Research article

An averaging principle for stochastic evolution equations with jumps and random time delays

Min Han\(^1\) and Bin Pei\(^2,\ast\)

\(^1\) School of Mathematics and Statistics, Northwestern Polytechnical University, Xi’an, 710072, China
\(^2\) School of Mathematical Sciences, Fudan University, Shanghai, 200433, China

* Correspondence: Email: binpei@hotmail.com.

Abstract: This paper investigates an averaging principle for stochastic evolution equations with jumps and random time delays modulated by two-time-scale Markov switching processes in which both fast and slow components co-exist. We prove that there exists a limit process (averaged equation) being substantially simpler than that of the original one.

Keywords: averaging principle; stochastic evolution equations; jumps; random time delays; two-time-scale Markov switching processes.

Mathematics Subject Classification: 60H15, 70K70, 34C29.

1. Introduction

The theory of evolution equations has attracted researchers’ great interest stimulated by its numerous practical applications in the areas such as physics, population dynamics, electrical engineering, medicine biology, ecology [1, 7]. In fact, much evidence has been gathered that Poisson jumps are ubiquitous in many fields of science. By now, it is well established that stochastic systems driven by Poisson jumps are more suitable for capturing sudden bursty fluctuations, some large moves and unpredictable events than classical diffusion models [2, 4, 13]. Therefore, it is necessary to study stochastic evolution equation (SEEs) driven by jumps. For example, in the areas of population dynamics, Rathinasamy, Yin and Yasodha [20] discussed a stochastic age-dependent population equation with Markovian switching

\[
\begin{aligned}
\frac{\partial P}{\partial t} + \frac{\partial P}{\partial a} &= -\mu(t, a)P + f(r(t), P) + g(r(t), P)\frac{dW(t)}{dt} + h(r(t), P)\frac{dN(t)}{dt}, \quad (t, a) \in Q, \\
P(0, a) &= P_0(a), \quad r(0) = r_0, \quad a \in [0, \bar{A}], \\
P(t, 0) &= \int_0^\bar{A} \beta(t, a)P(t, a)da, \quad t \in [0, T],
\end{aligned}
\]  

(1.1)
where $T > 0, \hat{A} > 0, Q := (0, T) \times (0, \hat{A}), P = P(t, a)$ is the population density of age $a$ at time $t$, $r(t) \geq 0$ is a continuous-time Markov chain, $\mu(r(t), a)$ denotes the mortality rate of age $a$ at time $t$, $\beta(t, a)$ denotes the fertility rate of females of age $a$ at time $t$ and $f(\cdot, \cdot)$ denotes the effects of the external environment on the population system. $g(\cdot, \cdot)$ is a diffusion coefficient, $W$ is a Brownian motion. $h(\cdot, \cdot)$ is a jump coefficient, $N$ is a scalar Poisson process with intensity $\lambda$.

This paper considers SEEs with jumps and random time delays modulated by Markov chains. To illustrate, consider the following SEEs with random time delays

$$dX(t) = (AX(t) + f(X(t), X(t - r(t))))dt + \int_U h(X(t), z)\tilde{N}(dt, dz),$$  

where $U \in B_r(\mathbb{R} - \{0\})$ which will be defined in next section, $r(t) \geq 0$ is a continuous-time Markov chain in a finite state space $\mathbb{S} := \{r_1, \ldots, r_n\}$ with generator $\tilde{Q} = (\tilde{q}_{ij}) \in \mathbb{R}^{n \times n}$. Without loss of generality, assume that $r_1 < r_2 < \cdots < r_n$. Recall that the generator $\tilde{Q}$ is weakly irreducible, if the system of equation

$$\begin{cases}
\sum_{i=1}^n \bar{v}_i = 1,
\bar{v} \tilde{Q} = 0,
\end{cases}$$

has a unique solution $\bar{v} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$ satisfying $\bar{v}_i \geq 0$ for all $i = 1, \ldots, n$. The solution $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n)$ is termed as the stationary measure.

This system (1.2) is a switching system among the following $n$ fixed delay subsystems

$$dX_i(t) = (AX_i(t) + f(X_i(t), X(t - r_i)))dt + \int_U h(X_i(t), z)\tilde{N}(dt, dz), \quad i = 1, 2 \cdots n.$$  

The random switching is governed by the Markov chain $r(t) \geq 0$. In our setup, the Markov chain has both fast and slow motions and involves strong and weak interactions. To reflect the fast and slow motions of the Markov chain, we introduce a small parameter $\varepsilon > 0$ and rewrite the Markov chain $r(t)$ as $r^\varepsilon(t)$ and the generator $\tilde{Q}$ as $Q^\varepsilon$. Thus, the Markov chain displays two-time-scales. That is, a usual running time $t$ and a stretched (fast) time $t/\varepsilon$. Suppose that the generator of the Markov chain is given by

$$Q^\varepsilon := \frac{Q}{\varepsilon} + \tilde{Q},$$

where both $Q$ and $\tilde{Q}$ are generators of suitable continuous-time Markov chains such that $Q/\varepsilon$ and $\tilde{Q}$ represent the fast-varying and slowly-changing parts, respectively. Throughout the paper, we assume that $Q$ is weakly irreducible in the sense defined in (1.3). Denote the stationary measure associated with $\tilde{Q}$ as $v = (v_1, v_2, \ldots, v_n)$.

Now, we consider the following SEEs driven by jumps with random time delays modulated by two-time-scale Markov switching processes, that is, we can rewrite Eq.(1.2) as

$$dX^\varepsilon(t) = (AX^\varepsilon(t) + f(X^\varepsilon(t), X^\varepsilon(t - r^\varepsilon(t))))dt + \int_U h(X^\varepsilon(t), z)\tilde{N}(dt, dz), \quad t \in [0, T],$$

where $X_0 = \psi(\cdot) \in D^b_{\tilde{\mathbb{F}}_0}([-r_n, 0]; \mathbb{H})$, $\psi(\cdot)$ satisfies

$$\mathbb{E}\left[\sup_{-r_n \leq s \leq 0} |\psi(s)|^2_{\mathbb{H}}\right] < +\infty,$$  

where $\tilde{\mathbb{F}}_0$ is the filtration generated by the jump process and $D^b_{\tilde{\mathbb{F}}_0}$ is the space of $L^2$-continuous adapted processes.
and the Lipschitz condition, namely, there exists a constant $K_1 > 0$ such that

$$|\psi(t_1) - \psi(t_2)| \leq K_1 |t_1 - t_2|.$$  \hfill (1.7)

An interesting question is: Whether or when the averaging principle still holds for the overall system consisting of $n$ fixed delay subsystems, can an averaged system be obtained? Such a question is the main concern of this paper. It is well known that there is an extensive literature on averaging principles for singularly perturbed stochastic (partial) differential equations [5, 6, 10, 16, 22, 24, 25]. Cerrai and Freidlin [6] presented an averaged result for stochastic parabolic equations. Bréhier [5] derived explicit convergence rates in strong and weak convergence for averaging of stochastic parabolic equations. Pei et al. [14, 17] obtained an averaging principle for fast-slow stochastic partial differential equations (SPDEs) in which the slow variable is driven by a fractional Brownian motion (fBm) and the fast variable is driven by an additive Brownian motion. Xu, Miao and Liu [23] obtained an averaging principle for two-time-scale SPDEs driven by Poisson random measures in the sense of mean-square. Xu and Liu [22] proved a stochastic averaging principle for two-time-scale jump-diffusion SDEs under the non-Lipschitz coefficients. Pei, Xu and Wu [15] studied the existence, uniqueness and averaging principles for two-time-scales hyperbolic-parabolic equations driven by Poisson random measures. Bao, Yin and Yuan [3] obtained averaging principles for SPDEs driven by $\alpha$-stable noises with two-time-scale Markov switching. As for random delays cases, Wu, Yin, and Wang [21] considered a class of nonlinear systems with random time delays modeled by a two-time-scale Markov chain. Pei, Xu, Yin and Zhang [19] examined averaging principles for functional SPDEs driven by a fBm modulated by two-time-scale Markov switching process. Pei, Xu and Yin [18] considered SPDEs driven by fBms with random time delays modulated by two-time-scale Markov switching processes.

Therefore, it is quite natural to ask whether or when an averaging principle for SEEs driven by jumps with random time delays still holds. On the one hand, to the authors’ knowledge, the averaging principle for SEEs driven by jumps with random time delays modulated by two-time-scale Markov switching processes has not been considered. Therefore, it is necessary to obtain the averaging principle for SEEs driven by jumps with random time delays. On the other hand, since introducing randomly switching time delays, it is much more difficult to make system analysis and control design. However, in our setup, we can employ a two-time scale framework to describe the dynamic relationships of the switching time delay and the underlying system dynamics. Thus, it will be very useful to consider delay-dependent stability by the two-time-scale approach whose main advantage is that the stability can be examined by an “average” system as a bridge. Thus, in this paper, my aim is to obtain the averaging principle for SEEs driven by jumps with random time delays. Then, in forthcoming work, it will be possible to consider delay-dependent stability involving jumps.

The rest of the paper is arranged as follows. Section 2 presents preliminary results that are needed in the subsequent sections. In Section 3, we established an averaging principle for the case of SEEs driven by jumps with random time delays.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\mathbb{H}$, $\mathbb{K}$ be two separable Hilbert spaces and we denote by $\langle \cdot, \cdot \rangle_{\mathbb{H}}$, $\langle \cdot, \cdot \rangle_{\mathbb{K}}$ their inner products and by $\| \cdot \|_{\mathbb{H}}$, $\| \cdot \|_{\mathbb{K}}$ their norms,
AIMS Mathematics

∥ϕ\right-continuous functions with left-hand limits ν arise. Then, we assume that the operator \( S \) is the infinitesimal generator of an analytic semigroup of bounded linear operators \( \left\| A \right\| \leq M e^{-\lambda t} \), \( t \geq 0 \), for some constants \( M \geq 1 \) and \( \lambda > 0 \).

**Lemma 2.1.** Suppose that (H1) holds. Then for any \( \theta \in (0, 1) \), the following equality holds:

\[
S(t)(-A)^\theta x = (-A)^\theta S(t)x, \quad x \in D((-A)^{-\theta}),
\]

and there exists a positive constant \( M_\theta > 0 \) such that

\[
\left\| (-A)^\theta S(t) \right\| \leq M_\theta e^{-\lambda t}, \quad t > 0,
\]

\[
\left\| (S(t) - I)(-A)^{-\theta} \right\| \leq M_\theta^\frac{1}{\theta}, \quad t > 0,
\]

where I is the identity operator.

3. Stochastic evolution equations with jumps and random time delays

Throughout this paper, we assume that the following conditions are satisfied.

(H2) There exist constants \( K_2, K_3 > 0 \), for any \( x_1, y_1, x_2, y_2 \in \mathbb{H} \) and \( q \geq 2 \), such that

\[
\left\| f(x_1, y_1) - f(x_2, y_2) \right\|_H^2 \leq K_2(|x_1 - x_2|_H^2 + |y_1 - y_2|_H^2),
\]

\[
\int_U |h(x_1, z) - h(x_2, z)|_H^q \nu(dz) \leq K_3|x_1 - x_2|_H^q.
\]
Theorem 3.1. Let the condition

\[ |f(x, y)|^2_H \leq K_4(1 + |x|^2_H + |y|^2_H), \]

\[ \int_U |h(x, z)|^q_H v(dz) \leq K_5(1 + |x|^q_H). \]

To proceed, we show that as \( \varepsilon \to 0 \), \( X^\varepsilon \) to Eq. (1.5) has a limit \( \bar{X} \) which satisfies the following SEEs that an averaged system

\[ d\bar{X}(t) = \left( A\bar{X}(t) + \sum_{i=1}^n f(\bar{X}(t), \bar{X}(t - r_i))v_i \right)dt + \int_U h(\bar{X}(t), z)\tilde{N}(dt, dz), \]  

(3.1)

in the mild sense, with initial value \( \bar{X}_0 = \psi(\cdot) \in D^{\theta}_{F_0}([-r_n, 0]; \mathbb{H}) \). Here, as \( \varepsilon \to 0 \), \( \sum_{i=1}^n f(\bar{X}(t), \bar{X}(t - r_i))v_i \) is the averaged drift term of \( f(X^\varepsilon(t), X^\varepsilon(t - r^\varepsilon(t))) \) in (1.5) based on the stationary measure \( \nu \) associated with the generator \( Q \).

Following the discretization techniques inspired by Khasminskii in [10], we formulate our main result of averaging principle.

**Theorem 3.1.** Let the condition (H1)–(H3) hold and assume further that \( \psi(0) \in D((-A)\theta) \) with \( \theta \in (0, \frac{1}{2}) \). Then, for sufficiently small \( \varepsilon \in (0, 1) \), we have

\[ \mathbb{E}\left[ \sup_{t \in [0, T]} |X^\varepsilon(t) - \bar{X}(t)|^2_H \right] \leq C_{p,n}(\varepsilon^p + \varepsilon^{p\theta}) \to 0, \quad \text{as} \quad \varepsilon \to 0, \]

where \( C_{p,n} > 0 \) is a constant which is independent of \( \varepsilon \) and \( p \in [1, \frac{1}{2\theta}) \), \( 0 < \rho < \beta < 1 \).

In this section, we firstly prove the existence and uniqueness of mild solution to Eq. (1.5). Then, we give some estimations on the solution process. Finally, we prove the averaging principle result (Theorem 3.1).

### 3.1. The existence and uniqueness of mild solution and some estimations on the solution process

**Definition 3.2.** (cf. [11] for example) An \( \mathbb{H} \)-valued process \( X^\varepsilon(t), t \in [0, T] \) is called a mild solution of Eq (1.5), if

1. \( X^\varepsilon(t) \) is \( \mathcal{F}_t \)-adapted and cadlag;

2. \( X^\varepsilon(t) \) satisfies the following integral equation:

\[ X^\varepsilon(t) = S(t)\psi(0) + \int_0^t S(t - s)f(X^\varepsilon(s), X^\varepsilon(s - r^\varepsilon(s)))ds \]

\[ + \int_0^t \int_U S(t - s)h(\bar{X}(s), z)\tilde{N}(ds, dz), \quad t \in [0, T], \]  

(3.2)

where \( r^\varepsilon(t) \) is a two-time-scales Markov chain taking values in \( \mathbb{S} \).

By the classical method [11, 12], these assumptions (H1)–(H3) can guarantee the existence and uniqueness of the mild solution to Eq (1.5).
By Definition 3.2, the integral form of Eq (3.1) can be written as

$$
\bar{X}(t) = S(t)\psi(0) + \sum_{i=1}^{n} \int_{0}^{\tau} S(t-s) f(\bar{X}(s), \bar{X}(s-r_i)) v_i ds \\
+ \int_{0}^{\tau} \int_{U} S(t-s) h(\bar{X}(s), z) \tilde{N}(ds, dz), \quad t \in [0, T],
$$

$$
\bar{X}(s) = \psi(s), \quad s \in [-r_n, 0], \quad r_n > 0,
$$

(3.3)

where $r_i \in \mathbb{S}, i = 1, 2, \ldots, n$.

**Lemma 3.3.** Let (H1)–(H3) hold. Then, for $p \geq 1$, we have

$$
\mathbb{E}[ \sup_{t \in [-r_n, T]} |\bar{X}(t)|_{2^p}^{2p} + |X^p(t)|_{2^p}^{2p} ] \leq C_{p,n},
$$

where $C_{p,n} > 0$ is a constant which is independent of $\varepsilon$.

**Proof:** Taking expectation and employing the elementary inequality, we get from Eq (3.3) that

$$
\mathbb{E}[ \sup_{t \in [-r_n, T]} |\bar{X}(t)|_{2^p}^{2p} ] \leq C_p \mathbb{E}[ \sup_{-r_n \leq t \leq 0} |\psi(t)|_{2^p}^{2p} ] + C_p \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \sum_{i=1}^{n} \int_{0}^{\tau} S(t-s) f(\bar{X}(s), \bar{X}(s-r_i)) v_i ds \right|_{2^p}^{2p} \right] \\
+ C_p \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_{0}^{\tau} \int_{U} S(t-s) h(X^p(s), z) \tilde{N}(ds, dz) \right|_{2^p}^{2p} \right] \\
=: I_1 + I_2 + I_3.
$$

Firstly, for $I_1$, by (1.6), it is easy to know there exists a constant $C_{p,n} > 0$ such that

$$
I_1 \leq C_{p,n}.
$$

Next, for $I_2$, we have

$$
I_2 \leq C_{p,n} \sum_{i=1}^{n} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_{0}^{\tau} S(t-s) f(\bar{X}(s), \bar{X}(s-r_i)) v_i ds \right|_{2^p}^{2p} \right] \\
\leq C_{p,n} \sum_{i=1}^{n} \int_{0}^{T} \mathbb{E} [|f(\bar{X}(s), \bar{X}(s-r_i))|_{2^p}^{2p}] ds \\
\leq C_{p,n} \sum_{i=1}^{n} \int_{0}^{T} (1 + \mathbb{E}[|\bar{X}(s)|_{2^p}^{2p} + |\bar{X}(s-r_i)|_{2^p}^{2p}]) ds \\
\leq C_{p,n} + C_{p,n} \int_{0}^{T} \mathbb{E} \left[ \sup_{s \in [-r_n, s]} |\bar{X}(s)|_{2^p}^{2p} \right] ds.
$$

Then, for $I_3$, by [8, Lemma 3.1], we have

$$
I_3 = C_p \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_{0}^{\tau} \int_{U} S(t-s) h(\bar{X}(s), z) \tilde{N}(ds, dz) \right|_{2^p}^{2p} \right] \\
\leq C_p \mathbb{E} \left[ \left( \int_{0}^{T} \int_{U} |h(\bar{X}(s), z)|_{2^p}^{2p} \nu(dz) ds \right)^{\frac{1}{2p}} \right] + C_p \mathbb{E} \left[ \int_{0}^{\tau} \int_{U} |h(\bar{X}(s), z)|_{2^p}^{2p} \nu(dz) ds \right].
$$
Lemma 3.4. Let (H1)–(H3) hold and assume further that \( \psi(0) \in \mathcal{D}((-A)^\theta) \) with \( \theta \in (0, \frac{1}{2}) \). Then, for \( 0 \leq t < t + \delta \leq T, \delta \in (0, 1), p \in [1, \frac{2}{1-\theta}) \), we have

\[
\sup_{t \in [0, T]} \mathbb{E}[|\bar{X}(t + \delta) - \bar{X}(t)|^2] \leq C_{p,n} \delta^{2p\theta}.
\]

Proof: From Eq (1.5) and Eq (3.1), one has

\[
\mathbb{E}[|\bar{X}(t + \delta) - \bar{X}(t)|^2] \leq C_p^2 \mathbb{E}[|S(\delta) - I|S(t)\psi(0)|^2]
\]

\[
+ C_p \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_0^{t+\delta} S(t+\delta-s) f(\bar{X}(s), \bar{X}(s-r_i)) ds \right|^2 \right] \right]
\]

\[
- \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_0^{t} S(t-s) f(\bar{X}(s), \bar{X}(s-r_i)) ds \right|^2 \right] \right]
\]

\[
+ C_p \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_U S(t+\delta-s) h(\bar{X}(s), z) N(ds,dz) \right|^2 \right] \right]
\]

\[
- \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_U S(t-s) h(\bar{X}(s), z) N(ds,dz) \right|^2 \right] \right]
\]

\[
=: J_1 + J_2 + J_3.
\]

Firstly, for \( J_1 \), by Lemma 2.1 and the condition \( \psi(0) \in \mathcal{D}((-A)^\theta) \), one has

\[
J_1 \leq C_p \|S(\delta) - I\|_{L^2}^2 \|S(t)\|_{L^2}^2 \mathbb{E}[|(-A)^\theta \psi(0)|^2]
\]

\[
\leq C_p \delta^{2p\theta}.
\]

Then, for \( J_2 \), we have

\[
J_2 \leq C_{p,n} \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_0^{t+\delta} S(t+\delta-s) f(\bar{X}(s), \bar{X}(s-r_i)) ds \right|^2 \right] \right]
\]

\[
\leq C_{p,n} \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_0^{t} S(t-s) f(\bar{X}(s), \bar{X}(s-r_i)) ds \right|^2 \right] \right]
\]

\[
+ C_{p,n} \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_U S(t+\delta-s) f(\bar{X}(s), \bar{X}(s-r_i)) ds \right|^2 \right] \right]
\]

where \( C_{p,n} > 0 \) is a constant.

Therefore, by Gromwall’s inequality, we have

\[
\mathbb{E}[\sup_{t \in [-r_n, T]} |\bar{X}(t)|^2_p] \leq C_{p,n},
\]

where \( C_{p,n} > 0 \) is a constant which is independent of \( \varepsilon \). This completes the proof. \( \square \)

\[\text{Lemma 3.4. Let (H1)–(H3) hold and assume further that } \psi(0) \in \mathcal{D}((-A)^\theta) \text{ with } \theta \in (0, \frac{1}{2}). \text{ Then, for } \]

\[\text{where } C_{p,n} > 0 \text{ is a constant which is independent of } \varepsilon. \text{ This completes the proof. } \]

\[\text{Proof: From Eq (1.5) and Eq (3.1), one has }\]

\[\mathbb{E}[|\bar{X}(t + \delta) - \bar{X}(t)|^2] \leq C_p^2 \mathbb{E}[|S(\delta) - I|S(t)\psi(0)|^2]
\]

\[\leq C_p \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_0^{t+\delta} S(t+\delta-s) f(\bar{X}(s), \bar{X}(s-r_i)) ds \right|^2 \right] \right]
\]

\[\leq C_p \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_0^{t} S(t-s) f(\bar{X}(s), \bar{X}(s-r_i)) ds \right|^2 \right] \right]
\]

\[+ C_p \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_U S(t+\delta-s) h(\bar{X}(s), z) N(ds,dz) \right|^2 \right] \right]
\]

\[\leq C_p \delta^{2p\theta}.
\]

\[\text{Firstly, for } J_1, \text{ by Lemma 2.1 and the condition } \psi(0) \in \mathcal{D}((-A)^\theta), \text{ one has}
\]

\[J_1 \leq C_p \|S(\delta) - I\|_{L^2}^2 \|S(t)\|_{L^2}^2 \mathbb{E}[|(-A)^\theta \psi(0)|^2]
\]

\[\leq C_p \delta^{2p\theta}.
\]

\[\text{Then, for } J_2, \text{ we have}
\]

\[J_2 \leq C_{p,n} \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_0^{t+\delta} S(t+\delta-s) f(\bar{X}(s), \bar{X}(s-r_i)) ds \right|^2 \right] \right]
\]

\[\leq C_{p,n} \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_0^{t} S(t-s) f(\bar{X}(s), \bar{X}(s-r_i)) ds \right|^2 \right] \right]
\]

\[+ C_{p,n} \sum_{i=1}^{n} \left[ \mathbb{E}\left[ \left| \int_U S(t+\delta-s) f(\bar{X}(s), \bar{X}(s-r_i)) ds \right|^2 \right] \right]
\]

\[\leq C_{p,n} \delta^{2p\theta}.
\]
\[ J_{21} \leq C_{p,n} \left( \int_0^\infty \| (S(\delta) - \mathbf{1})(-A)^{-\theta} || S(t-s)(-A)^{-\theta} || \right) ds \]
\[ \times \sum_{i=1}^n \int_0^\infty \| (S(\delta) - \mathbf{1})(-A)^{-\theta} || S(t-s)(-A)^{-\theta} || \| [f(\tilde{X}(s), \tilde{X}(s-r_i))]_{H}^{2p} ds, \]
\[ \leq C_{p,n} M_{\theta}^2 \delta^{(2p-1)} \sum_{i=1}^n \int_0^\infty (t-s)^{-\theta}(1 + \mathbb{E}[||\tilde{X}(s)||_{H}^{2p}]) ds \]
\[ \leq C_{p,n} M_{\theta}^2 \delta^{2p \theta}. \] (3.5)

For \( J_{22} \), one has
\[ J_{22} \leq C_{p,n} \sum_{i=1}^n \mathbb{E} \left[ \left( \int_t^{t+\delta} \int_U S(t+s-r_i)\tilde{X}(s), \tilde{X}(s-r_i) v ds ds \right)^{2p} \right] \]
\[ \leq C_{p,n} \delta^{2p-1} \sum_{i=1}^n \int_t^{t+\delta} \mathbb{E}[||\tilde{X}(s), \tilde{X}(s-r_i)||_{H}^{2p}] ds \]
\[ \leq C_{p,n} \delta^{2p-1} \int_t^{t+\delta} \mathbb{E}[||\tilde{X}(s), \tilde{X}(s-r_i)||_{H}^{2p}] ds \]
\[ \leq C_{p,n} \delta^{2p}. \]

Then, for \( J_3 \), by [8, Lemma 3.1], Lemma 2.1 and Lemma 3.3, we have
\[ J_3 = C_p \mathbb{E} \left[ \left( \int_0^{t+\delta} \int_U S(t+s-r_i)h(\tilde{X}(s), \tilde{X}(s-r_i))N(ds, dz) - \int_0^t \int_U S(t-s)h(\tilde{X}(s), \tilde{X}(s-r_i))N(ds, dz) \right)^{2p} \right] \]
\[ \leq C_p \mathbb{E} \left[ \left( \int_0^t \int_U (S(t-s) - S(t-s))h(\tilde{X}(s), \tilde{X}(s-r_i))N(ds, dz) \right)^{2p} \right] \]
\[ + C_p \mathbb{E} \left[ \left( \int_0^t \int_U S(t-s)h(\tilde{X}(s), \tilde{X}(s-r_i))N(ds, dz) \right)^{2p} \right] \]
\[ \leq C_p \mathbb{E} \left[ \left( \int_0^t \| (S(\delta) - \mathbf{1})S(t-s) || \right)^{2p} \int_U |h(\tilde{X}(s), \tilde{X}(s-r_i))|_{H}^{2p} V(dz) ds \right] \]
\[ + C_p \mathbb{E} \left[ \left( \int_0^t \| (S(\delta) - \mathbf{1})S(t-s) || \right)^{2p} \int_U \mathbb{E}[|h(\tilde{X}(s), \tilde{X}(s-r_i))|_{H}^{2p}] V(dz) ds \right] \]
\[ + C_p \mathbb{E} \left[ \left( \int_0^t \| (S(t-s) - S(t-s)) || \right)^{2p} \int_U |h(\tilde{X}(s), \tilde{X}(s-r_i))|_{H}^{2p} V(dz) ds \right] \]
\[ + C_p \mathbb{E} \left[ \left( \int_0^t \| (S(t-s) - S(t-s)) || \right)^{2p} \int_U \mathbb{E}[|h(\tilde{X}(s), \tilde{X}(s-r_i))|_{H}^{2p}] V(dz) ds \right] \]
\[ \leq C_{p,n} \delta^{2p \theta}. \]

Here, we need the condition \( p \in [1, \frac{1}{2\theta}) \).

Finally, putting above results together, it yields that
\[ \sup_{t \in [0, T]} \mathbb{E}[||\tilde{X}(t) + \delta - \tilde{X}(t)||_{H}^{2p}] = C_{p,n} \delta^{2p \theta}. \]

This completes the proof. \( \square \)
3.2. Proof of main result

By Hölder’s inequality and (H2), it follows from (1.5) and (3.1) that

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} |X^x(t) - \tilde{X}(t)|_H^{2p} \right] \leq \sum_{i=1}^{n} \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^t S(t-s) (f(X^x(s), X^x(s-r_i)) - f(\tilde{X}(s), \tilde{X}(s-r_i))) I_{[r^x(s)=r_i]} ds \right|^{2p} \right]
\]

\[ + C_{p,n} \sum_{i=1}^{n} \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^t S(t-s) f(\tilde{X}(s), \tilde{X}(s-r_i)) I_{[r^x(s)=r_i]} - v_i \right| ds \right|^{2p} \]

\[ + C_p \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^t \int_U S(t-s) h(X^x(s), z) \tilde{N}(ds, dz) \right|^{2p} \right] =: C_{p,n} (\Psi_1 + \sum_{i=1}^{n} \Psi_{2i}) + C_p \Psi_3.
\]

Now, let us evaluate \( \Psi_1, \Psi_3 \), by (H2), we have

\[
\Psi_1 = \sum_{i=1}^{n} \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^t S(t-s) (f(X^x(s), X^x(s-r_i)) - f(\tilde{X}(s), \tilde{X}(s-r_i))) I_{[r^x(s)=r_i]} ds \right|^{2p} \right]
\]

\[ \leq C_p \sum_{i=1}^{n} \int_0^T \mathbb{E}[|f(X^x(s), X^x(s-r_i)) - f(\tilde{X}(s), \tilde{X}(s-r_i))|^2] ds
\]

\[ \leq C_p \sum_{i=1}^{n} \int_0^T \mathbb{E}[|X^x(s) - \tilde{X}(s)|^2 + |X^x(s-r_i) - \tilde{X}(s-r_i)|^2] ds
\]

\[ \leq C_{p,n} \int_0^T \mathbb{E}[|X^x(s) - \tilde{X}(s)|^2] ds.
\]

For \( \Psi_3 \), using [8, Lemma 3.1], we get

\[
\Psi_3 \leq C_p \mathbb{E}\left[ \left( \int_0^T \int_U |h(X^x(s), z) - h(\tilde{X}(s), z)|^2 \nu(dz) ds \right)^{p/2} \right]
\]

\[ + C_p \mathbb{E}\left[ \int_0^T \int_U |h(X^x(s), z) - h(\tilde{X}(s), z)|^2 \nu(dz) ds \right]
\]

\[ \leq C_p \int_0^T \mathbb{E}[|X^x(s) - \tilde{X}(s)|^2] ds.
\]

Now, let \([s] := \lfloor \frac{s}{\varepsilon} \rfloor \varepsilon, 0 < \varepsilon < 1, \rho > 0 \) with \( \lfloor \frac{s}{\varepsilon} \rfloor \) denoting the integer part of \( \frac{s}{\varepsilon} \). Then, one has

\[ \Psi_{2i} \leq C_p \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^T S(t-s) (f(X^x(s), X^x(s-[s])) f(\tilde{X}(s), \tilde{X}(s-[s])) \right|^{2p} \right]
\]

\[ + C_{p,n} \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^T S(t-[s]) f(\tilde{X}(s), \tilde{X}(s-[s])) - f(\tilde{X}([s]), \tilde{X}([s]-r_i)) \right|^{2p} \right]
\]

\[ + C_{p,n} \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^T (S(t-[s]) - S(t-s)) f(\tilde{X}([s]), \tilde{X}([s]-r_i)) \right|^{2p} \right]
\]
\[ + C_p \mathbb{E} \left[ \sup_{t \in [0, T)} \left| \int_0^t S(t-s) \nu_i [f(\bar{X}(s), \bar{X}(s) - r_i))] - f(\bar{X}(s), \bar{X}(s) - r_i))ds \right|^{2p} \right] \]
\[ + C_p \mathbb{E} \left[ \sup_{t \in [0, T)} \left| \int_0^t S(t-[s]) f(\bar{X}(s), \bar{X}(s) - r_i)) \mathbb{I}_{[r_i(s)=r_i]} - \nu_i)ds \right|^{2p} \right] \]
\[ =: \sum_{k=1}^{s} \Pi_k(t). \]

Next, let \( t_j := j \varepsilon \rho, j = 0, 1, ..., \lfloor \frac{T}{\varepsilon \rho} \rfloor \) and \( t_{j+1} := t, \rho > 0. \) Then, similarly to the proof of \( J_{21}, \) for \( \Pi_i(t) \) and \( \Pi_3(t) \), using Hölder’s inequality, we have
\[ \Pi_i(t) + \Pi_3(t) \leq C_{p,n}(\varepsilon \rho)^{2p} \).

For \( \Pi_2(t) \) and \( \Pi_4(t) \), by (H2) and Lemma 3.4, we have
\[ \Pi_2(t) \leq C_p \mathbb{E} \left[ \sup_{t \in [0, T)} \left| \int_0^t \| S(t-[s]) f(\bar{X}(s), \bar{X}(s) - r_i)) - f(\bar{X}(s), \bar{X}(s) - r_i) \|^{2p} ds \right| \right] \]
\[ \leq C_p \mathbb{E} \left[ \sup_{t \in [0, T)} \left| \int_0^t (|\bar{X}(s)| - \bar{X}(s))^{2p} + |\bar{X}(s)| - \bar{X}(s) \|^{2p} ds \right| \right] \]
\[ \leq C_p \int_0^T \mathbb{E} \left[ |\bar{X}(s)| - \bar{X}(s) \|^{2p} ds \right] + C_p \int_0^T \mathbb{E} \left[ |\bar{X}(s)| - \bar{X}(s) \|^{2p} ds \right]. \]

Then, by the initial condition on \( \psi(\cdot) \) (1.7) and Lemma 3.4, we obtain
\[ \Pi_2(t) + \Pi_4(t) \leq C_{p,n}(\varepsilon \rho)^{2p} \).

To proceed, let us estimate \( \Pi_5(t) \). By Hölder’s inequality, (H3), Lemma 3.3 and the boundedness of \( |\mathbb{I}_{[r_i(s)=r_i]} - \nu_i| \), we have
\[ \Pi_5(t) \leq \frac{C_p}{(\varepsilon \rho)^{2p-1}} \mathbb{E} \left[ \sup_{t \in [0, T)} \left( \sum_{j=0}^{\lfloor \frac{T}{\varepsilon \rho} \rfloor} \left( \mathbb{E} \left[ \left| \int_{t_j}^{t_{j+1}} |\mathbb{I}_{[r_i(s)=r_i]} - \nu_i| ds \right|^{4p} \right] \right)^{\frac{1}{2}} \right) \right] \]
\[ \leq \frac{C_p}{(\varepsilon \rho)^{2p-1}} \mathbb{E} \left[ \sup_{t \in [0, T)} \left( \sum_{j=0}^{\lfloor \frac{T}{\varepsilon \rho} \rfloor} e^{-2p(t-t_j)} 1 + \mathbb{E} \left[ \left| \int_{t_j}^{t_{j+1}} |\mathbb{I}_{[r_i(s)=r_i]} - \nu_i| ds \right|^{4p} \right] \right)^{\frac{1}{2}} \right] \]
\[ \leq \frac{C_p}{(\varepsilon \rho)^{2p-1}} \mathbb{E} \left[ \sup_{t \in [0, T)} \left( \sum_{j=0}^{\lfloor \frac{T}{\varepsilon \rho} \rfloor} e^{-2p(t-t_j)} 1 + \mathbb{E} \left[ \left| \int_{t_j}^{t_{j+1}} |\mathbb{I}_{[r_i(s)=r_i]} - \nu_i| ds \right|^{4p} \right] \right)^{\frac{1}{2}} \right] \]
\[ \leq C_p(\varepsilon \rho)^{2p} - 1)^{-1} \max_{0 \leq j \leq \lfloor \frac{T}{\varepsilon \rho} \rfloor} \left( \mathbb{E} \left[ \left| \int_{t_j}^{t_{j+1}} |\mathbb{I}_{[r_i(s)=r_i]} - \nu_i| ds \right|^{2p} \right] \right)^{\frac{1}{2}}. \]

(3.6)

To show (3.6), we adopt an argument similar to that of Yin and Zhang [26, Theorem 7.2, page 170] and Bao, Yin and Yuan [3, Proof of Theorem 2.1, (2.24), pp. 653]. Let
\[ \eta'(u) := \frac{1}{2} \mathbb{E}\left[ \left| \int_{t_j}^{u} (\mathbf{1}_{[r^\varepsilon(s)=i]} - v_i) ds \right|^2 \right], \quad u \in [t_j, t_{j+1}]. \]

Then, it is easy to see from the chain rule that
\[ \frac{d\eta'}{du} = \mathbb{E}\left[ \int_{t_j}^{u} (\mathbf{1}_{[r^\varepsilon(u)=i]} - v_i)(\mathbf{1}_{[r^\varepsilon(s)=i]} - v_i) ds \right], \quad u \in [t_j, t_{j+1}]. \]

Let \( \bar{t}_k := k \varepsilon^\beta, k = 0, 1, \ldots, \lfloor (u - t_j)/\varepsilon^\beta \rfloor + 1 \), where \( \bar{t}_0 := t_j \) and \( \bar{t}_{\lfloor (u - t_j)/\varepsilon^\beta \rfloor + 1} =: u \). Thus, by [3, Proof of Theorem 2.1, (2.24), pp. 653], we have
\[ \mathbb{E}\left[ \left| \int_{t_j}^{t_{j+1}} (\mathbf{1}_{[r^\varepsilon(s)\leq r_i]} - v_i) ds \right|^2 \right] = O(\varepsilon^{\alpha+\beta}), \]
where \( \beta \in (\rho, 1) \). Next, by Taylor’s formula, it is easy to know \( (e^{2\rho \lambda(\varepsilon^\rho) - 1}) = O(\varepsilon^\rho) \) for sufficiently small \( \varepsilon^\rho \in (0, 1) \). Thus, we get
\[ \Pi_5(t) \leq C_p \varepsilon^{\frac{\beta}{\rho}}. \]

Finally, by Gronwall’s inequality, we have
\[ \mathbb{E}\left[ \sup_{t \in [0,T]} |X^\varepsilon(t) - \tilde{X}(t)|^2 \right] \leq C_{p,n}(\varepsilon^\rho)^{2\rho} + \varepsilon^{\frac{\beta}{\rho}}, \]
with \( \beta \in (\rho, 1) \). Thus, we have
\[ \lim_{\varepsilon \to 0} \mathbb{E}\left[ \sup_{t \in [0,T]} |X^\varepsilon(t) - \tilde{X}(t)|^2 \right] = 0, \]
This completes the proof. \( \square \)

4. Conclusions

In this paper, by focusing on SEEs with jumps and random time delays modulated by two-time-scale Markov switching processes in which both fast and slow components co-exist, we establish an averaging principle under suitable conditions. For future research, delay-dependent stability involving two-time-scale structure and jumps is both interesting and important. It deserves to be further investigated in the future.

Acknowledgments

This work was partially supported by the NSF of China (11802216) and the China Postdoctoral Science Foundation (2019M651334). The authors would like to thank the referees for their careful reading of the manuscript and for the clarification comments which lead to an improvement of the presentation of the paper.
References


©2021 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)