



Research article

Tikhonov-type regularization method for a sideways problem of the time-fractional diffusion equation

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Abstract: A sideways problem of the time-fractional diffusion equation is investigated. The solution of this problem does not depend on the given data. In view of this, this article uses a Tikhonov-type regularized method to construct an approximate solution and overcome the ill-posedness of considered problem. The a-posteriori convergence estimates of logarithmic and double logarithmic types for the regularized method are derived. Finally, for smooth and non-smooth cases we respectively verify the effectiveness of proposed method by doing the corresponding numerical experiments.

Keywords: sideways problem; time-fractional diffusion equation; Tikhonov-type regularization method; a-posteriori convergence estimate; numerical implementation

Mathematics Subject Classification: 65N20, 65N21

1. Introduction

In some scientific fields, the fractional diffusion equation has many wide applications, such as biology, mechanical engineering, physical, signal processing and systems identification, fractional dynamics, chemical, finance, control theory, and so on, see [1–8]. In the past years, the direct problems for the fractional diffusion equation have been studied extensively. Recent years, more and more people are focusing on the inverse problems for this kind of equation, which usually include parameter identification problem, inverse initial value problem, Cauchy problem, inverse heat conduction problem, inverse source problem, inverse boundary condition problem, and so on.

The time-fractional diffusion equation is derived by replacing the classical time derivative with fractional derivative, and it can be used to describe superdiffusion and subdiffusion phenomena. In this

paper, we consider the sideways problem of the time-fractional diffusion equation in the following

$$\begin{cases} D_t^\alpha u(x, t) - u_{xx}(x, t) = 0, & x > 0, t > 0, \\ u(x, 0) = 0, & x \geq 0, \\ u(1, t) = g(t), & t \geq 0, \\ u(x, t)|_{x \rightarrow \infty} \text{ bounded,} \end{cases} \quad (1.1)$$

here, D_t^α denotes the Caputo fractional derivative defined by [4]

$$D_t^\alpha u(x, t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(x, \tau)}{(t-\tau)^\alpha} d\tau, & 0 < \alpha < 1, \\ u_t(x, t), & \alpha = 1, \end{cases} \quad (1.2)$$

and the Fourier transform with respect to t is

$$\mathcal{F}\{D_t^\alpha f(t); \xi\} = (i\xi)^\alpha \widehat{f}(\xi), \quad i = \sqrt{-1}. \quad (1.3)$$

We know that, as $\alpha = 1$, problem (1.1) just is the standard inverse heat conduction problem(IHCP). Let $\delta > 0$ be the measured error bound, given the measured data $g^\delta(t)$ that satisfies $\|g^\delta(t) - g(t)\|_{L^2(\mathbb{R})} \leq \delta$, the task of this paper is to seek $u(x, t)$ ($0 \leq x < 1$) from (1.1). This problem is ill-posed in the sense that the solution does not depend continuously on the given data, so the regularized method is required to overcome its ill-posedness and recover the stability of the solution. The early papers which considered the sideways problem for time-fractional diffusion equation can be found in [9, 10]. After that, some regularized methods have been presented and used to solve some similar problems, such as spectral truncation method [11–15], convolution method [16, 17], and so on.

In the ordinary Tikhonov method, the regularization item is imposed as $\|u(x, t)\|_{L^2(\mathbb{R})}^2$. In this paper, to solve problem (1.1), we construct a Tikhonov-type regularized method by adding the regularized item as $\|u(0, t)\|_{H^s(\mathbb{R})}^2$ ($0 \leq s < p$, $p > 0$ is a constant), and under the a-posteriori selection of regularized parameter we derive the convergence estimates of logarithmic and double logarithmic types. Ultimately, by doing some numerical experiments we verify the simulation effectiveness of Tikhonov-type method. Note that, the considered problem is defined in the unbounded region, some traditional numerical methods (such as finite difference and finite element methods) can not be applied directly, so many scholars developed the artificial boundary method or translated the unbounded problem into a bounded one by adopting certain transformation, but these methods may lead to artificial errors and singularity. On the other hand, since the singularity of the time fractional derivative can produce a full numerical discrete matrix, this leads to a lot of computation and storage. In view of this, based on the Fourier translation we adopt the Tikhonov-type regularized method to give the explicit expression of regularized solution in frequency space, and use the fast discrete Fourier transform to compute the regularized solution, this technology is simple and convenient, and it can overcome the weak singularity near the initial time.

The writing motivation and innovation of this work are as below: we not only design a stable regularization method to overcome the ill-posedness of the considered problem, but also shall derive the a-posteriori convergence result for the regularized solution under an a-posteriori selection rule for the regularized parameter, which has not been considered in the existing references [9–17], etc, so the

work is relatively novel in solving the sideways problem of time-fractional diffusion equation. It can be seen as an extension and supplement on the existing works.

The writing arrangement of the article are as below. Section 2 constructs the regularized method based on the result of conditional stability. Section 3 is devoted to derive the convergence estimate of a-posteriori type for regularized solution. In Section 4, we make some numerical experiments and simulations to verify the calculation effect. Section 5 gives the corresponding summary.

2. Conditional stability and regularization method

2.1. Conditional stability

Let $f \in L^2(\mathbb{R})$, we respectively define the Fourier transform(FT) and inverse Fourier transform(IFT) as

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)e^{-i\xi y} dy, \quad \xi \in \mathbb{R}. \quad (2.1)$$

$$f(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{i\xi y} d\xi, \quad y \in \mathbb{R}. \quad (2.2)$$

For $\xi \in \mathbb{R}$, we take the Fourier transform with regard to t , in the frequency domain the solution of (1.1) can be expressed as

$$\widehat{u}(x, \xi) = e^{\psi_\alpha(\xi)(1-x)} \widehat{g}(\xi), \quad (2.3)$$

here, $\psi_\alpha(\xi) = |\xi|^{\frac{\alpha}{2}} \left(\cos\left(\frac{\alpha\pi}{4}\right) + i \operatorname{sign}(\xi) \sin\left(\frac{\alpha\pi}{4}\right) \right)$. hence, we can obtain that the exact solution of (1.1) is

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\psi_\alpha(\xi)(1-x)} \widehat{g}(\xi) e^{i\xi t} d\xi. \quad (2.4)$$

Equation (2.4) means that, when $|\xi| \rightarrow \infty$, the function $e^{|\xi|^{\frac{\alpha}{2}}(1-x)\cos\frac{\alpha\pi}{4}}$ tends to infinity, i.e., the solution of (1.1) is not stable (it does not depend continuously on the given data). But if it satisfies certain a-priori condition, one can establish the stability of solution, i.e., the conditional stability.

Let $E > 0$, we suppose that the solution of (1.1) satisfies the a-priori bound

$$\|u(0, \cdot)\|_p \leq E, \quad p > 0, \quad (2.5)$$

where the norm of Sobolev space H^p is defined as

$$\|u(0, \cdot)\|_p = \left(\int_{-\infty}^{\infty} (1 + \xi^2)^p |\widehat{u}(0, \cdot)|^2 d\xi \right)^{1/2}, \quad (2.6)$$

we know that, as $p = 0$, this is the L^2 -norm. Throughout this paper, we denote $\|\cdot\|$ as the L^2 -norm. In [18], by using the interpolation method the authors established the following result of condition stability under the assumption of the a-priori bound condition (2.5).

Theorem 1. [18] Suppose that the a-priori bound condition (2.5) is valid, then for the fixed $0 \leq x < 1$, there holds the estimate of conditional stability

$$\|u(x, \cdot)\| \leq E^{1-x} \|g(\cdot)\|^x \left(\frac{1}{\cos(\alpha\pi/4)} \ln \frac{1}{\|g(\cdot)\|} \right)^{\frac{-2p(1-x)}{\alpha}}. \quad (2.7)$$

2.2. Tikhonov-type regularization method

From the Eq (2.4) we can see that, in order to recover the consecutive dependence of solution for problem (1.1), one need design a stable approximation solution of it by removing the part of high frequency of function $e^{|\xi|^{\frac{\alpha}{2}}(1-x)\cos\frac{\alpha\pi}{4}}$. In the following, we give a concise description for the designed procedure of the regularized method.

For the fixed $0 \leq x < 1$, according to (2.4) we can equivalently transform (1.1) into the operator equation below

$$K(x)u(x, t) = g(t), \quad (2.8)$$

and based on (2.3), $\widehat{K}(x) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is a linear bounded multiplication operator with $\widehat{K}(x) = e^{-\psi_\alpha(\xi)(1-x)} = e^{-|\xi|^{\frac{\alpha}{2}}(1-x)\cos(\frac{\alpha\pi}{4}) - i|\xi|^{\frac{\alpha}{2}}(1-x)\text{sign}(\xi)\sin(\frac{\alpha\pi}{4})}$, by the basic calculation, we know that the adjoint operator of it can be written as $\widehat{K}^*(x) = e^{-|\xi|^{\frac{\alpha}{2}}(1-x)\cos(\frac{\alpha\pi}{4}) + i|\xi|^{\frac{\alpha}{2}}(1-x)\text{sign}(\xi)\sin(\frac{\alpha\pi}{4})}$.

We assume g^δ satisfy the following error bound

$$\|g^\delta - g\| \leq \delta. \quad (2.9)$$

Let $\mu > 0$ denote the regularization parameter, to construct the regularized solution, we consider the variational problem

$$\min_{u \in L^2(\mathbb{R})} \|K(x)u(x, t) - g^\delta(t)\|^2 + \mu \|u(0, t)\|_{H^s(\mathbb{R})}^2, \quad 0 \leq s < p, \quad (2.10)$$

By (2.3) and using the Parseval identity, we know that problem (2.10) becomes as

$$\min_{\widehat{u} \in L^2(\mathbb{R})} \left\| e^{-\psi_\alpha(\xi)(1-x)} \widehat{u}(x, \xi) - \widehat{g}^\delta(\xi) \right\|^2 + \mu \left\| (1 + |\xi|^2)^{s/2} \frac{e^{-\psi_\alpha(\xi)(1-x)}}{e^{-\psi_\alpha(\xi)}} \widehat{u}(x, \xi) \right\|^2. \quad (2.11)$$

Now, we denote $\widehat{u}_\mu^\delta(x, \xi)$ be the regularization solution for (1.1) in the frequency domain, then $\widehat{u}_\mu^\delta(x, \xi)$ satisfies the normal equation

$$\left(e^{-2|\xi|^{\frac{\alpha}{2}}(1-x)\cos(\frac{\alpha\pi}{4})} + \mu (1 + |\xi|^2)^s \frac{e^{-2|\xi|^{\frac{\alpha}{2}}(1-x)\cos(\frac{\alpha\pi}{4})}}{e^{-2|\xi|^{\frac{\alpha}{2}}\cos(\frac{\alpha\pi}{4})}} \right) \widehat{u}_\mu^\delta(\xi, t) = e^{-|\xi|^{\frac{\alpha}{2}}(1-x)\cos(\frac{\alpha\pi}{4}) + i|\xi|^{\frac{\alpha}{2}}(1-x)\text{sign}(\xi)\sin(\frac{\alpha\pi}{4})} \widehat{g}^\delta(\xi), \quad (2.12)$$

according to (2.12), we obtain that $\widehat{u}_\mu^\delta(\xi, t)$ holds the following formulation

$$\widehat{u}_\mu^\delta(x, \xi) = \frac{e^{\psi_\alpha(\xi)(1-x)} \widehat{g}^\delta(\xi)}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}}\cos(\frac{\alpha\pi}{4})}}, \quad (2.13)$$

finally, we can write the expression of regularized solution for (1.1) as

$$u_\mu^\delta(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{\psi_\alpha(\xi)(1-x)} \widehat{g}^\delta(\xi) e^{i\xi t}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}}\cos(\frac{\alpha\pi}{4})}} d\xi. \quad (2.14)$$

Remark 1. In the construction procedure of the regularized solution (2.14), in (2.10) we add the penalty item in the sense of H^s -norm. We know that, in the standard Tikhonov method, the penalty item

is usually imposed in the sense of L^2 -norm, i.e., $\|u(x, t)\|_{L^2(\mathbb{R})}^2$, and the Tikhonov regularized solution can be expressed as

$$u_{Tik\mu}^\delta(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{\psi_\alpha(\xi)(1-x)} \widehat{g^\delta}(\xi) e^{i\xi t}}{1 + \mu e^{2(1-x)|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} d\xi. \quad (2.15)$$

And if setting $x = 0$ and making a modification on (2.15), then we can obtain a simplified Tikhonov regularized solution

$$u_{STik\mu}^\delta(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{\psi_\alpha(\xi)(1-x)} \widehat{g^\delta}(\xi) e^{i\xi t}}{1 + \mu e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} d\xi. \quad (2.16)$$

3. An a-posteriori convergence result

This section is devoted to derive the convergence estimate for the regularized method, we select the regularized parameter μ by a kind of a-posteriori rule that is presented in [19]. Setting $h(\delta) > \delta$, here μ is found by solving the equation

$$\|u_\mu^\delta(1, t) - g^\delta(t)\| = h(\delta). \quad (3.1)$$

On the a-posteriori selection rule of regularized parameter, we can refer to [20] which gives the general description for the a-posteriori rule of regularized parameter. We give two Lemmas that will be used in the following.

Lemma 1. Let $\varrho(\mu) = \|u_\mu^\delta(1, t) - g^\delta(t)\|$, $0 < h(\delta) < \|g^\delta\|$, then for $\mu \in (0, +\infty)$, $\varrho(\mu)$ is a continuous and strictly increasing function; and $\lim_{\mu \rightarrow 0} \varrho(\mu) = 0$, $\lim_{\mu \rightarrow +\infty} \varrho(\mu) = \|g^\delta\|$.

Proof of Lemma 1. By taking

$$\varrho(u) = \left\| \frac{\mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})} \widehat{g^\delta}(\xi)}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} \right\|, \quad (3.2)$$

one can easily prove it.

From Lemma 1, we can know that the Eq (3.1) has a unique solution as $0 < h(\delta) < \|g^\delta\|$.

Lemma 2. We suppose (2.5) hold, then the regularized parameter $\mu = \mu(\delta, g^\delta)$ determined by (3.1) satisfies $\frac{1}{\mu} \leq \frac{C_1 E^2}{4(h(\delta) - \delta)^2}$, here C_1 is a positive constant.

Proof of Lemma 2. According to (3.1), it can be gotten that

$$\begin{aligned} h(\delta) &= \left\| \frac{\mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})} \widehat{g^\delta}(\xi)}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} \right\| \\ &\leq \left\| \frac{\mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} (\widehat{g^\delta}(\xi) - \widehat{g}(\xi)) \right\| + \left\| \frac{\mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} \widehat{g}(\xi) \right\| \\ &\leq \delta + \left\| \frac{\mu(1 + |\xi|^2)^s e^{|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})} (1 + |\xi|^2)^{-\frac{p}{2}} (1 + |\xi|^2)^{\frac{p}{2}} e^{|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} \widehat{g}(\xi) \right\| \\ &\leq \delta + E \sup_{\xi \in \mathbb{R}} B(\xi), \end{aligned} \quad (3.3)$$

using the mean value inequality and by the simple computation, we can obtain that

$$\begin{aligned}
 B(\xi) &= \frac{\mu(1 + |\xi|^2)^s e^{|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} (1 + |\xi|^2)^{-\frac{p}{2}} \\
 &= \frac{\sqrt{\mu}(1 + |\xi|^2)^{-\frac{p}{2}}}{\frac{1}{\sqrt{\mu}} \cdot (1 + |\xi|^2)^{-s} e^{-|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})} + \sqrt{\mu} e^{|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} \leq \frac{\sqrt{\mu}}{2} \cdot (1 + |\xi|^2)^{-\frac{p-s}{2}}.
 \end{aligned}
 \tag{3.4}$$

Since $0 \leq s < p$, and $\lim_{|\xi| \rightarrow 0^+} (1 + |\xi|^2)^{-\frac{p-s}{2}} = 1$, $\lim_{|\xi| \rightarrow +\infty} (1 + |\xi|^2)^{-\frac{p-s}{2}} = 0$, then there exists $C_1 > 0$, such that $B(\xi) \leq C_1 \sqrt{\mu}/2$. Now combining (3.3), we can derive the result of Lemma 2.

Theorem 2. Let the exact and observed datum g, g_δ satisfy (2.9), the a priori assumption (2.5) is valid. Denote u, u_μ^δ given in (2.4), (2.14) are the exact and regularized solutions of the considered problem, respectively.

(i) If we select the regularized parameter μ by the Eq (3.1) with $h(\delta) = \delta + \delta^{\frac{1-\gamma}{2}}$ ($0 < \gamma < 1$), then the below convergence result of logarithmic type can be established

$$\|u_\mu^\delta(x, t) - u(x, t)\| \leq \frac{C_2 C_1^2 E^2 \delta^\gamma}{4} + E^{1-x} \left(2\delta + \delta^{\frac{1-\gamma}{2}}\right)^x \left(\frac{1}{\cos(\alpha\pi/4)} \ln \frac{1}{(2\delta + \delta^{\frac{1-\gamma}{2}})}\right)^{\frac{-2p(1-x)}{\alpha}}; \tag{3.5}$$

(ii) If we select the regularized parameter μ by the Eq (3.1) with $h(\delta) = \delta + \sqrt{\delta} \ln \frac{1}{\delta}$, then the following convergence estimate of double logarithmic type can be derived

$$\|u_\mu^\delta(x, t) - u(x, t)\| \leq \frac{C_2 C_1^2 E^2}{4 \left(\ln \frac{1}{\delta}\right)^2} + E^{1-x} \left(2\delta + \sqrt{\delta} \ln \frac{1}{\delta}\right)^x \left(\frac{1}{\cos(\alpha\pi/4)} \ln \frac{1}{(2\delta + \sqrt{\delta} \ln \frac{1}{\delta})}\right)^{\frac{-2p(1-x)}{\alpha}}, \tag{3.6}$$

where, $C_2 > 0$ is a constant, C_1 is given in Lemma 2.

Proof of Theorem 2. By using the Parseval theorem, we have

$$\|u_\mu^\delta(x, \cdot) - u(x, \cdot)\| \leq \|\widehat{u}_\mu^\delta(x, \cdot) - \widehat{u}_\mu(x, \cdot)\| + \|\widehat{u}_\mu(x, \cdot) - \widehat{u}(x, \cdot)\|. \tag{3.7}$$

From (2.9), for $\mu \in (0, 1)$, one can derive that

$$\begin{aligned}
 \|\widehat{u}_\mu^\delta(x, \cdot) - \widehat{u}_\mu(x, \cdot)\| &= \left\| \frac{e^{\psi_\alpha(\xi)(1-x)}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} (\widehat{g}^\delta(\xi) - \widehat{g}(\xi)) \right\| \\
 &\leq \frac{\delta}{\mu \left(e^{-(1-x)|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})} + (1 + |\xi|^2)^s e^{(1+x)|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})} \right)}.
 \end{aligned}
 \tag{3.8}$$

We notice that, $\lim_{|\xi| \rightarrow 0^+} e^{-(1-x)|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})} + (1 + |\xi|^2)^s e^{(1+x)|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})} = 2$, and $\lim_{|\xi| \rightarrow +\infty} e^{-(1-x)|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})} + (1 + |\xi|^2)^s e^{(1+x)|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})} = +\infty$, then there exists a positive number C_2 , such that

$$\|\widehat{u}_\mu^\delta(x, \cdot) - \widehat{u}_\mu(x, \cdot)\| \leq C_2 \delta / \mu. \tag{3.9}$$

From (3.9) and Lemma 2, we have

$$\|u_\mu^\delta(x, \cdot) - u_\mu(x, \cdot)\| \leq \frac{C_2 C_1^2 E^2 \delta}{4(h(\delta) - \delta)^2}. \tag{3.10}$$

On the other hand, it can be noticed that

$$\begin{aligned}
 \|K(x)(u_\mu(x, t) - u(x, t))\| &= \|\widehat{K}(x)(\widehat{u}_\mu(x, t) - \widehat{u}(x, t))\| \\
 &= \left\| \frac{\mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} \widehat{g}(\xi) \right\| \\
 &\leq \left\| \frac{\mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} (\widehat{g}^\delta(\xi) - \widehat{g}(\xi)) \right\| \\
 &\quad + \left\| \frac{\mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} \widehat{g}^\delta(\xi) \right\| \\
 &\leq \delta + h(\delta).
 \end{aligned} \tag{3.11}$$

Meanwhile, using the a-priori assumption (2.5), one can derive that

$$\begin{aligned}
 \|u_\mu(x, t) - u(x, t)\|_{HP(\mathbb{R})} &= \left\| (1 + |\xi|^2)^{\frac{p}{2}} \frac{\mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} e^{\psi_\alpha(\xi)(1-x)} \widehat{g}(\xi) \right\| \\
 &\leq \left\| (1 + |\xi|^2)^{\frac{p}{2}} \frac{\mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}}{1 + \mu(1 + |\xi|^2)^s e^{2|\xi|^{\frac{\alpha}{2}} \cos(\frac{\alpha\pi}{4})}} e^{\psi_\alpha(\xi)} \widehat{g}(\xi) \right\| \\
 &\leq \left\| (1 + |\xi|^2)^{\frac{p}{2}} \widehat{u}(0, \xi) \right\| = \|u(0, x)\|_{HP(\mathbb{R})} \leq E.
 \end{aligned} \tag{3.12}$$

Using the result of the conditional stability (2.7), we can get that

$$\|u_\mu(x, t) - u(x, t)\| \leq E^{1-x} (\delta + h(\delta))^x \left(\frac{1}{\cos(\alpha\pi/4)} \ln \frac{1}{(\delta + h(\delta))} \right)^{\frac{-2p(1-x)}{\alpha}}. \tag{3.13}$$

Combining with (3.10) and (3.13), it can be obtain that

$$\|u_\mu^\delta(x, t) - u(x, t)\| \leq \frac{C_2 C_1^2 E^2 \delta}{4(h(\delta) - \delta)^2} + E^{1-x} (\delta + h(\delta))^x \left(\frac{1}{\cos(\alpha\pi/4)} \ln \frac{1}{(\delta + h(\delta))} \right)^{\frac{-2p(1-x)}{\alpha}}. \tag{3.14}$$

Ultimately, the convergence results (3.5) and (3.6) can be established, respectively.

Remark 2. We find that, if adopting the normal discrepancy principle [20] to select the regularized parameter μ , i.e., $\|u_\mu^\delta(1, t) - g^\delta(t)\| = \tau\delta$ ($\tau > 1$), the convergence result of regularized method can not be easily established. So here we adopt a modified version (3.1) to choose the regularized parameter and derive the corresponding convergence result.

Remark 3. We point out that this method and the corresponding analysis can be extended to multi-dimensional models. For instance, the two-dimensional problem in a semi-infinite slab

$$\begin{cases} D_t^\alpha u(x, y, t) - u_{xx}(x, y, t) - u_{yy}(x, y, t) = 0, 0 < x < 1, y > 0, t > 0, \\ u(x, y, 0) = 0, 0 \leq x \leq 1, y \geq 0, \\ u(x, 0, t) = 0, 0 \leq x \leq 1, t \geq 0, \\ u(1, y, t) = g(y, t), y \geq 0, t \geq 0, \\ u_x(1, y, t) = 0, y \geq 0, t \geq 0. \end{cases} \tag{3.15}$$

We can adopt the similar procedure to determine the distribution $u(x, y, t)$ for $0 \leq x < 1$ from the measured data $g^\delta(y, t)$.

4. Numerical simulations

In this section, for smooth and non-smooth cases we respectively verify the effectiveness of proposed method by doing some numerical experiments. We use the fast discrete Fourier transform (DFT) and inverse Fourier transform (IFT) to complete our numerical experiments. Since the analytic solution of problem (1.1) is generally difficult to be expressed explicitly, we construct the final data $g(t)$ by solving the following direct problem.

$$\begin{cases} D_t^\alpha u(x, t) - u_{xx}(x, t) = 0, x > 0, t > 0, \\ u(x, 0) = 0, x \geq 0, \\ u(0, t) = f(t), t \geq 0. \end{cases} \quad (4.1)$$

This is a well-posed problem and the solution at $x = 1$ can be given as

$$g(t) := u(1, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\psi_\alpha(\xi)} \widehat{f}(\xi) e^{i\xi t} d\xi, \quad (4.2)$$

which is taken as the exact data. The measured data is selected by the following random form

$$g^\delta(x) = g(x) + \varepsilon \text{rand}(\text{size}(g(x))), \quad (4.3)$$

here ε denotes the noisy level, the function $\text{rand}(\text{size}(g))$ returns an array of random entries that is the same size as g .

The exact and regularized solutions are calculated by (2.4) and (2.14), respectively. For $\delta > 0$, we fix $\gamma = 0.1$, the regularized parameter is chosen by the a-posteriori rule (3.1) with $h(\delta) = \delta + \delta^{\frac{1-\gamma}{2}}$. In order to make the sensitivity analysis for numerical results, we calculate the relative error by

$$\epsilon(u) = \|u(x, t) - u_\mu^\delta(x, t)\| / \|u(x, t)\|. \quad (4.4)$$

We select the interval $[0, 4]$ to make the numerical experiment, and suppose the function value be equal to zero for $t \in (-\infty, 0) \cup (4, +\infty)$.

Example 1. In the direct problem (4.1), we take smooth function $f(t) = 2e^{-2t^2}$.

For $\alpha = 0.2$, $s = 1$, $\gamma = 0.1$, $\varepsilon = 0.01$, the exact and regularized solutions at $x = 0.9, 0.5, 0.3, 0$ with $\mu = 6.2383e - 06$ are shown in Figure 1. At $x = 0$, we also investigate the influences of α and s on numerical result. For $s = 1$, $\gamma = 0.1$, $\varepsilon = 0.01$, the relative errors for various α are given in Table 1. For $\alpha = 0.2$, $\gamma = 0.1$, $\varepsilon = 0.01$, the relative errors for various s are presented in Table 2.

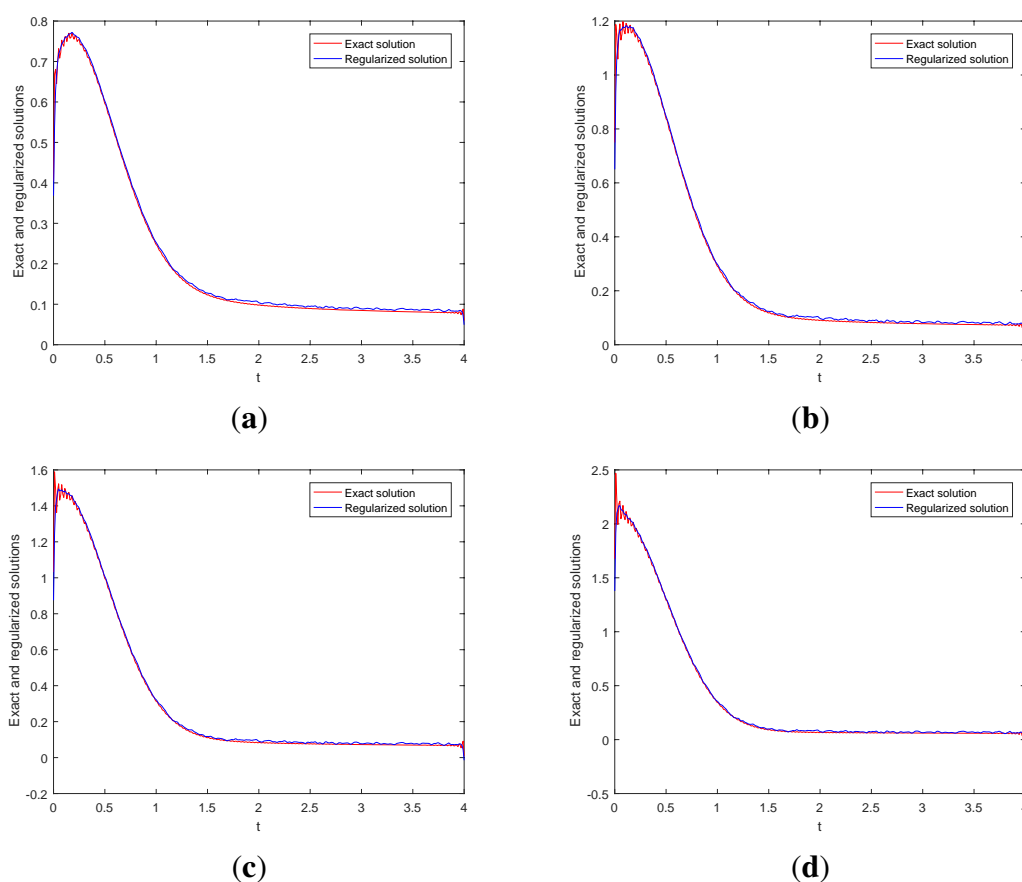


Figure 1. $\alpha = 0.2$, $s = 1$, $\gamma = 0.1$, $\varepsilon = 0.01$: Exact and regularized solutions with $\mu=6.2383e-06$. (a): $x = 0.9$, (b): $x = 0.5$, (c): $x = 0.3$, (d): $x = 0$.

Table 1. $s = 1$, $\gamma = 0.1$, $\varepsilon = 0.01$: the relative errors for various α at $x = 0$.

α	0.05	0.1	0.3	0.5	0.7	0.9
μ	5.6612e-06	5.6303e-06	8.6958e-06	2.1836e-05	3.7929e-05	4.5216e-05
$\epsilon(u)$	0.0334	0.0375	0.0694	0.1219	0.3108	0.9472

Table 2. $\alpha = 0.2$, $\gamma = 0.1$, $\varepsilon = 0.01$: the relative errors for various s at $x = 0$.

s	0.25	0.5	0.75	1.0	2.0	3.0
μ	0.0016	3.7939e-04	5.3591e-05	6.2383e-06	7.3903e-10	7.1920e-14
$\epsilon(u)$	0.0363	0.0456	0.0489	0.0500	0.0512	0.0516

Figure 1 shows that the simulation effect of this method is feasible and acceptable in solving the considered problem. Table 1 means that the numerical result becomes well as α tends to zero. Table 2 shows that, the better numerical results are, the smaller as s is taken. Meanwhile, as $s = 0$, our method

just is the simplified Tikhonov method.

Example 2. In (4.1), we take the non-smooth function $f(t)$ with

$$f(t) = \begin{cases} 0.5t, & 0 \leq t \leq 2, \\ 0.5(4-t), & 2 \leq t \leq 4. \end{cases} \quad (4.5)$$

For $\alpha = 0.2$, $s = 1$, $\gamma = 0.1$, $\varepsilon = 0.01$, numerical results at $x = 0.9, 0.5, 0.3, 0$ with $\mu = 1.1539e-04$ are shown in Figure 2. For $s = 1$, $\gamma = 0.1$, $\varepsilon = 0.01$, the relative errors for various α are given in Table 3. The relative errors for various s are presented in Table 4 with $\alpha = 0.2$, $\gamma = 0.1$, $\varepsilon = 0.01$.

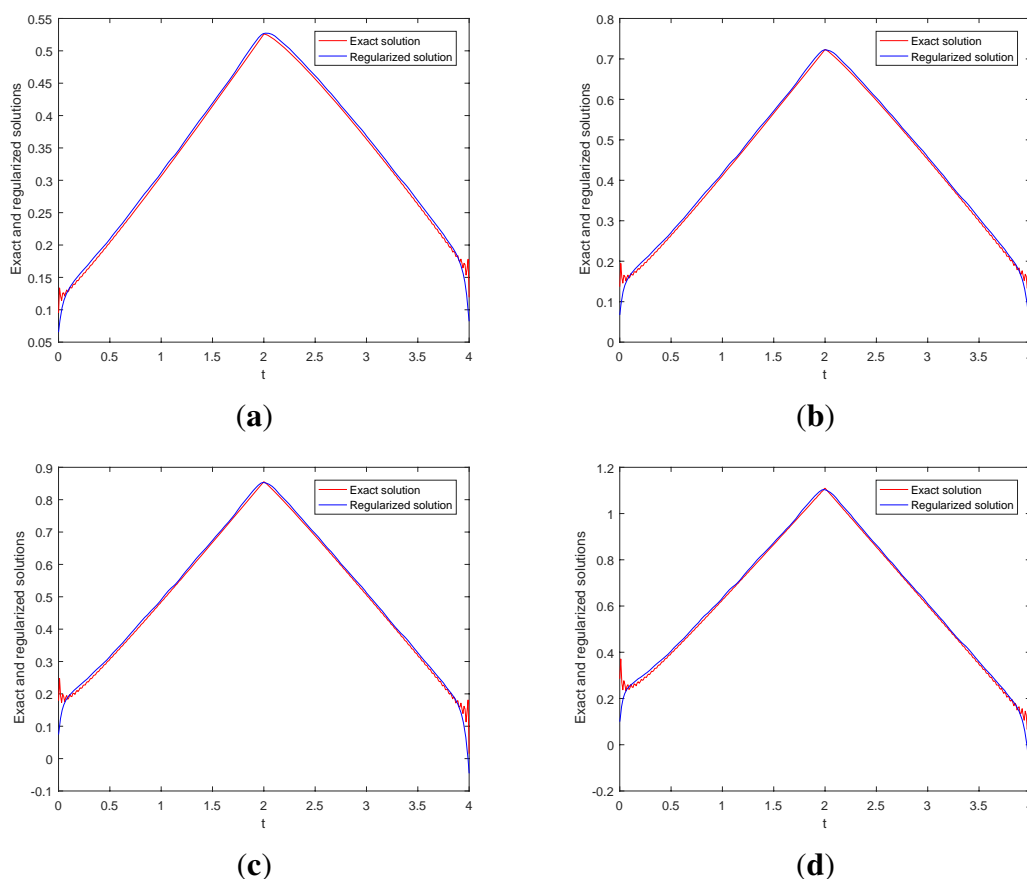


Figure 2. $\alpha = 0.2$, $s = 1$, $\gamma = 0.1$, $\varepsilon = 0.01$: Exact and regularized solutions with $\mu = 1.1539e-04$. (a): $x = 0.9$, (b): $x = 0.5$, (c): $x = 0.3$, (d): $x = 0$.

Table 3. $s = 1$, $\gamma = 0.1$, $\varepsilon = 0.01$: the relative errors for various α at $x = 0$.

α	0.05	0.1	0.3	0.5	0.7	0.9
μ	2.6555e-04	2.0622e-04	5.8400e-05	1.0613e-05	1.0663e-06	4.8697e-08
$\epsilon(u)$	0.0281	0.0314	0.0548	0.1567	0.6489	0.9875

Table 4. $\alpha = 0.2, \gamma = 0.1, \varepsilon = 0.01$: the relative errors for various s at $x = 0$.

s	0.25	0.5	0.75	1.0	2.0	3.0
μ	0.0034	0.0018	5.7902e-04	1.1539e-04	1.2494e-07	1.2513e-10
$\epsilon(u)$	0.0203	0.0306	0.0378	0.0403	0.0419	0.0422

Figure 2 and Tables 3, 4 mean the similar simulation results with the case of Example 1, which indicate that the computation effect of this method is also satisfied and acceptable in solving the non-smooth case.

5. Conclusions

A sideways problem of the time-fractional diffusion equation is investigated. We design a Tikhonov-type regularized method to overcome the ill-posedness and recover the continuous dependence of the solution on the given data, based on the result of conditional stability, we derive the convergence estimates of logarithmic and double logarithmic types for the regularized method by adopting an a-posteriori choice rule of regularized parameter. Finally, we verify the convergence and stability for this method by doing some numerical experiments.

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Conflict of interest

The authors declare no conflict of interest.

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