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Research article

Hermite-Hadamard inequality for new generalized conformable fractional operators

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Abstract: This paper is concerned to establish an advanced form of the well-known Hermite-Hadamard (HH) inequality for recently-defined Generalized Conformable (GC) fractional operators. This form of the HH inequality combines various versions (new and old) of this inequality, containing operators of the types Katugampula, Hadamard, Riemann-Liouville, conformable and Riemann, into a single form. Moreover, a novel identity containing the new GC fractional integral operators is proved. By using this identity, a bound for the absolute of the difference between the two rightmost terms in the newly-established Hermite-Hadamard inequality is obtained. Also, some relations of our results with the already existing results are presented. Conclusion and future works are presented in the last section.

Keywords: generalized conformable fractional operators; Riemann-Liouville operators; conformable integral; Hermite-Hadamard inequality
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1. Introduction and preliminaries

1.1. Fractional calculus

Fractional calculus has come out as one of the most applicable subjects of mathematics [1]. Its importance is evident from the fact that many real-world phenomena can be best interpreted and modeled using this theory. It is also a fact that many disciplines of engineering and science have been influenced by the tools and techniques of fractional calculus. Its emergence can easily be traced and linked with the famous correspondence between the two mathematicians, L'Hospital and Leibnitz, which was made on 30th September 1695. After that, many researchers tried to explore the concept of fractional calculus, which is based on the generalization of nth order derivatives or n-fold integration [2–4].
Recently, Khan and Khan [5] have discovered novel definitions of fractional integral and derivative operators. These operators enjoy interesting properties such as continuity, boundedeness, linearity etc. The integral operators, they presented, are stated as under:

Definition 1 ([5]). Let $h \in L_\theta[s,t]$(conformable integrable on $[s,t] \subseteq [0,\infty)$). The left-sided and right-sided generalized conformable fractional integrals ${}^{\tau}_{\theta}K^{\nu}_{s^+}$ and ${}^{\tau}_{\theta}K^{\nu}_{t^-}$ of order $\nu > 0$ with $\theta \in (0,1]$, $\tau \in \mathbb{R}$, $\theta + \tau \neq 0$ are defined by:

$$ {}^{\tau}_{\theta}K^{\nu}_{s^+}h(r) = \frac{1}{\Gamma(\nu)} \int_s^r \left(\frac{r^{\tau+\theta} - w^{\tau+\theta}}{\tau+\theta} \right)^{\nu-1} h(w) w^{\tau} d_\theta w, \quad r > s, \tag{1.1} $$

and

$$ {}^{\tau}_{\theta}K^{\nu}_{t^-}h(r) = \frac{1}{\Gamma(\nu)} \int_r^t \left(\frac{w^{\tau+\theta} - r^{\tau+\theta}}{\tau+\theta} \right)^{\nu-1} h(w) w^{\tau} d_\theta w, \quad t > r, \tag{1.2} $$

respectively, and ${}^{\tau}_{\theta}K^{0}_{s^+}h(r) = {}^{\tau}_{\theta}K^{0}_{t^-}h(r) = h(r)$. Here Γ denotes the well-known Gamma function.

Here the integral $\int_s^t d_\theta w$ represents the conformable integration, defined as:

$$ \int_s^t h(w) d_\theta w = \int_s^t h(w) w^{\theta-1} dw. \tag{1.3} $$

The operators defined in Definition 1 are in generalized form and contain few important operators in themselves. Here, only the left-sided operators are presented, the corresponding right-sided operators may be deduced in the similar way. Moreover, to understand the theory of conformable fractional calculus, one can see [5–7]. Also, the basic theory of fractional calculus can be found in the books [1, 8, 9] and for the latest research in this field one can see [3,4, 10–12] and the references there in.

Remark 1. 1) For $\theta = 1$ in the Definition 1, the following Katugampula fractional integral operator is obtained [13]:

$$ {}^{\tau}_{1}K^{\nu}_{s^+}h(r) = \frac{1}{\Gamma(\nu)} \int_s^r \left(\frac{r^{\tau+1} - w^{\tau+1}}{\tau + 1} \right)^{\nu-1} h(w)dw, \quad r > s. \tag{1.4} $$

2) For $\tau = 0$ in the Definition 1, the New Riemann Liouville type conformable fractional integral operator is obtained as given below:

$$ {}^{0}_{\theta}K^{\nu}_{s^+}h(r) = \frac{1}{\Gamma(\nu)} \int_s^r \left(\frac{r^{\theta} - w^{\theta}}{\theta} \right)^{\nu-1} h(w)d_{\theta}w, \quad r > s. \tag{1.5} $$

3) Using the definition of conformable integral given in (1.3) and L'Hospital rule, it is straightforward that when $\theta \to 0$ in (1.5), we get the Hadamard fractional integral operator as follows:

$$ {}^{0}_{0^+}K^{\nu}_{s^+}h(r) = \frac{1}{\Gamma(\nu)} \int_s^r \left(\log \frac{r}{w} \right)^{\nu-1} h(w)\frac{dw}{w}, \quad r > s. \tag{1.6} $$

4) For $\theta = 1$ in (1.5), the well-known Riemann-Liouville fractional integral operator is obtained as follows:

$$ {}^{0}_{1}K^{\nu}_{s^+}h(r) = \frac{1}{\Gamma(\nu)} \int_s^r (r - w)^{\nu-1} h(w)dw, \quad r > s. \tag{1.7} $$

5) For the case $\nu = 1, \tau = 0$ in Definition 1, we get the conformable fractional integrals. And when $\theta = \nu = 1$, $\tau = 0$, we get the classical Riemann integrals.

1.2. Hermite-Hadamard inequalities

This subsection is devoted to start with the definition of convex function, which plays a very important role in establishment of various kinds of inequalities [14]. This definition is given as follows [15]:

Definition 2. A function $h : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on I if the inequality

$$h(\eta s + (1-\eta)t) \leq \eta h(s) + (1-\eta)h(t) \tag{1.8}$$

holds for all $s, t \in I$ and $0 \leq \eta \leq 1$. The function h is said to be concave on I if the inequality given in (1.8) holds in the reverse direction.

Associated with the Definition 2 of convex functions the following double inequality is well-known and it has been playing a key role in various fields of science and engineering [15].

Theorem 1. *Let $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $s, t \in I$ with $s < t$. Then we have the following Hermite-Hadamard inequality:*

$$h\left(\frac{s+t}{2}\right) \leq \frac{1}{t-s}\int_s^t h(\tau)d\tau \leq \frac{h(s)+h(t)}{2}. \tag{1.9}$$

This inequality (1.9) appears in a reversed order if the function h is supposed to be concave. Also, the relation (1.9) provides upper and lower estimates for the integral mean of the convex function h. The inequality (1.9) has various versions (extensions or generalizations) corresponding to different integral operators [16–25] each version has further forms with respect to various kinds of convexities [26–32] or with respect to different bounds obtained for the absolute difference of the two leftmost or rightmost terms in the Hermite-Hadamard inequality.

By using the Riemann-Liouville fractional integral operators, Sirikaye et al. have proved the following Hermite-Hadamard inequality [33].

Theorem 2 ([33])**.** *Let $h : [s, t] \to \mathbb{R}$ be a function such that $0 \leq s < t$ and $h \in L[s, t]$. If h is convex on $[s, t]$, then the following double inequality holds:*

$$h\left(\frac{s+t}{2}\right) \leq \frac{\Gamma(\nu+1)}{2(t-s)^{\nu}}\left[{}_1^0K_{s^+}^{\nu}h(t) + {}_1^0K_{t^-}^{\nu}h(s)\right] \leq \frac{h(s)+h(t)}{2}. \tag{1.10}$$

For more recent research related to generalized Hermite-Hadamard inequality one can see [34–42] and the references therein.

Motivated from the Riemann-Liouville version of Hermite-Hadamard inequality (given above in (1.10)), we prove the same inequality for newly introduced generalized conformable fractional operators. As a result we get a more generalized inequality, containing different versions of Hermite-Hadamard inequality in single form. We also prove an identity for generalized conformable fractional operators and establish a bound for the absolute difference of two rightmost terms in the newly obtained Hermite-Hadamard inequality. We point out some relations of our results with those of other results from the past. At the end we present conclusion, where directions for future research are also mentioned.

2. Main results

In the following theorem the well-known Hermite-Hadamard inequality for the newly defined integral operators is proved.

Theorem 3. *Let $\nu > 0$ and $\tau \in \mathbb{R}, \theta \in (0,1]$ such that $\tau + \theta > 0$. Let $h : [s,t] \subseteq [0,\infty) \to \mathbb{R}$ be a function such that $h \in L_\theta[s,t]$(conformal integrable on [s,t]). If h is also a convex function on $[s,t]$, then the following Hermite-Hadamard inequality for generalized conformable fractional Integrals ${}^{\tau}_{\theta}K^{\nu}_{s^+}$ and ${}^{\tau}_{\theta}K^{\nu}_{t^-}$ holds:*

$$h\left(\frac{s+t}{2}\right) \le \frac{(\tau+\theta)^{\nu}\Gamma(\nu+1)}{4(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}\left[{}^{\tau}_{\theta}K^{\nu}_{s^+}H(t) + {}^{\tau}_{\theta}K^{\nu}_{t^-}H(s)\right] \le \frac{h(s)+h(t)}{2}, \tag{2.1}$$

where $H(x) = h(x) + \tilde{h}(x)$, $\tilde{h}(x) = h(s+t-x)$.

Proof. Let $\eta \in [0,1]$. Consider $x, y \in [s,t]$, defined by $x = \eta s + (1-\eta)t, y = (1-\eta)s + \eta t$. Since h is a convex function on $[s,t]$, we have

$$h\left(\frac{s+t}{2}\right) = h\left(\frac{x+y}{2}\right) \le \frac{h(x)+h(y)}{2} = \frac{h(\eta s + (1-\eta)t) + h((1-\eta)s + \eta t)}{2}. \tag{2.2}$$

Multiplying both sides of (2.2) by

$$\frac{(t-s)(\tau+\theta)^{1-\nu}((1-\eta)s+\eta t)^{\tau+\theta-1}}{\Gamma(\nu)[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}},$$

and integrating with respect to η, we get

$$\begin{aligned}
&\frac{(t-s)(\tau+\theta)^{1-\nu}}{\Gamma(\nu)}h\left(\frac{s+t}{2}\right)\int_0^1 \frac{((1-\eta)s+\eta t)^{\tau+\theta-1}}{[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}}d\eta \\
\le&\frac{(t-s)(\tau+\theta)^{1-\nu}}{\Gamma(\nu)}\frac{1}{2}\left\{\int_0^1 \frac{((1-\eta)s+\eta t)^{\tau+\theta-1}}{[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}}h(\eta s+(1-\eta)t)d\eta\right. \\
&+\left.\int_0^1 \frac{(1-\eta)s+\eta t)^{\tau+\theta-1}}{[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}}h((1-\eta)s+\eta t)d\eta\right\}.
\end{aligned} \tag{2.3}$$

Note that we have

$$\int_0^1 \frac{((1-\eta)s+\eta t)^{\tau+\theta-1}}{[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}}d\eta = \frac{1}{\nu(\tau+\theta)(t-s)}(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}.$$

Also, by using the identity $\tilde{h}((1-\eta)s+\eta t) = h(\eta s+(1-\eta)t)$, and making substitution $(1-\eta)s+\eta t = w$, we get

$$\begin{aligned}
&\frac{(t-s)(\tau+\theta)^{1-\nu}}{\Gamma(\nu)}\int_0^1 \frac{((1-\eta)s+\eta t)^{\tau+\theta-1}}{[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}}h(\eta s+(1-\eta)t)d\eta \\
&= \frac{(\tau+\theta)^{1-\nu}}{\Gamma(\nu)}\int_s^t \frac{w^{\tau+\theta-1}}{[t^{\tau+\theta}-w^{\tau+\theta}]^{1-\nu}}\tilde{h}(w)dw \\
&= \frac{(\tau+\theta)^{1-\nu}}{\Gamma(\nu)}\int_s^t \frac{w^{\tau}}{[t^{\tau+\theta}-w^{\tau+\theta}]^{1-\nu}}\tilde{h}(w)d_\theta w \\
&= {}^{\tau}_{\theta}K^{\nu}_{s^+}\tilde{h}(t).
\end{aligned} \tag{2.4}$$

Similarly

$$\frac{(t-s)(\tau+\theta)^{1-\nu}}{\Gamma(\nu)}\int_0^1 \frac{((1-\eta)s+\eta t)^{\tau+\theta-1}}{[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}}h(\eta t+(1-\eta)s)d\eta = {}^{\tau}_{\theta}K^{\nu}_{s^+}h(t). \tag{2.5}$$

By substituting these values in (2.3), we get

$$\frac{(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}{\Gamma(\nu+1)(\tau+\theta)^{\nu}}h\left(\frac{s+t}{2}\right) \leq \frac{{}^{\tau}_{\theta}K^{\nu}_{s^+}H(t)}{2}. \tag{2.6}$$

Again, by multiplying both sides of (2.2) by

$$\frac{(t-s)(\tau+\theta)^{1-\nu}((1-\eta)s+\eta t)^{\tau+\theta-1}}{\Gamma(\nu)[((1-\eta)s+\eta t)^{\tau+\theta}-s^{\tau+\theta}]^{1-\nu}},$$

and then integrating with respect to η and by using the same techniques used above, we can obtain:

$$\frac{(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}{\Gamma(\nu+1)(\tau+\theta)^{\nu}}h\left(\frac{s+t}{2}\right) \leq \frac{{}^{\tau}_{\theta}K^{\nu}_{t^-}H(s)}{2}. \tag{2.7}$$

Adding (2.7) and (2.6), we get:

$$h\left(\frac{s+t}{2}\right) \leq \frac{\Gamma(\nu+1)(\tau+\theta)^{\nu}}{4(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}\left[{}^{\tau}_{\theta}K^{\nu}_{s^+}H(t)+{}^{\tau}_{\theta}K^{\nu}_{t^-}H(s)\right]. \tag{2.8}$$

Hence the left-hand side of the inequality (2.1) is established.

Also since h is convex, we have:

$$h(\eta s+(1-\eta)t)+h((1-\eta)s+\eta t) \leq h(s)+h(t). \tag{2.9}$$

Multiplying both sides

$$\frac{(t-s)(\tau+\theta)^{1-\nu}((1-\eta)s+\eta t)^{\tau+\theta-1}}{\Gamma(\nu)[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}},$$

and integrating with respect to η we get

$$\begin{aligned}&\frac{(t-s)(\tau+\theta)^{1-\nu}}{\Gamma(\nu)}\int_0^1 \frac{((1-\eta)s+\eta t)^{\tau+\theta-1}}{[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}}h(\eta s+(1-\eta)t)d\eta\\ &+\frac{(t-s)(\tau+\theta)^{1-\nu}}{\Gamma(\nu)}\int_0^1 \frac{((1-\eta)s+\eta t)^{\tau+\theta-1}}{[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}}h(\eta t+(1-\eta)s)d\eta \qquad (2.10)\\ \leq&\frac{(t-s)(\tau+\theta)^{1-\nu}}{\Gamma(\nu)}[h(s)+h(t)]\int_0^1 \frac{(1-\eta)s+\eta t)^{\tau+\theta-1}}{[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^{1-\nu}}d\eta,\end{aligned}$$

that is,

$$ {}^{\tau}_{\theta}K^{\nu}_{s^+}H(t) \leq \frac{(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}{\Gamma(\nu+1)(\tau+\theta)^{\nu}}[h(s)+h(t)]. \qquad (2.11)$$

Similarly multiplying both sides of (2.9) by

$$\frac{(t-s)(\tau+\theta)^{1-\nu}((1-\eta)s+\eta t)^{\tau+\theta-1}}{\Gamma(\nu)[((1-\eta)s+\eta t)^{\tau+\theta}-s^{\tau+\theta}]^{1-\nu}},$$

and integrating with respect to η, we can obtain

$$ {}^{\tau}_{\theta}K^{\nu}_{t^-}H(s) \leq \frac{(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}{\Gamma(\nu+1)(\tau+\theta)^{\nu}}[h(s)+h(t)]. \qquad (2.12)$$

Adding the inequalities (2.11) and (2.12), we get:

$$\frac{\Gamma(\nu+1)(\tau+\theta)^{\nu}}{4(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}\left[{}^{\tau}_{\theta}K^{\nu}_{t^-}H(s)+{}^{\tau}_{\theta}K^{\nu}_{s^+}H(t)\right] \leq \frac{h(s)+h(t)}{2}. \qquad (2.13)$$

Combining (2.8) and (2.13), we get the required result. □

The inequality in (2.1) is in compact form containing few inequalities for different integrals in it. The following remark tells us about that fact.

Remark 2. **1)** For $\theta = 1$ in (2.1), we get Hermite-Hadamard inequality for Katugampola fractional integral operators, as follows [38]:

$$h\left(\frac{s+t}{2}\right) \leq \frac{(\tau+1)^{\nu}\Gamma(\nu+1)}{4(t^{\tau+1}-s^{\tau+1})^{\nu}}\left[{}_{1}^{\tau}K_{s^+}^{\nu}H(t) + {}_{1}^{\tau}K_{t^-}^{\nu}H(s)\right] \leq \frac{h(s)+h(t)}{2}, \tag{2.14}$$

where $H(x) = h(x) + \tilde{h}(x)$, $\tilde{h}(x) = h(s+t-x)$.
2) For $\tau = 0$ in (2.1), we get Hermite-Hadamard inequality for newly obtained Riemann Liouville type conformable fractional integral operators, as follows:

$$h\left(\frac{s+t}{2}\right) \leq \frac{\theta^{\nu}\Gamma(\nu+1)}{4(t^{\theta}-s^{\theta})^{\nu}}\left[{}_{\theta}^{0}K_{s^+}^{\nu}H(t) + {}_{\theta}^{0}K_{t^-}^{\nu}H(s)\right] \leq \frac{h(s)+h(t)}{2}, \tag{2.15}$$

where $H(x) = h(x) + \tilde{h}(x)$, $\tilde{h}(x) = h(s+t-x)$.
3) For $\tau+\theta \to 0$, in (2.1), applying L'Hospital rule and the relation (1.3), we get Hermite-Hadamard inequality for Hadamard fractional integral operators, as follows:

$$h\left(\frac{s+t}{2}\right) \leq \frac{\Gamma(\nu+1)}{2(\ln\frac{t}{s})^{\nu}}\left[{}_{0^+}^{0}K_{s^+}^{\nu}h(t) + {}_{0^+}^{0}K_{t^-}^{\nu}h(s)\right] \leq \frac{h(s)+h(t)}{2}. \tag{2.16}$$

4) For $\tau + \theta = 1$ in (2.1), the Hermite-Hadamard inequality is obtained for Riemann-Liouville fractional integrals [33]:

$$h\left(\frac{s+t}{2}\right) \leq \frac{\Gamma(\nu+1)}{2(t-s)^{\nu}}\left[{}_{1}^{0}K_{s^+}^{\nu}h(t) + {}_{1}^{0}K_{t^-}^{\nu}h(s)\right] \leq \frac{h(s)+h(t)}{2}. \tag{2.17}$$

5) For the case $\nu = 1, \tau = 0$ in (2.1), the Hermite-Hadamard inequality is obtained for the conformable fractional integrals as follows:

$$h\left(\frac{s+t}{2}\right) \leq \frac{\theta}{2(t^{\theta}-s^{\theta})}\int_{s}^{t} H(w)d_{\theta}w \leq \frac{h(s)+h(t)}{2}. \tag{2.18}$$

6) When $\theta = \nu = 1$, $\tau = 0$ the Hermite-Hadamard inequality is obtained for classical Riemann integrals [15]:

$$h\left(\frac{s+t}{2}\right) \leq \frac{1}{t-s}\int_{s}^{t} h(w)dw \leq \frac{h(s)+h(t)}{2}. \tag{2.19}$$

To bound the difference of two rightmost terms in the main inequality (2.1), we need to establish the following Lemma.

Lemma 1. *Let* $\tau+\theta>0$ *and* $\nu>0$*. If* $h\in L_\theta[s,t]$ *, then*

$$\begin{aligned}&\frac{h(s)+h(t)}{2}-\frac{(\tau+\theta)^\nu\Gamma(\nu+1)}{4(t^{\tau+\theta}-s^{\tau+\theta})^\nu}\left[{}^{\tau}_{\theta}K^{\nu}_{s^+}H(t)+{}^{\tau}_{\theta}K^{\nu}_{t^-}H(s)\right]\\&=\frac{t-s}{4(t^{\tau+\theta}-s^{\tau+\theta})^\nu}\int_0^1\Delta^\nu_{\tau+\theta}(\eta)h'(\eta s+(1-\eta)t)d\eta,\end{aligned}\tag{2.20}$$

where

$$\begin{aligned}\Delta^\nu_{\tau+\theta}(\eta)&=[(\eta s+(1-\eta)t)^{\tau+\theta}-s^{\tau+\theta}]^\nu-[(\eta t+(1-\eta)s)^{\tau+\theta}-s^{\tau+\theta}]^\nu\\&+[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^\nu-[t^{\tau+\theta}-((1-\eta)t+\eta s)^{\tau+\theta}]^\nu.\end{aligned}$$

Proof. With the help of integration by parts, we have

$$\begin{aligned}{}^{\tau}_{\theta}K^{\nu}_{s^+}H(t)&=\frac{(t^{\tau+\theta}-s^{\tau+\theta})^\nu}{(\tau+\theta)^\nu\Gamma(\nu+1)}H(s)\\&+\frac{(t-s)^\nu}{(\tau+\theta)^\nu\Gamma(\nu+1)}\int_0^1[t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^\nu H'(\eta t+(1-\eta)s)d\eta.\end{aligned}\tag{2.21}$$

Similarly, we have

$$\begin{aligned}{}^{\tau}_{\theta}K^{\nu}_{t^-}H(s)&=\frac{(t^{\tau+\theta}-s^{\tau+\theta})^\nu}{(\tau+\theta)^\nu\Gamma(\nu+1)}H(t)\\&-\frac{(t-s)^\nu}{(\tau+\theta)^\nu\Gamma(\nu+1)}\int_0^1[((1-\eta)s+\eta t)^{\tau+\theta}-s^{\tau+\theta}]^\nu H'(\eta t+(1-\eta)s)d\eta.\end{aligned}\tag{2.22}$$

Using (2.21) and (2.22) we have

$$\begin{aligned}&\frac{4(t^{\tau+\theta}-s^{\tau+\theta})^\nu}{t-s}\left(\frac{h(s)+h(t)}{2}-\frac{(\tau+\theta)^\nu\Gamma(\nu+1)}{4(t^{\tau+\theta}-s^{\tau+\theta})^\nu}\left[{}^{\tau}_{\theta}K^{\nu}_{t^-}H(s)+{}^{\tau}_{\theta}K^{\nu}_{s^+}H(t)\right]\right)\\&=\int_0^1\left([((1-\eta)s+\eta t)^{\tau+\theta}-s^{\tau+\theta}]^\nu-[(t^{\tau+\theta}-((1-\eta)s+\eta t)^{\tau+\theta}]^\nu\right)H'(\eta t+(1-\eta)s)d\eta.\end{aligned}\tag{2.23}$$

Also, we have

$$H'(\eta t+(1-\eta)s)=h'(\eta t+(1-\eta)s)-h'(\eta s+(1-\eta)t),\quad \eta\in[0,1].\tag{2.24}$$

And

$$\int_0^1 [((1-\eta)s+\eta t)^{\tau+\theta} - s^{\tau+\theta}]^{\nu} H'(\eta t+(1-\eta)s)d\eta$$
$$= \int_0^1 [((1-\eta)t+\eta s)^{\tau+\theta} - s^{\tau+\theta}]^{\nu} h'(\eta s+(1-\eta)t)d\eta \tag{2.25}$$
$$- \int_0^1 [((1-\eta)s+\eta t)^{\tau+\theta} - s^{\tau+\theta}]^{\nu} h'(\eta s+(1-\eta)t)d\eta.$$

Also, we have

$$\int_0^1 [t^{\tau+\theta} - ((1-\eta)s+\eta t)^{\tau+\theta}]^{\nu} H'(\eta t+(1-\eta)s)d\eta$$
$$= \int_0^1 [t^{\tau+\theta} - ((1-\eta)t+\eta s)^{\tau+\theta}]^{\nu} h'(\eta s+(1-\eta)t)d\eta \tag{2.26}$$
$$- \int_0^1 [t^{\tau+\theta} - ((1-\eta)s+\eta t)^{\tau+\theta}]^{\nu} h'(\eta s+(1-\eta)t)d\eta.$$

Using (2.23), (2.25) and (2.26) we get the required result. □

Remark 3. When $\tau+\theta = 1$ in Lemma 1, we get the Lemma 2 in [33].

Definition 3. For $\nu > 0$, we define the operators

$$\Omega_1^{\nu}(x,y,\tau+\theta) = \int_s^{\frac{s+t}{2}} |x-w||y^{\tau+\theta}-w^{\tau+\theta}|^{\nu} dw - \int_{\frac{s+t}{2}}^{t} |x-w||y^{\tau+\theta}-w^{\tau+\theta}|^{\nu} dw, \tag{2.27}$$

and

$$\Omega_2^{\nu}(x,y,\tau+\theta) = \int_s^{\frac{s+t}{2}} |x-w||w^{\tau+\theta}-y^{\tau+\theta}|^{\nu} dw - \int_{\frac{s+t}{2}}^{t} |x-w||w^{\tau+\theta}-y^{\tau+\theta}|^{\nu} dw, \tag{2.28}$$

where $x, y \in [s,t] \subseteq [0,\infty)$ and $\tau+\theta > 0$.

Theorem 4. *Let h be a conformable integrable function over* $[s,t]$ *such that* $|h'|$ *is convex function. Then for* $\nu>0$ *and* $\tau+\theta>0$ *we have:*

$$\begin{aligned}&\left|\frac{h(s)+h(t)}{2}-\frac{(\tau+\theta)^{\nu}\Gamma(\nu+1)}{4(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}\left[{}_{\theta}^{\tau}K_{s^+}^{\nu}H(t)+{}_{\theta}^{\tau}K_{t^-}^{\nu}H(s)\right]\right|\\&\leq\frac{K_{\tau+\theta}^{\nu}(s,t)}{4(t-s)(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}(|h'(s)|+|h'(t)|),\end{aligned}\tag{2.29}$$

where $K_{\tau+\theta}^{\nu}(s,t)=\Omega_1^{\nu}(t,t,\tau+\theta)+\Omega_2^{\nu}(s,s,\tau+\theta)-\Omega_2^{\nu}(t,s,\tau+\theta)-\Omega_1^{\nu}(s,t,\tau+\theta)$.

Proof. Using Lemma 1 and convexity of $|h'|$, we have:

$$\begin{aligned}&\left|\frac{h(s)+h(t)}{2}-\frac{(\tau+\theta)^{\nu}\Gamma(\nu+1)}{4(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}\left[{}_{\theta}^{\tau}K_{s^+}^{\nu}H(t)+{}_{\theta}^{\tau}K_{t^-}^{\nu}H(s)\right]\right|\\&\leq\frac{t-s}{4(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}\int_0^1|\Delta_{\tau+\theta}^{\nu}(\eta)||h'(\eta s+(1-\eta)t)|d\eta\\&\leq\frac{t-s}{4(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}}\left(|h'(s)|\int_0^1\eta\left|\Delta_{\tau+\theta}^{\nu}(\eta)\right|d\eta+|h'(t)|\int_0^1(1-\eta)\left|\Delta_{\tau+\theta}^{\nu}(\eta)\right|d\eta\right).\end{aligned}\tag{2.30}$$

Here $\int_0^1\eta\left|\Delta_{\tau+\theta}^{\nu}(\eta)\right|d\eta=\frac{1}{(t-s)^2}\int_s^t|\psi(u)|(t-u)du$,
and $\psi(u)=(u^{\tau+\theta}-s^{\tau+\theta})^{\nu}-((t+s-u)^{\tau+\theta}-s^{\tau+\theta})^{\nu}+(t^{\tau+\theta}-(s+t-u)^{\tau+\theta})^{\nu}-(t^{\tau+\theta}-u^{\tau+\theta})^{\nu}$.
We observe that ψ is a nondecreasing function on $[s,t]$. Moreover, we have:

$$\psi(s)=-2(t^{\tau+\theta}-s^{\tau+\theta})^{\nu}<0,$$

and also $\psi\left(\frac{s+t}{2}\right)=0$. As a consequence, we have

$$\begin{cases}\psi(u)\leq 0, & if\quad s\leq u\leq\frac{s+t}{2},\\ \psi(u)>0, & if\quad \frac{s+t}{2}<u\leq t.\end{cases}$$

Thus we get

$$\begin{aligned}\int_0^1\eta\left|\Delta_{\tau+\theta}^{\nu}(\eta)\right|d\eta&=\frac{1}{(t-s)^2}\int_s^t|\psi(u)|(t-u)du\\&=\frac{1}{(t-s)^2}\left[-\int_s^{\frac{s+t}{2}}\psi(u)(t-u)du+\int_{\frac{s+t}{2}}^t\psi(u)(t-u)du\right]\\&=\frac{1}{(t-s)^2}\left[K_1+K_2+K_3+K_4\right],\end{aligned}\tag{2.31}$$

where

$$K_1 = -\int_s^{\frac{s+t}{2}} (t-u)(u^{\tau+\theta} - s^{\tau+\theta})^\nu du + \int_{\frac{s+t}{2}}^t (t-u)(u^{\tau+\theta} - s^{\tau+\theta})^\nu du, \tag{2.32}$$

$$K_2 = \int_s^{\frac{s+t}{2}} (t-u)((t+s-u)^{\tau+\theta} - s^{\tau+\theta})^\nu du - \int_{\frac{s+t}{2}}^t (t-u)((t+s-u)^{\tau+\theta} - s^{\tau+\theta})^\nu du, \tag{2.33}$$

$$K_3 = -\int_s^{\frac{s+t}{2}} (t-u)(t^{\tau+\theta} - (s+t-u)^{\tau+\theta})^\nu du + \int_{\frac{s+t}{2}}^t (t-u)(t^{\tau+\theta} - (s+t-u)^{\tau+\theta})^\nu du, \tag{2.34}$$

and

$$K_4 = \int_s^{\frac{s+t}{2}} (t-u)(t^{\tau+\theta} - u^{\tau+\theta})^\nu du - \int_{\frac{s+t}{2}}^t (t-u)(t^{\tau+\theta} - u^{\tau+\theta})^\nu du. \tag{2.35}$$

We can see here that $K_1 = -\Omega_2^\nu(t, s, \tau+\theta)$, $K_4 = \Omega_1^\nu(t, t, \tau+\theta)$.

Also, by using of change of the variables $v = s + t - u$, we get

$$K_2 = \Omega_2^\nu(s, s, \tau+\theta),\ K_3 = -\Omega_1^\nu(s, t, \tau+\theta). \tag{2.36}$$

By substituting these values in (2.31), we get

$$\int_0^1 \eta \Delta_{\tau+\theta}^\nu(\eta) d\eta = \frac{-\Omega_2^\nu(t, s, \tau+\theta) + \Omega_1^\nu(t, t, \tau+\theta) + \Omega_2^\nu(s, s, \tau+\theta) - \Omega_1^\nu(s, t, \tau+\theta)}{(t-s)^2}. \tag{2.37}$$

Similarly, we can find

$$\int_0^1 (1-\eta) \Delta_{\tau+\theta}^\nu(\eta) d\eta = \frac{\Omega_2^\nu(s, s, \tau+\theta) - \Omega_2^\nu(t, s, \tau+\theta) + \Omega_1^\nu(t, t, \tau+\theta) - \Omega_1^\nu(s, t, \tau+\theta)}{(t-s)^2}. \tag{2.38}$$

Finally, by using (2.30), (2.37) and (2.38) we get the required result. □

Remark 4. when $\tau + \theta = 1$ in (2.29), we obtain

$$\left| \frac{h(s) + h(t)}{2} - \frac{\Gamma(\nu+1)}{2(t-s)^\nu} \left[{}_1^0K_{t^-}^\nu h(s) + {}_1^0K_{s^+}^\nu h(t) \right] \right| \leq \frac{(t-s)}{2(\nu+1)} \left(1 - \frac{1}{2^\nu} \right) \left[h'(s) + h'(t) \right],$$

which is Theorem 3 in [33].

3. Conclusion and future works

A generalized version of Hermite-Hadamard inequality via newly introduced GC fractional operators has been acquired successfully. This result combines several versions (new and old) of the Hermite-Hadamard inequality into a single form, each one has been discussed by fixing parameters in the newly established version of the Hermite-Hadamard inequality. Moreover, an identity containing the GC fractional integral operators has been proved. By using this identity, a bound for the absolute of the difference between the two rightmost terms in the newly established Hermite-Hadamard inequality has been presented. Also, some relations of our results with those of already existing results have been pointed out. Since this is a fact that there exist more than one definitions for fractional derivatives [2] which makes it difficult to choose a convenient operator for solving a given problem. Thus, in the present paper, the GC fractional operators (containing various previously defined fractional operators into a single form) have been used in order to overcome the problem of choosing a suitable fractional operator and to provide a unique platform for researchers working with different operators in this field. Also, by making use of GC fractional operators one can follow the research work which has been performed for the two versions (1.9) and (1.10) of Hermite-Hadamard inequality.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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