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Research article

Weak and pseudo-solutions of an arbitrary (fractional) orders differential equation in nonreflexive Banach space

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Abstract: In this paper, we establish some existence results of weak solutions and pseudo-solutions for the initial value problem of the arbitrary (fractional) orders differential equation

$$\begin{aligned} \frac{dx}{dt} &= f(t, D^{\gamma} x(t)),\ \gamma \in (0,1),\ \ t \in [0,T] = \mathbb{I} \\ x(0) &= x_0. \end{aligned}$$

in nonreflexive Banach spaces E, where $D^{\gamma} x(\cdot)$ is a fractional derivative of the function $x(\cdot) : \mathbb{I} \to E$ of order γ. The function $f(t,x) : \mathbb{I} \times E \to E$ will be assumed to be weakly sequentially continuous in x for each $t \in \mathbb{I}$ and Pettis integrable in t on $\mathbb{I}$ for each $x \in C[\mathbb{I}, E]$. Also, a weak noncompactness type condition (expressed in terms of measure of noncompactness) will be imposed.

Keywords: measure of weak noncompactness; weakly continuous solution; pseudo-solution; weakly relatively compact; fractional Pettis integral
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1. Introduction and preliminaries

Let E be nonreflexive Banach space with norm $\| \ . \ \|$ with its dual E^*, and we will denote by $E_\omega = (E, \omega) = (E, \sigma(E, E^*))$ the space E with its weak topology. Let $L^1(\mathbb{I})$ be the space of Lebesgue integrable functions on the interval $\mathbb{I} = [0,T]$. Denote by $C[\mathbb{I}, E]$ the Banach space of strongly continuous functions $x : \mathbb{I} \to E$ with the sup-norm $\| \cdot \|_0$. Also, we consider the space $C(\mathbb{I}, E)$ with its weak topology $\sigma(C(\mathbb{I}, E), C(\mathbb{I}, E)^*)$. Denote by $L^\infty(\mathbb{I})$ the space of all measurable and essential bounded real functions defined on $\mathbb{I}$. Let $C_{\omega}(\mathbb{I}, E)$ denotes the space of all weakly continuous functions from $\mathbb{I}$ into E_w endowed with the topology of weak uniform convergence.

The existence of weak solutions or pseudo-solutions for ordinary differential equations in Banach spaces has been investigated in many papers. For example, Cichoń ([6, 8]), Cramer et al. [9], Knight [20], Kubiaczyk, Szufla [21], O'Regan ([26, 27]) and for fractional order differential equations in Banach spaces (see Agarwal et al. [2, 3], Salem et al. [34] and the references therein), for quadratic integral equations in reflexive Banach algebra (see Banaś et al [5]).

Consider the initial value problem

$$\begin{aligned} \frac{dx}{dt} &= f(t, D^{\gamma}x(t)),\ \gamma \in (0,1),\ \ t\ \in\ \mathbb{I} \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where $D^{\gamma}x(\cdot)$ is a fractional derivative of the function $x(\cdot) : \mathbb{I} \to E$ of order γ.
We remark the following:

- for real-valued function and the function f is independent of the fractional derivatives, then we have the problems studied in, for example [10, 31].
- for real-valued function with $\gamma \in (0, 1)$ we have the problem studied in [16] with nonlocal and integral condition.
- in abstract spaces with conditions related to the weak topology on E and when E is reflexive Banach space, then we have the problem studied in [33].
- in abstract spaces with conditions related to the weak topology on E and the function f is independent of the fractional derivatives, then we have the problem studied in [6, 7].

Motivated by the above results, in this paper we investigate the case where f is a vector-valued Pettis integrable function. The assumptions in the existence theorem are expressed in terms of the weak topology, and a weak noncompactness type condition will be considered.

Here we prove the existence of a weak solution $x \in C[\mathbb{I}, E]$ of the the initial value problem (1.1) in a nonreflexive Banach space E. For this aim, we consider firstly the following integral equation of fractional type

$$x(t) = p(t) + \lambda\, I^{\alpha} f(t, x(t)),\ t \in \mathbb{I},\ 0 < \alpha < 1. \tag{1.2}$$

Now, let us recall the following basic facts.
Let E be a Banach space and let $x : \mathbb{I} \to E$, then x is said to be Pettis integrable on some interval $\mathbb{I}$ if and only if there is an element $x_J \in E$ corresponding to each $J \subset \mathbb{I}$ such that

$$\phi(x_J) = \int_J \phi(x(s))\, ds \quad for\ all\ \ \phi \in E^*,$$

where the integral on the right is supposed to exist in the sense of Lebesgue.

In a Banach space, both Pettis integrable functions and weakly continuous functions are weakly measurable. Moreover, in reflexive Banach space (even in Banach spaces without a copy of c_0) it is true that "the weakly measurable function $x(\cdot)$ is Pettis integrable on $\mathbb{I}$ if and only if $\phi(x(\cdot))$ is Lebesgue-integrable on $\mathbb{I}$, for every $\phi \in E^*$" (see Diestel and Uhl [11]). Also, it can be easily proved that weak differentiability implies weak continuity.

Let us denote by $P^{\infty}(\mathbb{I}, E)$ the space of all weakly measurable and Pettis integrable functions $x(\cdot) : \mathbb{I} \to E$ with the property that $\langle \phi, x(\cdot)\rangle \in L^{\infty}(\mathbb{I})$, for every $\phi \in E^*$. Since for each $t \in \mathbb{I}$ the real valued function $s \mapsto (t-s)^{\alpha-1}$ is Lebesgue integrable on $[0, t]$, the fractional Pettis integral [2]

$$I^{\alpha}x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, \ t \in \mathbb{I},$$

exists, for every function $x(\cdot) \in P^{\infty}(\mathbb{I}, E)$, as a function from $\mathbb{I}$ into E (see [32]). Moreover, we have

$$\langle \phi, I^{\alpha}x(t)\rangle = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle \phi, x(s)\rangle ds, \ t \in \mathbb{I},$$

for every $\phi \in E^*$, and the real function $t \mapsto \langle \phi, I^{\alpha}x(t)\rangle$ is continuous (in fact, bounded and uniformly continuous on $\mathbb{I}$ if $\mathbb{I} = R$) on $\mathbb{I}$, for every $\phi \in E^*$ ([4], Proposition 1.3.2).
In the following, consider $\alpha \in (0, 1)$ and for a given function $x(\cdot) \in P^{\infty}(\mathbb{I}, E)$ we also denote by $x_{1-\alpha}(t)$ the fractional Pettis integral

$$I^{1-\alpha}x(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(\alpha)} x(s)ds, \ t \in \mathbb{I}.$$

Lemma 1. *[32] The fractional Pettis integral is a linear operator from $P^{\infty}(\mathbb{I}, E)$ into $P^{\infty}(\mathbb{I}, E)$. Moreover, if $x(\cdot) \in P^{\infty}(\mathbb{I}, E)$, then for $\alpha, \beta > 0$ we have*

(a) $I^{\alpha}I^{\beta}x(t) = I^{\alpha+\beta}x(t), \ t \in \mathbb{I}$;
(b) $lim_{\alpha\to 1}I^{\alpha}x(t) = I^1x(t) = x(t)-x(0)$ *weakly uniformly on* $\mathbb{I}$;
(c) $lim_{\alpha\to 0}I^{\alpha}x(t) = x(t)$ *weakly on* $\mathbb{I}$.

If $y(\cdot) : \mathbb{I} \to E$ is a pseudo-differentiable function on $\mathbb{I}$ with a pseudo-derivative $x(\cdot) \in P^{\infty}(\mathbb{I}, E)$, then the fractional Pettis integral $I^{1-\alpha}x(t)$ exists on $\mathbb{I}$. The fractional Pettis integral $I^{1-\alpha}x(t)$ is called a fractional pseudo-derivative of $y(\cdot)$ on $\mathbb{I}$ and it will be denoted by $D^{\alpha}y(\cdot)$; that is,

$$D^{\alpha}y(t) = I^{1-\alpha}x(t), \ t \in \mathbb{I}.$$

Usually, $D^{\alpha}y(\cdot)$ is called the Caputo fractional pseudo-derivative of $y(\cdot)$.
For the properties of the fractional integral in Banach spaces (see [33, 34, 2]).
Now, we give the definition of the weak derivative of fractional order.

Definition 1. *Let $x : \mathbb{I} \to E$ be a weakly differentiable function and let x' be a weakly continuous. Then the weak derivative of x of order $\beta \in (0, 1]$ by*

$$D^{\beta}x(t) = I^{1-\beta}Dx(t)$$

where D is the weakly differential operator.

Recall that a function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E. In reflexive Banach spaces, both Pettis-integrable and weakly continuous functions are weakly measurable (see [11, 14, 15, 18]). The following results are due to Pettis (see [29]).

Proposition 1. *If $x(\cdot)$ is Pettis integrable and $h(\cdot)$ is a measurable and essentially bounded real-valued function, then $x(\cdot)h(\cdot)$ is Pettis integrable.*

The following result follows from the Hahn-Banach theorem.

Proposition 2. *Let E be a normed space with $x_0 \neq 0$. Then there exists a $\phi \in E^*$ with $\| \phi \| = 1$ and $\phi(x_0) = \| x_0 \|$.*

Our main condition that guarantees the existence of weak solutions of (1.2) will be formulated in terms of a measure of weak noncompactness β introduced by De Blasi in [12]. Further on, denote by m_E the family of nonempty and bounded subsets of E.
Let us recall that for any subset $A \in m_E$ of a Banach space E,

$$\beta(A) = inf\{r > 0 : \text{ there exists a weakly compact set } C \text{ such that } A \subset C + rB_1\};$$

where B_1 is the closed unit ball in E. The measure of weak noncompactness will be understood as a function $\beta : m_E \to [0, \infty)$ such that $(A,\ B \in m_E)$ [12]

(a_1) $\beta(A) = 0 \Leftrightarrow A$ is relatively weakly compact in E,
(a_2) $\beta(A) = \beta(\overline{conv}A)$,
(a_3) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,
(a_4) $\beta(A \cup \{x\}) = \beta(A),\ x \in E$,
(a_5) $\beta(\lambda A) = |\lambda|.\beta(A),\ \lambda \in R$,
(a_6) $\beta(A + B) \leq \beta(A) + \beta(B)$,
(a_7) $\beta(A \cup B) = \max\ (\beta(A), \beta(B))$.

It is necessary to remark that if β has these properties, then the following lemma is true:

Lemma 2. *([1, 6, 25])*
Let $H \subset C[\mathbb{I}, E]$ be a family of bounded and equicontinuous functions. Then the function $t \mapsto \upsilon(t) = \beta(H(t))$ is continuous and $\beta(H(\mathbb{I})) = \sup\{\beta(H(t)) : t \in \mathbb{I}\}$, where $\beta(\cdot)$ denotes the weak noncompactness measure on $C(\mathbb{I}, E)$ and $H(t) = \{u(t), u \in H\},\ t \in \mathbb{I}$.

Now we have the following theorem that will be needed in this paper.

Theorem 1. *[22] Let Ω be a closed convex and equicontinuous subset of a metrizable locally convex vector space E and let A be a weakly sequentially continuous mapping of Ω into itself. If for some $x \in \Omega$ the implication*

$$\overline{V} = \overline{conv}(A(V) \cup \{x\}) \Rightarrow V \text{ is relatively weakly compact}$$

holds, for every subset V of Ω, then A has a fixed point.

2. Main result

Let $\alpha \in (0, 1)$. In this section we study the existence of solutions of the equation (1.2) in a nonreflexive Banach space E, it will be investigated under the assumptions:

$(\Im)$ $p \in C[\mathbb{I}, E]$;

($\Im\Im$) $f(\cdot,\cdot) : \mathbb{I}\times E \to E$ is a function such that:

(i) for each $t \in \mathbb{I}$, $f(t,\cdot)$ is weakly sequentially continuous in x;

(ii) for each $x \in C[\mathbb{I}, E]$, $f(\cdot, x(\cdot)) \in P^\infty(\mathbb{I}, E)$;

(iii) for any $r_0 > 0$, there exists a nonnegative constant M_{r_0} with $\|f(s, x(s))\| \le M_{r_0}$ for all $t \in \mathbb{I}$ and all $x \in E$ with $\|x\| \le r_0$.

Definition 2. *By a solution to (1.2) we mean a function $x \in C[\mathbb{I}, E]$ which satisfies the integral equation (1.2). This is equivalent to finding $x \in C[\mathbb{I}, E]$ with*

$$\phi(x(t)) \ = \ \phi(p(t) + \lambda I^\alpha f(t, x(t))),\ t \in \mathbb{I},\ 0 < \alpha < 1,$$

for all $\phi \in E^*$.

Now, we shall prove the following existence theorem

Theorem 2. *Let the assumptions ($\Im$) and ($\Im\Im$) be satisfied, and if*

$$\beta(f(\mathbb{I}\times X)) \le K\,\beta(X), \quad K \ge 0$$

for each bounded subset X of E, then there exists at least one weak solution $x \in C[\mathbb{I}, E]$ for the equation (1.2), for each $\lambda \in R$ such that $|\lambda| < \rho$, $\rho > 0$.

Proof.
Let $r(H)$ be the spectral radius of the integral operator H defined as

$$Hu(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Ku(s)ds,\ u \in C[\mathbb{I}, R],\ t \in \mathbb{I},$$

and let

$$\rho = \min\left(\sup_{r>0} \frac{r - \| p \|_0}{\frac{M_{r_0} T^\alpha}{\Gamma(\alpha+1)}}, \frac{1}{r(H)} \right)$$

Fix $\lambda \in R$, $|\lambda| < \rho$ and choose $r_0 > 0$ such that

$$\| p \|_0 + |\lambda| \frac{M_{r_0} T^\alpha}{\Gamma(\alpha+1)} \le r_0. \tag{2.1}$$

Let us define the operator A by

$$(Ax)(t) = p(t) + \lambda\, I^\alpha f(t, x(t)),\ t \in \mathbb{I},\ 0 < \alpha < 1.$$

First, note that assumption (ii) implies that for each $x \in C[\mathbb{I}, E]$, $f(\cdot, x(\cdot))$ is Pettis integrable on $\mathbb{I}$ then $\phi(f(\cdot, x(\cdot)))$ is Lebesgue integrable on $\mathbb{I}$ for every $\phi \in E^*$. Also, $f(\cdot, x(\cdot))$ is fractionally Pettis integrable for all $t \in \mathbb{I}$ which implies that the fractional Pettis integral of the function f is weakly continuous and thus the operator A makes sense.
Now, define the set Ω as follows:

$$\Omega = \{x \in C[\mathbb{I}, E] : \| x \|_0 \le r_0,\ \forall t_1,\ t_2 \in I : [\| x(t_2) - x(t_1) \| \le \| p(t_2) - p(t_1) \|$$

$$+\frac{|\lambda| M_{r_0}}{\Gamma(\alpha+1)}(|t_2^\alpha - t_1^\alpha| + 2(t_2-t_1)^\alpha)]\}.$$

Note that Ω is closed, bounded, convex and equicontinuous subset of $C[\mathbb{I}, E]$.
We shall show that A satisfies the assumptions of Theorem 1. The proof will be given in four steps.

Step 1 : The operator A maps $C[\mathbb{I}, E]$ into itself.
Let $t_1, t_2 \in \mathbb{I}, t_2 > t_1$, without loss of generality, assume $Ax(t_2) - Ax(t_1) \neq 0$, then there exists $\phi \in E^*$ with $\| \phi \| = 1$ and

$$\| Ax(t_2) - Ax(t_1) \| = \phi(Ax(t_2) - Ax(t_1)).$$

Thus

$$\begin{aligned}
\| Ax(t_2) - Ax(t_1) \| &\leq |\phi(p(t_2) - p(t_1))| + |\lambda| \ \left| \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s,x(s)))\, ds \right. \\
&- \left. \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s,x(s)))\, ds \ \right| \\
&\leq \| p(t_2) - p(t_1) \| + |\lambda| \ \left| \int_0^{t_1} \left(\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \phi(f(s,x(s)))\, ds \right| \\
&+ \ |\lambda| \ \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(f(s,x(s)))\, ds \ \right|. \\
&\leq \| p(t_2) - p(t_1) \| + |\lambda| M_{r_0} \int_0^{t_1} \left| \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \right| ds \\
&+ \ |\lambda| M_{r_0} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds \\
&\leq \| p(t_2) - p(t_1) \| + |\lambda| M_{r_0} \left(\frac{|t_2^\alpha - t_1^\alpha|}{\Gamma(\alpha+1)} + 2\frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)} \right) \\
&\leq \| p(t_2) - p(t_1) \| + \frac{|\lambda| M_{r_0}}{\Gamma(\alpha+1)}(|t_2^\alpha - t_1^\alpha| + 2(t_2-t_1)^\alpha).
\end{aligned}$$

Hence

$$\| Ax(t_2) - Ax(t_1) \| \leq \| p(t_2) - p(t_1) \| + \frac{|\lambda| M_{r_0}}{\Gamma(\alpha+1)}(|t_2^\alpha - t_1^\alpha| + 2(t_2-t_1)^\alpha). \tag{2.2}$$

This estimation shows that A maps $C[\mathbb{I}, E]$ into itself.

Step 2 : The operator A maps Ω into itself.
To see this, take $x \in \Omega$; without loss of generality; assume $I^\alpha f(t, x(t)) \neq 0$, then there exists (by Proposition 2) $\phi \in E^*$ with $\|\phi\| = 1$ and $\|I^\alpha f(t, x(t))\| = \phi(I^\alpha f(t, x(t)))$ and by (4) $\phi(I^\alpha f(t, x(t))) = I^\alpha \phi(f(t, x(t)))$. Thus

$$\begin{aligned}
\| Ax(t) \| &\leq \phi(p(t)) + \phi(\lambda I^\alpha f(t,x(t))) \leq \| p(t) \| + |\lambda| I^\alpha \phi(f(t,x(t))) \\
&\leq \| p(t) \| + |\lambda| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\phi(f(s,x(s)))| \ ds
\end{aligned}$$

$$\begin{aligned}
&\leq \quad \| p(t) \| + | \lambda | M_{r_0} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&\leq \quad \| p(t) \| + \frac{| \lambda | M_{r_0} t^{\alpha}}{\Gamma(\alpha+1)} \\
&\leq \quad \| p \|_0 + \frac{| \lambda | M_{r_0} T^{\alpha}}{\Gamma(\alpha+1)} \leq r_0,
\end{aligned}$$

therefore, $\| Ax \|_0 = \sup_{t\in\mathbb{I}} \| Ax(t) \| \leq r_0$, and by using (2.2) we get $A : \Omega \to \Omega$.

Step 3 : The operator A is weakly sequentially continuous on Ω.
To see this, let $(x_n(\cdot))$ be a sequence in Ω such that $x_n(\cdot)$ converges weakly to $x(\cdot)$ in Ω. Then $x_n(t) \to x(t)$ in E_ω for each $t \in \mathbb{I}$. Fix $t \in \mathbb{I}$. Then by the weak sequential continuity of $f(t,.)$ it follows that $f(s, x_n(s))$ converges weakly to $f(s, x(s))$ for each $s \in \mathbb{I}$, therefore
$\phi(f(s, x_n(s)))$ converges to $\phi(f(s, x(s)))$ for each $s \in \mathbb{I}$.
By the Lebesgue dominated convergence theorem [17] we have $I^\alpha f(s, x_n(s)) \to I^\alpha f(s, x(s))$ and then $Ax_n(t) \to Ax(t)$ in E_ω for each $t \in \mathbb{I}$, so A is weakly sequentially continuous on Ω.

Step 4 : $V \subset \Omega$ is relatively weakly compact.
Put $Ax(t) = p(t) + Fx(t)$ where $Fx(t) = \lambda\, I^\alpha f(t, x(t))$ for $x \in \Omega,\ t \in \mathbb{I}$.

Suppose that $V \subset \Omega$ such that $\overline{V} \subset \overline{conv}(A(V) \bigcup \{0\})$. We will show that V is weakly relatively compact in $C[\mathbb{I}, E]$.
Put $N = F(V),\ \upsilon(t) = \beta(V(t)),\ \nu(t) = \beta(N(t))$ for $t \in \mathbb{I}$.
Obviously $V(t) \subset \overline{conv}(A(V)(t) \bigcup \{0\}),\ t \in \mathbb{I}$. Using the properties of β we have

$$\upsilon(t) \leq \beta(A(V)(t) \bigcup \{0\}) = \beta(A(V)(t)) = \beta(F(V)(t)) = \nu(t),\ t \in \mathbb{I}.$$

As $V \subset \Omega$ is equi-continuous, by Lemma 2 the function $t \mapsto \nu(t)$ is continuous on $[0, t)$. It follows that $s \mapsto (t-s)^{\alpha-1}\nu(s)$ is continuous on $[0, t)$.
Hence there exists $\delta > 0,\ 0 < \epsilon < 1$ such that

$$\| (t-\tau)^{\alpha-1}\nu(\tau) - (t-s)^{\alpha-1}\nu(s) \| < \frac{\epsilon}{2},$$

and

$$\| \nu(\zeta) - \nu(\tau) \| < \frac{\epsilon}{2(t_i - t_{i-1})^{\alpha-1}},$$

for $| \tau - s | < \delta$ and $| \tau - \zeta | < \delta$ with $\tau, \zeta, s \in [0, t)$, it follows that

$$| (t-\tau)^{\alpha-1}\nu(\zeta) - (t-s)^{\alpha-1}\nu(s) | \leq | (t-\tau)^{\alpha-1}\nu(\tau) - (t-s)^{\alpha-1}\nu(s) | + (t-\tau)^{\alpha-1} | \nu(\zeta) - \nu(\tau) |$$

that is

$$| (t-\tau)^{\alpha-1}\nu(\zeta) - (t-s)^{\alpha-1}\nu(s) | < \epsilon \tag{2.3}$$

for all $\tau, \zeta, s \in [0, t)$ with $| \tau - s | < \delta,\ \ | \tau - \zeta | < \delta$. Fix $t \in \mathbb{I}$, divide the interval $[0, t)$ into n parts $0 = t_0 < t_1 < ... < t_n = t,\ t_i - t_{i-1} < \delta,\ i = 1, 2, 3, ..., n$. Put $T_i = [t_{i-1}, t_i]$. In view of Lemma 2 it follows that for each $i \in [1, 2, ..., n]$ there exists $\tau_i \in T_i$ such that

$$\beta(N(T_i)) = \nu(\tau_i),\ \ i = 1, ..., n.$$

By the Mean Value theorem we have

$$\begin{aligned} Fx(t) &\leq\mid \lambda \mid \sum_{i=1}^{n} \int_{T_i} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, ds \\ &\in \frac{\mid \lambda \mid}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1}) \overline{conv}((t-s)^{\alpha-1} f(T_i \times V(T_i))) \end{aligned}$$

where $f(T_i \times V(T_i)) = \{f(s, x(s)) : s \in T_i,\ x \in V\}$. Then

$$FV(t) \subset \frac{\mid \lambda \mid}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1}) \overline{conv}((t-s)^{\alpha-1} f(T_i \times V(T_i)))$$

for some $\tau_i \in T_i$. Hence

$$\begin{aligned} v(t) &\leq \frac{\mid \lambda \mid}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha-1} \beta(f(T_i \times V(T_i))) \\ &\leq \frac{\mid \lambda \mid}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha-1} K\beta(V(T_i)) \\ &\leq \frac{\mid \lambda \mid}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha-1} K\beta(N(T_i)) \\ &\leq \frac{\mid \lambda \mid}{\Gamma(\alpha)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha-1} K v(\tau_i). \end{aligned}$$

Moreover as

$$\mid (t - t_i)^{\alpha-1} v(\tau_i) - (t-s)^{\alpha-1} v(s) \mid < \epsilon\, \Gamma(\alpha),\ \ s \in T_i,$$

we have

$$(t - t_i)^{\alpha-1} v(\tau_i)(t_i - t_{i-1}) \leq \int_{T_i} (t-s)^{\alpha-1} v(s)\, ds + \epsilon\Gamma(\alpha)(t_i - t_{i-1}).$$

Thus

$$v(t) \leq \frac{\mid \lambda \mid}{\Gamma(\alpha)} \int_{T_i} (t-s)^{\alpha-1} K v(s)\, ds + \mid \lambda \mid K \sum_{i=1}^{n} (t_i - t_{i-1})\, \epsilon.$$

As ϵ is arbitrary, and letting $n \to \infty$ we get

$$v(t) \leq\mid \lambda \mid \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} K v(s)\, ds.$$

Since $\mid \lambda \mid r(H) < 1$, it follows that $v(t) = 0 \Rightarrow \upsilon(t) = 0$ for $t \in \mathbb{I}$. Hence $V(t)$ is weakly relatively compact in E. Applying now Theorem (1) we deduce that A has a fixed point. ∎

Remark:

Now, If E is reflexive, it is not necessary to assume any compactness condition on the function f because, a subset of reflexive Banach space is weakly compact if and only if it is weakly closed and bounded in norm.

3. Initial value problem

In this section we shall study existence theorems of weak solutions and pseudo-solutions for the initial value problem (1.1).

3.1. Weak solution for the initial value problem

As a particular case of Theorem 2 we can obtain a theorem on the existence of solutions belonging to the space $C(\mathbb{I}, E)$ for the initial value problem (1.1).
Consider the following assumption:

(i)* $f(\cdot, x(\cdot))$ is weakly-weakly continuous.

Theorem 3. *If the assumptions of Theorem 2 be satisfied and replace the assumptions (i) and (ii) by the assumption* $(i)^*$ *. Then the initial value problem (1.1) has at least one weak solution* $x \in C[\mathbb{I}, E]$.

Proof.
Putting $\alpha = 1 - \gamma$, $p(t) = 0$ and $\lambda = 1$ in the equation (1.2) and considering a solution $y : \mathbb{I} \to E$, then $y(\cdot)$ satisfies

$$y(t) = I^{1-\gamma} f(t, y(t)), \tag{3.1}$$

and

$$I^{\gamma} y(t) = I^{1} f(t, y(t)),\ t \in \mathbb{I},$$

since f is weakly continuous in t, then it is weakly differentiable with respect to the right end point of the integration interval and its derivative equals the integrand at that point [25], therefore

$$\frac{d}{dt} I^{\gamma} y(t) = f(t, y(t)),\ t \in \mathbb{I}.$$

Set

$$x(t) = x_0 + I^{\gamma} y(t) = x_0 + I^{1} f(t, y(t)). \tag{3.2}$$

Then $x(\cdot)$ is weakly differentiable and $x(0) = x_0$, $\frac{dx}{dt} = f(t, y(t))$,
since f is weakly continuous in t, $I^{1-\gamma}\frac{dx}{dt}$ exists and

$$D^{\gamma} x(t) = I^{1-\gamma} \frac{dx}{dt} = I^{1-\gamma} f(t, y(t)) = y(t).$$

Then any solution of (3.1) will be a solution of (1.1), this solution is given by (3.2). This completes the proof.■

3.2. Pseudo-solutions for the initial value problem

In this subsection, we are looking for sufficient conditions to prove the existence of pseudo-solution to the initial value problem (1.1) under the Pettis integrability assumption imposed on f.
The existence of pseudo-solution in Banach space for the initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0 \tag{3.3}$$

is proved in [6, 7]. The function $f : \mathbb{I} \times E \to E$ will assumed to be Pettis integrable.

The existence of pseudo-solutions of (3.3) is equivalent to the existence of solutions to the integral equation (see [29])

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds \tag{3.4}$$

Definition 3. *[20] A function $x : \mathbb{I} \to E$ is said to be a pseudo-solution of the initial value problem (3.3) if*
(a) $x(\cdot)$ is absolutely continuous and $x(0) = x_0$,
(b) for each $\phi \in E^*$ there exists a null set $N(\phi)$ (i.e. N is depending on ϕ and $mes(N(\phi)) = 0$) such that for each $t \notin N(\phi)$

$$(\phi x)'(t) \; = \; \phi(f(t, x(t)))$$

where x' denotes the pseudo-derivative (see Pettis [29] or [6]).

The following lemma will be needed.

Lemma 3. *Let $x(\cdot) : \mathbb{I} \to E$ be a weakly measurable function.*
If $x(\cdot)$ is Pettis integrable on $\mathbb{I}$, then the indefinite Pettis integral

$$y(t) \; = \; \int_0^t x(s) ds, \quad t \in \mathbb{I}$$

is absolutely continuous on $\mathbb{I}$ and $x(\cdot)$ is a pseudo-derivative of $y(\cdot)$.

Now, we can prove the following theorem.

Theorem 4. *Let the assumptions of Theorem 2 be satisfied. Then the initial value problem (3.3) has at least one pseudo-solution.*

Proof:
For any $x \in C[\mathbb{I}, E]$ and by a direct application of Theorem 2 (with $\alpha = 1$), it can be easily seen that the equation (3.4) has a weak solution $x \in C[\mathbb{I}, E]$.
Let x be a weak solution of (3.4). Then for any $\phi \in E^*$ we have

$$\begin{aligned} \phi x(t) \; &= \; \phi(x_0 + \int_0^t f(s, x(s))\, ds) \\ &= \; \phi(x_0) + \phi(\int_0^t f(s, x(s))\, ds) \\ &= \; x_0 + \int_0^t \phi(f(s, x(s)))\, ds. \end{aligned}$$

By differentiating both sides, we obtain

$$\frac{d}{dx}\phi x(t) = \phi(f(s, x(s))) \;\; a.e. \text{ on } \mathbb{I}$$

and

$$\lim_{t \to 0^+} \phi x(t) = \lim_{t \to 0^+} [\phi(x_0) + \int_0^t \phi(f(s, x(s))) ds] = x_0.$$

That is, $x(t)$ has a pseudo-derivative and satisfies

$$\frac{d}{dt}x(t) = f(t, x(t)) \text{ on } \mathbb{I}.$$

Now we shall study the existence of pseudo-solution for the initial value problem (1.1).

Definition 4. *A function $x : \mathbb{I} \to E$ is called pseudo-solution of the problem (1.1) if $x \in C[\mathbb{I}, E]$ has a pseudo-derivative, $x(0) = x_0$ and satisfies*

$$\phi(\frac{dx}{dt}) = \phi(f(t, D^{\gamma}x(t))) \quad a.e. \text{ on } \mathbb{I} \text{ for each } \phi \in E^*.$$

The following Lemma is needed.

Lemma 4. *If $y \in C[\mathbb{I}, E]$ is a solution to the problem*

$$y(t) = I^{1-\gamma}f(t, y(t)), \quad \gamma \in (0, 1), \quad t \in \mathbb{I} \tag{3.5}$$

then $x(t) = x_0 + I^{\gamma}y(t)$ is a pseudo-solution for the problem (1.1).

Proof:
Let $y \in C[\mathbb{I}, E]$ be a solution to the problem (3.5). For $x(t) = x_0 + I^{\gamma}y(t)$, then $x(t) \in C[\mathbb{I}, E]$ and the real function ϕx is continuous for every $\phi \in E^*$; moreover

$$\begin{aligned}
\lim_{t\to 0^+} \phi x(t) &= \lim_{t\to 0^+}[x_0 + (I^{\gamma}\phi y)(t)] \\
&= x_0 + \lim_{t\to 0^+}(I^{\gamma}\phi y)(t) \\
&= x_0 + \lim_{t\to 0^+}\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\phi(y(s))\,ds \\
&= x_0.
\end{aligned}$$

Thus $\phi x(0) = x_0$ for $\phi \in E^*$. That is $x(0) = x_0$.
Then we have

$$\begin{aligned}
D^{\gamma}x(t) &= D^{\gamma}[x_0 + I^{\gamma}y(t)] \\
&= 0 + D^{\gamma}I^{\gamma}y(t) \\
&= D^0 y(t) \\
&= y(t). \blacksquare
\end{aligned}$$

Theorem 5. *Let the assumptions of Theorem 2 be satisfied. Then the initial value problem (1.1) has at least one pseudo-solution $x \in C[\mathbb{I}, E]$.*

Proof
Firstly, observe that $x(t) = x_0 + I^{\gamma}y(t)$ makes sense for any $y \in C[\mathbb{I}, E]$.
According to Theorem 2 it can be easily seen that the integral equation (3.5) has a solution $y \in C[\mathbb{I}, E]$.
Let y be a weak solution of (3.5). Then for any $\phi \in E^*$ we have

$$\phi y(t) = \phi(I^{1-\gamma}f(t, y(t))) = I^{1-\gamma}\phi(f(t, y(t))). \tag{3.6}$$

Operating by I^{γ} on both sides of (3.6) and using the properties of fractional calculus in the space $L^1[\mathbb{I}]$ (see [23, 28]) result in

$$I^{\gamma}\phi y(t) = I^1\phi(f(t, y(t))).$$

Therefore

$$\phi(I^{\gamma}y(t)) = I^1\phi(f(t, y(t))),$$

$$\phi(x(t) - x_0) = \phi(x(t)) - x_0 = I^1\phi(f(t, y(t))).$$

Thus

$$\frac{d}{dt}\phi(x(t)) = \phi(f(t, D^{\gamma}x(t)),$$

and

$$\frac{d}{dt}\phi(x(t)) = \phi(f(t, D^{\gamma}x(t)) \ \ a.e. \ \text{on} \ \ \mathbb{I}$$

That is, $x(t)$ has the pseudo-derivative and satisfies

$$\frac{dx(t)}{dt} = f(t, D^{\gamma}x(t)) \quad \text{on} \ \mathbb{I}.\blacksquare$$

4. Conclusions

In this paper, we established some existence results of weak solutions and pseudo-solutions for the initial value problem of the arbitrary (fractional) orders differential equation in nonreflexive Banach spaces, we investigate the case of a vector-valued Pettis integrable function. The assumptions in the existence theorem are expressed in terms of the weak topology using a weak noncompactness type condition (expressed in terms of measure of noncompactness) .
For real-valued function and the function f is independent of the fractional derivatives, we have the problems studied in [10, 31]. In abstract spaces with conditions related to the weak topology on a reflexive Banach space, we have the problem studied in [33]. In abstract spaces with conditions related to the weak topology on a Banach space and the function f is independent of the fractional derivatives, we have the problem studied in [6, 7].

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Conflict of interest

The authors declare that they have no competing interests.

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