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Research article

On geometry of isophote curves in Galilean space

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Abstract: In this paper, we introduce isophote curves on surfaces in Galilean 3-space. Apart from the general concept of isophotes, we split our studies into two cases to get the axis d of isophote curves lying on a surface such that d is an isotropic or a non-isotropic vector. We also give a method to compute isophote curves of surfaces of revolution. Subsequently, we show the relationship between isophote curves and slant (general) helices on surfaces of revolution obtained by revolving a curve by Euclidean rotations. Finally, we give some characterizations for isophote curves lying on surfaces of revolution.

Keywords: Galilean space; isophote curve; surfaces of revolution
Mathematics Subject Classification: 53A35, 53Z05

1. Introduction

The isophote curve method is one of the most efficient methods that can be used to analyze and visualize surfaces by lines of equal light intensity. Isophote curve whose normal vectors make a constant angle with a fixed vector (the axis) is one of the curves to characterize surfaces such as parameter, geodesics, and asymptotic curves or lines of curvature. Moreover, this curve is used in computer graphics and it is also interesting to study for geometry.

The isophote curve of a given surface is calculated in two steps: firstly the normal vector field $n(s,t)$ of the surface is computed, and secondly the surface point is traced as

$$\frac{\langle n(s,t), d\rangle}{\|n(s,t)\|} = \cos\beta,$$

where β is a constant angle ($0 \le \beta \le \frac{\pi}{2}$).

Isophote curve is called a silhouette curve when

$$\frac{\langle n(s,t), d\rangle}{\|n(s,t)\|} = 0,$$

where d is the unit fixed vector.

From past to present, there have been a lot of researches about isophote curves and their characterizations in [3, 4, 6, 7].

In this paper, our aim is to investigate isophote curves on surfaces in Galilean space and find its axis d such that it is an isotropic and a non-isotropic vector through the Galilean Darboux frame. According to the axis d, we split our studies into two cases to find the axis of isophote curves lying on a surface in Galilean space. Moreover, we give a method to compute isophote curves of surfaces of revolution obtained by revolving a curve by Euclidean and isotropic rotations.

2. Preliminaries

Following the Erlangen Program, due to F. Klein, each geometry is associated with a group of transformations, and hence there are as many geometries as groups of transformations. Associated with group of transformations that in physics guarantees the invariance of many mechanical systems, the Galilei group, is the so-called Galilean geometry G_3. That is, Galilean geometry is one of the nine Cayley-Klein geometries with projective signature $(0, 0, +, +)$. The absolute of the Galilean geometry is an ordered triple $\{\omega, f, I\}$, where ω is the ideal (absolute) plane, f the line in ω and I the fixed elliptic involution of f as in [11].

A plane is called Euclidean if it contains f, otherwise it is called isotropic or i.e., plane x=constant is Euclidean, and so is the plane ω. Other planes are isotropic. In other words, an isotropic plane does not involve any isotropic direction. A vector $u = (u_1, u_2, u_3)$ is called non-isotropic if $u_1 \neq 0$, otherwise it is called isotropic vector. All unit non-isotropic vectors are of the form $u = (1, u_2, u_3)$ [11].

A Galilean scalar product of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in the Galilean 3-space G_3 is defined as

$$\langle u, v\rangle = \begin{cases} u_1 v_1, & \text{if } \ u_1 \neq 0 \quad \text{or} \quad v_1 \neq 0, \\ u_2 v_2 + u_3 v_3, & \text{if } \ u_1 = 0 \quad \text{and} \quad v_1 = 0 \end{cases}$$

and a Galilean norm of u is given by

$$\|u\| = \begin{cases} |u_1|, & \text{if } \ u_1 \neq 0, \\ \sqrt{u_2^2 + u_3^2}, & \text{if } \ u_1 = 0. \end{cases}$$

A Galilean cross product of u and v on G_3 is defined by

$$u \times v = \begin{vmatrix} 0 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ [8, 10].

A curve given in parametric form $\alpha(t) = (f(t), g(t), h(t))$ in the Galilean 3-space G_3 is called admissible if nowhere its tangent vector is isotropic, that is $f'(t) = \frac{df}{dt} \neq 0$.

Let α be a unit-speed admissible curve of the class C^∞ in G_3, and parametrized by the invariant parameter s, defined by

$$\alpha(s) = (s, g(s), h(s)).$$

Then the Frenet frame fields of $\alpha(s)$ are given by

$$T(s) = \alpha'(s),$$
$$N(s) = \frac{1}{\kappa(s)}\alpha''(s),$$
$$B(s) = T(s) \times N(s),$$

where the curvature $\kappa(s)$ and the torsion $\tau(s)$ of $\alpha(s)$ are written as, respectively,

$$\kappa(s) = \sqrt{g''(s)^2 + h''(s)^2},$$
$$\tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)}.$$

Here T, N and B are the tangent, principal normal and binormal vectors of $\alpha(s)$.
On the other hand, the Frenet formula of the curve is given by (cf. [9])

$$\begin{aligned} T' &= \kappa N, \\ N' &= \tau B, \\ B' &= -\tau N. \end{aligned} \tag{2.1}$$

Consider a C^r-regular surface M, $r \geq 1$, in G_3 parameterized by

$$\mathbf{X}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).$$

We denote by x_{u_i}, y_{u_i} and z_{u_i} the partial derivatives of the functions x, y and z with respect to u_i $(i = 1, 2)$, respectively. Besides $\mathbf{X}$ is said to be admissible if nowhere it has Euclidean tangent planes, i.e., $x_{u_i} \neq 0$ for some $i = 1, 2$.

On the other hand, the matrix of the first fundamental form ds^2 of a surface M in G_3 is given by

$$ds^2 = \begin{pmatrix} ds_1^2 & 0 \\ 0 & ds_2^2 \end{pmatrix},$$

where $ds_1^2 = (g_1 du_1 + g_2 du_2)^2$ and $ds_2^2 = h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2$. Here $g_i = x_{u_i}$ and $h_{ij} = \langle \tilde{\mathbf{X}}_{u_i}, \tilde{\mathbf{X}}_{u_j} \rangle$ $(i, j = 1, 2)$ means the Euclidean scalar product of the projections $\tilde{\mathbf{X}}_{u_i}$ of vectors $\mathbf{X}_{u_i}$ onto the yz-plane.

The unit normal vector field n of a surface M is defined by

$$n = \frac{1}{\omega}(0, x_{u_2} z_{u_1} - x_{u_1} z_{u_2}, x_{u_1} y_{u_2} - x_{u_2} y_{u_1}),$$

where the positive function ω is given by

$$\omega = \sqrt{(x_{u_2} z_{u_1} - x_{u_1} z_{u_2})^2 + (x_{u_1} y_{u_2} - x_{u_2} y_{u_1})^2}.$$

Let $\{T, Q, n\}$ be a Galilean Darboux frame of $\alpha(s)$ with T as the tangent vector of a curve $\alpha(s)$ in G_3 and n be the unit normal to a surface and $Q = n \times T$. Then the Galilean Darboux frame is expressed as

$$\begin{aligned} T' &= k_g Q + k_n n, \\ Q' &= \tau_g n, \\ n' &= -\tau_g Q, \end{aligned} \tag{2.2}$$

where k_g, k_n and τ_g are the geodesic curvature, normal curvature and geodesic torsion of $\alpha(s)$ on M, respectively. Also, (2.2) implies

$$\begin{aligned} &\kappa^2 = k_g^2 + k_n^2, \quad \tau = -\tau_g + \frac{k_g' k_n - k_g k_n'}{k_g^2 + k_n^2}, \\ &k_g = k \cos\phi \text{ and } k_n = -k \sin\phi, \end{aligned} \tag{2.3}$$

where ϕ is an angle between the surface normal vector n and the binormal vector B of α [12]. A curve $\alpha(s)$ is a geodesic (an asymptotic curve or a line of curvature) if and only if k_g (k_n or τ_g) vanishes, respectively.

On the other hand, the usual transformation between the Galilean Frenet frames and the Darboux frames takes the form

$$\begin{aligned} Q &= \cos\phi N + \sin\phi B, \\ n &= -\sin\phi N + \cos\phi B. \end{aligned} \tag{2.4}$$

Artykbaev introduced an angle between two vectors in Galilean space as follows:

Definition 2.1. ([1]) Let $x = (1, x_2, x_3)$ and $y = (1, y_2, y_3)$ be two unit non-isotropic vectors in G_3. Then an angle ϑ between x and y is defined by

$$\vartheta = \sqrt{(y_2 - x_2)^2 + (y_3 - x_3)^2}. \tag{2.5}$$

Definition 2.2. ([1]) An angle between a unit non-isotropic vector $x = (1, x_2, x_3)$ and an isotropic vector $y = (0, y_2, y_3)$ in G_3 is defined by

$$\varphi = \frac{x_2 y_2 + x_3 y_3}{\sqrt{{y_2}^2 + y_3^2}}. \tag{2.6}$$

Definition 2.3. ([1]) An angle θ between two isotropic vectors $x = (0, x_2, x_3)$ and $y = (0, y_2, y_3)$ parallel to the Euclidean plane in G_3 is equal to the Euclidean angle between them. That is,

$$\cos\theta = \frac{x_2 y_2 + x_3 y_3}{\sqrt{{x_2}^2 + x_3^2}\sqrt{{y_2}^2 + y_3^2}}. \tag{2.7}$$

3. The axis of an isophote curve in Galilean space

The starting point of this section is to get the unit fixed vector d of an isophote curve via its Galilean Darboux frame.

Let M be an admissible regular surface and $\alpha : I \subset R \to M$ be an unit-speed curve parametrized by $\alpha(s) = (s, \alpha_2(s), \alpha_3(s))$ as an isophote curve for some $s \in I$.

In order to prove the results, we split it into two cases according to the fixed vector d.

Case 1. Considering d is a unit fixed isotropic vector.
Since n is the unit isotropic normal vector of a surface M, using the Definition 2.3, we have

$$\langle n, d\rangle = \cos\theta = \text{constant}, \tag{3.1}$$

differentiating the above equation, we get

$$\tau_g \langle Q, d\rangle = 0. \tag{3.2}$$

That is, $\tau_g = 0$ or $\langle Q, d\rangle = 0$. By differentiating $\langle Q, d\rangle = 0$, we get

$$\tau_g\langle n, d\rangle = 0.$$

It means that $\tau_g = 0$. So

$$\tau_g = 0. \tag{3.3}$$

On the other hand, if $k_g \neq 0$, using the differentiating $\langle T, d\rangle = 0$ with respect to s, we get

$$k_g \langle Q, d\rangle + k_n\langle n, d\rangle = 0, \tag{3.4}$$

$$\langle Q, d\rangle = -\frac{k_n}{k_g}\cos\theta. \tag{3.5}$$

Then d can be written as

$$d = -\frac{k_n}{k_g}\cos\theta Q + \cos\theta n. \tag{3.6}$$

Since $\|d\| = 1$, we get

$$\frac{k_n}{k_g} = \pm\tan\theta. \tag{3.7}$$

In this situation, we conclude that $\phi = \pm\theta$ or $\phi = \pi \pm \theta$.

From (2.3) and (2.4) in terms of the Galilean Frenet frame, we get

$$d = (-\frac{k_n}{k}\cos\theta - \frac{k_g}{k}\sin\theta)N + (-\frac{k_n}{k}\sin\theta + \frac{k_g}{k}\cos\theta)B. \tag{3.8}$$

If we differentiate (3.6) using (3.7) and (3.3), we get $d' = 0$, that is, d is a constant isotropic vector. From now on, we suppose if α is a unit-speed isophote curve, then α is also a line of curvature.

Theorem 3.1. *Let α be a unit-speed isophote curve on a surface M in G_3 with a unit fixed isotropic vector d as the axis of the isophote curve. In that case, we have the following:*

i) If α is a geodesic curve, then α is a straight line.

ii) If α is an asymptotic curve on M, then it is a plane curve, and the fixed vector d is parallel to B.

Proof. *i*) If $k_g = 0$, then $\tau = -\tau_g = 0$ from (2.3) and (3.3). Also from (3.4), $k_n = 0$. Then from (2.3), we get $\kappa = 0$. Hence α is a straight line.

ii) If α is an asymptotic curve, then $k_n = 0$. Trivially, if $k_g = 0$, it is trivial that α is a plane curve. If $k_g \neq 0$, then from (3.6), we get

$$d = \cos\theta n. \tag{3.9}$$

So from (2.3), $\tau = 0$. That is, α is a plane curve. Also using (2.3) and (2.4) into (3.9), it can be easily shown that d is parallel to B. □

Theorem 3.2. *Let α be a unit-speed isophote curve on a surface M in G_3 with a unit fixed isotropic vector d as the axis of the isophote curve. The axis d is perpendicular to the principal normal line of α if and only if either α is a straight line or an asymptotic curve on M or α is a curve with $\frac{k_n}{k_g} = -\tan\theta$.*

Proof. If α is a unit-speed isophote curve, then from (3.8), we get

$$\langle N, d\rangle = -2\frac{k_g}{k}\sin\theta = 0,$$

from this equation, we have $k_g = 0$ or $\sin\theta = 0$.

If $k_g = 0$ then, from Theorem 3.1, α is a straight line.

If $\sin\theta = 0$, then $k_n = 0$, that is, α is an asymptotic curve.

If we take $\frac{k_n}{k_g} = -\tan\theta$, then we can easily get $\langle N, d\rangle = 0$. □

Theorem 3.3. *Let α be a unit-speed isophote curve on a surface M in G_3 with a unit fixed isotropic vector d as the axis of the isophote curve. The axis d is perpendicular to the principal binormal line of α such that $\dfrac{k_n}{k_g} = \tan\theta$ if and only if $k_n = k_g$*

Proof. If α is a unit-speed isophote curve with $\frac{k_n}{k_g} = \tan\theta$, then from (3.8), we get

$$\langle B, d\rangle = \frac{k_g}{k}(-\sin^2\theta + \cos^2\theta) = 0.$$

Since α is a non-geodesic curve, $-\sin^2\theta + \cos^2\theta = 0$. So, $\tan\theta = 1$. We know that $0 \leq \theta \leq \dfrac{\pi}{2}$, then we get $\theta = \dfrac{\pi}{4}$. □

Theorem 3.4. *If α is a silhouette curve on M, and d is a unit isotropic vector such that it is parallel to Q, then the curve α is a plane curve.*

Proof. If a fixed vector d is a unit isotropic vector and is parallel to Q, then we have

$$d = \pm Q, \quad \langle T, d\rangle = 0.$$

By differentiating above equations with respect to s, we obtain

$$\tau_g n = 0, \quad k_g\langle Q, d\rangle + k_n\langle n, d\rangle = 0.$$

Since α is a silhouette curve with $\langle n, d\rangle = 0$, we get

$$\tau_g = 0, \quad k_g = 0,$$

from this, we have $\tau = 0$. It means that α is a plane curve. □

Case 2. Considering d is a unit fixed non-isotropic vector.

Since n is the unit isotropic normal vector of a surface M, using the Definition 2.2, we have

$$\langle n, d\rangle = \varphi = \text{constant}, \tag{3.10}$$

where $0 \leq \varphi \leq 1$.

By differentiating the above equation, we get

$$\tau_g \langle Q, d\rangle = 0. \tag{3.11}$$

That is, $\tau_g = 0$ or $\langle Q, d\rangle = 0$. By differentiating $\langle Q, d\rangle = 0$, we get

$$\tau_g\langle n, d\rangle = 0.$$

That means $\tau_g = 0$. So

$$\tau_g = 0. \tag{3.12}$$

On the other hand, if $k_g \neq 0$, using the differentiating $\langle T, d\rangle = 1$ with respect to s, we get

$$k_g \langle Q, d\rangle + k_n\langle n, d\rangle = 0, \tag{3.13}$$

$$\langle Q, d\rangle = -\frac{k_n}{k_g}\varphi. \tag{3.14}$$

Then the unit non-isotropic vector d can be written as

$$d = T - \frac{k_n}{k_g}\varphi Q + \varphi n. \tag{3.15}$$

Since d is a constant vector, we get $k_g = k_n = 0$. Trivially, it is a conflict.

On the other hand considering $k_n = 0$, from (3.13) and using (3.12), d can be written as

$$d = T + \varphi n$$

Since d is a constant vector, we can easily get $k_g = 0$.

Thus, we have the following result:

Theorem 3.5. *Let α be a unit-speed isophote curve on a surface M in G_3 with a unit fixed non-isotropic vector d as the axis of the isophote curve. Then α is a straight line.*

Theorem 3.6. *Let α be a silhouette curve on M and d be a unit non-isotropic vector.*

i) If d lies in the plane spanned by T and Q, then α is a plane curve.

ii) If the axis d is parallel to T, then α is a geodesic curve.

Proof. *i*) Since α is a silhouette curve and d is a unit non-isotropic vector, we get

$$\langle T, d\rangle = 1. \tag{3.16}$$

If we differentiate (3.16) with respect to s, then we get

$$k_g \langle Q, d\rangle = 0.$$

Since d is lied in the plane spanned by T and Q, we get $k_g = 0$. Also, if we differentiate $\langle n, d\rangle = 0$ with respect to s, we get

$$\tau_g \langle Q, d\rangle = 0,$$

it follows that $\tau_g = 0$.

Also, by substituting $\tau_g = 0$ and $k_g = 0$ into (2.3), we get $\tau = 0$. Thus, α is a plane curve.

ii) If d is parallel to T, then we get

$$d = T.$$

If we differentiate the above equation, then $d' = k_g Q$, it follows that $k_g = 0$, that is, the curve is a geodesic curve. □

4. Applications for isophote curves

We investigate an isophote curve among surfaces of revolution in Galilean space and give some characterization for isophote curves on these surfaces. To see this, notice that in G_3 surfaces of revolution are obtained by revolving a curve by Euclidean or isotropic rotations as follows, respectively,

$$\begin{aligned} \overline{x} &= x, \\ \overline{y} &= y\cos t + z\sin t, \\ \overline{z} &= -y\sin t + z\cos t, \end{aligned} \tag{4.1}$$

where t is the Euclidean angle and

$$\begin{aligned} \overline{x} &= x + ct, \\ \overline{y} &= y + xt + c\frac{t^2}{2}, \\ \overline{z} &= z, \end{aligned} \tag{4.2}$$

where $t \in \mathbb{R}$ and $c = \text{constant} > 0$.

The trajectory of a single point under a Euclidean rotation is a Euclidean circle

$$x = \text{constant}, \quad y^2 + z^2 = r^2, \quad r \in \mathbb{R}.$$

The invariant r is the radius of the circle.

The trajectory of a point under isotropic rotation is an isotropic circle whose normal form is

$$z = \text{constant}, \quad y = \frac{x^2}{2c}.$$

The invariant c is the radius of the circle. The fixed line of the isotropic rotation is the absolute line f [11]. For some more studies, see [2, 5].

If a curve $\alpha(s) = (f(s), 0, g(s))$, $(g(s) > 0)$ is rotated by Euclidean rotations, then a surface of revolution is parametrized by

$$S(s,t) = (f(s), g(s)\sin t, g(s)\cos t). \tag{4.3}$$

If a curve $\alpha(s)$ is parametrized by the arc-length, then we take $f(s) = s$. In this case, the unit isotropic normal vector field $n(s,t)$ of S is defined by

$$n(s,t) = \frac{S_s \times S_t}{\|S_s \times S_t\|}, \tag{4.4}$$

where S_s and S_t are the partial differentiations with respect to s and t, respectively. Then, the isotropic normal vector is given by

$$n(s,t) = (0, \sin t, \cos t),$$

it becomes in terms of the Frenet frame as follows:

$$n(s,t) = -\sin tB + \cos tN. \tag{4.5}$$

Proposition 4.1. *Let $\alpha(s)$ be a general helix with the isotropic axis d. Then, for $t_0 = (\frac{2k+1}{2})\pi$ $(k \in \mathbb{Z})$, the curve $\alpha(s)$ on surfaces of revolution given by* (4.3) *is an isophote curve with the axis d.*

Proof. Substituting t_0 into (4.5), we get

$$n(s,t_0) = \mp B.$$

If $\alpha(s)$ is a general helix with the axis d, then $\langle B, d\rangle$ =constant. Therefore, we get

$$\langle n(s,t_0), d\rangle = \mp \langle B, d\rangle = \text{constant}.$$

Thus $\alpha(s)$ is an isophote curve with the axis d on the surfaces of revolution. □

Proposition 4.2. *Let $\alpha(s)$ be a slant helix with the isotropic axis d. Then, for $t_0 = k\pi$ $(k \in \mathbb{Z})$, the curve $\alpha(s)$ on surfaces of revolution given by* (4.3) *is an isophote curve with the axis d.*

Proof. Substituting t_0 into (4.5), we get

$$n(s,t_0) = \mp N.$$

If $\alpha(s)$ is a slant helix with the axis d, then $\langle N, d\rangle$ =constant. Therefore, we get

$$\langle n(s,t_0), d\rangle = \mp \langle N, d\rangle = \text{constant}.$$

Thus $\alpha(s)$ is an isophote curve with the axis d on the surfaces of revolution. □

If a curve $\alpha(s) = (f(s), 0, g(s))$, $(g(s) > 0)$ is rotated by isotropic rotations, then a surface of revolution is parametrized by

$$S(s,t) = (f(s) + ct, f(s)t + c\frac{t^2}{2}, g(s)). \tag{4.6}$$

If a curve $\alpha(s)$ is parametrized by the arc-length, then we take $f(s) = s$. In this case, the isotropic surface normal is given by

$$n = \frac{1}{\sqrt{(g'(s)c)^2 + s^2}}(0, g'(s)c, s),$$

it becomes in terms of the Frenet frame as follows:

$$n = \frac{1}{\sqrt{(g'(s)c)^2 + s^2}}\left(-g'(s)cB + sN\right). \tag{4.7}$$

Theorem 4.3. *Let d be an isotropic axis given by (0,d_y, d_z).*

i) If $d_y = 0$ and $g(s) = \frac{s^2}{2c} + b_1$, ($b_1 \in \mathbb{R}$), then the curve $\alpha(s)$ on surfaces of revolution given by (4.6) *is an isophote curve.*

ii) If $d_z = 0$ and $g(s) = \frac{s^2}{2c} + b_1$, ($b_1 \in \mathbb{R}$), then the curve $\alpha(s)$ on surfaces of revolution given by (4.6) *is an isophote curve.*

Proof. i) If $d_y = 0$, then we get $d = \lambda_1 N$, ($\lambda_1 \in \mathbb{R} - \{0\}$, it folllows that (4.7) implies

$$\langle n, d\rangle = \frac{\lambda_1 s}{\sqrt{(g'(s)c)^2 + s^2}}.$$

If we take $g(s) = \frac{s^2}{2c} + b_1$, ($b_1 \in \mathbb{R}$), then we obtain $\langle n, d\rangle = \frac{\lambda_1}{\sqrt{2}}$. So the curve α is an isophote curve.

ii) If $d_z = 0$, then we get $d = -\lambda_2 B$, ($\lambda_2 \in \mathbb{R} - \{0\}$). From (4.7) we get

$$\langle n, d\rangle = \frac{\lambda_2 g'(s)c}{\sqrt{(g'(s)c)^2 + s^2}}.$$

If we consider $g(s) = \frac{s^2}{2c} + b_1$, ($b_1 \in \mathbb{R}$), then we obtain $\langle n, d\rangle = \frac{\lambda_2}{\sqrt{2}}$, It means that α is an isophote curve. □

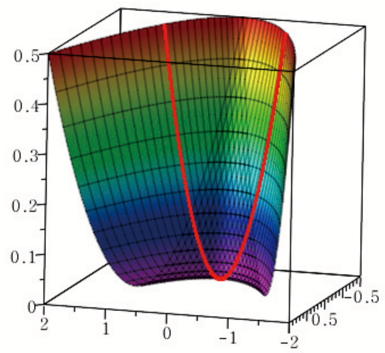

Figure 1. The red curve is isophote on isotropic surface of revolution.

Corollary 4.4. The generating curve $\alpha(s) = (f(s), 0, g(s))$ on surfaces of revolution given by (4.6) becomes both a general helix and a slant helix when the axis $d = (0, d_y, d_z)$.

Remark 4.5. The isophote curve $\alpha(s)$ in Theorem 4.3 is an isotropic circle on surfaces of revolution given by (4.6). Figure 1 is shown an isophote curve on the isotropic surface of revolution with $c = 1$ and $b_1 = 0$ in Theorem 4.3 ii).

5. Conclusion

In this paper, we investigated isophote curves on surfaces in Galilean space G_3 and obtained its axis d such that it is an isotropic and a non-isotropic vector. Furthermore, we presented some characterizations for isophote curves lying on surfaces of revolution.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. A. Artykbaev, Total angle about the vertex of a cone in Galilean space, *Math. Notes*, **43** (1988), 379–382.
2. M. Dede, C. Ekici and W. Goemans, Surfaces of revolution with vanishing curvature in Galilean 3-space, *J. Math. Phys. Anal. Geo.*, **14** (2018), 141–152.
3. F. Doğan and Y. Yaylı, On isophote curves and their characterizations, *Turkish J. Math.*, **39** (2015), 650–664.
4. F. Doğan and Y. Yaylı, Isophote curves on spacelike surfaces in Lorentz-Minkowski space E_1^3, *arXiv preprint arXiv:1203.4388*, 2012.
5. A. Kazan and H. B. Karadag, Weighted Minimal and Weighted Flat Surfaces of Revolution in Galilean 3-Space with Density, *Int. J. Anal. Appl.*, **16** (2018), 414–426.
6. K. J. Kim and I. K. Lee, Computing isophotes of surface of revolution and canal surface, *Comput-Aided Des.*, **35** (2003), 215–223.
7. J. J. Koenderink and A. J. van Doorn, Photometric invariants related to solid shape, *J. Modern Opt.*, **27** (1980), 981–996.
8. E. Molnar, The projective interpretation of the eight 3-dimensional Homogeneous geometries, *Beitr. Algebra Geom.*, **38** (1997), 261–288.
9. B. J. Pavkovic and I. Kamenarovic, The equiform differential geometry of curves in the Galilean space G_3, *Glas. Mat.*, **22** (1987), 449–457.
10. O. Röschel, *Die Geometrie des Galileischen raumes,* Habilitationsschrift, Leoben, 1984.
11. Z. M. Sipus, Ruled Weingarten surfaces in Galilean space, *Period. Math. Hungar*, **56** (2008), 213–225.
12. T. Şahin, Intrinsic equations for a generalized relaxed elastic line on an oriented surface in the Galilean space, *Acta Math. Sci.*, **33** (2013), 701–711.