80}|y_1(\tau) - y_2(\tau)| + \frac{9}{400}|z_1(\tau) - z_2(\tau)|\Big)$$
$$|g(\tau, y_1(\tau), z_1(\tau)) - g(\tau, y_2(\tau), z_2(\tau))| = \Big(\frac{4}{125}|y_1(\tau) - y_2(\tau)| + \frac{21}{500}|z_1(\tau) - z_2(\tau)|\Big),$$

we have $\kappa_1 = \frac{1}{80}$, $\kappa_2 = \frac{9}{400}$, $\widehat{\kappa_1} = \frac{4}{125}$, $\widehat{\kappa_2} = \frac{21}{500}$.
With the data given, we find that $\nu_1 \approx 0.4804530139182014$, $\nu_2 \approx 0.01960022943737283$, $\nu_3 \approx 0.02532387378450272$, $\nu \approx 0.23033874484666408$, $\mathcal{P}_1 \approx 0.16113944169300537$, $\mathcal{Q}_1 \approx 0.15010321156298798$, $\mathcal{P}_2 \approx 0.00596663953967596$, $\mathcal{Q}_2 \approx 0.2995844504200921$ *and* $(\mathcal{P}_1 + \mathcal{P}_2)(\kappa_1 + \kappa_2) + (\mathcal{Q}_1 + \mathcal{Q}_2)(\widehat{\kappa_1} + \widehat{\kappa_2}) \approx 0.03912559982989178 < 1$. *Therefore all of the Theorem 3.2 assumptions are fulfilled. Consequently, on* $[1, 2]$ *a unique solution exists for the problem (4.3) and (4.4) by Theorem 3.2.*

5. Stability results for the problem (1.1) and (1.2)

The stability of the solutions given by

$$y(\tau) = \Upsilon_1(y, z)(\tau), \;\; z(\tau) \; = \Upsilon_2(y, z)(\tau), \tag{5.1}$$

Hyers-Ulam for BVP (1.1) and (1.2) is discussed in this section. Where Υ_1 and Υ_2 are defined by (3.2) and (3.3). Let us define nonlinear operators in the following $\mathcal{S}_1, \mathcal{S}_2 \in C(\mathcal{K}, \mathbb{R}) \times C(\mathcal{K}, \mathbb{R}) \to C(\mathcal{K}, \mathbb{R})$;

$$\begin{cases} {}^{C}\mathcal{D}^{\varrho} y(\tau) - f(\tau, y(\tau), z(\tau)) = \mathcal{S}_1(y, z)(\tau), \;\; \tau \in \mathcal{K}, \\ {}^{C}\mathcal{D}^{\varsigma} z(\tau) - g(\tau, y(\tau), z(\tau)) = \mathcal{S}_2(y, z)(\tau), \;\; \tau \in \mathcal{K}. \end{cases}$$

For some $\mu_1, \mu_2 > 0$, it considered the following inequalities:

$$\|\mathcal{S}_1(y, z)\| \leq \mu_1, \;\; \|\mathcal{S}_2(y, z)\| \leq \mu_2. \tag{5.2}$$

Definition 5.1. *The coupled system (1.1) and (1.2) is said to be stable in Hyers-Ulam, if* $\mathcal{H}_1, \mathcal{H}_2 > 0$ *exists such that there is a unique solution* $(y, z) \in C(\mathcal{K}, \mathbb{R}) \times C(\mathcal{K}, \mathbb{R})$ *of problems (1.1) and (1.2) with*

$$\|(y, z) - (y^*, z^*)\| \leq \mathcal{H}_1\mu_1 + \mathcal{H}_2\mu_2,$$

for every solution $(y^*, z^*) \in C(\mathcal{K}, \mathbb{R}) \times C(\mathcal{K}, \mathbb{R})$ *of inequality (5.2).*

Theorem 5.2. *Suppose that ($\mathcal{E}_2$) hold. Then the BVP (1.1) and (1.2) is Hyers-Ulam-stable.*

Proof. Let $(y,z) \in C(\mathcal{K},\mathbb{R}) \times C(\mathcal{K},\mathbb{R})$ be the (1.1) and (1.2) the solution of the problems that satisfy (3.2) and (3.3). Let (y^*,z^*) be any satisfying solution (5.2):

$$\begin{cases} {}^C\mathcal{D}^{\varrho}y(\tau) = f(\tau,y(\tau),z(\tau)) + \mathcal{S}_1(y,z)(\tau), & \tau \in \mathcal{K}, \\ {}^C\mathcal{D}^{\varsigma}z(\tau) = g(\tau,y(\tau),z(\tau)) + \mathcal{S}_2(y,z)(\tau), & \tau \in \mathcal{K}, \end{cases}$$

So,

$$\begin{aligned} y^*(\tau) = \Upsilon_1(y^*,z^*)(\tau) &+ \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\mathcal{S}_1(y,z)(\theta)\frac{d\theta}{\theta} \\ &+ \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}\mathcal{S}_2(y,z)(\theta)\frac{d\theta}{\theta} \\ &+ \beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}\mathcal{S}_2(y,z)(\theta)\frac{d\theta}{\theta} \\ &- \frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\mathcal{S}_1(y,z)(\theta)\frac{d\theta}{\theta}\Bigg\} \\ &+ \nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}\mathcal{S}_1(y,z)(\theta)\frac{d\theta}{\theta} \\ &+ \beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}\mathcal{S}_1(y,z)(\theta)\frac{d\theta}{\theta} \\ &- \frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}\mathcal{S}_2(y,z)(\theta)\frac{d\theta}{\theta}\Bigg\}\Bigg]. \end{aligned}$$

It follows that

$$\begin{aligned} |\Upsilon_1(y^*,z^*)(\tau) - y^*(\tau)| \le{} & \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\mu_1\frac{d\theta}{\theta} \\ &+ \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1}\mu_2\frac{d\theta}{\theta} \\ &+ \beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1}\mu_2\frac{d\theta}{\theta} \\ &+ \frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\mu_1\frac{d\theta}{\theta}\Bigg\} \\ &+ \nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\omega_j}\Big(\log\frac{\omega_j}{\theta}\Big)^{\varrho-1}\mu_1\frac{d\theta}{\theta} \\ &+ \beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\varphi}\Big(\log\frac{\varphi}{\theta}\Big)^{\varrho+\varrho_1-1}\mu_1\frac{d\theta}{\theta} \\ &+ \frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1}\mu_2\frac{d\theta}{\theta}\Bigg\}\Bigg] \end{aligned}$$

$$
\begin{aligned}
&\le \left[\frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \frac{(\log T)^2}{\nu}\left\{\frac{\nu_1(\log T)^{\varrho}}{\Gamma(\varrho+1)} + \nu_3\left(\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)}\right.\right.\right.\\
&\quad \left.\left.\left.+\beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\right)\right\}\right]\mu_1 + \left[\frac{(\log T)^2}{\nu}\left\{\frac{\nu_3(\log T)^{\varsigma}}{\Gamma(\varsigma+1)}\right.\right.\\
&\quad \left.\left.+\nu_1\left(\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)} + \beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\right)\right\}\right]\mu_2\\
&\le \mathcal{P}_1\mu_1 + \mathcal{Q}_1\mu_2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|\Upsilon_2(y^*,z^*)(\tau) - z^*(\tau)| &\le \left[\frac{(\log T)^{\varsigma}}{\Gamma(\varsigma+1)} + \frac{(\log T)^2}{\nu}\left\{\frac{\nu_1(\log T)^{\varsigma}}{\Gamma(\varsigma+1)} + \nu_2\left(\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^{\varsigma}}{\Gamma(\varsigma+1)}\right.\right.\right.\\
&\quad \left.\left.\left.+\beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\right)\right]\mu_2 + \frac{(\log T)^2}{\nu}\left[\left\{\frac{\nu_2(\log T)^{\varrho}}{\Gamma(\varrho+1)}\right.\right.\\
&\quad \left.\left.+\nu_1\left(\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\omega_j)^{\varrho}}{\Gamma(\varrho+1)} + \beta_2\frac{(\log\varphi)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\right)\right\}\right]\mu_1\\
&\le \mathcal{Q}_2\mu_2 + \mathcal{P}_2\mu_1,
\end{aligned}
$$

where in (3.4)–(3.7) is described $\mathcal{P}_1$,$\mathcal{P}_2$,$\mathcal{Q}_1$ and $\mathcal{Q}_2$. The Υ operator, given by (3.2) and (3.3), can therefore be excluded as follows from the fixed point property.

$$
\begin{aligned}
|y(\tau) - y^*(\tau)| &= |y(\tau) - \Upsilon_1(y^*,z^*)(\tau) + \Upsilon_1(y^*,z^*)(\tau) - y^*(\tau)|\\
&\le |\Upsilon_1(y,z)(\tau) - \Upsilon_1(y^*,z^*)(\tau)| + |\Upsilon_1(y^*,z^*)(\tau) - y^*(\tau)|\\
&\le (\mathcal{P}_1\kappa_1 + \mathcal{Q}_1\widehat{\kappa_1}) + (\mathcal{P}_1\kappa_2 + \mathcal{Q}_1\widehat{\kappa_2})\|(y,z) - (y^* - z^*)\| + \mathcal{P}_1\mu_1 + \mathcal{Q}_1\mu_2 \qquad (5.3)
\end{aligned}
$$

$$
\begin{aligned}
|z(\tau) - z^*(\tau)| &= |z(\tau) - \Upsilon_2(y^*,z^*)(\tau) + \Upsilon_2(y^*,z^*)(\tau) - z^*(\tau)|\\
&\le |\Upsilon_2(y,z)(\tau) - \Upsilon_2(y^*,z^*)(\tau)| + |\Upsilon_2(y^*,z^*)(\tau) - z^*(\tau)|\\
&\le (\mathcal{Q}_2\widehat{\kappa_1} + \mathcal{P}_2\kappa_1) + (\mathcal{Q}_2\widehat{\kappa_2} + \mathcal{P}_2\kappa_2)\|(y,z) - (y^* - z^*)\| + \mathcal{Q}_2\mu_2 + \mathcal{P}_2\mu_1. \qquad (5.4)
\end{aligned}
$$

From (5.3) and (5.4) it follows that

$$
\begin{aligned}
\|(y,z) - (y^* - z^*)\| \le (\mathcal{P}_1 + \mathcal{P}_2)\mu_1 + (\mathcal{Q}_1 + \mathcal{Q}_2)\mu_2 + (\mathcal{P}_1 + \mathcal{P}_2)(\kappa_1 + \kappa_2) + (\mathcal{Q}_1 + \mathcal{Q}_2)(\widehat{\kappa_1} + \widehat{\kappa_2})\\
\times \|(y,z) - (y^* - z^*)\|
\end{aligned}
$$

$$
\begin{aligned}
\|(y,z) - (y^* - z^*)\| &\le \frac{(\mathcal{P}_1 + \mathcal{P}_2)\mu_1 + (\mathcal{Q}_1 + \mathcal{Q}_2)\mu_2}{1 - ((\mathcal{P}_1 + \mathcal{P}_2)(\kappa_1 + \kappa_2) + (\mathcal{Q}_1 + \mathcal{Q}_2)(\widehat{\kappa_1} + \widehat{\kappa_2}))}\\
&\le \mathcal{H}_1\mu_1 + \mathcal{H}_2\mu_2,
\end{aligned}
$$

with

$$\mathcal{H}_1 = \frac{(\mathcal{P}_1 + \mathcal{P}_2)}{1 - ((\mathcal{P}_1 + \mathcal{P}_2)(\kappa_1 + \kappa_2) + (\mathcal{Q}_1 + \mathcal{Q}_2)(\widehat{\kappa_1} + \widehat{\kappa_2}))},$$
$$\mathcal{H}_2 = \frac{(\mathcal{Q}_1 + \mathcal{Q}_2)}{1 - ((\mathcal{P}_1 + \mathcal{P}_2)(\kappa_1 + \kappa_2) + (\mathcal{Q}_1 + \mathcal{Q}_2)(\widehat{\kappa_1} + \widehat{\kappa_2}))}.$$

Therefore, the BVP (1.1)–(1.2) is Hyers-Ulam stable. □

6. Example

Example 6.1. *Consider the following coupled system of Caputo-Hadamard type FDEs:*

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{97}{40}} y(\tau) = \dfrac{e^{-\log\tau}}{4} + \dfrac{2}{5(\tau+16)}\dfrac{|y(\tau)|}{1+|y(\tau)|} + \dfrac{7}{200}\cos(z(\tau)), \quad \tau \in [1,2], \\ {}^{C}\mathcal{D}^{\frac{41}{15}} z(\tau) = \dfrac{1}{(\tau+2)^2} + \dfrac{11}{200}\sin(y(\tau)) + \dfrac{3}{7(\tau^2+27)}\dfrac{|z(\tau)|}{1+|z(\tau)|}, \quad \tau \in [1,2], \end{cases} \tag{6.1}$$

equipped with coupled boundary conditions:

$$\begin{cases} y(1) = 0, \quad y'(1) = 0, \quad y(T) = \dfrac{17}{400}\sum_{j=1}^{4} \xi_j z(\zeta_j) + \dfrac{13}{250}{}^{H}\mathcal{I}^{\frac{9}{20}} z(\frac{93}{50}), \\ z(1) = 0, \quad z'(1) = 0, \quad z(T) = \dfrac{8}{125}\sum_{j=1}^{4} \upsilon_j y(\omega_j) + \dfrac{3}{40}{}^{H}\mathcal{I}^{\frac{17}{20}} y(\frac{73}{50}). \end{cases} \tag{6.2}$$

Here, $\varrho = \frac{97}{40}$, $\varsigma = \frac{41}{15}$, $\varrho_1 = \frac{17}{20}$, $\varsigma_1 = \frac{9}{20}$, $\xi_1 = \frac{1}{8}$, $\xi_2 = \frac{9}{40}$, $\xi_3 = \frac{13}{40}$, $\xi_4 = \frac{17}{40}$, $\upsilon_1 = \frac{3}{25}$, $\upsilon_2 = \frac{11}{50}$, $\upsilon_3 = \frac{8}{25}$, $\upsilon_4 = \frac{21}{50}$, $\zeta_1 = \frac{36}{25}$, $\zeta_2 = \frac{79}{50}$, $\zeta_3 = \frac{44}{25}$, $\zeta_4 = \frac{97}{50}$, $\omega_1 = \frac{63}{50}$, $\omega_2 = \frac{34}{25}$, $\omega_3 = \frac{83}{50}$, $\omega_4 = \frac{47}{25}$, $\alpha_1 = \frac{17}{400}$, $\alpha_2 = \frac{8}{125}$, $\beta_1 = \frac{13}{250}$, $\beta_2 = \frac{3}{40}$, $T = 2$, $\vartheta = \frac{93}{50}$, $\varphi = \frac{73}{50}$,

$$|f(\tau, y_1(\tau), z_1(\tau)) - f(\tau, y_2(\tau), z_2(\tau))| = \Big(\frac{2}{85}|y_1(\tau) - y_2(\tau)| + \frac{7}{200}|z_1(\tau) - z_2(\tau)|\Big)$$
$$|g(\tau, y_1(\tau), z_1(\tau)) - g(\tau, y_2(\tau), z_2(\tau))| = \Big(\frac{11}{200}|y_1(\tau) - y_2(\tau)| + \frac{3}{196}|z_1(\tau) - z_2(\tau)|\Big),$$

we have $\kappa_1 = \frac{2}{85}$, $\kappa_2 = \frac{7}{200}$, $\widehat{\kappa_1} = \frac{11}{200}$, $\widehat{\kappa_2} = \frac{3}{196}$.

With the data given, we find that $\nu_1 \approx 0.4804530139182014$, $\nu_2 \approx 0.01960022943737283$, $\nu_3 \approx 0.02532387378450272$, $\nu \approx 0.23033874484666408$, $\mathcal{P}_1 \approx 0.2690687091717075$, $\mathcal{Q}_1 \approx 0.08501392524416244$, $\mathcal{P}_2 \approx 0.010361690715239952$, $\mathcal{Q}_2 \approx 0.16968755312542472$. *Then problem (6.1) and (6.2) has a unique solution for* $[1,2]$, *which is stable for Hyers-Ulam, with* $(\mathcal{P}_1 + \mathcal{P}_2)(\kappa_1 + \kappa_2) + (\mathcal{Q}_1 + \mathcal{Q}_2)(\widehat{\kappa_1} + \widehat{\kappa_2}) \approx 0.0342619702607479 < 1$, *so all requirements of Theorem 5.2.*

7. Existence results for the problem (1.1) and (1.3)

Lemma 7.1. *Let $\hat{f}, \hat{g} \in \mathcal{AC}_\delta^n[1,T]$. Then, the linear system solution of FDEs*

$$\begin{cases} {}^C\mathcal{D}^\varrho y(\tau) = \hat{f}(\tau), \\ {}^C\mathcal{D}^\varsigma z(\tau) = \hat{g}(\tau), \end{cases} \tag{7.1}$$

enhanced with the boundary conditions:

$$\begin{cases} y(1) = 0, \;\; y^{'}(1) = 0, \;\; y(T) = \alpha_1 \sum_{j=1}^{k-2} \xi_j z(\zeta_j) + \beta_1 {}^H\mathcal{I}^{\varsigma_1} z(\vartheta), \\ z(1) = 0, \;\; z^{'}(1) = 0, \;\; z(T) = \alpha_2 \sum_{j=1}^{k-2} \upsilon_j y(\zeta_j) + \beta_2 {}^H\mathcal{I}^{\varrho_1} y(\vartheta), \\ 1 < \vartheta < \zeta_1 < \zeta_2 < \cdots < \zeta_{k-2} < T. \end{cases} \tag{7.2}$$

$$\begin{aligned} y(\tau) = {}^H\mathcal{I}^\varrho \hat{f}(\tau) + \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1 \sum_{j=1}^{k-2} \xi_j {}^H\mathcal{I}^\varsigma \hat{g}(\zeta_j) + \beta_1 {}^H\mathcal{I}^{\varsigma+\varsigma_1} \hat{g}(\vartheta) - {}^H\mathcal{I}^\varrho \hat{f}(T)\Bigg\} \\ + \nu_3\Bigg\{\alpha_2 \sum_{j=1}^{k-2} \upsilon_j {}^H\mathcal{I}^\varrho \hat{f}(\zeta_j) + \beta_2 {}^H\mathcal{I}^{\varrho+\varrho_1} \hat{f}(\vartheta) - {}^H\mathcal{I}^\varsigma \hat{g}(T)\Bigg\}\Bigg] \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} z(\tau) = {}^H\mathcal{I}^\varsigma \hat{g}(\tau) + \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_2 \sum_{j=1}^{k-2} \upsilon_j {}^H\mathcal{I}^\varrho \hat{f}(\zeta_j) + \beta_2 {}^H\mathcal{I}^{\varrho+\varrho_1} \hat{f}(\vartheta) - {}^H\mathcal{I}^\varsigma \hat{g}(T)\Bigg\} \\ + \nu_2\Bigg\{\alpha_1 \sum_{j=1}^{k-2} \xi_j {}^H\mathcal{I}^\varsigma \hat{g}(\zeta_j) + \beta_1 {}^H\mathcal{I}^{\varsigma+\varsigma_1} \hat{g}(\vartheta) - {}^H\mathcal{I}^\varrho \hat{f}(T)\Bigg\}\Bigg] \end{aligned} \tag{7.4}$$

where

$$\nu_1 = (\log T)^2, \;\; \nu_2 = \alpha_2 \sum_{j=1}^{k-2} \upsilon_j (\log\zeta_j)^2 + \frac{2\beta_2(\log\vartheta)^{2+\varrho_1}}{\Gamma(3+\varrho_1)}, \tag{7.5}$$

$$\nu_3 = \alpha_1 \sum_{j=1}^{k-2} \xi_j (\log\zeta_j)^2 + \frac{2\beta_1(\log\vartheta)^{2+\varsigma_1}}{\Gamma(3+\varsigma_1)}, \;\; \nu = \nu_1^2 - \nu_2\nu_3. \tag{7.6}$$

Proof. Solving the FDEs (7.1) in a standard manner, we get

$$y(\tau) = {}^H\mathcal{I}^\varrho \hat{f}(\tau) + a_0 + a_1 \log\tau + a_2(\log\tau)^2, \tag{7.7}$$

$$z(\tau) = {}^H\mathcal{I}^\varsigma \hat{g}(\tau) + b_0 + b_1 \log\tau + b_2(\log\tau)^2, \tag{7.8}$$

where $a_i, b_i \in \mathbb{R}$, $i = 0,1,2$, are arbitrary constants. Using the boundary conditions (7.2) in (7.7) and (7.8), we obtain $a_0 = a_1 = 0$, $b_0 = b_1 = 0$, and

$$a_2\nu_1 - b_2\nu_3 = \alpha_1 \sum_{j=1}^{k-2} \xi_j {}^H\mathcal{I}^\varsigma \hat{g}(\zeta_j) + \beta_1 {}^H\mathcal{I}^{\varsigma+\varsigma_1} \hat{g}(\vartheta) - {}^H\mathcal{I}^\varrho \hat{f}(T), \tag{7.9}$$

$$b_2\nu_1 - a_2\nu_2 = \alpha_2 \sum_{j=1}^{k-2} \upsilon_j {}^H\mathcal{I}^{\varrho} \hat{f}(\zeta_j) + \beta_2 {}^H\mathcal{I}^{\varrho+\varrho_1} \hat{f}(\vartheta) - {}^H\mathcal{I}^{\varsigma} \hat{g}(T). \tag{7.10}$$

Solving the system (7.9)–(7.10) for a_2, b_2, we get

$$\begin{aligned} a_2 &= \nu_1\Big(\alpha_1 \sum_{j=1}^{k-2} \xi_j {}^H\mathcal{I}^{\varsigma} \hat{g}(\zeta_j) + \beta_1 {}^H\mathcal{I}^{\varsigma+\varsigma_1} \hat{g}(\vartheta) - {}^H\mathcal{I}^{\varrho} \hat{f}(T)\Big) \\ &\quad + \nu_3\Big(\alpha_2 \sum_{j=1}^{k-2} \upsilon_j {}^H\mathcal{I}^{\varrho} \hat{f}(\zeta_j) + \beta_2 {}^H\mathcal{I}^{\varrho+\varrho_1} \hat{f}(\vartheta) - {}^H\mathcal{I}^{\varsigma} \hat{g}(T)\Big), \qquad (7.11) \\ b_2 &= \nu_1\Big(\alpha_2 \sum_{j=1}^{k-2} \upsilon_j {}^H\mathcal{I}^{\varrho} \hat{f}(\zeta_j) + \beta_2 {}^H\mathcal{I}^{\varrho+\varrho_1} \hat{f}(\vartheta) - {}^H\mathcal{I}^{\varsigma} \hat{g}(T)\Big) \\ &\quad + \nu_2\Big(\alpha_1 \sum_{j=1}^{k-2} \xi_j {}^H\mathcal{I}^{\varsigma} \hat{g}(\zeta_j) + \beta_1 {}^H\mathcal{I}^{\varsigma+\varsigma_1} \hat{g}(\vartheta) - {}^H\mathcal{I}^{\varrho} \hat{f}(T)\Big), \qquad (7.12) \end{aligned}$$

where ν_1, ν_2, ν_3, ν are given by (7.5) and (7.6) respectively. Substituting the values of a_2, b_2 in (7.7) and (7.8), we obtain the solutions (7.3) and (7.4). □

Next, we define an operator

$$\Upsilon(y, z)(\tau) = (\Upsilon_1(y, z)(\tau), \Upsilon_2(y, z)(\tau)), \tag{7.13}$$

in relation to problem (1.1) and (1.3), with

$$\begin{aligned} \Upsilon_1(y, z)(\tau) &= \frac{1}{\Gamma(\varrho)} \int_1^{\tau} \Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1} f(\theta, y(\theta), z(\theta)) \frac{d\theta}{\theta} \\ &+ \frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_1 \sum_{j=1}^{k-2} \xi_j \frac{1}{\Gamma(\varsigma)} \int_1^{\zeta_j} \Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1} g(\theta, y(\theta), z(\theta)) \frac{d\theta}{\theta} \\ &+ \beta_1 \frac{1}{\Gamma(\varsigma+\varsigma_1)} \int_1^{\vartheta} \Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1} g(\theta, y(\theta), z(\theta)) \frac{d\theta}{\theta} \\ &- \frac{1}{\Gamma(\varrho)} \int_1^{T} \Big(\log\frac{T}{\theta}\Big)^{\varrho-1} f(\theta, y(\theta), z(\theta)) \frac{d\theta}{\theta}\Bigg\} \\ &+ \nu_3\Bigg\{\alpha_2 \sum_{j=1}^{k-2} \upsilon_j \frac{1}{\Gamma(\varrho)} \int_1^{\zeta_j} \Big(\log\frac{\zeta_j}{\theta}\Big)^{\varrho-1} f(\theta, y(\theta), z(\theta)) \frac{d\theta}{\theta} \\ &+ \beta_2 \frac{1}{\Gamma(\varrho+\varrho_1)} \int_1^{\vartheta} \Big(\log\frac{\vartheta}{\theta}\Big)^{\varrho+\varrho_1-1} f(\theta, y(\theta), z(\theta)) \frac{d\theta}{\theta} \\ &- \frac{1}{\Gamma(\varsigma)} \int_1^{T} \Big(\log\frac{T}{\theta}\Big)^{\varsigma-1} g(\theta, y(\theta), z(\theta)) \frac{d\theta}{\theta}\Bigg\}\Bigg], \end{aligned} \tag{7.14}$$

$$\begin{aligned}\Upsilon_2(y,z)(\tau) = &\frac{1}{\Gamma(\varsigma)}\int_1^\tau \Big(\log\frac{\tau}{\theta}\Big)^{\varsigma-1} g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\
&+\frac{(\log\tau)^2}{\nu}\Bigg[\nu_1\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{1}{\Gamma(\varrho)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varrho-1} f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\
&+\beta_2\frac{1}{\Gamma(\varrho+\varrho_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varrho+\varrho_1-1} f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\
&-\frac{1}{\Gamma(\varsigma)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varsigma-1} g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\Bigg\}\\
&+\nu_2\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{1}{\Gamma(\varsigma)}\int_1^{\zeta_j}\Big(\log\frac{\zeta_j}{\theta}\Big)^{\varsigma-1} g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\
&+\beta_1\frac{1}{\Gamma(\varsigma+\varsigma_1)}\int_1^{\vartheta}\Big(\log\frac{\vartheta}{\theta}\Big)^{\varsigma+\varsigma_1-1} g(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\\
&-\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1} f(\theta,y(\theta),z(\theta))\frac{d\theta}{\theta}\Bigg\}\Bigg]. \qquad (7.15)\end{aligned}$$

For the convenience of computation, we set

$$\mathcal{P}_1 = \frac{(\log T)^\varrho}{\Gamma(\varrho+1)}+\frac{(\log T)^2}{\nu}\Bigg[\frac{\nu_1(\log T)^\varrho}{\Gamma(\varrho+1)}+\nu_3\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\zeta_j)^\varrho}{\Gamma(\varrho+1)}+\beta_2\frac{(\log\vartheta)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Bigg\}\Bigg] \qquad (7.16)$$

$$\mathcal{Q}_1 = \frac{(\log T)^2}{\nu}\Bigg[\frac{\nu_3(\log T)^\varsigma}{\Gamma(\varsigma+1)}+\nu_1\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^\varsigma}{\Gamma(\varsigma+1)}+\beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Bigg\}\Bigg] \qquad (7.17)$$

$$\mathcal{P}_2 = \frac{(\log T)^2}{\nu}\Bigg[\frac{\nu_2(\log T)^\varrho}{\Gamma(\varrho+1)}+\nu_1\Bigg\{\alpha_2\sum_{j=1}^{k-2}\upsilon_j\frac{(\log\zeta_j)^\varrho}{\Gamma(\varrho+1)}+\beta_2\frac{(\log\vartheta)^{\varrho+\varrho_1}}{\Gamma(\varrho+\varrho_1+1)}\Bigg\}\Bigg] \qquad (7.18)$$

$$\mathcal{Q}_2 = \frac{(\log T)^\varsigma}{\Gamma(\varsigma+1)}+\frac{(\log T)^2}{\nu}\Bigg[\frac{\nu_1(\log T)^\varsigma}{\Gamma(\varsigma+1)}+\nu_2\Bigg\{\alpha_1\sum_{j=1}^{k-2}\xi_j\frac{(\log\zeta_j)^\varsigma}{\Gamma(\varsigma+1)}+\beta_1\frac{(\log\vartheta)^{\varsigma+\varsigma_1}}{\Gamma(\varsigma+\varsigma_1+1)}\Bigg\}\Bigg]. \qquad (7.19)$$

Now for the problem (1.1) and (1.3), we state the results of existence, uniqueness, and stability. We are not providing the proof as it is similar to those in Section 3, Section 4, Section 5, Section 6.

Theorem 7.2. *Suppose that ($\mathcal{E}_1$) hold. If*

$$\lambda_1(\mathcal{P}_1+\mathcal{P}_2)+\widehat{\lambda_1}(\mathcal{Q}_1+\mathcal{Q}_2)<1,\ \ \lambda_2(\mathcal{P}_1+\mathcal{P}_2)+\widehat{\lambda_2}(\mathcal{Q}_1+\mathcal{Q}_2)<1. \qquad (7.20)$$

Then there exists at least one solution for problem (1.1) and (1.3) on $\mathcal{K}$, where $\mathcal{P}_1$, $\mathcal{Q}_1$, $\mathcal{P}_2$, and $\mathcal{Q}_2$ are given by (7.16)–(7.19) respectively.

Theorem 7.3. *Suppose that ($\mathcal{E}_2$) hold. Then the BVP (1.1) and (1.3) has a unique solution on $\mathcal{K}$, provided that*

$$(\mathcal{P}_1+\mathcal{P}_2)(\kappa_1+\kappa_2)+(\mathcal{Q}_1+\mathcal{Q}_2)(\widehat{\kappa_1}+\widehat{\kappa_2})<1, \qquad (7.21)$$

where $\mathcal{P}_1$, $\mathcal{Q}_1$, $\mathcal{P}_2$, and $\mathcal{Q}_2$ are given by (7.16)–(7.19).

Theorem 7.4. *Suppose that* $(\mathcal{E}_2)$ *hold. In addition,* $\exists$ *positive constants* $\mathcal{T}_1$, $\mathcal{T}_2$ *such that* $\forall\ \tau \in \mathcal{K}$ *and* $y, z \in \mathbb{R}$,

$$|f(\tau, y, z)| \leq \mathcal{T}_1, \quad |g(\tau, y, z)| \leq \mathcal{T}_2. \tag{7.22}$$

Then the BVP (1.1) and (1.3) has at least one solution on $\mathcal{K}$*, if*

$$\frac{(\log T)^{\varrho}(\kappa_1 + \kappa_2)}{\Gamma(\varrho + 1)} + \frac{(\log T)^{\varsigma}(\widehat{\kappa_1} + \widehat{\kappa_2})}{\Gamma(\varsigma + 1)} < 1. \tag{7.23}$$

Theorem 7.5. *Suppose that* $(\mathcal{E}_2)$ *hold. Then the BVP (1.1) and (1.3) is Hyers-Ulam-stable.*

8. Conclusions

We have studied the existence, uniqueness, and stability of solutions for a coupled system of Caputo-Hadamard-type FDEs augmented by Hadamard fractional integral and multi-point conditions via the alternatives of Leray-Schauder, Banach, fixed-point theorems of Krasnoselskii, Hyer-Ulam stable. The work presented in this paper is new and significantly contributes to the existing literature on the topic. When the parameters involved in problem $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ were set, our results corresponded to some special problems. Suppose we present the problems (1.1) and (1.2) with the form: to take $\alpha_1 = \alpha_2 = 0$ in the results provided;

$$\begin{cases} y(1) = 0, \quad y'(1) = 0, \quad y(T) = \beta_1 {}^{H}\mathcal{I}^{\varsigma_1} z(\vartheta), \\ z(1) = 0, \quad z'(1) = 0, \quad z(T) = \beta_2 {}^{H}\mathcal{I}^{\varrho_1} y(\varphi), \\ 1 < \vartheta < \varphi < T, \end{cases}$$

while the results are:

$$\begin{cases} y(1) = 0, \quad y'(1) = 0, \quad y(T) = \alpha_1 \sum_{j=1}^{k-2} \xi_j z(\zeta_j), \\ z(1) = 0, \quad z'(1) = 0, \quad z(T) = \alpha_2 \sum_{j=1}^{k-2} \upsilon_j y(\omega_j), \\ 1 < \zeta_1 < \omega_1 < \zeta_2 < \omega_2 < \cdots < \zeta_{k-2} < \omega_{k-2} < T, \end{cases}$$

followed by $\beta_1 = \beta_2 = 0$. We can solve above problems similar to problem (1.1) and (1.2) by using the methodology employed in the previous section.

Acknowledgments

We thank the reviewers for their constructive remarks on our work.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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