Mathematics

Research article

On the extinction of continuous-state branching processes in random environments

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Abstract: This paper establishes a model of continuous-state branching processes with time inhomogeneous competition in Lévy random environments. Some results on extinction are presented, including the distribution of the extinction time, the limiting distribution conditioned on large extinction times and the asymptotic behavior near extinction. This paper also provides a new time-space transformation which can be used for further exploration in similar models. The results are applied to an epidemic model to describe the dynamics of infectious population and a virus model to describe the dynamics of viral load.

1. Introduction

Branching processes have wide applications and have raised the interest of many researchers, see [10]. The model was first established by Galton and Watson [8] in order to study the demographic phenomenon that a large proportion of families were continually dying out. Decades later, Feller [7] formalized the model to the modern version. He provided the recursion of the population of the n-th generation Z_n as follows,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}, \qquad (1.1)$$

where $\xi_{n,i}$ are random variables representing the number of offspring of the i-th individual in the n-th generation. In this model, time is discretized by generations. However, in real life, population develops continuously in time. Therefore, researchers improved the model to a continuous-time

version. The first general formulation of the continuous-time discrete-state branching processes was given by Kolmogorov and Dmitrive, and has been one of the most popular demographic models, see [21].

In microbiology, the population is large and the events of birth and death occur very frequently. Scientists usually use terminologies like *density* or *viral load* to describe the amount, and the number is often not integers, see [20]. In this situation, a branching model of continuous-time and continuous-state is more appropriate and convenient to approximate the dynamics. Ba and Pardoux [1] proved that the continuous-state branching processes can be considered as the scaling limit of the continuous-time discrete-state branching processes. They considered the interactions between individuals and established the limiting stochastic equation,

$$Y_t = 1 + \int_0^t f(Y_s)\mathrm{d}s + \int_0^t \sqrt{2cY_s}\mathrm{d}B(s), \qquad (1.2)$$

where $x \mapsto f(x)$ is a function on $[0, \infty)$ representing the interactions, c is a positive constant and $B(s)$ is a standard Brownian motion. This has the general form of a Feller branching diffusion.

There are plenty of biological articles focusing on how the population is influenced by the environment, see [5] for an example. But the investigation on the mathematical models of the continuous-state branching process in random environment, which is usually abbreviated as *CBRE-process*, is just in its infancy. Pioneering works on CBRE-processes include [2] and [9]. To describe the drastic reduction in population by catastrophes, Bansaye et al. [2] added a Lévy process with bounded variation into the continuous-state branching model. They used the following equation to model infected cells,

$$Y_t = 1 + \int_0^t bY_s\mathrm{d}s + \int_0^t \sqrt{2cY_s}\mathrm{d}B(s) + \int_0^t \int_0^1 (\theta - 1)Y_{s-}N(\mathrm{d}s, \mathrm{d}\theta), \qquad (1.3)$$

where $c > 0, b$ are constants, $B(s)$ is a Brownian motion and $N(\mathrm{d}s, \mathrm{d}\theta)$ is Poisson random measure representing the catastrophes. They studied the behavior of $\mathbf{P}(Y_t > 0)$ as $t \to \infty$. This work was generalised by Li and Xu [17], who set the random environment to be a Lévy process $\{L(t) : t \geq 0\}$ with no jump less than -1. This jump restriction prevents the population from becoming negative. They constructed an equation for $(\alpha + 1)$-*stable branching processes in Lévy random environment (CBRE-processes)* as follows,

$$X(t) = x + \int_0^t \sqrt[(1+\alpha)]{(1 + \alpha)cX(s-)}\mathrm{d}Z_\alpha(s) + \int_0^t X(s-)\mathrm{d}L(s),$$

where $c > 0, \alpha \in (0, 1]$ are constants, $\{Z_\alpha(t) : t \geq 0\}$ is a spectrally positive $(\alpha + 1)$-stable process.

Motivated by the models above, we construct an $(\alpha + 1)$-stable continuous-state branching process with both the effect of random environments and a time inhomogeneous competition, which will be abbreviated as the $(\alpha + 1)$-*stable CBCRE-process*. The equation of our model is

$$X(t) = x + \int_0^t \sqrt[(1+\alpha)]{(1 + \alpha)cX(s-)}\mathrm{d}Z_\alpha(s) - \int_0^t f(s)X(s)\mathrm{d}s + \int_0^t X(s-)\mathrm{d}L(s), \qquad (1.4)$$

where $s \mapsto f(s)$ is a locally bounded non-negative function on $[0, \infty)$ describing the competition between individuals. When $f \equiv 0$, our model degenerates to the model of $(\alpha + 1)$-*stable CBRE-processes* in [17]. Furthermore, [17] studied the asymptotic behavior of the survival probability.

But our results include the asymptotic behavior of the process itself. In other words, we study the asymptotic property of the paths instead of probability. When $\alpha = 1$, $f \equiv 0$ and $\{L(t) : t \geq 0\}$ takes the special form of $\int_0^t \int_0^1 (1 - \theta)N(\mathrm{d}s, \mathrm{d}\theta)$, our model degenerates to the model of [2]. When $\alpha = 1$ and $L(t) \equiv 0$, it becomes a model with only interactions but no environments, like [1].

In this paper, we define the *extinction time* of the process as the first time that the process reaches zero, which is denoted by T_0. We provide the distribution of T_0, and give a necessary condition for T_0 to be almost surely finite. The long-time behavior is an important topic for stochastic processes, see [11–13]. Our results also include the long-time behavior of the model. We present the limit distribution of CBCRE-processes conditioned on large extinction times. Furthermore, we find a determining function g to describe the asymptotic behavior near extinction of the $(\alpha+1)$-stable CBCRE-process $\{X(t) : t \geq 0\}$. In other words,

$$\limsup_{t \to 0} \frac{X(T_0 - t)}{g(t)} = 1, \; almost \; surely.$$

This is a result more accurate than [2] and [17], since we describe the behavior of the paths rather than the extinction and survival probability. And this result is new even in CBRE-processes without competition.

The paper is organized as follows, in Section 2, we establish the model, present the stochastic equation and state the main results. In Section 3, we give the proofs of the main results including a time-space transformation. In Section 4, we apply the results to two mathematical biological models. In the first part of Section 4, we improved the stochastic Susceptible-Infected-Recovered (SIR) epidemic model by considering the influence of environment. In the second part of Section 4, we apply our results to a virus elimination model and conclude that the differences between the immune systems of different hosts only affects the distribution of their recovery time. The asymptotic behavior of the viral load near the host's recovery time (equivalently, the extinction time of virus) depends only on the characteristics of the virus.

2. Preliminaries and main results

In this section, we establish the model of $(\alpha + 1)$-stable CBCRE-processes on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$. Suppose that constants $b \in \mathbb{R}$, $c > 0$, $\alpha \in (0, 1]$. A stable branching mechanism is a continuous function ϕ on $[0, \infty)$ with the representation

$$\phi(\lambda) = c\lambda^{\alpha+1}, \; \lambda > 0. \tag{2.1}$$

Suppose that $(1 \wedge z^2)n(\mathrm{d}z)$ is a finite measure on $(0, \infty)$, $\{L(t) : t \geq 0\}$ is an $(\mathcal{F}_t)$-Lévy process with Lévy-Itô decomposition:

$$L(t) = bt + \sigma W(t) + \int_0^t \int_{[-1,1]} (e^z - 1)\tilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{[-1,1]^c} (e^z - 1)N(\mathrm{d}s, \mathrm{d}z), \tag{2.2}$$

where $\{W(t) : t \geq 0\}$ is an $(\mathcal{F}_t)$-Brownian motion, and $N(\mathrm{d}s, \mathrm{d}z)$ is an $(\mathcal{F}_t)$-Poisson random measure on $(0, \infty) \times \mathbb{R}$ with intensity $\mathrm{d}sn(\mathrm{d}z)$, whose compensated measure is denoted by $\tilde{N}(\mathrm{d}s, \mathrm{d}z)$. Let $s \mapsto f(s)$ be a locally bounded nonnegative function on $[0, \infty)$. For any positive constant x, consider the following stochastic integral equation,

$$X(t) = x + \int_0^t \sqrt[(1+\alpha)]{(1 + \alpha)cX(s-)}\mathrm{d}Z_\alpha(s) - \int_0^t f(s)X(s)\mathrm{d}s + \int_0^t X(s-)\mathrm{d}L(s), \tag{2.3}$$

where $\{Z_\alpha(t) : t \geq 0\}$ is a spectrally positive $(\alpha + 1)$-stable process with $\alpha \in (0, 1]$. When $\alpha = 1$, this can be regarded as a Brownian motion. When $\alpha \in (0, 1)$, its Lévy measure is

$$\frac{\alpha \mathbf{1}_{z>0}\mathrm{d}z}{\Gamma(1 - \alpha)z^{2+\alpha}}, \tag{2.4}$$

where $\Gamma(1 - \alpha) = \int_0^\infty t^{-\alpha}e^{-t}\mathrm{d}t$. In this equation, $\int_0^t f(s)X(s)\mathrm{d}s$ describes the influence of competition on the population dynamics. We may consider $f(s)$ as a competition rate. It changes with time and it stays non-negative, since at a fixed time point, the competition becomes fiercer as the population becomes larger. In Eq (2.3), there are two kinds of noise: $\int_0^t \sqrt[(1+\alpha)]{(1 + \alpha)cX(s-)}\mathrm{d}Z_\alpha(s)$ represents the randomness of branching, which is the same as the noise term in the equation of traditional stable branching processes, see P.55 of [16]; and $\int_0^t X(s-)\mathrm{d}L(s)$ represents the effect of random environments, see [9] and [17].

For convenience, define another $(\mathcal{F}_t)$-Lévy process $\{\xi(t) : t \geq 0\}$ by

$$\xi(t) = at + \sigma W(t) + \int_0^t \int_{[-1,1]} z\tilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{[-1,1]^c} zN(\mathrm{d}s, \mathrm{d}z),$$

where

$$a = b - \frac{\sigma^2}{2} - \int_{[-1,1]} (e^z - 1 - z)n(\mathrm{d}z). \tag{2.5}$$

In the situation where $f \equiv 0$, the unique strong solution of (2.3) is the CBRE-process with branching mechanism ϕ and random environment $\{\xi(t) : t \geq 0\}$ defined by [9]. Denote

$$\bar{\xi}(t) := \xi(t) - \int_0^t f(s)\mathrm{d}s. \tag{2.6}$$

Theorem 2.1. *The Eq (2.3) has a positive unique strong solution $\{X(t) : t \geq 0\}$. And the solution is a conservative Markov process on $[0, \infty)$ with transition semigroup $(Q_t)_{t\geq0}$ defined by*

$$\int_0^\infty e^{-\lambda y}Q_t(x, \mathrm{d}y) = \mathbf{P}e^{-xv_{0,t}^{\bar{\xi}}(\lambda)}, \qquad \lambda, x \geq 0, \tag{2.7}$$

where $r \mapsto v_{r,t}^{\bar{\xi}}(\lambda)$ is the unique positive solution of

$$v_{r,t}^{\bar{\xi}}(\lambda) = e^{\bar{\xi}(t)-\bar{\xi}(r)}\lambda - \int_r^t e^{\bar{\xi}(s)-\bar{\xi}(r)}\phi(v_{s,t}^{\bar{\xi}}(\lambda))\mathrm{d}s, \qquad t \geq r \geq 0. \tag{2.8}$$

Proof. According to Theorem 1.3 of [6],

$$X(t) = x + \int_0^t \sqrt[(1+\alpha)]{(1 + \alpha)cX(s-)}\mathrm{d}Z_\alpha(s) - \int_0^t f(s)X(s)\mathrm{d}s, \tag{2.9}$$

has a unique strong solution. Following by similar arguments in [19], the existence and uniqueness of Eq (2.3) can be proved. The remaining procedures can be obtained by repeating the calculations in the proof of Theorem 3.6 in [9], with ξ in there replaced by $\bar{\xi}$. □

The unique strong solution $\{X(t) : t \geq 0\}$ of Eq (2.3) is a CBCRE-process with initial value x, branching mechanism ϕ, and competition f in random environment ξ, which is abbreviated as $(\alpha + 1)$-*stable CBCRE-process*. By the definition in He et al. [9], the *extinction time* $T_0 := \inf\{t \geq 0 : X(t) = 0\}$ is obviously an $\mathcal{F}_t$-stopping time. Again, by replacing parameters as in the proof of Theorem 3.3, we have the following conclusion paralleling to that in Section 5 of [17].

Proposition 2.2. *The distribution of the extinction time T_0 is provided as*

$$\mathbf{P}(T_0 \leq t) = \mathbf{P}(X(t) = 0) = \mathbf{P}(\mathrm{e}^{-x\bar{v}_{0,t}^{\bar{\xi}}}), \tag{2.10}$$

where $\bar{v}_{0,t}^{\bar{\xi}} := [\int_0^t c\alpha \mathrm{e}^{-\alpha\bar{\xi}(s)}\mathrm{d}s]^{-\frac{1}{\alpha}}$.

Furthermore, if we assume that $\liminf_{t\to\infty} \bar{\xi}(t) = -\infty$, then we have

$$\liminf_{t\to\infty} \bar{v}_{0,t}^{\bar{\xi}} = \liminf_{t\to\infty}[\int_0^t c\alpha \mathrm{e}^{-\alpha\bar{\xi}(s)}\mathrm{d}s]^{-\frac{1}{\alpha}} = 0. \tag{2.11}$$

From (2.10) it is obvious that $\bar{v}_{0,t}^{\bar{\xi}}$ is decreasing in t. Thus,

$$\limsup_{t\to\infty} \bar{v}_{0,t}^{\bar{\xi}} = \liminf_{t\to\infty} \bar{v}_{0,t}^{\bar{\xi}} = 0. \tag{2.12}$$

Using Fatou's Lemma we get,

$$\liminf_{t\to\infty} \mathbf{P}(T_0 \leq t) \geq \mathbf{P}\liminf_{t\to\infty} \mathrm{e}^{-x\bar{v}_{0,t}^{\bar{\xi}}} = 1. \tag{2.13}$$

Thus,

$$\mathbf{P}(T_0 = \infty) = \lim_{t\to\infty} \mathbf{P}(T_0 > t) = 0.$$

Therefore, $\liminf_{t\to\infty} \bar{\xi}(t) = -\infty$ is the necessary condition for the stable CBRE-process $\{X(t) : t \geq 0\}$ to extinct in a finite time almost surely. The main result of this paper is the extinction behavior of the process, including the limit distribution conditioned on large extinction times and the asymptotic behavior near extinction.

Theorem 2.3. *[Conditional Limit Theorem] Let $x > 0$ and $t \geq 0$. $\{X(t) : t \geq 0\}$ is a stable CBCRE-process with initial value x and extinction time T_0. Then for any $\mathcal{F}_t$-measurable bounded random variable F,*

$$\bar{\mathbf{P}}[F] = \lim_{r\to\infty} \mathbf{P}[F|T_0 > r + t],$$

where $\bar{\mathbf{P}}$ is a probability measure such that $\bar{\mathbf{P}}[F] = \mathbf{P}[M_t(x)F]$ with

$$M_t(x) = \frac{1}{x}e^{-\bar{\xi}(t)}X(t).$$

The above result provides the distribution of $\{X(t) : t \geq 0\}$ conditioned on large extinction times. Under this new probability measure, $\{X(t) : t \geq 0\}$ is still a Markov process in $(0, \infty)$. See [15] for the similar conditional limit theorems of continuous-state branching processes with neither environment nor competition.

Theorem 2.4. *[Asymptotic Behavior Theorem] Let $\{X(t) : t \geq 0\}$ be an $(\alpha + 1)$-stable CBCRE-process with competition f and random environment ξ. Its extinction time is denoted by T_0. Suppose that*

$$\liminf_{t\to\infty} \left(\xi(t) - \int_0^t f(s)\mathrm{d}s \right) = -\infty.$$

Then,

$$\limsup_{t\to 0} \frac{X(T_0 - t)}{(c\alpha t)^{\frac{1}{\alpha}}(\log\log\frac{1}{t})^{-\alpha}} = 1, \ \mathbf{P}\text{-}a.s.$$

3. Proofs of main results

In this section, we prove the main results and provide an interesting time-space transformation.

Proposition 3.1. *The process $t \to M_t(x) := \frac{1}{x}e^{-\bar{\xi}(t)}X(t)$ is a positive $(\mathcal{F}_t)$-martingale with $\mathbf{P}M_0(x) = 1$.*

Proof. Define $M_t = e^{-\bar{\xi}_t}X(t)$, where $X(t)$ is the strong solution of (2.3). By Itô's formula,

$$
\begin{aligned}
M_t \ &= \ x + \int_0^t e^{-\bar{\xi}(s-)}\mathrm{d}X(s) - \int_0^t X(s-)e^{-\bar{\xi}(s-)}\mathrm{d}\bar{\xi}(s) \\
&\quad + \frac{\sigma^2}{2} \int_0^t X(s)e^{-\bar{\xi}(s)}\mathrm{d}s - \int_0^t \sigma^2 X(s)e^{-\bar{\xi}(s)}\mathrm{d}s \\
&\quad + \int_0^t \int_{-\infty}^{\infty} \{[X(s-) + X(s-)(e^z - 1)]e^{-\bar{\xi}(s-)+z} - X(s-)e^{-\bar{\xi}(s-)} \\
&\qquad\qquad -e^{-\bar{\xi}(s-)}X(s-)(e^z - 1) + X(s)e^{-\bar{\xi}(s-)}z\}N(\mathrm{d}s,\mathrm{d}z) \\
&= \ x + \int_0^t e^{-\bar{\xi}(s-)} \sqrt[(1+\alpha)]{(1+\alpha)cX(s-)}\mathrm{d}Z_\alpha(s).
\end{aligned}
\tag{3.1}
$$

It is easy to see that $\{M_t : t \geq 0\}$ is a positive martingale with $\mathbf{P}M_t = x$. The conclusion is obvious. $\square$

Proof of Theorem 2.3. By Proposition 3.1 and Girsanov's theorem, there is a unique probability measure $\bar{\mathbf{P}}$ on $(\Omega, \mathcal{F})$ such that

$$\bar{\mathbf{P}}[F] = \mathbf{P}[M_t(x)F],$$

for any $\mathcal{F}_t$-measurable bounded random variable F. Now it suffices to show that $\bar{\mathbf{P}}[e^{-\lambda X(t)}] = \lim_{r\to\infty} \mathbf{P}[e^{-\lambda X(t)}|T_0 > r + t]$. Define $\mathbf{P}^{\bar{\xi}}$ to be the quenched law conditioned on the random environment,

$$
\begin{aligned}
\mathbf{P}^{\bar{\xi}}[e^{-\lambda X(t)}|T_0 > t + r] \ &= \ \frac{\mathbf{P}^{\bar{\xi}}[e^{-\lambda X(t)}1_{\{T_0>r+t\}}]}{\mathbf{P}^{\bar{\xi}}(T_0 > r + t)} \\
&= \ \lim_{\theta\to\infty} \frac{\mathbf{P}^{\bar{\xi}}[e^{-\lambda X(t)}(1 - e^{-\theta X_{t+r}})]}{\mathbf{P}^{\bar{\xi}}(1 - e^{-\theta X_{t+r}})} \\
&= \ \frac{\mathbf{P}^{\bar{\xi}}[e^{-\lambda X(t)}(1 - e^{-X(t)\bar{v}^{\bar{\xi}}_{t,t+r}})]}{1 - e^{-x\bar{v}^{\bar{\xi}}_{0,t+r}}}.
\end{aligned}
$$

It is not difficult to get $\bar{v}^{\bar{\xi}}_{0,r+t} = v^{\bar{\xi}}_{0,t}(\bar{v}^{\bar{\xi}}_{t,t+r})$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}v^{\bar{\xi}}_{0,t}(\lambda)\Big|_{\lambda=0} = e^{\bar{\xi}_t}$. By the above and (2.13) we have, $\lim\limits_{r\to\infty}\bar{v}^{\bar{\xi}}_{t,t+r} = 0$ almost surely. Then

$$
\begin{aligned}
\lim_{r\to\infty}\mathbf{P}^{\bar{\xi}}[e^{-\lambda X(t)}|T_0 > t+r] &= \lim_{r\to\infty}\frac{\mathbf{P}^{\bar{\xi}}[e^{-\lambda X(t)}(1 - e^{-X(t)\bar{v}^{\bar{\xi}}_{t,t+r}})]}{1 - e^{-x\bar{v}^{\bar{\xi}}_{0,t+r}}} \\
&= \lim_{r\to\infty}\frac{\mathbf{P}^{\bar{\xi}}[e^{-\lambda X(t)}(\bar{v}^{\bar{\xi}}_{t,t+r})^{-1}(1 - e^{-X(t)\bar{v}^{\bar{\xi}}_{t,t+r}})]}{(\bar{v}^{\bar{\xi}}_{t,t+r})^{-1}(1 - e^{-x\bar{v}^{\bar{\xi}}_{0,t+r}})} \\
&= \frac{1}{x}e^{-\bar{\xi}_t}\mathbf{P}^{\bar{\xi}}[X(t)e^{-\lambda X(t)}].
\end{aligned}
$$

By bounded convergence theorem, it is easy to see that

$$
\lim_{r\to\infty}\mathbf{P}[e^{-\lambda X(t)}|T_0 > t+r] = \frac{1}{x}\mathbf{P}[X(t)e^{-\lambda X(t)-\bar{\xi}_t}].
$$

That completes the proof. □

Consider the stochastic integral equation

$$
Y(t) = x + \int_0^t \sqrt[(1+\alpha)]{(1 + \alpha)cY(s-)}\mathrm{d}Z_\alpha(s). \tag{3.2}
$$

The unique strong solution of (3.2) is a continuous-state branching process with $(\alpha+1)$-stable branching mechanism. We denote the process by $\{Y(t) : t \geq 0\}$. There is a nice relationship between the process $\{Y(t) : t \geq 0\}$ and $\{X(t) : t \geq 0\}$.

Proposition 3.2. *[Time-space Transformation] A stable CBCRE-process can be obtained by a time-space transformation of a classical continuous-state branching process with the same branching mechanism. In other words, $Y(A(t))e^{\bar{\xi}(t)}$ is a weak solution of (2.3), where $A(t) = \int_0^t e^{-\alpha\bar{\xi}(s)}\mathrm{d}s$.*

Proof. By Itô's formula and integration by parts,

$$
\begin{aligned}
Y(A(t))e^{\bar{\xi}(t)} &= x + \int_0^t e^{\bar{\xi}(s)}\mathrm{d}Y(A(s)) + \int_0^t Y(A(s))\mathrm{d}e^{\bar{\xi}(s)} \\
&= x + \int_0^t Y(A(s))e^{\bar{\xi}(s)}(a + \frac{\sigma^2}{2})\mathrm{d}s + \int_0^t Y(A(s))e^{\bar{\xi}(s)}\sigma\mathrm{d}W(s) \\
&\quad + \int_0^t e^{\bar{\xi}(s)}\sqrt[(1+\alpha)]{(1 + \alpha)cY(A(s))}\mathrm{d}Z_\alpha(A(s)) - \int_0^t e^{\bar{\xi}(s)}Y(A(s))f(s)\mathrm{d}s \\
&\quad + \int_0^t\int_{[-1,1]^c} Y(A(s))e^{\bar{\xi}(s)}(e^z - 1)N(\mathrm{d}s, \mathrm{d}z) \\
&\quad + \int_0^t\int_{[-1,1]} Y(A(s))e^{\bar{\xi}(s)}(e^z - 1)\tilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&\quad + \int_0^t\int_{[-1,1]} Y(A(s))e^{\bar{\xi}(s)}(e^z - 1 - z)\mathrm{d}s n(\mathrm{d}z)
\end{aligned}
$$

$$= x + \int_0^t e^{\bar{\xi}(s)} \sqrt[(1+\alpha)]{(1+\alpha)cY(A(s))} \mathrm{d}Z_\alpha(A(s))$$

$$- \int_0^t f(s)Y(A(s))e^{\bar{\xi}(s)}\mathrm{d}s + \int_0^t Y(A(s))e^{\bar{\xi}(s)}\mathrm{d}L(s). \tag{3.3}$$

Let $\tilde{Z}(t) := \int_0^t e^{\frac{\alpha}{1+\alpha}\bar{\xi}(s)}\mathrm{d}Z_\alpha(A(s))$. Then $\tilde{Z}$ is also an $(\alpha + 1)$-stable process with Lévy measure $m(\mathrm{d}z)$ defined by (2.4). Thus,

$$Y(A(t))e^{\bar{\xi}(t)} = Y(0) + \int_0^t \sqrt[(1+\alpha)]{(1+\alpha)cY(A(s))e^{\bar{\xi}(s)}}\mathrm{d}\tilde{Z}(A(s))$$

$$- \int_0^t f(s)Y(A(s))e^{\bar{\xi}(s)}\mathrm{d}s + \int_0^t Y(A(s))e^{\bar{\xi}(s)}\mathrm{d}L(s).$$

$\square$

We denote the extinction time of $\{Y(t) : t \geq 0\}$ by $T_0' := \inf\{t \geq 0 : Y(t) = 0\}$. There is a result on the asymptotic behavior near extinction of $Y(t)$, which we state as the Lemma below.

Lemma 3.3. *[Kyprianou and Pardo [14]] Suppose that $\{Y(t) : t \geq 0\}$ is a continuous-state branching process with branching mechanism ϕ given by* (2.1). *Then*

$$\limsup_{t\to 0+} \frac{Y(T_0' - t)}{g(t)} = 1, \mathbf{P}\text{-}a.s.,$$

where $g(t) = (c\alpha t)^{\frac{1}{\alpha}}(\log\log\frac{1}{t})^{-\alpha}$.

Now we are able to give the asymptotic behavior near extinction of the $(\alpha + 1)$-stable CBCRE-process $\{X(t) : t \geq 0\}$.

Proof of Theorem 2.4. Without loss of generality, assume that $\{X(t) : t \geq 0\}$ is the unique strong solution of (2.3). By Theorem 3.2 we get, $X(t)$ has the same distribution to $Y(A(t))e^{\bar{\xi}(t)}$. Thus,

$$\lim_{t\to 0+} Y(A(T_0 - t)) = 0.$$

According to Lemma 3.3,

$$\limsup_{t\to 0+} \frac{Y(A(T_0 - t))}{g(A(T_0) - A(T_0 - t))} = 1, \mathbf{P}\text{-}a.s. \tag{3.4}$$

Thus,

$$\limsup_{t\to 0+} \frac{X(T_0 - t)}{g(te^{-\alpha\bar{\xi}(T_0)})e^{\bar{\xi}(T_0)}} = 1, \ \mathbf{P}\text{-}a.s.$$

Notice that the limitation is in the sense of $\mathbf{P}$-*a.s.*, and $\alpha\bar{\xi}(T_0)$ is pathwisely constant. Thus, when $t \to 0+$, the influence of $\bar{\xi}(T_0)$ can be ignored compared to $\log\frac{1}{t}$. Hence we get the conclusion. $\square$

The time-space transformation that we use in the proof builds a strong bridge between CBCRE-processes and traditional continuous-state branching processes. This novel approach contributes to the study of continuous-state branching processes in random environments and can be used for further exploration in similar models.

4. Applications

In this section we apply our model to mathematical biology. We consider a particle system with $X(0)$ particles at time $t = 0$. After each unit of time, each particle in the system splits into a random number of new particles following a fixed reproduction law. This procedure can approximate the reproduction of a virus at the micro level, as well as the increase in the infectious population at the macro level. If we denote the population of the n-th generation by Z_n, we can get

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}, \tag{4.1}$$

where $\xi_{n,i}$ are random variables representing the number of offspring of the i-th individual in the n-th generation. This is the definition of classic branching processes, or GW-processes. However, the discrete setting is idealized. This is problematic since the reproduction of the virus and the speed of contagion are always quite fast, and the quantities we discuss are always quite large. Thus, it is more appropriate and convenient to compress the original process defined by (4.1) both in time and state. It was provided in 36 and 37 pages of [4] that the scaling limits of a series of GW-processes with a fixed reproduction law are continuous-time and continuous-state stable branching processes. Moreover, when we consider the random environment and competition, we can have stable CBCRE-processes as the scaling limits, see [3]. Thus, it is reasonable to apply our model to the study of epidemic and viral load.

4.1. Application to an epidemic model

In this part, we apply our results to an improved stochastic Susceptible-Infected-Recovered (SIR) epidemic model. The stochastic SIR-model was introduced in [18] as the nonlinear stochastic differential equations:

$$
\begin{aligned}
\mathrm{d}S(t) &= -\beta I(t)\frac{S(t)}{N(t)}\mathrm{d}t - \sqrt{\beta I(t)\frac{S(t)}{N(t)}}\mathrm{d}B_1(t) \\
\mathrm{d}I(t) &= [\beta I(t)\frac{S(t)}{N(t)} - (\alpha + \gamma)I(t)]\mathrm{d}t + \sqrt{\beta I(t)\frac{S(t)}{N(t)}}\mathrm{d}B_1(t) \\
&\quad - \sqrt{\gamma I(t)}\mathrm{d}B_2(t) + \sqrt{\alpha I(t)}\mathrm{d}B_3(t) \\
\mathrm{d}R(t) &= \gamma I(t)\mathrm{d}t + \sqrt{\gamma I(t)}\mathrm{d}B_2(t).
\end{aligned}
\tag{4.2}
$$

In this model, the whole population $N(t)$ is divided into three parts: the susceptible part $S(t)$, the infectious part $I(t)$ and the recovered part $R(t)$. Parameters α, β and γ denote the disease mortality rate, the transmission rate and the recovery rate, respectively. $B_1(t), B_2(t)$ and $B_3(t)$ are mutually independent standard Brownian motions.

To better approximate the actual conditions, we consider environmental influences on the epidemic model. The environment may affect the dynamics of an infectious population via fast traffic tools like airplanes and high-speed railways, and policies like quarantine and lock-down, among others.

We describe the environment by the Lévy processes $\xi(t)$ and $L(t)$ defined in Section 2 and build the following equations:

$$
\begin{aligned}
dS(t) &= -\beta I(t)\frac{S(t)}{N(t)}dt - \sqrt{\beta I(t)\frac{S(t)}{N(t)}}dB_1(t) \\
dI(t) &= [\beta I(t)\frac{S(t)}{N(t)} - (\alpha + \gamma)I(t)]dt + \sqrt{\beta I(t)\frac{S(t)}{N(t)}}dB_1(t) \\
&\quad - \sqrt{\gamma I(t)}dB_2(t) + \sqrt{\alpha I(t)}dB_3(t) + I(t)dL(t) \\
dR(t) &= \gamma I(t)dt + \sqrt{\gamma I(t)}dB_2(t).
\end{aligned}
\tag{4.3}
$$

When the infectious population is small compared to the large whole population, the infectious population can be approximated by a branching process defined by

$$
I(t) = I(0) + \int_0^t (\beta - \alpha - \gamma)I(s)ds + \int_0^t \sqrt{(\beta + \alpha + \gamma)I(s)}dB(s) + \int_0^t X(s)dL(s),
\tag{4.4}
$$

where $B(s)$ is a standard Brownian motion and $L(t)$ is defined by (2.2). According to the results in previous sections, if the condition $\liminf_{t\to\infty}[\xi(t) + (\beta - \alpha - \gamma)t] = -\infty$ is satisfied, the epidemic will be eliminated in a finite amount of time T. The distribution of T is given by

$$
\mathbf{P}(T \le t) = \mathbf{P}\exp\left(-2I(0)\left(\int_0^t (\alpha + \beta + \gamma)e^{-\xi(s)+(\alpha+\gamma-\beta)s}ds\right)^{-1}\right).
$$

Moreover, the infectious population will tend toward zero as $t \to T$ in the behavior described by the function

$$
\frac{(\alpha + \beta + \gamma)(T - t)}{2\log\log\frac{1}{(T-t)}}.
$$

4.2. Application to a virus elimination model

The continuous-state branching process is a natural model for population which die and multiply fast and randomly, see Bansaye et al. [2]. Assume that at time 0 a person is infected with initial viral load x. Then the viruses in the person replicate with the branching mechanism

$$
\phi(\lambda) = c\lambda^{\alpha+1}, \ \lambda > 0,
$$

where $c > 0$, $\alpha \in (0, 1]$. The immune cells of the host are produced according to a Poisson distribution with parameter $r > 0$, and kill a random fraction $\theta \in (0, 1)$ of the viruses. The distribution of the killing fraction θ is denoted by ν, where ν satisfies $\int_0^1 \theta\nu(d\theta) < \infty$.

Hence, the dynamics of the viral load $\{X(t) : t \ge 0\}$ is the unique positive strong solution of the stochastic integral equation on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$,

$$
X(t) = x + \int_0^t \sqrt[(1+\alpha)]{(1 + \alpha)cX(s-)}dZ_\alpha(s) - \int_0^t \int_0^1 \theta X(s-)M(ds, d\theta).
\tag{4.5}
$$

In this equation, $\{Z_\alpha(t) : t \geq 0\}$ is a spectrally positive $(\alpha + 1)$-stable process with $\alpha \in (0, 1]$, as defined in Section 2. $M(\mathrm{d}s, \mathrm{d}\theta)$ is a Poisson random measure with intensity $r\mathrm{d}s\nu(\mathrm{d}\theta)$.

A host *recovers* when the viral load reaches zero. And the time it takes to recover is denoted by T, which amounts to the extinction time of $\{X(t) : t \geq 0\}$. According to the conclusions in Section 2, T is a random time with distribution

$$\mathbf{P}(T \leq t) = \mathbf{P}(\mathrm{e}^{-x\eta_{0,t}}), \tag{4.6}$$

where

$$\eta_{0,t} := [\int_0^t c\alpha \exp\{-\alpha \int_0^s \int_0^1 \ln(1 - \theta)M(\mathrm{d}u, \mathrm{d}\theta)\}ds]^{-\frac{1}{\alpha}} < \infty, \ \mathbf{P}\text{-}a.s. \tag{4.7}$$

Moreover, the viral load tends toward zero near the recovery moment T in a behavior that can be approximated by function

$$(c\alpha(T - t))^{\alpha^{-1}}(\log\log\frac{1}{(T - t)})^{-\alpha}.$$

These results demonstrate

- The fewer viruses with which a host was infected at beginning, the faster the host is expected to recover;
- Differences in the sensitivity and effectiveness between the immune systems of different hosts (corresponding to r and ν in the model) result in different lengths of recovery time;
- The proliferative ability of the virus (corresponding to c, α in the model) affects both the recovery time and the behavior of the viruses being eliminated. Indeed, it is the only factor determining this behavior.

Acknowledgments

Conflict of interest

This work does not have any conflicts of interest.

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